I. S. GRADSHTEYN
I. M. RYZHIK

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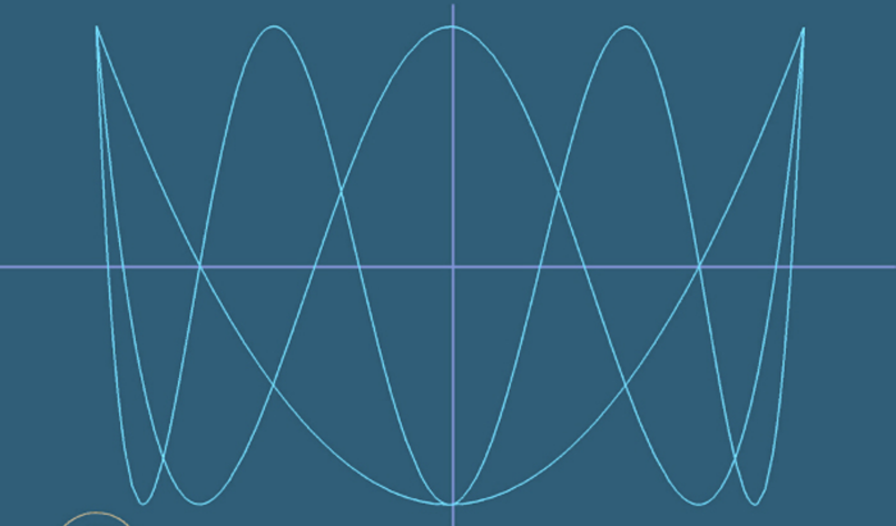
TABLE OF INTEGRALS, SERIES, AND PRODUCTS

SEVENTH EDITION



Edited by Alan Jeffrey and Daniel Zwillinger

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I.S. Gradshteyn and I.M. Ryzhik

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Preface to the Seventh Edition

Since the publication in 2000 of the completely reset sixth edition of Gradshteyn and Ryzhik, users of the reference work have continued to submit corrections, new results that extend the work, and suggestions for changes that improve the presentation of existing entries. It is a matter of regret to us that the structure of the book makes it impossible to acknowledge these individual contributions, so, as usual, the names of the many new contributors have been added to the acknowledgment list at the front of the book.

This seventh edition contains the corrections received since the publication of the sixth edition in 2000, together with a considerable amount of new material acquired from isolated sources. Following our previous conventions, an amended entry has a superscript "11" added to its entry reference number, where the equivalent superscript number for the sixth edition was "10." Similarly, an asterisk on an entry's reference number indicates a new result. When, for technical reasons, an entry in a previous edition has been removed, to preserve the continuity of numbering between the new and older editions the subsequent entries have not been renumbered, so the numbering will jump.

We wish to express our gratitude to all who have been in contact with us with the object of improving and extending the book, and we want to give special thanks to Dr. Victor H. Moll for his interest in the book and for the many contributions he has made over an extended period of time. We also wish to acknowledge the contributions made by Dr. Francis J. O'Brien Jr. of the Naval Station in Newport, in particular for results involving integrands where exponentials are combined with algebraic functions.

Experience over many years has shown that each new edition of Gradshteyn and Ryzhik generates a fresh supply of suggestions for new entries, and for the improvement of the presentation of existing entries and errata. In view of this, we do not expect this new edition to be free from errors, so all users of this reference work who identify errors, or who wish to propose new entries, are invited to contact the authors, whose email addresses are listed below. Corrections will be posted on the web site `www.az-tec.com/gr/errata`.

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The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x)\,dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula. We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one. Sometimes, several complicated formulas were thereby reduced to a single, simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of Chapter Two and the Newton–Leibniz formula, or to an integral of the form

$$\int_{-a}^{a} f(x)\,dx,$$

where $f(x)$ is an odd function. In such cases, the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\pi/4} \frac{(\cot x - 1)^{p-1}}{\sin^2 x} \ln\tan x\,dx = -\frac{\pi}{p}\operatorname{cosec} p\pi. \tag{0.1}$$

By making the natural substitution $u = \cot x - 1$, we obtain

$$\int_0^{\infty} u^{p-1} \ln(1+u)\,du = \frac{\pi}{p}\operatorname{cosec} p\pi. \tag{0.2}$$

Integrals similar to formula (0.1) are omitted in this new edition. Instead, we have formula (0.2).

As a second example, let us take

$$I = \int_0^{\pi/2} \ln\left(\tan^p x + \cot^p x\right) \ln \tan x \, dx = 0.$$

The substitution $u = \tan x$ yields

$$I = \int_0^{\infty} \frac{\ln\left(u^p + u^{-p}\right) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{ve^v}{1 + e^{2v}} \ln\left(e^{pv} + e^{-pv}\right) \, dv = \int_{-\infty}^{\infty} v \, \frac{\ln\left(2\cosh pv\right)}{2\cosh v} \, dv.$$

The integrand is odd, and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows: First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite integrals.)
7. The inverse trigonometric functions.

Then follow the special functions:

8. Elliptic integrals.
9. Elliptic functions.
10. The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.
11. Probability integrals and Fresnel's integrals.
12. The gamma function and related functions.
13. Bessel functions.
14. Mathieu functions.
15. Legendre functions.
16. Orthogonal polynomials.
17. Hypergeometric functions.
18. Degenerate hypergeometric functions.
19. Parabolic cylinder functions.
20. Meijer's and MacRobert's functions.
21. Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine function in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* We shall distinguish between all these functions and those listed above, and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x+\cos x}{\sin x-\cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic operators (expressions of the form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions [whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$] are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \tan x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear [in the order determined by the position of $\varphi_2(x)$ in the list] after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions are the trigonometric and power functions [that is, $\varphi_2(x) = x$]. Then come

- combinations of trigonometric and exponential functions,
- combinations of trigonometric functions, exponential functions, and powers, etc.,
- combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function $e^{\frac{1}{x}}$ comes after e^x as regards complexity, but $\ln x$ and $\ln \dfrac{1}{x}$ are equally complex since $\ln \dfrac{1}{x} = -\ln x$. In the section on "powers and algebraic functions," polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

*For any natural number n, the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

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Use of the Tables*

For the effective use of the tables contained in this book, it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled *The Order of Presentation of the Formulas* (see page xxvii) and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function, and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Gradshteyn and Ryzhik's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals, and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are α, β, γ, δ, t, u, z, z_k, and Δ. The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k)$, $E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second, and third kinds.

The four inverse hyperbolic functions $\operatorname{arcsinh} z$, $\operatorname{arccosh} z$, $\operatorname{arctanh} z$, and $\operatorname{arccoth} z$ are introduced through the definitions

$$
\begin{aligned}
\arcsin z &= \frac{1}{i} \operatorname{arcsinh}(iz) \\
\arccos z &= \frac{1}{i} \operatorname{arccosh}(z) \\
\arctan z &= \frac{1}{i} \operatorname{arctanh}(iz) \\
\operatorname{arccot} z &= i \operatorname{arccoth}(iz)
\end{aligned}
$$

*Prepared by Alan Jeffrey for the English language edition.

or

$$\begin{aligned}\operatorname{arcsinh} z &= \frac{1}{i}\arcsin(iz)\\ \operatorname{arccosh} z &= i\arccos z\\ \operatorname{arctanh} z &= \frac{1}{i}\arctan(iz)\\ \operatorname{arccoth} z &= \frac{1}{i}\operatorname{arccot}(-iz)\end{aligned}$$

The numerical constants $\boldsymbol{C}$ and $\boldsymbol{G}$ which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant $\boldsymbol{C}$ is defined by the limit

$$\boldsymbol{C} = \lim_{s\to\infty}\left(\sum_{m=1}^{s}\frac{1}{m} - \ln s\right) = 0.577215\ldots.$$

On occasion, other writers denote Euler's constant by the symbol γ, but this is also often used instead to denote the constant

$$\gamma = e^{\boldsymbol{C}} = 1.781072\ldots.$$

Catalan's constant $\boldsymbol{G}$ is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

by the expression

$$\boldsymbol{G} = \frac{1}{2}\int_0^1 \mathbf{K}\,dk = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2} = 0.915965\ldots.$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the *Index of Special Functions and Notation* on page xxxix, and by then referring to the defining formula or section number listed there. We now present a brief discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n, $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \qquad \text{for } n = 0, 1, \ldots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots.$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n\, E_n\left(\frac{1}{2}\right) \qquad \text{for } n = 0, 1, \ldots$$

The E_n are all integral, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, ...

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^*, uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \ldots.$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66,$$
$$B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \quad \ldots.$$

These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, E_k(2x) \qquad n = 0, 1, \ldots$$

$$E_{n-1}(x) = \frac{2^n}{n} \left\{ B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right\}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n\, B_n\left(\frac{x}{2}\right) \right\} \qquad n = 1, 2, \ldots$$

and

$$E_{n-2}(x) = 2\binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} \left(2^{n-k} - 1\right) B_{n-k}\, B_n(x) \qquad n = 2, 3, \ldots$$

There are also alternative definitions of the Euler polynomial of order n, and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1}\right) \left\{ B_n(x) - 2^n\, B_n\left(\frac{x}{2}\right) \right\}$$

the Euler polynomial of order n.

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u} \qquad \operatorname{nc} u = \frac{1}{\operatorname{cn} u} \qquad \operatorname{nd} u = \frac{1}{\operatorname{dn} u}$$
$$\operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u}$$
$$\operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}$$

The elliptic integral of the third kind is defined by Gradshteyn and Ryzhik to be

$$\begin{aligned}\Pi\left(\varphi, n^2, k\right) &= \int_0^{\varphi} \frac{da}{\left(1-n^2\sin^2 a\right)\sqrt{1-k^2\sin^2 a}} \\ &= \int_0^{\sin\varphi} \frac{dx}{\left(1-n^2x^2\right)\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}\end{aligned} \qquad \left(-\infty < n^2 < \infty\right)$$

The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\operatorname{zn}(u,k)$, frequently written $Z(u)$, is defined by the relation

$$\operatorname{zn}(u,k) = Z(u) = \int_0^u \left\{\operatorname{dn}^2 v - \frac{\boldsymbol{E}}{\boldsymbol{K}}\right\} dv = \boldsymbol{E}(u) - \frac{\boldsymbol{E}}{\boldsymbol{K}}u.$$

This is related to the theta functions by the relationship

$$\operatorname{zn}(u,k) = \frac{\partial}{\partial u}\ln\Theta(u)$$

giving

(i). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_1'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}$$

(ii). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_2'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u}$$

(iii). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_3'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_3\left(\frac{\pi u}{2\boldsymbol{K}}\right)} - k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

(iv). $$\operatorname{zn}(u,k) = \frac{\pi}{2\boldsymbol{K}} \frac{\vartheta_4'\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ.

The Factorial (Gamma) Function

In older reference texts, the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt,$$

is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions, the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d\,(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -\boldsymbol{C} + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$\Psi'(z+1) = \Psi'(z) - \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^\infty e^{-zt} t^{-n}\, dt \qquad (n = 0, 1, \ldots, \quad \operatorname{Re} z > 0)\,.$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $\operatorname{Ei}(z)$ through the expressions

$$E_1(z) = -\operatorname{Ei}(-z) = \int_z^\infty e^{-t} t^{-1}\, dt$$

and

$$\operatorname{li}(z) = \int_0^z \frac{dt}{\ln t} = \operatorname{Ei}(\ln z) \qquad [z > 1]\,.$$

The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \left\{ e^{-z} - z\, E_{n-1}(z) \right\} \qquad [n > 1]$$

and

$$E_n'(z) = -\,E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \cdots \right\} \qquad \left[|\arg z| < \frac{3\pi}{2} \right]$$

while for large n,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n\left(6x^2 - 8nx + n^2\right)}{(x+n)^6} + R(n,x) \right\},$$

where

$$-0.36 n^{-4} \le R(n,x) \le \left(1 + \frac{1}{x+n-1} \right) n^{-4} \qquad [x > 0]\,.$$

The sine and cosine integrals $\operatorname{si}(x)$ and $\operatorname{ci}(x)$ are related to the functions $\operatorname{Si}(x)$ and $\operatorname{Ci}(x)$ by the integrals

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt = \operatorname{si}(x) + \frac{\pi}{2}$$

and

$$\operatorname{Ci}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t}\, dt.$$

The hyperbolic sine and cosine integrals $\operatorname{shi}(x)$ and $\operatorname{chi}(x)$ are defined by the relations

$$\operatorname{shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt$$

and

$$\operatorname{chi}(x) = \boldsymbol{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t}\, dt.$$

Some authors write

$$\operatorname{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t}\, dt$$

so that

$$\operatorname{Cin}(x) = -\operatorname{Ci}(x) + \ln x + \boldsymbol{C}.$$

The error function $\operatorname{erf}(x)$ is defined by the relation

$$\operatorname{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and the complementary error function $\operatorname{erfc}(x)$ is related to the error function $\operatorname{erfc}(x)$ and to $\Phi(x)$ by the expression

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Gradshteyn and Ryzhik as

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2\, dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2\, dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2}\, dt, \qquad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2}\, dt,$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\, dt, \qquad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\, dt.$$

These are related by the expressions

$$S(x) = S_1\left(x\sqrt{\frac{2}{\pi}}\right) = S_2\left(x^2\right)$$

and

$$C(x) = C_1\left(x\sqrt{\frac{2}{\pi}}\right) = C_2\left(x^2\right)$$

Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $He_n(x)$ by the relations

$$He_n(x) = 2^{-n/2}\, H_n\left(\frac{x}{\sqrt{2}}\right)$$

and

$$H_n(x) = 2^{n/2}\, He_n\left(x\sqrt{2}\right).$$

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x\frac{d H_n}{dx} + 2n H_n = 0$$

and

$$\frac{d^2 He_n}{dx^2} - x\frac{d He_n}{dx} + n He_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2x H_n - 2n H_{n-1}$$

and

$$He_{n+1} = x He_n - n He_{n-1}.$$

The first six orthogonal polynomials He_n are

$He_0 = 1,\quad He_1 = x,\quad He_2 = x^2 - 1,\quad He_3 = x^3 - 3x,\quad He_4 = x^4 - 6x^2 + 3,\quad He_5 = x^5 - 10x^3 + 15x.$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involve the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n}\Gamma(n+1)\, J_n(z).$$

In the book by Gray, Mathews, and MacRobert, the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi\, Y_n(z) + (\ln 2 - \mathbf{C})\, J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $Hs_\nu(z)$ and $Hi_\nu(z)$, and some authors write $G_\nu(z) = \left(\frac{1}{2}\right)\pi i\, H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n+1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z)\, O_0(t) + 2\sum_{k=1}^{\infty} J_k(z)\, O_k(t),$$

giving

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{[n/2]}\frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \ldots,$$

where $\left[\frac{1}{2}n\right]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1)\, O_{n+1}(t) + (n+1)\, O_{n-1}(t) - \frac{2\left(n^2-1\right)}{t}\, O_n(t) = \frac{2n}{t}\sin^2\frac{n\pi}{2}.$$

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\mathrm{Ai}(z) = \frac{1}{3}\sqrt{z}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\} = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}\left(\frac{2}{3}z^{3/2}\right)$$

$$\mathrm{Ai}(-z) = \frac{1}{3}\sqrt{z}\left\{J_{1/3}\left(\frac{2}{3}z^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}$$

and by

$$\mathrm{Bi}(z) = \sqrt{\frac{z}{3}}\left\{I_{-1/3}\left(\frac{2}{3}z^{3/2}\right) + I_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\},$$

$$\mathrm{Bi}(-z) = \sqrt{\frac{z}{3}}\left\{J_{-1/3}\left(\frac{2}{3}z^{3/2}\right) - J_{1/3}\left(\frac{2}{3}z^{3/2}\right)\right\}.$$

Parabolic Cylinder Functions and Whittaker Functions

The differential equation

$$\frac{d^2y}{dz^2} + \left(az^2 + bz + c\right)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left(\frac{1}{4}z^2 + a\right)y = 0 \text{ and } \frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0,$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0$$

are sometimes written $U(a,z)$ and $V(a,z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a,z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a,z) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + a\right)\left\{D_{-a-\frac{1}{2}}(-z) + (\sin\pi a)\,D_{-a-\frac{1}{2}}(z)\right\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Gradshteyn and Ryzhik is

$$\frac{d^2y}{dz^2} + \left(a - 2k^2\cos 2z\right)y = 0 \qquad \text{with } k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ, λ and h^2, and b and $c = 2\sqrt{q}$, respectively. The periodic solutions $\mathrm{se}_n(z,q)$ and $\mathrm{ce}_n(z,q)$ and the modified periodic solutions $\mathrm{Se}_n(z,q)$ and $\mathrm{Ce}_n(z,q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: *Tables Relating to Mathieu Functions.* National Bureau of Standards, Columbia University Press, New York, 1951.

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$F_\Lambda(\alpha:\beta_1,\dots,\beta_n; \gamma_1,\dots\dots,\gamma_n : z_1,\dots,z_n)$	Hypergeometric function of several variables	9.19
F_1, F_2, F_3, F_4	Hypergeometric functions of two variables	9.18
$\left\{ \begin{array}{c} \mathrm{fe}_n(z,q), \mathrm{Fe}_n(z,q)\dots \\ \mathrm{Fey}_n(z,q), \mathrm{Fek}_n(z,q)\dots \end{array} \right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$\boldsymbol{G}$	Catalan constant	9.73
g_2, g_3	Invariants of the $\wp(u)$-function	8.161
$\mathrm{gd}\, x$	Gudermannian	1.49
$\left\{ \begin{array}{c} \mathrm{ge}_n(z,q), \mathrm{Ge}_n(z,q) \\ \mathrm{Gey}_n(z,q), \mathrm{Gek}_n(z,q) \end{array} \right\}$	Other nonperiodic solutions of Mathieu's equation	8.64, 8.663
$G^{m,n}_{p,q}\left(x \,\middle\vert\, \begin{array}{l} a_1,\dots,a_p \\ b_1,\dots,b_q \end{array}\right)$	Meijer's functions	9.3

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Notation	Name of the function and the number of the formula containing its definition	
$h(n)$	Unit integer function	18.1
$\mathrm{hei}_\nu(z), \quad \mathrm{her}_\nu(z)$	Thomson functions	8.56
$H_\nu^{(1)}(z), \quad H_\nu^{(2)}(z)$	Hankel functions of the first and second kinds	8.405, 8.42
$H(u) = \vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)$	Theta function	8.192
$H_1(u) = \vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)$	Theta function	8.192
$H_n(z)$	Hermite polynomials	8.95
$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Bessel functions of an imaginary argument	8.406, 8.43
$I_x(p,q)$	Normalized incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger function	8.58
$\mathrm{k}_\nu(x)$	Bateman's function	9.210 3
$\mathbf{K}(\mathbf{k}) = \boldsymbol{K}, \quad \mathbf{K}(\mathbf{k}') = \boldsymbol{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Bessel functions of imaginary argument	8.407, 8.43
$\mathrm{kei}(z), \quad \mathrm{ker}(z)$	Thomson functions	8.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\mathrm{li}(x)$	Logarithm integral	8.24
$M_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$O_n(x)$	Neumann's polynomials	8.59
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_\nu(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$	Riemann's differential equation	9.160
$P_n^{(\alpha,\beta)}(x)$	Jacobi's polynomials	8.96
$Q_\nu^\mu(z), \quad Q_\nu^\mu(x)$	Associated Legendre functions of the second kind	8.7, 8.8
$Q_\nu(z), \quad Q_\nu(x)$	Legendre functions of the second kind	8.82, 8.83
$S(x)$	Fresnel sine integral	8.25
$S_n(x)$	Schläfli's polynomials	8.59
$s_{\mu,\nu}(z), \quad S_{\mu,\nu}(z)$	Lommel functions	8.57
$\mathrm{se}_{2n+1}(z,q), \quad \mathrm{se}_{2n+2}(z,q)$	Periodic Mathieu functions	8.61
$\mathrm{Se}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+2}(z,q)$	Mathieu functions of an imaginary argument	8.63
$\mathrm{shi}(x)$	Hyperbolic sine integral	8.22
$\mathrm{si}(x)$	Sine integral	8.23
$\mathrm{sn}\, u$	Sine amplitude	8.14
$T_n(x)$	Chebyshev polynomial of the 1st kind	8.94
$U_n(x)$	Chebyshev polynomials of the 2nd kind	8.94

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Notation	Name of the function and the number of the formula containing its definition	
$U_\nu(w,z), \quad V_\nu(w,z)$	Lommel functions of two variables	8.578
$W_{\lambda,\mu}(z)$	Whittaker functions	9.22, 9.23
$Y_\nu(z)$	Neumann functions	8.403, 8.41
$Z_\nu(z)$	Bessel functions	8.401
$\mathfrak{Z}_\nu(z)$	Bessel functions	

Notation

Symbol	Meaning
$\lfloor x \rfloor$	The integral part of the real number x (also denoted by $[x]$)
$\int_a^{(b+)} \quad \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a, proceeding along the original path in the opposite direction.
$\int_C$	Line integral along the curve C
PV $\int$	Principal value integral
$\overline{z} = x - iy$	The complex conjugate of $z = x + iy$
$n!$	$= 1 \cdot 2 \cdot 3 \ldots n, \qquad 0! = 1$
$(2n+1)!!$	$= 1 \cdot 3 \ldots (2n+1)$. (double factorial notation)
$(2n)!!$	$= 2 \cdot 4 \ldots (2n)$. (double factorial notation)
$0!! = 1$ and $(-1)!! = 1$	(cf. 3.372 for $n = 0$)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$)
$\binom{p}{n}$	$= \dfrac{p(p-1)\ldots(p-n+1)}{1 \cdot 2 \ldots n} = \dfrac{p!}{n!(p-n)!}, \quad \binom{p}{0} = 1, \quad \binom{p}{n} = \dfrac{p!}{n!(p-n)!}$ $[n = 1, 2, \ldots, p \geq n]$
$\binom{x}{n}$	$= x(x-1)\ldots(x-n+1)/n!$ $[n = 0, 1, \ldots]$
$(a)_n$	$= a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (Pochhammer symbol)
$\sum_{k=m}^{n} u_k$	$= u_m + u_{m+1} + \ldots + u_n$. If $n < m$, we define $\sum_{k=m}^{n} u_k = 0$
$\sum_n{}', \quad \sum_{m,n}{}'$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$\sum, \quad \prod$	An empty $\sum$ has value 0, and an empty $\prod$ has value 1

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Symbol	Meaning
$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$	Kronecker delta
τ	Theta function parameter (cf. 8.18)
$\times$ and $\wedge$	Vector product (cf. 10.11)
$\cdot$	Scalar product (cf. 10.11)
∇ or "del"	Vector operator (cf. 10.21)
∇^2	Laplacian (cf. 10.31)
$\sim$	Asymptotically equal to
$\arg z$	The argument of the complex number $z = x + iy$
curl or rot	Vector operator (cf. 10.21)
div	Vector operator (divergence) (cf. 10.21)
$\mathcal{F}$	Fourier transform (cf. 17.21)
$\mathcal{F}_c$	Fourier cosine transform (cf. 17.31)
$\mathcal{F}_s$	Fourier sine transform (cf. 17.31)
grad	Vector operator (gradient) (cf. 10.21)
h_i and g_{ij}	Metric coefficients (cf. 10.51)
$^{\mathrm{H}}$	Hermitian transpose of a vector or matrix (cf. 13.123)
$\mathrm{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$	Heaviside step function
$\operatorname{Im} z \equiv y$	The imaginary part of the complex number $z = x + iy$
k	The letter k (when not used as an index of summation) denotes a number in the interval $[0, 1]$. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1-k^2}$ is denoted by k'.
$\mathcal{L}$	Laplace transform (cf. 17.11)
$\mathcal{M}$	Mellin transform (cf. 17.41)
$\mathbb{N}$	The natural numbers $(0, 1, 2, \dots)$
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0. If there exists an $M > 0$ such that $\lvert g(z)\rvert \leq M\lvert f(z)\rvert$ in some sufficiently small neighborhood of the point z_0, we write $g(z) = O(f(z))$.

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Symbol	Meaning
q	The nome, a theta function parameter (cf. 8.18)
$\mathbb{R}$	The real numbers
$R(x)$	A rational function
$\operatorname{Re} z \equiv x$	The real part of the complex number $z = x + iy$
S_n^m	Stirling number of the first kind (cd. 9.74)
$\mathfrak{S}_n^m$	Stirling number of the second kind (cd. 9.74)
$\operatorname{sign} x = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$	The sign (signum) of the real number x
$^{\mathrm{T}}$	Transpose of a vector or matrix (cf. 13.115)
$\mathbb{Z}$	The integers $(0, \pm 1, \pm 2, \dots)$
Z_b	Bilateral z transform (cf. 18.1)
Z_u	Unilateral z transform (cf. 18.1)

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Note on the Bibliographic References

The letters and numbers following equations refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography beginning on page 1141. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example, we may use the reference to equation 3.354–5:

ET I 118 (1) *a*

The key on page 1141 indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms.*

The Roman numeral denotes volume one of the work; 118 is the page on which the formula will be found; (1) refers to the number of the formula in this source; and the *a* indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases, the editors have used Russian editions of works published in other languages. Under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

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0 Introduction

0.1 Finite Sums

0.11 Progressions

0.111 Arithmetic progression.

$$\sum_{k=0}^{n-1}(a+kr) = \frac{n}{2}[2a+(n-1)r] = \frac{n}{2}(a+l) \qquad [l = a+(n-1)r \text{ is the last term}]$$

0.112 Geometric progression.

$$\sum_{k=1}^{n} aq^{k-1} = \frac{a\left(q^n-1\right)}{q-1} \qquad [q \neq 1]$$

0.113 Arithmetic-geometric progression.

$$\sum_{k=0}^{n-1}(a+kr)q^k = \frac{a-[a+(n-1)r]q^n}{1-q} + \frac{rq\left(1-q^{n-1}\right)}{(1-q)^2}$$

$$[q \neq 1, \quad n > 1]$$ JO (5)

0.114[8] $$\sum_{k=1}^{n-1} k^2x^k = \frac{\left(-n^2+2n-1\right)x^{n+2} + \left(2n^2-2n-1\right)x^{n+1} - n^2x^n + x^2 + x}{(1-x)^3}$$

0.12 Sums of powers of natural numbers

0.121 $$\sum_{k=1}^{n} k^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2}\binom{q}{1}B_2n^{q-1} + \frac{1}{4}\binom{q}{3}B_4n^{q-3} + \frac{1}{6}\binom{q}{5}B_6n^{q-5} + \cdots$$

$$= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720}n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240}n^{q-5} - \cdots$$

[last term contains either n or n^2] CE 332

1. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ CE 333

2. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$ CE 333

3. $$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$ CE 333

4. $$\sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$$ CE 333

5. $$\sum_{k=1}^{n} k^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1)$$ CE 333

6. $$\sum_{k=1}^{n} k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$ CE 333

7. $$\sum_{k=1}^{n} k^7 = \frac{1}{24} n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$$ CE 333

0.122 $$\sum_{k=1}^{n} (2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2}\binom{q}{1} 2^{q-1} B_2 n^{q-1} - \frac{1}{4}\binom{q}{3} 2^{q-3}\left(2^3-1\right) B_4 n^{q-3} - \cdots$$

[last term contains either n or n^2.]

1. $$\sum_{k=1}^{n} (2k-1) = n^2$$

2. $$\sum_{k=1}^{n} (2k-1)^2 = \frac{1}{3} n(4n^2-1)$$ JO (32a)

3. $$\sum_{k=1}^{n} (2k-1)^3 = n^2(2n^2-1)$$ JO (32b)

4.[11] $$\sum_{k=1}^{n} (mk-1) = \frac{n}{2}[m(n+1)-2]$$

5.[10] $$\sum_{k=1}^{n} (mk-1)^2 = \frac{1}{6} n[m^2(n+1)(2n+1)-6m(n+1)+6]$$

6.[10] $$\sum_{k=1}^{n} (mk-1)^3 = \frac{1}{4} n[m^3 n(n+1)^2 - 2m^2(n+1)(2n+1)+6m(n+1)-4]$$

0.123 $$\sum_{k=1}^{n} k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

0.124

1. $$\sum_{k=1}^{q} k\left(n^2-k^2\right) = \frac{1}{4} q(q+1)\left(2n^2-q^2-q\right)$$ $[q = 1, 2, \ldots]$

2.[10] $$\sum_{k=1}^{n} k(k+1)^3 = \frac{1}{60} n(n+1)\left(12n^3+63n^2+107n+58\right)$$

0.125 $$\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1$$ AD (188.1)

0.126 $$\sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}}\, K_{n+\frac{1}{2}}\left(\frac{1}{2}\right)$$ WA 94

0.13 Sums of reciprocals of natural numbers

0.131[11] $$\sum_{k=1}^{n} \frac{1}{k} = \boldsymbol{C} + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1)\dots(n+k-1)},$$ JO (59), AD (1876)

where

$$A_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)(3-x)\cdots(k-1-x)\,dx$$
$$A_2 = \frac{1}{12}, \qquad A_3 = \frac{1}{12}$$
$$A_4 = \frac{19}{120}, \qquad A_5 = \frac{9}{20},$$

0.132[7] $$\sum_{k=1}^{n} \frac{1}{2k-1} = \frac{1}{2}\left(\boldsymbol{C} + \ln n\right) + \ln 2 + \frac{B_2}{8n^2} + \frac{\left(2^3-1\right)B_4}{64n^4} + \dots$$ JO (71a)a

0.133 $$\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$ JO (184f)

0.14 Sums of products of reciprocals of natural numbers

1. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)} = \frac{n}{p(p+nq)}$$ GI III (64)a

2. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)[p+(k+1)q]} = \frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]}$$ GI III (65)a

3. $$\sum_{k=1}^{n} \frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]}$$
$$= \frac{1}{(l+1)q}\left\{\frac{1}{p(p+q)\dots(p+lq)} - \frac{1}{(p+nq)[p+(n+1)q]\dots[p+(n+l)q]}\right\}$$ AD (1856)a

4. $$\sum_{k=1}^{n} \frac{1}{[1+(k-1)q][1+(k-l)q+p]} = \frac{1}{p}\left[\sum_{k=1}^{n} \frac{1}{1+(k-1)q} - \sum_{k=1}^{n} \frac{1}{1+(k-1)q+p}\right]$$ GI III (66)a

0.142 $$\sum_{k=1}^{n} \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!}$$ JO (157)

0.15 Sums of the binomial coefficients

Notation: n is a natural number.

1. $$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}$$ KR 64 (70.1)

2. $$1 + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}$$ KR 62 (58.1)

3. $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots=2^{n-1}$ KR 62 (58.1)

4. $\sum_{k=0}^{m}(-1)^k\binom{n}{k}=(-1)^m\binom{n-1}{m}$ $[n\geq 1]$ KR 64 (70.2)

0.152

1. $\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$ KR 62 (59.1)

2. $\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right)$ KR 62 (59.2)

3. $\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-4)\pi}{3}\right)$ KR 62 (59.3)

0.153

1. $\binom{n}{0}+\binom{n}{4}+\binom{n}{8}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right)$ KR 63 (60.1)

2. $\binom{n}{1}+\binom{n}{5}+\binom{n}{9}+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right)$ KR 63 (60.2)

3. $\binom{n}{2}+\binom{n}{6}+\binom{n}{10}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right)$ KR 63 (60.3)

4. $\binom{n}{3}+\binom{n}{7}+\binom{n}{11}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right)$ KR 63 (60.4)

0.154

1. $\sum_{k=0}^{n}(k+1)\binom{n}{k}=2^{n-1}(n+2)$ $[n\geq 0]$ KR 63 (66.1)

2. $\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k}=0$ $[n\geq 2]$ KR 63 (66.2)

3. $\sum_{k=0}^{N}(-1)^k\binom{N}{k}k^{n-1}=0$ $[N\geq n\geq 1;\quad 0^0\equiv 1]$

4. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^n=(-1)^n n!$ $[n\geq 0;\quad 0^0\equiv 1]$

5. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha+k)^n=(-1)^n n!$ $[n\geq 0;\quad 0^0\equiv 1]$

6. $\sum_{k=0}^{N}(-1)^k\binom{N}{k}(\alpha+k)^{n-1}=0$ $[N\geq n\geq 1,\quad 0^0\equiv 1\quad N,\,n\in N^+]$

0.155

1. $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k}=\frac{n}{n+1}$ KR 63 (67)

2. $$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$ KR 63 (68.1)

3. $$\sum_{k=0}^{n} \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1}-1}{n+1}$$ KR 63 (68.2)

4. $$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^{n} \frac{1}{m}$$ KR 64 (69)

0.156

1. $$\sum_{k=0}^{p} \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p}$$ [m is a natural number] KR 64 (71.1)

2. $$\sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}$$ KR 64 (71.2)

0.157

1. $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$ KR 64 (72.1)

2. $$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$$ KR 64 (72.2)

3. $$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0$$ KR 64 (72.3)

4. $$\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}$$ KR 64 (72.4)

0.158[10]

1. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k = 4^n - \binom{2n}{n}$$

2. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^2 = 4^n - \binom{2n}{n} 3 \cdot 4^n$$

3. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^3 = (6n+13)4^n - 18n \binom{2n}{n}$$

4. $$\sum_{k=1}^{n} \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^4 = (32n^2+104n) \binom{2n}{n} - (60n+75)4^n$$

0.159[10]

1. $$\sum_{k=0}^{n} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k = \frac{1}{2} \left[4^n - \binom{2n}{n} \right]$$

2. $$\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^2=\frac{1}{2}\left[(2n+1)\binom{2n}{n}-4^n\right]$$

3. $$\sum_{k=0}^{n}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]k^3=\frac{(3n+2)}{4}\cdot 4^n-\frac{1}{2}\binom{2n}{n}(3n+1)$$

0.160[10]

1. $$\sum_{k=n+1}^{2n}\binom{2n}{k}\alpha^k+\frac{1}{2}\binom{2n}{n}\alpha^n+\frac{(1+\alpha)^{2n-1}(1-\alpha)}{2}\sum_{k=0}^{n-1}\binom{2k}{k}\left[\frac{\alpha}{(1+\alpha)^2}\right]^k=\frac{1}{2}(1+\alpha)^{2n}$$

2. $$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{\Gamma(r+b)}{\Gamma(r+a)}=\frac{\mathrm{B}(n+a-b,b)}{\Gamma(a-b)}$$

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

The series

0.211 $$\sum_{k=1}^{\infty}u_k=u_1+u_2+u_3+\dots$$

is said to *converge absolutely* if the series

0.212 $$\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\cdots,$$

composed of the absolute values of its terms converges. If the series **0.211** converges and the series **0.212** diverges, the series **0.211** is said to *converge conditionally*. Every absolutely convergent series converges.

0.22 Convergence tests

Suppose that

$$\lim_{k\to\infty}|u_k|^{1/k}=q$$

If $q<1$, the series **0.211** converges absolutely. On the other hand, if $q>1$, the series **0.211** diverges. (Cauchy)

0.222 Suppose that

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|=q$$

Here, if $q<1$, the series **0.211** converges absolutely. If $q>1$, the series **0.211** diverges. If $\left|\frac{u_{k+1}}{u_k}\right|$ approaches 1 but remains greater than unity, then the series **0.211** diverges. (d'Alembert)

0.223 Suppose that

$$\lim_{k\to\infty}k\left\{\left|\frac{u_k}{u_{k+1}}\right|-1\right\}=q$$

Here, if $q>1$, the series **0.211** converges absolutely. If $q<1$, the series **0.211** diverges. (Raabe)

0.224 Suppose that $f(x)$ is a positive decreasing function and that

$$\lim_{k\to\infty} \frac{e^k f\left(e^k\right)}{f(k)} = q$$

for natural k. If $q < 1$, the series $\sum_{k=1}^{\infty} f(k)$ converges. If $q > 1$, this series diverges. (Ermakov)

0.225 Suppose that

$$\left|\frac{u_k}{u_{k+1}}\right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

where $p > 1$ and the $|v_k|$ are bounded, that is, the $|v_k|$ are all less than some M, which is independent of k. Here, if $q > 1$, the series **0.211** converges absolutely. If $q \le 1$, this series diverges. (Gauss)

0.226 Suppose that a function $f(x)$ defined for $x \ge q \ge 1$ is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges accordingly as the integral

$$\int_q^{\infty} f(x)\, dx$$

converges or diverges (the Cauchy integral test).

0.227 Suppose that all terms of a sequence $u_1, u_2, \ldots, u_n$ are positive. In such a case, the series

1. $$\sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \ldots$$

is called an *alternating series.*

If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

2. $u_{k+1} < u_k$ and $\lim\limits_{k\to\infty} u_k = 0,$

the series **0.227** 1 converges. Here, the remainder of the series is

3. $$\sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left|\sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^{n} (-1)^{k+1} u_k < u_{n+1}\right|$$ (Leibniz)

0.228 If the series

1. $$\sum_{k=1}^{\infty} v_k = v_1 + v_2 + \ldots + v_k + \ldots$$

converges and the numbers u_k form a monotonic bounded sequence, that is, if $|u_k| < M$ for some number M and for all k, the series

2. $$\sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \ldots + u_k v_k + \ldots$$ FI II 354

converges. (Abel)

0.229 If the partial sums of the series **0.228** 1 are bounded and if the numbers u_k constitute a monotonic sequence that approaches zero, that is, if

$$\left|\sum_{k=1}^{n} v_k\right| < M \qquad [n = 1, 2, \ldots] \qquad \text{and } \lim_{k\to\infty} u_k = 0,$$ FI II 355

then the series **0.228** 2 converges (Dirichlet).

0.23–0.24 Examples of numerical series

0.231 Progressions

1. $$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} \qquad [|q| < 1]$$

2. $$\sum_{k=0}^{\infty} (a+kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} \qquad [|q| < 1]$$ (cf. **0.113**)

0.232

1. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = \ln 2$$ (cf. **1.511**)

2. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{2k-1} = 1 - 2\sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4}$$ (cf. **1.643**)

3.* $$\sum_{k=1}^{\infty} \frac{k^a}{b^k} = \frac{1}{(b-1)^{a+1}}\sum_{i=1}^{a}\left[\frac{1}{b^{a-i}}\sum_{j=0}^{i}\frac{(-1)^j(a+1)!(i-j)^a}{j!(a+1-j)!}\right]$$ $$[a = 1, 2, 3, \ldots, \quad b \neq 1]$$

0.233

1. $$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots = \zeta(p) \qquad [\operatorname{Re} p > 1]$$ WH

2. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k^p} = \left(1-2^{1-p}\right)\zeta(p) \qquad [\operatorname{Re} p > 0]$$ WH

3.[10] $$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \qquad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$ FI II 721

4. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k^{2n}} = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}|$$ JO (165)

5. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2\cdot(2n)!}|B_{2n}|$$ JO (184b)

6. $$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}|E_{2n}|$$ JO (184d)

0.234

1. $$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2} = \frac{\pi^2}{12}$$ EU

2. $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$ EU

3. $$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = \boldsymbol{G}$$ FI II 482

4. $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}$$ EU

5. $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{\pi^4}{96}$$ EU

6. $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^5} = \frac{5\pi^5}{1536}$$ EU

7. $$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2$$

8.[6] $$\sum_{k=1}^{\infty}\frac{1}{k(2k+1)} = 2 - 2\ln 2$$

9.* $$\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n^2\,\Gamma\left(n\right)} = \sqrt{\pi}\ln 4$$

0.235 $$S_n = \sum_{k=1}^{\infty}\frac{1}{\left(4k^2-1\right)^n}$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2-8}{16}, \quad S_3 = \frac{32-3\pi^2}{64}, \quad S_4 = \frac{\pi^4+30\pi^2-384}{768}$$

JO (186)

0.236

1. $$\sum_{k=1}^{\infty}\frac{1}{k\left(4k^2-1\right)} = 2\ln 2 - 1$$ BR 51a

2. $$\sum_{k=1}^{\infty}\frac{1}{k\left(9k^2-1\right)} = \frac{3}{2}\left(\ln 3 - 1\right)$$ BR 51a

3. $$\sum_{k=1}^{\infty}\frac{1}{k\left(36k^2-1\right)} = -3 + \frac{3}{2}\ln 3 + 2\ln 2$$ BR 52, AD (6913.3)

4. $$\sum_{k=1}^{\infty}\frac{k}{\left(4k^2-1\right)^2} = \frac{1}{8}$$ BR 52

5. $$\sum_{k=1}^{\infty}\frac{1}{k\left(4k^2-1\right)^2} = \frac{3}{2} - 2\ln 2$$ BR 52

6. $$\sum_{k=1}^{\infty} \frac{12k^2-1}{k\left(4k^2-1\right)^2} = 2\ln 2$$ AD (6917.3), BR 52

7.[6] $$\sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2\ln 2$$

0.237

1. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}$$ AD (6917.2), BR 52

2. $$\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}$$

3. $$\sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4}$$ [cf. **0.133**],

4. $$\sum_{k=1,k\neq m}^{\infty}{}' \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2}$$ [m is an integer] AD (6916.1)

5. $$\sum_{k=1,k\neq m}^{\infty}{}' \frac{(-1)^{k-1}}{(m-k)(m+k)} = \frac{3}{4m^2}$$ [m is an even number] AD (6916.2)

0.238

1. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2}$$ GI III (93)

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2}(1-\ln 2)$$ GI III (94)a

3. $$\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4}\ln 3 + \frac{\pi}{12\sqrt{3}}$$ GI III (95)

0.239

1.[11] $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-2} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2\right)$$ GI III (85), BR* 161 (1)

2.[7] $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} - \ln 2\right)$$ BR* 161 (1)

3. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}}\left[\pi + 2\ln\left(\sqrt{2}+1\right)\right]$$ BR* 161 (1)

4. $$\sum_{k=1}^{\infty} (-1)^{\left[\frac{k+3}{2}\right]} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2}\ln 2$$ GI III (87)

5. $$\sum_{k=1}^{\infty}(-1)^{\left[\frac{k+3}{2}\right]}\frac{1}{2k-1}=\frac{\pi}{2\sqrt{2}}$$

6. $$\sum_{k=1}^{\infty}(-1)^{\left[\frac{k+5}{3}\right]}\frac{1}{2k-1}=\frac{5\pi}{12}$$ GI III (88)

7. $$\sum_{k=1}^{\infty}\frac{1}{(8k-1)(8k+1)}=\frac{1}{2}-\frac{\pi}{16}\left(\sqrt{2}+1\right)$$

0.241

1. $$\sum_{k=1}^{\infty}\frac{1}{2^k k}=\ln 2$$ JO (172g)

2. $$\sum_{k=1}^{\infty}\frac{1}{2^k k^2}=\frac{\pi^2}{12}-\frac{1}{2}(\ln 2)^2$$ JO (174)

3.[11] $$\sum_{n=0}^{\infty}\binom{2n}{n}p^n=\frac{1}{\sqrt{1-4p}} \qquad \left[0\le p<\tfrac{1}{4}\right]$$

4.[10] $$\sum_{n=1}^{\infty}\frac{p^n}{n^2}=\frac{\pi^2}{6}-\int_1^p\frac{\ln(1-x)}{x}\,dx \qquad [0\le p\le 1]$$

5.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j=4^i-\binom{2i}{i}$$
$$\left[\binom{n}{m}=0, \quad m<0\right]$$

6.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^2=4i\binom{2i}{i}-3\cdot 4^i$$
$$\left[\binom{n}{m}=0, \quad m<0\right]$$

7.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^3=(6i+13)4^i-18i\binom{2i}{i}$$
$$\left[\binom{n}{m}=0, \quad m<0\right]$$

8.[10] $$\sum_{j=1}^{i}\left[2^j\binom{2i-j}{i-j}-2^{j+1}\binom{2i-(j+1)}{i-(j+1)}\right]j^4=\left(32i^2+104i\right)\binom{2i}{i}-(60i+75)4^i$$

9.[10] $$\sum_{j=n+1}^{2n}\binom{2n}{j}k^j+\frac{1}{2}\binom{2n}{n}k^n+\frac{(1+k)^{2n-1}(1-k)}{2}\sum_{i=0}^{n-1}\binom{2i}{i}\left[\frac{k}{(1+k)^2}\right]^i=\frac{1}{2}(1+k)^{2n}$$

10.[10] $$\sum_{k=0}^{i}\binom{i+k}{k}2^{i-k}=4^i$$

11.[10] $$\sum_{k=0}^{i} \binom{i+k}{h}^{i-k} k = (i+1)4^i - (2i+1)\binom{2i}{i}$$

12.[10] $$\sum_{k=0}^{i} \binom{2i}{k} = \frac{1}{2}\left[4^i + \binom{2i}{i}\right]$$

13.[10] $$\sum_{k=0}^{i} \binom{2i}{k} k = \frac{i}{2} 4^i$$

14.[10] $$\sum_{k=0}^{i} \binom{2i}{k} k^2 = (2i+1)i4^{i-1} - \frac{i^2}{2}\binom{2i}{i}$$

0.242 $$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{2k}} = \frac{n^2}{n^2+1} \qquad [n > 1]$$

0.243

1. $$\sum_{k=1}^{\infty} \frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]} = \frac{1}{(l+1)q} \frac{1}{p(p+q)\dots(p+lq)}$$

(see also **0.141** 3)

2.[7] $$\sum_{k=1}^{\infty} \frac{x^{k-1}}{[p+(k-1)q][p+(k-1)q+1][p+(k-1)q+2]\dots[p+(k-1)q+l]} = \frac{1}{l!}\int_0^i \frac{t^{p-1}(1-t)^t}{1-xt^q}\,dt$$

$[p > 0, \quad x^2 < 1]$ BR* 161 (2), AD (6.704)

3. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\left(\frac{1}{x}\tanh\left[\frac{(2k+1)\pi x}{2}\right] + x\tanh\left[\frac{(2k+1)\pi}{2x}\right]\right) = \frac{\pi^3}{16}$$

0.244

1. $$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p}\int_0^1 \frac{x^p - x^q}{1-x}\,dx \qquad [p > -1, \quad q > -1, \quad p \neq q]$$ GI III (90)

2. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{p+(k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q}\,dt \qquad [p > 0, \quad q > 0]$$ BR* 161 (1)

3.[10] $$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p}\sum_{m=p+1}^{q} \frac{1}{m} \qquad [q > p > -1, \quad p \text{ and } q \text{ integers}]$$

Summations of reciprocals of factorials

0.245

1. $$\sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828\dots$$

2.[11] $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{2e} \approx 0.1839397\dots$$

3. $$\sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787\ldots$$

4. $$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1$$

5. $$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2}\left(e + \frac{1}{e}\right) = 1.54308\ldots$$

6. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2}\left(e - \frac{1}{e}\right) = 1.17520\ldots$$

7. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030\ldots$$

8. $$\sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147\ldots$$

0.246

1. $$\sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530\ldots$$

2. $$\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855\ldots$$

3. $$\sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2)$$

4. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078\ldots$$

5. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} = J_1(2) = 0.57672481\ldots$$

6. $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} = J_n(2)$$

0.247 $$\sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2)\cdot(n-1)!}$$

0.248 $$\sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$S_1 = e, \qquad S_2 = 2e, \qquad S_3 = 5e, \qquad S_4 = 15e$$
$$S_5 = 52e, \qquad S_6 = 203e, \qquad S_7 = 877e, \qquad S_8 = 4140e$$

0.249[7] $$\sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e$$

0.25 Infinite products

0.250 Suppose that a sequence of numbers $a_1, a_2, \dots, a_k, \dots$ is given. If the limit $\lim_{n\to\infty} \prod_{k=1}^{n} (1 + a_k)$ exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product* $\prod_{k=1}^{\infty} (1 + a_k)$, and we write

1. $$\lim_{n\to\infty} \prod_{k=1}^{n} (1 + a_k) = \prod_{k=1}^{\infty} (1 + a_k)$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no a_k is equal to -1. FI II 400

0.251 For the infinite product **0.250** 1. to converge, it is necessary that $\lim_{k\to\infty} a_k = 0$. FI II 403

0.252 If $a_k > 0$ or $a_k < 0$ for all values of the index k starting with some particular value, then, for the product **0.250** 1 to converge, it is necessary and sufficient that the series $\sum_{k=1}^{\infty} a_k$ converge.

0.253 The product $\prod_{k=1}^{\infty} (1 + a_k)$ is said to converge absolutely if the product $\prod_{k=1}^{\infty} (1 + |a_k|)$ converges. FI II 403

0.254 Absolute convergence of an infinite product implies its convergence.

0.255 The product $\prod_{k=1}^{\infty} (1 + a_k)$ converges absolutely if, and only if, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. FI II 406

0.26 Examples of infinite products

0.261 $$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}$$ EU

0.262

1. $$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$$ FI II 401

2. $$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi}$$ FI II 401

3. $$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4}$$ FI II 401

0.263

1. $$e = \frac{2}{1} \cdot \left(\frac{4}{3}\right)^{1/2} \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{1/4} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{1/8} \dots$$

2.* $$e = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \dots$$

3.* $$\frac{\pi}{2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/4} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/8} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/16} \cdots$$

where the n^{th} factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^{n} (k+1)^{(-1)^{k+1}\binom{n}{k}}$.

0.264

1. $$e^{\boldsymbol{C}} = \prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}}$$ FI II 402

2.* $$e^{\boldsymbol{C}} = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \cdots$$

where the n^{th} factor is the $(n+1)^{\text{th}}$ root of the product $\prod_{k=0}^{n} (k+1)^{(-1)^{k+1}\binom{n}{k}}$. Here $\boldsymbol{C}$ is the Euler constant, denoted in other works by γ.

0.265 $$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$ FI II 402

0.266[8] $$\prod_{k=0}^{\infty} \left(1 + x^{2^k}\right) = \frac{1}{1-x} \qquad [0 < x < 1]$$ FI II 401

0.3 Functional Series

0.30 Definitions and theorems

0.301 The series

1. $$\sum_{k=1}^{\infty} f_k(x),$$

the terms of which are functions, is called a *functional series*. The set of values of the independent variable x for which the series **0.301** 1 converges constitutes what is called the *region of convergence* of that series.

0.302 A series that converges for all values of x in a region M is said to *converge uniformly* in that region if, for every $\varepsilon \geq 0$, there exists a number N such that, for $n > N$, the inequality

$$\left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \varepsilon$$

holds for *all* x in M.

0.303 If the terms of the functional series **0.301** 1 satisfy the inequalities:

$$|f_k(x)| < u_k \quad (k = 1, 2, 3, \ldots),$$

throughout the region M, where the u_k are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \ldots + u_k + \ldots,$$

the series **0.301** 1 converges uniformly in M. (Weierstrass)

0.304 Suppose that the series **0.301** 1 converges uniformly in a region M and that a set of functions $g_k(x)$ constitutes (for each x) a monotonic sequence, and that these functions are uniformly bounded; that is, suppose that a number L exists such that the inequalities

1. $$|g_n(x)| \le L$$

hold for all n and x. Then, the series

2. $$\sum_{k=1}^{\infty} f_k(x) g_k(x)$$

converges uniformly in the region M. (Abel) FI II 451

0.305 Suppose that the partial sums of the series **0.301** 1 are uniformly bounded; that is, suppose that, for some L and for all n and x in M, the inequalities

$$\left| \sum_{k=1}^{n} f_k(x) \right| < L$$

hold. Suppose also that for each x the functions $g_n(x)$ constitute a monotonic sequence that approaches zero uniformly in the region M. Then, the series **0.304** 2 converges uniformly in the region M. (Dirichlet) FI II 451

0.306[6] If the functions $f_k(x)$ (for $k = 1, 2, 3, \ldots$) are integrable on the interval $[a, b]$ and if the series **0.301** 1 made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\, dx \qquad [a \le x \le b]$$ FI II 459

0.307 Suppose that the functions $f_k(x)$ (for $k = 1, 2, 3, \ldots$) have continuous derivatives $f_k'(x)$ on the interval $[a, b]$. If the series **0.301** 1 converges on this interval and if the series $\sum_{k=1}^{\infty} f_k'(x)$ of these derivatives converges uniformly, the series **0.301** 1 may be differentiated termwise; that is,

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f_k'(x)$$ FI II 460

0.31 Power series

0.311 A functional series of the form

1. $$\sum_{k=0}^{\infty} a_k (x - \xi)^k = a_0 + a_1(x - \xi) + a_2(x - \xi)^2 + \ldots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point ξ and a radius equal to R; at every interior point of this circle, the power series **0.311** 1 converges absolutely, and outside this circle, it diverges. This circle is called the *circle of convergence*, and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ($R = +\infty$).

0.312 Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^{x}\left\{\sum_{k=0}^{\infty} a_k (x-\xi)^k\right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-\xi)^{k+1},$$

$$\frac{d}{dx}\left\{\sum_{k=0}^{\infty} a_k (x-\xi)^k\right\} = \sum_{k=1}^{\infty} k a_k (x-\xi)^{k-1}.$$

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

Operations on power series

0.313 Division of power series.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0}\sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0}\sum_{k=1}^{n} c_{n-k} a_k - b_n = 0,$$

or

$$c_n = \frac{(-1)^n}{a_0^n}\begin{bmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix} \qquad \text{AD (6360)}$$

0.314 Power series raised to powers.

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0}\sum_{k=1}^{m}(kn - m + k) a_k c_{m-k} \qquad \text{for } m \ge 1 \qquad [n \text{ is a natural number}] \qquad \text{AD (6361)}$$

0.315 The substitution of one series into another.

$$\sum_{k=1}^{\infty} b_k y^k = \sum_{k=1}^{\infty} c_k x^k \qquad y = \sum_{k=1}^{\infty} a_k x^k;$$

$$c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2a_1 a_2 b_2 + a_1^3 b_3,$$

$$c_4 = a_4 b_1 + a_2^2 b_2 + 2a_1 a_3 b_2 + 3a_1^2 a_2 b_3 + a_1^4 b_4, \quad \ldots \qquad \text{AD (6362)}$$

0.316 Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$ FI II 372

Taylor series

0.317 If a function $f(x)$ has derivatives of all orders throughout a neighborhood of a point ξ, then we may write the series

1. $$f(\xi) + \frac{(x-\xi)}{1!} f'(\xi) + \frac{(x-\xi)^2}{2!} f''(\xi) + \frac{(x-\xi)^3}{3!} f'''(\xi) + \dots,$$

which is known as the *Taylor series* of the function $f(x)$.

The Taylor series converges to the function $f(x)$ if the remainder

2. $$R_n(x) = f(x) - f(\xi) - \sum_{k=1}^{n} \frac{(x-\xi)^k}{k!} f^{(k)}(\xi)$$

approaches zero as $n \to \infty$.

The following are different forms for the remainder of a Taylor series:

3. $$R_n(x) = \frac{(x-\xi)^{n+1}}{(n+1)!} f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < \theta < 1]$$ (Lagrange)

4. $$R_n(x) = \frac{(x-\xi)^{n+1}}{n!} (1-\theta)^n f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < \theta < 1]$$ (Cauchy)

5. $$R_n(x) = \frac{\psi(x-\xi) - \psi(0)}{\psi'[(x-\xi)(1-\theta)]} \frac{(x-\xi)^n (1-\theta)^n}{n!} f^{(n+1)}(\xi + \theta(x-\xi))$$

$$[0 < \theta < 1],$$ (Schlömilch)

where $\psi(x)$ is an arbitrary function satisfying the following two conditions: (1) It and its derivative $\psi'(x)$ are continuous in the interval $(0, x-\xi)$; and (2) the derivative $\psi'(x)$ does not change sign in that interval. If we set $\psi(x) = x^{p+1}$, we obtain the following form for the remainder:

$$R_n(x) = \frac{(x-\xi)^{n+1}(1-\theta)^{n-p-1}}{(p+1)n!} f^{(n+1)}(\xi + \theta(x-\xi)) \qquad [0 < p \le n; \quad 0 < \theta < 1]$$ (Rouché)

6. $$R_n(x) = \frac{1}{n!} \int_{\xi}^{x} f^{(n+1)}(t)(x-t)^n \, dt$$

0.318 Other forms in which a Taylor series may be written:

1.[11] $$f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

2. $$\sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$ (Maclaurin series)

0.319 The Taylor series of functions of several variables:

$$f(x,y) = f(\xi,\eta) + (x-\xi)\frac{\partial f(\xi,\eta)}{\partial x} + (y-\eta)\frac{\partial f(\xi,\eta)}{\partial y} + \frac{1}{2!}\left\{(x-\xi)^2\frac{\partial^2 f(\xi,\eta)}{\partial x^2} + 2(x-\xi)(y-\eta)\frac{\partial^2 f(\xi,\eta)}{\partial x\,\partial y} + (y-\eta)^2\frac{\partial^2 f(\xi,\eta)}{\partial y^2}\right\} + \dots$$

0.32 Fourier series

0.320 Suppose that $f(x)$ is a *periodic* function of period $2l$ and that it is absolutely integrable (possibly improperly) over the interval $(-l,l)$. The following trigonometric series is called the *Fourier series* of $f(x)$:

1. $$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l},\right)$$

the coefficients of which (the Fourier coefficients) are given by the formulas

2. $$a_k = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\cos\frac{k\pi t}{l}\,dt \quad (k=0,1,2,\dots)$$

3.[11] $$b_k = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi t}{l}\,dt = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(t)\sin\frac{k\pi t}{l}\,dt \quad (k=1,2,\dots)$$

Convergence tests

0.321 The Fourier series of a function $f(x)$ at a point x_0 converges to the number

$$\frac{f(x_0+0)+f(x_0-0)}{2},$$

if, for some $h>0$, the integral

$$\int_0^h \frac{|f(x_0+t)+f(x_0-t)-f(x_0+0)-f(x_0-0)|}{t}\,dt$$

exists. Here, it is assumed that the function $f(x)$ either is continuous at the point x_0 or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits $f(x_0+0)$ and $f(x_0-0)$ exist. (Dini) FI III 524

0.322 The Fourier series of a periodic function $f(x)$ that satisfies the Dirichlet conditions on the interval $[a,b]$ converges at every point x_0 to the value $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$. (Dirichlet)

We say that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a,b]$ if it is bounded on that interval and if the interval $[a,b]$ can be partitioned into a finite number of subintervals inside each of which the function $f(x)$ is continuous and monotonic.

0.323 The Fourier series of a function $f(x)$ at a point x_0 converges to $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$ if $f(x)$ is of bounded variation in some interval (x_0-h, x_0+h) with center at x_0. (Jordan–Dirichlet) FI III 528

The definition of a function of bounded variation. Suppose that a function $f(x)$ is defined on some interval $[a,b]$, where $z<b$. Let us partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

and let us form the sum

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

Different partitions of the interval $[a, b]$ (that is, different choices of points of division x_i) yield, generally speaking, different sums. If the set of these sums is bounded above, we say that the function $f(x)$ is *of bounded variation* on the interval $[a, b]$. The least upper bound of these sums is called the *total variation* of the function $f(x)$ on the interval $[a, b]$.

0.324 Suppose that a function $f(x)$ is piecewise-continuous on the interval $[a, b]$ and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point x_0 of the interval $[a, b]$, the Fourier series of the function $f(x)$ converges to $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$.

0.325 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a cosine series of the form

1. $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

where

2. $$a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l}\, dt$$

0.326 A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a sine series of the form

1. $$\sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

where

2. $$b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l}\, dt$$

The convergence tests for the series **0.325** 1 and **0.326** 1 are analogous to the convergence tests for the series **0.320** 1 (see **0.321–0.324**).

0.327 The Fourier coefficients a_k and b_k (given by formulas **0.320** 2 and **0.320** 3) of an absolutely integrable function approach zero as $k \to \infty$.

If a function $f(x)$ is square-integrable on the interval $(-l, l)$, the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) = \frac{1}{l} \int_{-l}^{l} f^2(x)\, dx \qquad \text{(A. M. Lyapunov)}$$ FI III 705

0.328 Suppose that $f(x)$ and $\varphi(x)$ are two functions that are square-integrable on the interval $(-l, l)$ and that a_k, b_k and α_k, β_k are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0 \alpha_0}{2} + \sum_{k=1}^{\infty} (a_k \alpha_k + b_k \beta_k) = \frac{1}{l} \int_{-l}^{l} f(x) \varphi(x)\, dx$$ FI III 709

For examples of Fourier series, see **1.44** and **1.45**.

0.33 Asymptotic series

0.330 Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. **0.227** 3).

Asymptotic series have many properties that are analogous to the properties of convergent series, and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}$$

This is the definition of an asymptotic expansion. The divergent series $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ is called the *asymptotic expansion* of a function $f(z)$ in a given region of values of $\arg z$ if the expression $R_n(z) = z^n \left[f(z) - S_n(z)\right]$, where $S_n(z) = \sum_{k=0}^{n} \frac{A_k}{z^k}$, satisfies the condition $\lim_{|z|\to\infty} R_n(z) = 0$ for fixed n. FI II 820

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series.*

0.331 Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.
2. One asymptotic series can be divided by another, provided that the first term A_0 of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic. FI II 823-825
3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible. FI II 824
4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an asymptotic series in only one manner.

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

0.410 $$\frac{d}{da}\int_{\psi(a)}^{\varphi(a)} f(x,a)\,dx = f(\varphi(a),a)\frac{d\varphi(a)}{da} - f\left(\psi(a),a\right)\frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x,a)\,dx$$ FI II 680

0.411 In particular,

1. $$\frac{d}{da}\int_b^a f(x)\,dx = f(a)$$
2. $$\frac{d}{db}\int_b^a f(x)\,dx = -f(b)$$

0.42 The n^{th} derivative of a product (Leibniz's rule)

Suppose that u and v are n-times-differentiable functions of x. Then,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^n v}{dx^n} + \binom{n}{1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \binom{n}{2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \binom{n}{3}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \cdots + v\frac{d^n u}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u+v)^{(n)}$$ FI I 272

0.43 The n^{th} derivative of a composite function

0.430 If $f(x) = F(y)$and $y = \varphi(x)$, then

1. $$\frac{d^n}{dx^n}f(x) = \frac{U_1}{1!}F'(y) + \frac{U_2}{2!}F''(y) + \frac{U_3}{3!}F'''(y) + \ldots + \frac{U_n}{n!}F^{(n)}(y),$$

where

$$U_k = \frac{d^n}{dx^n}y^k - \frac{k}{1!}y\frac{d^n}{dx^n}y^{k-1} + \frac{k(k-1)}{2!}y^2\frac{d^n}{dx^n}y^{k-2} - \ldots + (-1)^{k-1}ky^{k-1}\frac{d^n y}{dx^n}$$ AD (7361) GO

2. $$\frac{d^n}{dx^n}f(x) = \sum \frac{n!}{i!j!h!\ldots k!}\frac{d^m F}{dy^m}\left(\frac{y'}{1!}\right)^i\left(\frac{y''}{2!}\right)^j\left(\frac{y'''}{3!}\right)^h\cdots\left(\frac{y^{(l)}}{l!}\right)^k,$$

Here, the symbol $\sum$ indicates summation over all solutions in non-negative integers of the equation $i + 2j + 3h + \ldots + lk = n$ and $m = i + j + h + \ldots + k$.

0.431

1. $$(-1)^n\frac{d^n}{dx^n}F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}}F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}}\frac{n}{1!}F^{(n-1)}\left(\frac{1}{x}\right) + \frac{(n-1)(n-2)}{x^{2n-2}}\frac{n(n-1)}{2!}F^{(n-2)}\left(\frac{1}{x}\right) + \ldots$$

AD (7362.1)

2. $$(-1)^n\frac{d^n}{dx^n}e^{\frac{a}{x}} = \frac{1}{x^n}e^{\frac{a}{x}}\left[\left(\frac{a}{x}\right)^n + (n-1)\binom{n}{1}\left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2)\binom{n}{2}\left(\frac{a}{x}\right)^{n-2} + (n-1)(n-2)(n-3)\binom{n}{3}\left(\frac{a}{x}\right)^{n-3} + \ldots\right]$$

AD (7362.2)

0.432

1. $$\frac{d^n}{dx^n}F\left(x^2\right) = (2x)^n F^{(n)}\left(x^2\right) + \frac{n(n-1)}{1!}(2x)^{n-2}F^{(n-1)}\left(x^2\right) + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}F^{(n-2)}\left(x^2\right) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}(2x)^{n-6}F^{(n-3)}\left(x^2\right) + \ldots$$

AD (7363.1)

2. $$\frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2} \left[1 + \frac{n(n-1)}{1!\,(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!\,(4ax^2)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!\,(4ax^2)^3} + \cdots\right]$$ AD (7363.2)

3. $$\frac{d^n}{dx^n} \left(1+ax^2\right)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times \left\{1 + \frac{n(n-1)}{1!(p-n+1)} \frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)} \left(\frac{1+ax^2}{4ax^2}\right)^2 + \dots\right\},$$ AD (7363.3)

4. $$\frac{d^{m-1}}{dx^{m-1}} \left(1-x^2\right)^{m-\frac{1}{2}} = (-1)^{m-1} \frac{(2m-1)!!}{m} \sin\left(m \arccos x\right)$$ AD (7363.4)

5. $$(-1)^n \frac{\partial^n}{\partial a^n} \left(\frac{a}{a^2+b^2}\right) = n! \left(\frac{a}{a^2+b^2}\right)^{n+1} \sum_{0 \le 2k \le n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{a}\right)^{2k}$$ (3.944.12)

6. $$(-1)^n \frac{\partial^n}{\partial a^n} \left(\frac{b}{a^2+b^2}\right) = n! \left(\frac{a}{a^2+b^2}\right)^{n+1} \sum_{0 \le 2k \le n} (-1)^k \binom{n+1}{2k+1} \left(\frac{b}{a}\right)^{2k+1}$$ (3.944.11)

0.433

1. $$\frac{d^n}{dx^n} F\left(\sqrt{x}\right) = \frac{F^{(n)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^n} - \frac{n(n-1)}{1!} \frac{F^{(n-1)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!} \frac{F^{(n-2)}\left(\sqrt{x}\right)}{\left(2\sqrt{x}\right)^{n+2}} - \dots$$ AD (7364.1)

2. $$\frac{d^n}{dx^n} \left(1+a\sqrt{x}\right)^{2n-1} = \frac{(2n-1)!!}{2^n} \frac{a}{\sqrt{x}} \left(a^2 - \frac{1}{x}\right)^{n-1}$$ AD (7364.2)

0.434 $$\frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{-\binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n\left(y^2\right)}{dx^n} - \dots\right\}$$ AD (737.1)

0.435 $$\frac{d^n}{dx^n} \ln y = \left\{\binom{n}{1} \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n\left(y^2\right)}{dx^n} + \frac{d^n\left(y^3\right)}{dx^n} x^n - \dots\right\}$$ AD (737.2)

0.44 Integration by substitution

0.440[11] Let $f(g(x))$ and $g(x)$ be continuous in $[a,b]$. Further, let $g'(x)$ exist and be continuous there. Then $\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

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1 Elementary Functions

1.1 Power of Binomials

1.11 Power series

1.110 $(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\ldots(q-k+1)}{k!}x^k + \cdots = \sum_{k=0}^{\infty} \binom{q}{k} x^k$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series converges for $q > -1$ and diverges for $q \le -1$. For $x = 1$, the series converges absolutely for $q > 0$. For $x = -1$, it converges absolutely for $q > 0$ and diverges for $q < 0$. If $q = n$ is a natural number, the series **1.110** is reduced to the finite sum **1.111**. FI II 425

1.111 $(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$

1.112

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$

(see also **1.121** 2)

2. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} k x^{k-1}$

3.[11] $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \ldots$

4. $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ldots$

1.113 $\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k$ $\left[x^2 < 1\right]$

1.114

1. $\left(1+\sqrt{1+x}\right)^q = 2^q \left[1 + \frac{q}{1!}\left(\frac{x}{4}\right) + \frac{q(q-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{q(q-4)(q-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots\right]$

$\left[x^2 < 1, \quad q \text{ is a real number}\right]$

AD (6351.1)

2. $$\left(x+\sqrt{1+x^2}\right)^q = 1+\sum_{k=0}^{\infty}\frac{q^2\left(q^2-2^2\right)\left(q^2-4^2\right)\dots\left[q^2-(2k)^2\right]x^{2k+2}}{(2k+2)!} + qx + q\sum_{k=1}^{\infty}\frac{\left(q^2-1^2\right)\left(q^2-3^2\right)\dots\left[q^2-(2k-1)^2\right]}{(2k+1)!}x^{2k+1}$$
$\left[x^2<1, \quad q \text{ is a real number}\right]$ AD(6351.2)

1.12 Series of rational fractions

1.121

1. $$\frac{x}{1-x}=\sum_{k=1}^{\infty}\frac{2^{k-1}x^{2^{k-1}}}{1+x^{2^{k-1}}}=\sum_{k=1}^{\infty}\frac{x^{2^{k-1}}}{1-x^{2^k}}$$ $\left[x^2<1\right]$ AD (6350.3)

2. $$\frac{1}{x-1}=\sum_{k=1}^{\infty}\frac{2^{k-1}}{x^{2^{k-1}}+1}$$ $\left[x^2>1\right]$ AD (6350.3)

1.2 The Exponential Function

1.21 Series representation

1.211

1.[11] $$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

2. $$a^x=\sum_{k=0}^{\infty}\frac{(x\ln a)^k}{k!}$$

3. $$e^{-x^2}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}$$

4.* $$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

1.212 $$e^x(1+x)=\sum_{k=0}^{\infty}\frac{x^k(k+1)}{k!}$$

1.213 $$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k)!}$$ $[x<2\pi]$ FI II 520

1.214 $$e^{e^x}=e\left(1+x+\frac{2x^2}{2!}+\frac{5x^3}{3!}+\frac{15x^4}{4!}+\dots\right)$$ AD (6460.3)

1.215

1. $$e^{\sin x}=1+x+\frac{x^2}{2!}-\frac{3x^4}{4!}-\frac{8x^5}{5!}-\frac{3x^6}{6!}+\frac{56x^7}{7!}+\dots$$ AD (6460.4)

2. $$e^{\cos x}=e\left(1-\frac{x^2}{2!}+\frac{4x^4}{4!}-\frac{31x^6}{6!}+\dots\right)$$ AD (6460.5)

3. $$e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots$$ AD (6460.6)

1.216

1. $$e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots$$ AD (6460.7)

2. $$e^{\arctan x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots$$ AD (6460.8)

1.217

1. $$\pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{1}{x^2 + k^2}$$ (cf. **1.421** 3) AD (6707.1)

2. $$\frac{2\pi}{e^{\pi x} - e^{-\pi x}} = x \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x^2 + k^2}$$ (cf. **1.422** 3) AD (6707.2)

1.22 Functional relations

1.221

1. $$a^x = e^{x \ln a}$$

2. $$a^{\log_a x} = a^{\frac{1}{\log_x a}} = x$$

1.222

1. $$e^x = \cosh x + \sinh x$$

2. $$e^{ix} = \cos x + i \sin x$$

1.223 $$e^{ax} - e^{bx} = (a-b)x \exp\left[\frac{1}{2}(a+b)x\right] \prod_{k=1}^{\infty} \left[1 + \frac{(a-b)^2 x^2}{2k^2 \pi^2}\right]$$ MO 216

1.23 Series of exponentials

1.231 $$\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1 - a^x} \qquad [a > 1 \text{ and } x < 0 \text{ or } 0 < a < 1 \text{ and } x > 0]$$

1.232

1. $$\tanh x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx}$$ $[x > 0]$

2. $$\operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x}$$ $[x > 0]$

3. $$\operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x}$$ $[x > 0]$

4.* $$\sin x = \exp\left[-\sum_{n=1}^{\infty} \frac{\cos^{2n} x}{2n}\right]$$ $[0 \le x \le \pi]$

1.3–1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\sinh x = \frac{1}{i}\sin(ix), \qquad \sin x = \frac{1}{i}\sinh(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\cosh x = \cos(ix), \qquad \cos x = \cosh(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful "pairs" are included in the list.

1.31 The basic functional relations

1.311

1. $$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -i\sinh(ix)$$

2. $$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) = -i\sin(ix)$$

3. $$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cosh(ix)$$

4. $$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cos(ix)$$

5. $$\tan x = \frac{\sin x}{\cos x} = \frac{1}{i}\tanh(ix)$$

6. $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{1}{i}\tan(ix)$$

7. $$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = i\coth(ix)$$

8. $$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = i\cot(ix)$$

1.312

1. $$\cos^2 x + \sin^2 x = 1$$

2. $\cosh^2 x - \sinh^2 x = 1$

1.313

1. $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
2. $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$
3. $\sin(x \pm iy) = \sin x \cosh y \pm i \sinh y \cos x$
4. $\sinh(x \pm iy) = \sinh x \cos y \pm i \sin y \cosh x$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
7. $\cos(x \pm iy) = \cos x \cosh y \mp i \sin x \sinh y$
8. $\cosh(x \pm iy) = \cosh x \cos y \pm i \sinh x \sin y$
9. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
10. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
11. $\tan(x \pm iy) = \dfrac{\tan x \pm i \tanh y}{1 \mp i \tan x \tanh y}$
12. $\tanh(x \pm iy) = \dfrac{\tanh x \pm i \tan y}{1 \pm i \tanh x \tan y}$

1.314

1. $\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$
2. $\sinh x \pm \sinh y = 2 \sinh \frac{1}{2}(x \pm y) \cosh \frac{1}{2}(x \mp y)$
3. $\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
4. $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
5. $\cos x - \cos y = 2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(y - x)$
6. $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$
7. $\tan x \pm \tan y = \dfrac{\sin(x \pm y)}{\cos x \cos y}$
8. $\tanh x \pm \tanh y = \dfrac{\sinh(x \pm y)}{\cosh x \cosh y}$

9.*
$$\begin{aligned}\sin x \pm \cos y &= \pm 2 \sin\left[\frac{1}{2}(x+y) \pm \frac{\pi}{4}\right] \sin\left[\frac{1}{2}(x-y) \pm \frac{\pi}{4}\right] \\ &= \pm 2 \cos\left[\frac{1}{2}(x+y) \mp \frac{\pi}{4}\right] \cos\left[\frac{1}{2}(x-y) \mp \frac{\pi}{4}\right] \\ &= 2 \sin\left[\frac{1}{2}(x \pm y) \pm \frac{\pi}{4}\right] \cos\left[\frac{1}{2}(x \mp y) \mp \frac{\pi}{4}\right]\end{aligned}$$

10.* $$a \sin x \pm b \cos x = a\sqrt{1+\left(\frac{b}{a}\right)^2} \sin\left[x \pm \arctan\left(\frac{b}{a}\right)\right]$$

$[a \neq 0]$

11.* $$\pm a \sin x + b \cos x = b\sqrt{1+\left(\frac{a}{b}\right)^2} \cos\left[x \mp \arctan\left(\frac{a}{b}\right)\right]$$

$[b \neq 0]$

12.* $$a \sin x \pm b \cos y = q\sqrt{1+\left(\frac{r}{q}\right)^2} \sin\left[\frac{1}{2}(x \pm y) + \arctan\left(\frac{r}{q}\right)\right]$$

$$q = (a+b)\cos\left[\frac{1}{2}(x \mp y)\right], \quad r = (a-b)\sin\left[\frac{1}{2}(x \mp y)\right] \qquad [q \neq 0]$$

13.* $$a \cos x + b \cos y = t\sqrt{1+\left(\frac{s}{t}\right)^2} \cos\left[\frac{1}{2}(x \mp y) + \arctan\left(\frac{s}{t}\right)\right] \qquad [t \neq 0]$$

$$= -s\sqrt{1+\left(\frac{t}{s}\right)^2} \cos\left[\frac{1}{2}(x \mp y) - \arctan\left(\frac{t}{s}\right)\right] \qquad [s \neq 0]$$

$$s = (a-b)\sin\left[\frac{1}{2}(x \pm y)\right], \quad t = (a+b)\cos\left[\frac{1}{2}(x \pm y)\right]$$

1.315

1. $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$
2. $\sinh^2 x - \sinh^2 y = \sinh(x+y)\sinh(x-y) = \cosh^2 x - \cosh^2 y$
3. $\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y) = \cos^2 y - \sin^2 x$
4. $\sinh^2 x + \cosh^2 y = \cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$

1.316

1. $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ [n is an integer]
2. $(\cosh x + \sinh x)^n = \sinh nx + \cosh nx$ [n is an integer]

1.317

1. $$\sin\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1-\cos x)}$$
2. $$\sinh\frac{x}{2} = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$$
3. $$\cos\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1+\cos x)}$$
4. $$\cosh\frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)}$$
5. $$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

6. $\tanh\frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$

The signs in front of the radical in formulas **1.317** 1, **1.317** 2, and **1.317** 3 are taken so as to agree with the signs of the left-hand members. The sign of the left hand members depends in turn on the value of x.

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.320

1. $\sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}$ KR 56 (10, 2)

2. $\sinh^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$

3. $\sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x$ KR 56 (10, 4)

4. $\sinh^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sinh(2n-2k-1)x$

5. $\cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}$ KR 56 (10, 1)

6. $\cosh^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cosh 2(n-k)x + \binom{2n}{n} \right\}$

7. $\cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos(2n-2k-1)x$ KR 56 (10, 3)

8. $\cosh^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cosh(2n-2k-1)x$

Special cases

1.321

1. $\sin^2 x = \frac{1}{2}(-\cos 2x + 1)$

2. $\sin^3 x = \frac{1}{4}(-\sin 3x + 3\sin x)$

3. $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$

4. $\sin^5 x = \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$

5. $\sin^6 x = \frac{1}{32}(-\cos 6x + 6\cos 4x - 15\cos 2x + 10)$

6. $\sin^7 x = \frac{1}{64}(-\sin 7x + 7\sin 5x - 21\sin 3x + 35\sin x)$

1.322

1. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$

2. $\sinh^3 x = \frac{1}{4}(\sinh 3x - 3\sinh x)$

3. $\sinh^4 x = \frac{1}{8}(\cosh 4x - 4\cosh 2x + 3)$

4. $\sinh^5 x = \frac{1}{16}(\sinh 5x - 5\sinh 3x + 10\sinh x)$

5. $\sinh^6 x = \frac{1}{32}(\cosh 6x - 6\cosh 4x + 15\cosh 2x + 10)$

6. $\sinh^7 x = \frac{1}{64}(\sinh 7x - 7\sinh 5x + 21\sinh 3x + 35\sinh x)$

1.323

1. $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

2. $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

3. $\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$

4. $\cos^5 x = \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x)$

5. $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$

6. $\cos^7 x = \frac{1}{64}(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x)$

1.324

1. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

2. $\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$

3. $\cosh^4 x = \frac{1}{8}(\cosh 4x + 4\cosh 2x + 3)$

4. $\cosh^5 x = \frac{1}{16}(\cosh 5x + 5\cosh 3x + 10\cosh x)$

5. $\cosh^6 x = \frac{1}{32}(\cosh 6x + 6\cosh 4x + 15\cosh 2x + 10)$

6. $\cosh^7 x = \frac{1}{64}(\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x)$

1.33 The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions

1.331

1.[7]
$$\sin nx = n\cos^{n-1}x\sin x - \binom{n}{3}\cos^{n-3}x\sin^3 x + \binom{n}{5}\cos^{n-5}x\sin^5 x - \dots;$$
$$= \sin x\left\{2^{n-1}\cos^{n-1}x - \binom{n-2}{1}2^{n-3}\cos^{n-3}x + \binom{n-3}{2}2^{n-5}\cos^{n-5}x - \binom{n-4}{3}2^{n-7}\cos^{n-7}x + \dots\right\}$$

AD (3.175)

2.
$$\sinh nx = x\sum_{k=1}^{[(n+1)/2]}\binom{n}{2k-1}\sinh^{2k-2}x\cosh^{n-2k+1}x$$
$$= \sinh x\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n-k-1}{k}2^{n-2k-1}\cosh^{n-2k-1}x$$

3.
$$\cos nx = \cos^n x - \binom{n}{2}\cos^{n-2}x\sin^2 x + \binom{n}{4}\cos^{n-4}x\sin^4 x - \dots;$$
$$= 2^{n-1}\cos^n x - \frac{n}{1}2^{n-3}\cos^{n-2}x + \frac{n}{2}\binom{n-3}{1}2^{n-5}\cos^{n-4}x - \frac{n}{3}\binom{n-4}{2}2^{n-7}\cos^{n-6}x + \dots$$

AD (3.175)

4.[3]
$$\cosh nx = \sum_{k=0}^{[n/2]}\binom{n}{2k}\sinh^{2k}x\cosh^{n-2k}x$$
$$= 2^{n-1}\cosh^n x + n\sum_{k=1}^{[n/2]}(-1)^k\frac{1}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\cosh^{n-2k}x$$

1.332

1.
$$\sin 2nx = 2n\cos x\left\{\sin x - \frac{4n^2-2^2}{3!}\sin^3 x + \frac{\left(4n^2-2^2\right)\left(4n^2-4^2\right)}{5!}\sin^5 x - \dots\right\}$$ AD (3.171)
$$= (-1)^{n-1}\cos x\left\{2^{2n-1}\sin^{2n-1}x - \frac{2n-2}{1!}2^{2n-3}\sin^{2n-3}x + \frac{(2n-3)(2n-4)}{2!}2^{2n-5}\sin^{2n-5}x - \frac{(2n-4)(2n-5)(2n-6)}{3!}2^{2n-7}\sin^{2n-7}x + \dots\right\}$$ AD (3.173)

2. $$\sin(2n-1)x = (2n-1)\left\{\sin x - \frac{(2n-1)^2-1^2}{3!}\sin^3 x + \frac{\left[(2n-1)^2-1^2\right]\left[(2n-1)^2-3^2\right]}{5!}\sin^5 x - \dots\right\}$$ AD (3.172)

$$= (-1)^{n-1}\left\{2^{2n-2}\sin^{2n-1} x - \frac{2n-1}{1!}2^{2n-4}\sin^{2n-3} x + \frac{(2n-1)(2n-4)}{2!}2^{2n-6}\sin^{2n-5} x - \frac{(2n-1)(2n-5)(2n-6)}{3!}2^{2n-8}\sin^{2n-7} x + \dots\right\}$$ AD (3.174)

3. $$\cos 2nx = 1 - \frac{4n^2}{2!}\sin^2 x + \frac{4n^2\left(4n^2-2^2\right)}{4!}\sin^4 x - \frac{4n^2\left(4n^2-2\right)\left(4n^2-4^2\right)}{6!}\sin^6 x + \dots$$ AD (3.171)

$$= (-1)^n\left\{2^{2n-1}\sin^{2n} x - \frac{2n}{1!}2^{2n-3}\sin^{2n-2} x + \frac{2n(2n-3)}{2!}2^{2n-5}\sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!}2^{2n-7}\sin^{2n-6} x + \dots\right\}$$ AD (3.173)a

4. $$\cos(2n-1)x = \cos x\left\{1 - \frac{(2n-1)^2-1^2}{2!}\sin^2 x + \frac{\left[(2n-1)^2-1^2\right]\left[(2n-1)^2-3^2\right]}{4!}\sin^4 x - \dots\right\}$$ AD (3.172)

$$= (-1)^{n-1}\cos x\left\{2^{2n-2}\sin^{2n-2} x - \frac{2n-3}{1!}2^{2n-4}\sin^{2n-4} x + \frac{(2n-4)(2n-5)}{2!}2^{2n-6}\sin^{2n-6} x - \frac{(2n-5)(2n-6)(2n-7)}{3!}2^{2n-8}\sin^{2n-8} x + \dots\right\}$$ AD (3.174)

By using the formulas and values of **1.30**, we can write formulas for $\sinh 2nx$, $\sinh(2n-1)x$, $\cosh 2nx$, and $\cosh(2n-1)x$ that are analogous to those of **1.332**, just as was done in the formulas in **1.331**.

Special cases

1.333

1. $\sin 2x = 2\sin x\cos x$
2. $\sin 3x = 3\sin x - 4\sin^3 x$
3. $\sin 4x = \cos x\left(4\sin x - 8\sin^3 x\right)$
4. $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$
5. $\sin 6x = \cos x\left(6\sin x - 32\sin^3 x + 32\sin^5 x\right)$

6. $\sin 7x = 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x$

1.334

1. $\sinh 2x = 2\sinh x \cosh x$
2. $\sinh 3x = 3\sinh x + 4\sinh^3 x$
3.[11] $\sinh 4x = \cosh x \left(4\sinh x + 8\sinh^3 x\right)$
4. $\sinh 5x = 5\sinh x + 20\sinh^3 x + 16\sinh^5 x$
5.[11] $\sinh 6x = \cosh x \left(6\sinh x + 32\sinh^3 x + 32\sinh^5 x\right)$
6. $\sinh 7x = 7\sinh x + 56\sinh^3 x + 112\sinh^5 x + 64\sinh^7 x$

1.335

1. $\cos 2x = 2\cos^2 x - 1$
2. $\cos 3x = 4\cos^3 x - 3\cos x$
3. $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$
4. $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$
5. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$
6. $\cos 7x = 64\cos^7 x - 112\cos^5 x + 56\cos^3 x - 7\cos x$

1.336

1. $\cosh 2x = 2\cosh^2 x - 1$
2. $\cosh 3x = 4\cosh^3 x - 3\cosh x$
3. $\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1$
4. $\cosh 5x = 16\cosh^5 x - 20\cosh^3 x + 5\cosh x$
5. $\cosh 6x = 32\cosh^6 x - 48\cosh^4 x + 18\cosh^2 x - 1$
6. $\cosh 7x = 64\cosh^7 x - 112\cosh^5 x + 56\cosh^3 x - 7\cosh x$

1.337

1.* $\dfrac{\cos 3x}{\cos^3 x} = 1 - 3\tan^2 x$
2.* $\dfrac{\cos 4x}{\cos^4 x} = 1 - 6\tan^2 x + \tan^4 x$
3.* $\dfrac{\cos 5x}{\cos^5 x} = 1 - 10\tan^2 x + 5\tan^4 x$
4.* $\dfrac{\cos 6x}{\cos^6 x} = 1 - 15\tan^2 x + 15\tan^4 x - \tan^6 x$
5.* $\dfrac{\sin 3x}{\cos^3 x} = 3\tan x - \tan^3 x$
6.* $\dfrac{\sin 4x}{\cos^4 x} = 4\tan x - 4\tan^3 x$

7.* $\dfrac{\sin 5x}{\cos^5 x} = 5\tan x - 10\tan^3 x + \tan^5 x$

8.* $\dfrac{\sin 6x}{\cos^6 x} = 6\tan x - 20\tan^3 x + 6\tan^5 x$

9.* $\dfrac{\cos 3x}{\sin^3 x} = \cot^3 x - 3\cot x$

10.* $\dfrac{\cos 4x}{\sin^4 x} = \cot^4 x - 6\cot^2 x + 1$

11.* $\dfrac{\cos 5x}{\sin^5 x} = \cot^5 x - 10\cot^3 x + 5\cot x$

12.* $\dfrac{\cos 6x}{\sin^6 x} = \cot^6 x - 15\cot^4 x + 15\cot^2 x - 1$

13.* $\dfrac{\sin 3x}{\sin^3 x} = 3\cot^2 x - 1$

14.* $\dfrac{\sin 4x}{\sin^4 x} = 4\cot^3 x - 4\cot x$

15.* $\dfrac{\sin 5x}{\sin^5 x} = 5\cot^4 x - 10\cot^2 x + 1$

16.* $\dfrac{\sin 6x}{\sin^6 x} = 6\cot^5 x - 20\cot^3 x + 6\cot x$

1.34 Certain sums of trigonometric and hyperbolic functions

1.341

1. $$\sum_{k=0}^{n-1} \sin(x+ky) = \sin\left(x + \frac{n-1}{2}y\right) \sin\frac{ny}{2} \operatorname{cosec}\frac{y}{2}$$ AD (361.8)

2. $$\sum_{k=0}^{n-1} \sinh(x+ky) = \sinh\left(x + \frac{n-1}{2}y\right) \sinh\frac{ny}{2} \frac{1}{\sinh\frac{y}{2}}$$

3. $$\sum_{k=0}^{n-1} \cos(x+ky) = \cos\left(x + \frac{n-1}{2}y\right) \sin\frac{ny}{2} \operatorname{cosec}\frac{y}{2}$$ AD (361.9)

4. $$\sum_{k=0}^{n-1} \cosh(x+ky) = \cosh\left(x + \frac{n-1}{2}y\right) \sinh\frac{ny}{2} \frac{1}{\sinh\frac{y}{2}}$$

5. $$\sum_{k=0}^{2n-1} (-1)^k \cos(x+ky) = \sin\left(x + \frac{2n-1}{2}y\right) \sin ny \sec\frac{y}{2}$$ JO (202)

6. $$\sum_{k=0}^{n-1} (-1)^k \sin(x+ky) = \sin\left(x + \frac{n-1}{2}(y+\pi)\right) \sin\frac{n(y+\pi)}{2} \sec\frac{y}{2}$$ AD (202a)

Special cases

1.342

1. $$\sum_{k=1}^{n} \sin kx = \sin \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$$ AD (361.1)

2.[10] $$\sum_{k=0}^{n} \cos kx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 = \cos \frac{nx}{2} \sin \frac{n+1}{2} x \operatorname{cosec} \frac{x}{2} = \frac{1}{2} \left(1 + \frac{\sin \left(n + \frac{1}{2} \right) x}{\sin \frac{x}{2}} \right)$$ AD (361.2)

3. $$\sum_{k=1}^{n} \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x$$ AD (361.7)

4. $$\sum_{k=1}^{n} \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x$$ JO (207)

1.343

1. $$\sum_{k=1}^{n} (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos\left(\frac{2n+1}{2} x\right)}{2 \cos \frac{x}{2}}$$ AD (361.11)

2. $$\sum_{k=1}^{n} (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x}$$ AD (361.10)

3. $$\sum_{k=1}^{n} \cos(4k-3)x + \sum_{k=1}^{n} \sin(4k-1)x = \sin 2nx \left(\cos 2nx + \sin 2nx\right) \left(\cos x + \sin x\right) \operatorname{cosec} 2x$$ JO (208)

1.344

1. $$\sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \cot \frac{\pi}{2n}$$ AD (361.19)

2. $$\sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right)$$ AD (361.18)

3. $$\sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right)$$ AD (361.17)

1.35 Sums of powers of trigonometric functions of multiple angles

1.351

1. $$\sum_{k=1}^{n} \sin^2 kx = \frac{1}{4} \left[(2n+1) \sin x - \sin(2n+1)x\right] \operatorname{cosec} x = \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x}$$ AD (361.3)

2. $$\sum_{k=1}^{n} \cos^2 kx = \frac{n-1}{2} + \frac{1}{2}\cos nx \sin(n+1)x \operatorname{cosec} x = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2\sin x}$$ AD (361.4)a

3. $$\sum_{k=1}^{n} \sin^3 kx = \frac{3}{4}\sin\frac{n+1}{2}x \sin\frac{nx}{2} \operatorname{cosec}\frac{x}{2} - \frac{1}{4}\sin\frac{3(n+1)x}{2}\sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}$$ JO (210)

4. $$\sum_{k=1}^{n} \cos^3 kx = \frac{3}{4}\cos\frac{n+1}{2}x \sin\frac{nx}{2} \operatorname{cosec}\frac{x}{2} + \frac{1}{4}\cos\frac{3(n+1)}{2}x\sin\frac{3nx}{2}\operatorname{cosec}\frac{3x}{2}$$ JO (211)a

5. $$\sum_{k=1}^{n} \sin^4 kx = \frac{1}{8}\left[3n - 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x\right]$$ JO (212)

6. $$\sum_{k=1}^{n} \cos^4 kx = \frac{1}{8}\left[3n + 4\cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x\right]$$ JO (213)

1.352

1.[11] $$\sum_{k=1}^{n-1} k\sin kx = \frac{\sin nx}{4\sin^2\frac{x}{2}} - \frac{n\cos\left(\frac{2n-1}{2}x\right)}{2\sin\frac{x}{2}}$$ AD (361.5)

2.[11] $$\sum_{k=1}^{n-1} k\cos kx = \frac{n\sin\left(\frac{2n-1}{2}x\right)}{2\sin\frac{x}{2}} - \frac{1-\cos nx}{4\sin^2\frac{x}{2}}$$ AD (361.6)

1.353

1. $$\sum_{k=1}^{n-1} p^k \sin kx = \frac{p\sin x - p^n \sin nx + p^{n+1}\sin(n-1)x}{1 - 2p\cos x + p^2}$$ AD (361.12)a

2. $$\sum_{k=1}^{n-1} p^k \sinh kx = \frac{p\sinh x - p^n \sinh nx + p^{n+1}\sinh(n-1)x}{1 - 2p\cosh x + p^2}$$

3. $$\sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p\cos x - p^n \cos nx + p^{n+1}\cos(n-1)x}{1 - 2p\cos x + p^2}$$ AD (361.13)a

4. $$\sum_{k=0}^{n-1} p^k \cosh kx = \frac{1 - p\cosh x - p^n \cosh nx + p^{n+1}\cosh(n-1)x}{1 - 2p\cosh x + p^2}$$ JO (396)

1.36 Sums of products of trigonometric functions of multiple angles

1.361

1. $$\sum_{k=1}^{n} \sin kx \sin(k+1)x = \frac{1}{4}\left[(n+1)\sin 2x - \sin 2(n+1)x\right]\operatorname{cosec} x$$ JO (214)

2. $$\sum_{k=1}^{n} \sin kx \sin(k+2)x = \frac{n}{2}\cos 2x - \frac{1}{2}\cos(n+3)x \sin nx \operatorname{cosec} x$$ JO (216)

3. $$2\sum_{k=1}^{n} \sin kx \cos(2k-1)y = \sin\left(ny + \frac{n+1}{2}x\right) \sin\frac{n(x+2y)}{2} \operatorname{cosec}\frac{x+2y}{2} - \sin\left(ny - \frac{n+1}{2}x\right) \sin\frac{n(2y-x)}{2} \operatorname{cosec}\frac{2y-x}{2}$$ JO (217)

1.362

1. $$\sum_{k=1}^{n}\left(2^k \sin^2\frac{x}{2^k}\right)^2 = \left(2^n \sin\frac{x}{2^n}\right)^2 - \sin^2 x$$ AD (361.15)

2. $$\sum_{k=1}^{n}\left(\frac{1}{2^k}\sec\frac{x}{2^k}\right)^2 = \operatorname{cosec}^2 x - \left(\frac{1}{2^n}\operatorname{cosec}\frac{x}{2^n}\right)^2$$ AD (361.14)

1.37 Sums of tangents of multiple angles

1.371

1. $$\sum_{k=0}^{n}\frac{1}{2^k}\tan\frac{x}{2^k} = \frac{1}{2^n}\cot\frac{x}{2^n} - 2\cot 2x$$ AD (361.16)

2. $$\sum_{k=0}^{n}\frac{1}{2^{2k}}\tan^2\frac{x}{2^k} = \frac{2^{2n+2}-1}{3\cdot 2^{2n-1}} + 4\cot^2 2x - \frac{1}{2^{2n}}\cot^2\frac{x}{2^n}$$ AD (361.20)

1.38 Sums leading to hyperbolic tangents and cotangents

1.381

1. $$\sum_{k=0}^{n-1}\frac{\tanh\left(x\dfrac{1}{n\sin^2\left(\dfrac{2k+1}{4n}\pi\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{2k+1}{4n}\pi\right)}} = \tanh(2nx)$$ JO (402)a

2. $$\sum_{k=1}^{n-1}\frac{\tanh\left(x\dfrac{1}{n\sin^2\left(\dfrac{k\pi}{2n}\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{k\pi}{2n}\right)}} = \coth(2nx) - \frac{1}{2n}(\tanh x + \coth x)$$ JO (403)

3. $$\sum_{k=0}^{n-1} \frac{\tanh\left(x \dfrac{2}{(2n+1)\sin^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}} = \tanh(2n+1)\,x - \frac{\tanh x}{2n+1}$$ JO (404)

4. $$\sum_{k=1}^{n} \frac{\tanh\left(x \dfrac{2}{(2n+1)\sin^2\left(\dfrac{k\pi}{2(2n+1)}\right)}\right)}{1+\dfrac{\tanh^2 x}{\tan^2\left(\dfrac{k\pi}{(2n+1)}\right)}} = \coth(2n+1)\,x - \frac{\coth x}{2n+1}$$ JO (405)

1.382

1. $$\sum_{k=0}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{2k+1}{4n}\pi\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = 2n\tanh(nx)$$ JO (406)

2. $$\sum_{k=1}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{k\pi}{2n}\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = 2n\coth(nx) - 2\coth x$$ JO (407)

3. $$\sum_{k=0}^{n-1} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{2k+1}{2(2n+1)}\pi\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = (2n+1)\tanh\left(\frac{(2n+1)x}{2}\right) - \tanh\frac{x}{2}$$ JO (408)

4. $$\sum_{k=1}^{n} \frac{1}{\left(\dfrac{\sin^2\left(\dfrac{k\pi}{2n+1}\right)}{\sinh x} + \dfrac{1}{2}\tanh\left(\dfrac{x}{2}\right)\right)} = (2n+1)\coth\left(\frac{(2n+1)x}{2}\right) - \coth\frac{x}{2}$$ JO (409)

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.391

1. $$\sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \qquad [n \text{ is even}]$$ JO (568)

2. $$\cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}}\right) \qquad [n \text{ is even}]$$ JO (569)

3. $$\sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \qquad [n \text{ is odd}]$$ JO (570)

4. $$\cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}}\right) \qquad [n \text{ is odd}]$$ JO (571)a

1.392

1. $$\sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(x + \frac{k\pi}{n}\right)$$ JO (548)

2. $$\cos nx = 2^{n-1} \prod_{k=1}^{n} \sin\left(x + \frac{2k-1}{2n}\pi\right)$$ JO (549)

1.393

1. $$\prod_{k=0}^{n-1} \cos\left(x + \frac{2k}{n}\pi\right) = \frac{1}{2^{n-1}} \cos nx \qquad [n \text{ odd}]$$
$$= \frac{1}{2^{n-1}} \left[(-1)^{\frac{n}{2}} - \cos nx\right] \qquad [n \text{ even}]$$
JO (543)

2.[11] $$\prod_{k=0}^{n-1} \sin\left(x + \frac{2k}{n}\pi\right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \qquad [n \text{ odd}]$$
$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \qquad [n \text{ even}]$$
JO (544)

1.394 $$\prod_{k=0}^{n-1} \left\{x^2 - 2xy \cos\left(\alpha + \frac{2k\pi}{n}\right) + y^2\right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}$$ JO (573)

1.395

1. $$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{\cos x - \cos\left(y + \frac{2k\pi}{n}\right)\right\}$$ JO (573)

2. $$\cosh nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cosh x - \cos\left(y + \frac{2k\pi}{n}\right) \right\}$$ JO (538)

1.396

1. $$\prod_{k=1}^{n-1} \left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right) = \frac{x^{2n}-1}{x^2-1}$$ KR 58 (28.1)

2. $$\prod_{k=1}^{n} \left(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\right) = \frac{x^{2n+1}-1}{x-1}$$ KR 58 (28.2)

3. $$\prod_{k=1}^{n} \left(x^2 + 2x\cos\frac{2k\pi}{2n+1} + 1\right) = \frac{x^{2n+1}-1}{x+1}$$ KR 58 (28.3)

4. $$\prod_{k=0}^{n-1} \left(x^2 - 2x\cos\frac{(2k+1)\pi}{2n} + 1\right) = x^{2n}+1$$ KR 58 (28.4)

1.41 The expansion of trigonometric and hyperbolic functions in power series

1.411

1. $$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

2. $$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

3. $$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

4. $$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

5. $$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)}{(2k)!} |B_{2k}| x^{2k-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$ FI II 523

6.[11] $$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315}x^7 + \cdots = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)}{(2k)!} B_{2k} x^{2k-1}$$
$$\left[x^2 < \frac{\pi^2}{4}\right]$$

7. $$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} \qquad \left[x^2 < \pi^2\right]$$ FI II 523a

8. $$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \cdots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1}$$
$$\left[x^2 < \pi^2\right]$$ FI II 522a

9. $\sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k}$ $\left[x^2 < \frac{\pi^2}{4}\right]$ CE 330a

10. $\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}$
$\left[x^2 < \frac{\pi^2}{4}\right]$ CE 330

11. $\operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1}-1\right)|B_{2k}|x^{2k-1}}{(2k)!}$ $\left[x^2 < \pi^2\right]$ CE 329a

12. $\operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1}-1\right)B_{2k}}{(2k)!} x^{2k-1}$
$\left[x^2 < \pi^2\right]$ JO (418)

1.412

1. $\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}$ JO (452)a

2. $\cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}$ JO (443)

3. $\sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1}-3}{(2k+1)!} x^{2k+1}$ JO (452a)a

4. $\cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^{k} \frac{\left(3^{2k}+3\right)x^{2k}}{(2k)!}$ JO (443a)

1.413

1. $\sinh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!}$ JO (508)

2. $\cosh x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^{k} \frac{2^{2k} x^{4k}}{(4k)!}$ JO (507)

3. $\sinh x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}$ JO (510)

4. $\cosh x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}$ JO (509)

1.414

1. $\cos\left[n \ln\left(x + \sqrt{1+x^2}\right)\right] = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{\left(n^2+0^2\right)\left(n^2+2^2\right)\ldots\left[n^2+(2k)^2\right]}{(2k+2)!} x^{2k+2}$
$\left[x^2 < 1\right]$ AD (6456.1)

2. $$\sin\left[n\ln\left(x+\sqrt{1+x^2}\right)\right] = nx - n\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(n^2+1^2\right)\left(n^2+3^2\right)\dots\left[n^2+(2k-1)^2\right]x^{2k+1}}{(2k+1)!}$$
$\left[x^2<1\right]$ AD (6456.2)

Power series for $\ln\sin x$, $\ln\cos x$, and $\ln\tan x$ see **1.518**.

1.42 Expansion in series of simple fractions

1.421

1. $$\tan\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}$$ BR* (191), AD (6495.1)

2.[10] $$\tanh\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2+x^2}$$

3. $$\cot\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2-k^2} = \frac{1}{\pi x}+\frac{x}{\pi}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{k(x-k)}$$ AD (6495.2), JO (450a)

4. $$\coth\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2+k^2}$$ (cf. **1.217** 1)

5. $$\tan^2\frac{\pi x}{2} = x^2\sum_{k=1}^{\infty}\frac{2(2k-1)^2-x^2}{\left(1^2-x^2\right)^2\left(3^2-x^2\right)^2\dots\left[(2k-1)^2-x^2\right]^2}$$ JO (450)

1.422

1. $$\sec\frac{\pi x}{2} = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k-1}{(2k-1)^2-x^2}$$ AD (6495.3)a

2. $$\sec^2\frac{\pi x}{2} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\left\{\frac{1}{(2k-1-x)^2}+\frac{1}{(2k-1+x)^2}\right\}$$ JO (451)a

3. $$\operatorname{cosec}\pi x = \frac{1}{\pi x}+\frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{x^2-k^2}$$ (see also **1.217** 2) AD (6495.4)a

4. $$\operatorname{cosec}^2\pi x = \frac{1}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(x-k)^2} = \frac{1}{\pi^2 x^2}+\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{x^2+k^2}{\left(x^2-k^2\right)^2}$$ JO (446)

5. $$\frac{1+x\operatorname{cosec}x}{2x^2} = \frac{1}{x^2}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\left(x^2-k^2\pi^2\right)}$$ JO (449)

6. $$\operatorname{cosec}\pi x = \frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{x^2-k^2}$$ JO (450b)

1.423 $$\frac{\pi^2}{4m^2}\operatorname{cosec}^2\frac{\pi}{m}+\frac{\pi}{4m}\cot\frac{\pi}{m}-\frac{1}{2} = \sum_{k=1}^{\infty}\frac{1}{\left(1-k^2m^2\right)^2}$$ JO (477)

1.43 Representation in the form of an infinite product

1.431

1. $$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)$$ EU

2. $$\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2\pi^2}\right)$$ EU

3. $$\cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right)$$ EU

4. $$\cosh x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2\pi^2}\right)$$ EU

1.432

1.[11] $$\cos x - \cos y = 2\left(1 - \frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{(2k\pi + y)^2}\right)\left(1 - \frac{x^2}{(2k\pi - y)^2}\right)$$ AD (653.2)

2. $$\cosh x - \cos y = 2\left(1 + \frac{x^2}{y^2}\right)\sin^2\frac{y}{2}\prod_{k=1}^{\infty}\left(1 + \frac{x^2}{(2k\pi + y)^2}\right)\left(1 + \frac{x^2}{(2k\pi - y)^2}\right)$$ AD (653.1)

1.433 $$\cos\frac{\pi x}{4} - \sin\frac{\pi x}{4} = \prod_{k=1}^{\infty}\left[1 + \frac{(-1)^k x}{2k-1}\right]$$ BR* 189

1.434 $$\cos^2 x = \frac{1}{4}(\pi + 2x)^2 \prod_{k=1}^{\infty}\left[1 - \left(\frac{\pi + 2x}{2k\pi}\right)^2\right]^2$$ MO 216

1.435 $$\frac{\sin\pi(x+a)}{\sin\pi a} = \frac{x+a}{a}\prod_{k=1}^{\infty}\left(1 - \frac{x}{k-a}\right)\left(1 + \frac{x}{k+a}\right)$$ MO 216

1.436 $$1 - \frac{\sin^2\pi x}{\sin^2\pi a} = \prod_{k=-\infty}^{\infty}\left[1 - \left(\frac{x}{k-a}\right)^2\right]$$ MO 216

1.437 $$\frac{\sin 3x}{\sin x} = -\prod_{k=-\infty}^{\infty}\left[1 - \left(\frac{2x}{x + k\pi}\right)^2\right]$$ MO 216

1.438 $$\frac{\cosh x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty}\left[1 + \left(\frac{x}{2k\pi + a}\right)^2\right]$$ MO 216

1.439

1. $$\sin x = x\prod_{k=1}^{\infty}\cos\frac{x}{2^k}$$ $[|x| < 1]$ AD (615), MO 216

2. $$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left[1 - \frac{4}{3}\sin^2\left(\frac{x}{3^k}\right)\right]$$ MO 216

1.44–1.45 Trigonometric (Fourier) series

1.441

1. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad [0 < x < 2\pi]$$ FI III 539

2. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\frac{1}{2} \ln \left[2\left(1 - \cos x\right)\right] \qquad [0 < x < 2\pi]$$ FI III 530a, AD (6814)

3. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \qquad [-\pi < x < \pi]$$ FI III 542

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln\left(2 \cos \frac{x}{2}\right) \qquad [-\pi < x < \pi]$$ FI III 550

1.442

1.[11] $$\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\pi}{4} \operatorname{sign} x \qquad [-\pi < x < \pi]$$ FI III 541

2. $$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{2k-1} = \frac{1}{2} \ln \cot \frac{x}{2} \qquad [0 < x < \pi]$$

BR* 168, JO (266), GI III(195)

3. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2} \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$ BR* 168, JO (268)a

4.[10] $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k-1)x}{2k-1} = \frac{\pi}{4} \qquad \left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$
$$= -\frac{\pi}{4} \qquad \left[\frac{\pi}{2} < x < \frac{3\pi}{2}\right]$$

BR* 168, JO (269)

1.443

1.[8] $$\sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} = (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \rho^k$$
$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n}\left(\frac{x}{2}\right)$$
$$\left[0 \le x \le 2, \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340, GE 71

2. $$\sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} = (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k$$
$$= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{x}{2}\right)$$
$$\left[0 < x < 1; \quad \rho = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor\right]$$ CE 340

3. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \qquad [0 \le x \le 2\pi]$$ FI III 547

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \qquad [-\pi \le x \le \pi]$$ FI III 544

5. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \qquad [0 \le x \le 2\pi]$$

6. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} \qquad [0 \le x \le 2\pi]$$ AD (6617)

7. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} \qquad [0 \le x \le 2\pi]$$ AD (6818)

1.444

1. $$\sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x)\sin^2 x - \sin x \cos x \ln\left(4\sin^2 x\right) \qquad [0 \le x \le \pi]$$ BR* 168, GI III (190)

2. $$\sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right)\sin 2x + \sin^2 x \ln\left(4\sin^2 x\right) \qquad [0 \le x \le \pi]$$ BR* 168

3. $$\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2}(1 + \cos x) - \sin x \ln\left|2\cos\frac{x}{2}\right|$$ MO 213

4. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2}\sin x - (1 + \cos x)\ln\left|2\cos\frac{x}{2}\right|$$ MO 213

5. $$\sum_{k=0}^{\infty} (-1)^k \frac{\sin(2k+1)x}{(2k+1)^2} = \frac{\pi}{4}x \qquad \left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$$
$$= \frac{\pi}{4}(\pi - x) \qquad \left[\frac{\pi}{2} \le x \le \frac{3}{2}\pi\right]$$
MO 213

6.[6] $$\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right) \qquad [-\pi \le x \le \pi]$$ FI III 546

7. $$\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4}\sin x \qquad \left[0 \le x \le \frac{\pi}{2}\right]$$ JO (591)

1.445

1. $$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha(\pi - x)}{\sinh \alpha\pi} \qquad [0 < x < 2\pi]$$ BR* 157, JO (411)

2. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha(\pi - x)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \qquad [0 \le x \le 2\pi]$$ BR* 257, JO (410)

3. $$\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2+\alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha x}{\sinh \alpha\pi} - \frac{1}{2\alpha^2}$$ $[-\pi \le x \le \pi]$ FI III 546

4. $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2+\alpha^2} = \frac{\pi}{2} \frac{\sinh \alpha x}{\sinh \alpha\pi}$$ $[-\pi < x < \pi]$ FI III, 546

5. $$\sum_{k=1}^{\infty} \frac{k \sin kx}{k^2-\alpha^2} = \pi \frac{\sin\{\alpha[(2m+1)\pi - x]\}}{2\sin \alpha\pi}$$ $\left[\text{if } x = 2m\pi, \text{ then } \sum \dots = 0\right]$

$[2m\pi < x < (2m+2)\pi, \quad \alpha \text{ not an integer}]$ MO 213

6. $$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos[\alpha\{(2m+1)\pi - x\}]}{\alpha \sin \alpha\pi}$$

$[2m\pi \le x \le (2m+2)\pi, \quad \alpha \text{ not an integer}]$ MO 213

7. $$\sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2-\alpha^2} = \pi \frac{\sin[\alpha(2m\pi - x)]}{2\sin\alpha\pi}$$ $\left[\text{if } x = (2m+1)\pi, \text{ then } \sum \dots = 0\right]$,

$[(2m-1)\pi < x < (2m+1)\pi, \alpha \text{ not an integer}]$ FI III 545a

8. $$\sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2} \frac{\cos[\alpha(2m\pi - x)]}{\alpha \sin \alpha\pi}$$

$[(2m-1)\pi \le x \le (2m+1)\pi, \alpha \text{ not an integer}]$ FI III 545a

9.* $$\sum_{n=-\infty}^{\infty} \frac{e^{in\alpha}}{(n-\beta)^2+\gamma^2} = \frac{\pi}{\gamma} \frac{e^{i\beta(\alpha-2\pi)}\sinh(\gamma\alpha) + e^{i\beta\alpha}\sinh[\gamma(2\pi-\alpha)]}{\cosh(2\pi\gamma) - \cos(2\pi\beta)}$$

$[0 \le \alpha \le 2\pi]$

1.446 $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\cos(2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8}\cos^2 x - \frac{1}{3}\cos x$$

$\left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$ BR* 256, GI III (189)

1.447

1. $$\sum_{k=1}^{\infty} p^k \sin kx = \frac{p \sin x}{1 - 2p\cos x + p^2}$$

$[|p| < 1]$ FI II 559

2. $$\sum_{k=0}^{\infty} p^k \cos kx = \frac{1 - p\cos x}{1 - 2p\cos x + p^2}$$

$[|p| < 1]$ FI II 559

3. $$1 + 2\sum_{k=1}^{\infty} p^k \cos kx = \frac{1-p^2}{1 - 2p\cos x + p^2}$$

$[|p| < 1]$ FI II 559a, MO 213

1.448

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} = \arctan \frac{p \sin x}{1 - p \cos x}$$
$[0 < x < 2\pi, \quad p^2 \leq 1]$ FI II 559

2. $$\sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} = -\frac{1}{2} \ln \left(1 - 2p \cos x + p^2\right)$$
$[0 < x < 2\pi, \quad p^2 \leq 1]$ FI II 559

3. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \sin x}{1 - p^2}$$
$[0 < x < 2\pi, \quad p^2 \leq 1]$ JO (594)

4. $$\sum_{k=1}^{\infty} \frac{p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2}$$
$[0 < x < 2\pi, \quad p^2 \leq 1]$ JO (259)

5. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \frac{1}{4} \ln \frac{1 + 2p \sin x + p^2}{1 - 2p \sin x + p^2}$$
$[0 < x < \pi, \quad p^2 \leq 1]$ JO (261)

6. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \frac{1}{2} \arctan \frac{2p \cos x}{1 - p^2}$$
$[0 < x < \pi, \quad p^2 \leq 1]$ JO (597)

1.449

1. $$\sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin (p \sin x)$$
$[p^2 \leq 1]$ JO (486)

2. $$\sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos (p \sin x)$$
$[p^2 \leq 1]$ JO (485)

Let $S(x) = -\frac{1}{x} \cos x + \frac{1}{x}$ and $C(x) = \frac{1}{x} \sin x$.

3.* $$\sum_{n=1}^{\infty} \frac{n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \left[C(ax) - \cot(\pi a) S(ax)\right]$$ $[0 < x < 2\pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

4.* $$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} C(nx) = \frac{1}{2a^2} - \frac{\pi}{2a} \left[S(ax) - \cot(\pi a) C(ax)\right]$$
$[0 \leq x \leq 2\pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

5.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2 - a^2} S(nx) = \frac{\pi}{2} \operatorname{cosec}(\pi a) S(ax)$$ $[-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

6.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2} C(nx) = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cosec}(\pi a) C(ax)$$ $[-\pi < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

7.* $$\sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4} \left[C(ax) + \tan\left(\frac{\pi a}{2}\right) S(ax)\right]$$ $[0 < x < \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

8.* $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2-a^2} C(nx) = -\frac{\pi}{4a} \left[S(ax) - \tan\left(\frac{\pi a}{2}\right) C(ax)\right]$$ $[0 \leq x \leq \pi, \quad a \neq 0, \pm 1, \pm 2, \ldots]$

9.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2-a^2} S(nx) = \frac{\pi}{4a} \sec\left(\frac{\pi a}{2}\right) S(ax)$$ $\left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$

10.* $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2-a^2} C(nx) = \frac{\pi}{4} \sec\left(\frac{\pi a}{2}\right) C(ax)$$ $\left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad a \neq 0, \pm 1, \pm 2, \ldots\right]$

Fourier expansions of hyperbolic functions

1.451

1. $$\sinh x = \cos x \sum_{k=0}^{\infty} \frac{\left(1^2+0^2\right)\left(1^2+2^2\right)\ldots\left[1^2+(2k)^2\right]}{(2k+1)!} \sin^{2k+1} x$$ JO (504)

2. $$\cosh x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{\left(1^2+1^2\right)\left(1^2+3^2\right)\ldots\left[1^2+(2k-1)^2\right]}{(2k)!} \sin^{2k} x$$ JO (503)

1.452

1. $$\sinh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\cos(2k+1)\theta}{(2k+1)!}$$ $\left[x^2 < 1\right]$ JO (391)

2. $$\cosh(x\cos\theta) = \sec(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k}\cos 2k\theta}{(2k)!}$$ $\left[x^2 < 1\right]$ JO (390)

3. $$\sinh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=1}^{\infty} \frac{x^{2k}\sin 2k\theta}{(2k)!}$$ $\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$ JO (393)

4. $$\cosh(x\cos\theta) = \operatorname{cosec}(x\sin\theta) \sum_{k=0}^{\infty} \frac{x^{2k+1}\sin(2k+1)\theta}{(2k+1)!}$$ $\left[x^2 < 1, \quad x\sin\theta \neq 0\right]$ JO (392)

1.46 Series of products of exponential and trigonometric functions

1.461

1. $$\sum_{k=0}^{\infty} e^{-kt} \sin kx = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x} \qquad [t > 0]$$ MO 213

2. $$1 + 2\sum_{k=1}^{\infty} e^{-kt} \cos kx = \frac{\sinh t}{\cosh t - \cos x} \qquad [t > 0]$$ MO 213

1.462[9] $$\sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[\frac{\sin^2 \frac{x+y}{2} + \sinh^2 t}{\sin^2 \frac{x-y}{2} + \sinh^2 t} \right]$$ MO 214

1.463

1. $$e^{x \cos\varphi} \cos(x \sin\varphi) = \sum_{n=0}^{\infty} \frac{x^n \cos n\varphi}{n!} \qquad \left[x^2 < 1\right]$$ AD (6476.1)

2. $$e^{x \cos\varphi} \sin(x \sin\varphi) = \sum_{n=1}^{\infty} \frac{x^n \sin n\varphi}{n!} \qquad \left[x^2 < 1\right]$$ AD (6476.2)

1.47 Series of hyperbolic functions

1.471

1. $$\sum_{k=1}^{\infty} \frac{\sinh kx}{k!} = e^{\cosh x} \sinh(\sinh x).$$ JO (395)

2. $$\sum_{k=0}^{\infty} \frac{\cosh kx}{k!} = e^{\cosh x} \cosh(\sinh x).$$ JO (394)

3. $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \tanh \frac{(2m+1)\pi x}{2} + x \tanh \frac{(2m+1)\pi}{2x} \right] = \frac{\pi^3}{16}$$

1.472

1. $$\sum_{k=1}^{\infty} p^k \sinh kx = \frac{p \sinh x}{1 - 2p\cosh x + p^2} \qquad \left[p^2 < 1\right]$$ JO (396)

2. $$\sum_{k=0}^{\infty} p^k \cosh kx = \frac{1 - p\cosh x}{1 - 2p\cosh x + p^2} \qquad \left[p^2 < 1\right]$$ JO (397)a

1.48 Lobachevskiy's "Angle of Parallelism" $\Pi(x)$

1.480 Definition.

1. $$\Pi(x) = 2 \operatorname{arccot} e^x = 2 \arctan e^{-x} \qquad [x \ge 0]$$ LO III 297, LOI 120

2. $\Pi(x) = \pi - \Pi(-x)$ $[x < 0]$ LO III 183, LOI 193

1.481 Functional relations

1. $\sin \Pi(x) = \dfrac{1}{\cosh x}$ LO III 297

2. $\cos \Pi(x) = \tanh x$ LO III 297

3. $\tan \Pi(x) = \dfrac{1}{\sinh x}$ LO III 297

4. $\cot \Pi(x) = \sinh x$ LO III 297

5. $\sin \Pi(x+y) = \dfrac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$ LO III 297

6. $\cos \Pi(x+y) = \dfrac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$ LO III 183

1.482 Connection with the Gudermannian.

$$\operatorname{gd}(-x) = \Pi(x) - \frac{\pi}{2}$$

(Definite) integral of the angle of parallelism: cf. **4.581** and **4.561**.

1.49 The hyperbolic amplitude (the Gudermannian) $\operatorname{gd} x$

1.490 Definition.

1. $\operatorname{gd} x = \displaystyle\int_0^x \frac{dt}{\cosh t} = 2 \arctan e^x - \frac{\pi}{2}$ JA

2. $x = \displaystyle\int_0^{\operatorname{gd} x} \frac{dt}{\cos t} = \ln \tan \left(\frac{\operatorname{gd} x}{2} + \frac{\pi}{4} \right)$ JA

1.491 Functional relations.

1. $\cosh x = \sec(\operatorname{gd} x)$ AD (343.1), JA

2. $\sinh x = \tan(\operatorname{gd} x)$ AD (343.2), JA

3. $e^x = \sec(\operatorname{gd} x) + \tan(\operatorname{gd} x) = \tan \left(\dfrac{\pi}{4} + \dfrac{\operatorname{gd} x}{2} \right) = \dfrac{1 + \sin(\operatorname{gd} x)}{\cos(\operatorname{gd} x)}$ AD (343.5), JA

4. $\tanh x = \sin(\operatorname{gd} x)$ AD (343.3), JA

5. $\tanh \dfrac{x}{2} = \tan \left(\dfrac{1}{2} \operatorname{gd} x \right)$ AD (343.4), JA

6. $\arctan (\tanh x) = \dfrac{1}{2} \operatorname{gd} 2x$ AD (343.6a)

1.492 If $\gamma = \operatorname{gd} x$, then $ix = \operatorname{gd} i\gamma$ JA

1.493 Series expansion.

1. $\dfrac{\operatorname{gd} x}{2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tanh^{2k+1} \frac{x}{2}$ JA

2. $$\frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \tan^{2k+1}\left(\frac{1}{2}\operatorname{gd} x\right)$$ JA

3. $$\operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \cdots$$ JA

4. $$x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \ldots \qquad \left[\operatorname{gd} x < \frac{\pi}{2}\right]$$ JA

1.5 The Logarithm

1.51 Series representation

1.511 $$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$$

$$[-1 < x \le 1]$$

1.512

1. $$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$$

$$[0 < x \le 2]$$

2. $$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right] = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}$$

$$[0 < x]$$

3. $$\ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x-1}{x}\right)^k$$

$$\left[x \ge \tfrac{1}{2}\right]$$ AD (644.6)

4.* $$\ln x = \lim_{\epsilon \to 0}\left(\frac{x^{\epsilon} - 1}{\epsilon}\right)$$

1.513

1. $$\ln\frac{1+x}{1-x} = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}x^{2k-1} \qquad \left[x^2 < 1\right]$$ FI II 421

2. $$\ln\frac{x+1}{x-1} = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)x^{2k-1}} \qquad \left[x^2 > 1\right]$$ AD (644.9)

3. $$\ln\frac{x}{x-1} = \sum_{k=1}^{\infty}\frac{1}{kx^k} \qquad [x \le -1 \text{ or } x > 1]$$ JO (88a)

4. $$\ln\frac{1}{1-x} = \sum_{k=1}^{\infty}\frac{x^k}{k} \qquad [-1 \le x < 1]$$ JO (88b)

5. $$\frac{1-x}{x}\ln\frac{1}{1-x} = 1 - \sum_{k=1}^{\infty}\frac{x^k}{k(k+1)} \qquad [-1 \le x < 1]$$ JO (102)

6. $$\frac{1}{1-x}\ln\frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^{k} \frac{1}{n} \qquad [x^2 < 1]$$ JO (88e)

7. $$\frac{(1-x)^2}{2x^3}\ln\frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \qquad [-1 \le x < 1]$$ AD (6445.1)

1.514 $$\ln\left(1 - 2x\cos\varphi + x^2\right) = -2\sum_{k=1}^{\infty}\frac{\cos k\varphi}{k}x^k; \quad \ln\left(x + \sqrt{1+x^2}\right) = \operatorname{arcsinh} x$$
(see **1.631, 1.641, 1.642, 1.646**) $[x^2 \le 1, \quad x\cos\varphi \ne 1]$ MO 98, FI II 485

1.515

1.[11] $$\ln\left(1 + \sqrt{1+x^2}\right) = \ln 2 + \frac{1\cdot 1}{2\cdot 2}x^2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}x^4 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}x^6 - \dots$$
$$= \ln 2 - \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)!}{2^{2k}\,(k!)^2}x^{2k}$$
$[x^2 \le 1]$ JO (91)

2. $$\ln\left(1 + \sqrt{1+x^2}\right) = \ln x + \frac{1}{x} - \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \dots$$
$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)!}{2^{2k-1}\cdot k!(k-1)!(2k+1)x^{2k+1}}$$
$[x^2 \ge 1]$ AD (644.4)

3. $$\sqrt{1+x^2}\ln\left(x + \sqrt{1+x^2}\right) = x - \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k-1}(k-1)!k!}{(2k+1)!}x^{2k+1}$$
$[x^2 \le 1]$ JO (93)

4. $$\frac{\ln\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}\,(k!)^2}{(2k+1)!}x^{2k+1} \qquad [x^2 \le 1]$$ JO (94)

1.516

1. $$\frac{1}{2}\left\{\ln(1 \pm x)\right\}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1}\,x^{k+1}}{k+1}\sum_{n=1}^{k}\frac{1}{n} \qquad [x^2 < 1]$$ JO (86), JO (85)

2. $$\frac{1}{6}\left\{\ln(1+x)\right\}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{k+2}}{k+2}\sum_{n=1}^{k}\frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m} \qquad [x^2 < 1]$$ AD (644.14)

3. $$-\ln(1+x)\cdot\ln(1-x) = \sum_{k=1}^{\infty}\frac{x^{2k}}{k}\sum_{n=1}^{2k-1}\frac{(-1)^{n+1}}{n} \qquad [x^2 < 1]$$ JO (87)

4. $$\frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}} + 2\ln(1-x)\right\} = \frac{1}{2x} + \sum_{k=1}^{\infty}\frac{x^{k-1}}{(2k-1)2k(2k+1)}$$
$[0 < x < 1]$ AD (6445.2)

1.517

1.[6] $$\frac{1}{2x}\left\{1-\ln(1+x)-\frac{1-x}{\sqrt{x}}\arctan\sqrt{x}\right\}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k-1}}{(2k-1)2k(2k+1)}$$

$[0 < x \le 1]$ AD (6445.3)

2. $$\frac{1}{2}\arctan x\ln\frac{1+x}{1-x}=\sum_{k=1}^{\infty}\frac{x^{4k-2}}{2k-1}\sum_{n=1}^{2k-1}\frac{(-1)^{n-1}}{2n-1}$$ $[x^2 < 1]$ BR* 163

3. $$\frac{1}{2}\arctan x\ln\left(1+x^2\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{2k+1}}{2k+1}\sum_{n=1}^{2k}\frac{1}{n}$$ $[x^2 \ge 1]$ AD (6455.3)

1.518

1. $$\ln\sin x=\ln x-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\dots$$
$$=\ln x+\sum_{k=1}^{\infty}\frac{(-1)^k2^{2k-1}B_{2k}x^{2k}}{k(2k)!}$$

$[0 < x < \pi]$ AD (643.1)a

2.[3] $$\ln\cos x=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\dots$$
$$=-\sum_{k=1}^{\infty}\frac{2^{2k-1}\left(2^{2k}-1\right)|B_{2k}|}{k(2k)!}x^{2k}=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin^{2k}x}{k}$$

$\left[x^2 < \frac{\pi^2}{4}\right]$ FI II 524

3. $$\ln\tan x=\ln x+\frac{x^2}{3}+\frac{7}{90}x^4+\frac{62}{2835}x^6+\frac{127}{18,900}x^8+\dots$$
$$=\ln x+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(2^{2k-1}-1\right)2^{2k}B_{2k}x^{2k}}{k(2k)!}$$

$\left[0 < x < \frac{\pi}{2}\right]$ AD (643.3)a

1.52 Series of logarithms (cf. 1.431)

1.521

1. $$\sum_{k=1}^{\infty}\ln\left(1-\frac{4x^2}{(2k-1)^2\pi^2}\right)=\ln\cos x$$ $\left[-\frac{\pi}{2} < x < \frac{\pi}{2}\right]$

2. $$\sum_{k=1}^{\infty}\ln\left(1-\frac{x^2}{k^2\pi^2}\right)=\ln\sin x-\ln x$$ $[0 < x < \pi]$

1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

1. $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}; \quad 0 \le \arccos x \le \pi$ $\quad [-1 \le x \le 1]$ FI II 553

2. $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}; \quad 0 < \operatorname{arccot} x < \pi$ $\quad [-\infty < x < +\infty]$ FI II 552

1.62–1.63 Functional relations

1.621 The relationship between the inverse and the direct trigonometric functions.

1. $\arcsin(\sin x) = x - 2n\pi \quad \left[2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}\right]$

   $= -x + (2n+1)\pi \quad \left[(2n+1)\pi - \frac{\pi}{2} \le x \le (2n+1)\pi + \frac{\pi}{2}\right]$

2. $\arccos(\cos x) = x - 2n\pi \quad [2n\pi \le x \le (2n+1)\pi]$

   $= -x + 2(n+1)\pi \quad [(2n+1)\pi \le x \le 2(n+1)\pi]$

3. $\arctan(\tan x) = x - n\pi \quad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right]$

4. $\operatorname{arccot}(\cot x) = x - n\pi \quad [n\pi < x < (n+1)\pi]$

1.622 The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

1. $\arcsin z = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) = \frac{1}{i}\operatorname{arcsinh}(iz)$

2. $\arccos z = \frac{1}{i}\ln\left(z + \sqrt{z^2-1}\right) = \frac{1}{i}\operatorname{arccosh} z$

3. $\arctan z = \frac{1}{2i}\ln\frac{1+iz}{1-iz} = \frac{1}{i}\operatorname{arctanh}(iz)$

4. $\operatorname{arccot} z = \frac{1}{2i}\ln\frac{iz-1}{iz+1} = i\operatorname{arccoth}(iz)$

5. $\operatorname{arcsinh} z = \ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{i}\arcsin(iz)$

6. $\operatorname{arccosh} z = \ln\left(z + \sqrt{z^2-1}\right) = i\arccos z$

7. $\operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z} = \frac{1}{i}\arctan(iz)$

8. $\operatorname{arccoth} z = \frac{1}{2}\ln\frac{z+1}{z-1} = \frac{1}{i}\operatorname{arccot}(-iz)$

Relations between different inverse trigonometric functions

1.623

1. $\arcsin x + \arccos x = \frac{\pi}{2}$ NV 43

2. $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ NV 43

1.624

1. $\arcsin x = \arccos \sqrt{1-x^2} \quad [0 \le x \le 1]$
   $= -\arccos \sqrt{1-x^2} \quad [-1 \le x \le 0]$ NV 47 (5), NV 46 (2)

2. $\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}} \quad [x^2 < 1]$

3. $\arcsin x = \operatorname{arccot} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \le 1]$
   $= \operatorname{arccot} \frac{\sqrt{1-x^2}}{x} - \pi \quad [-1 \le x < 0]$ NV 49 (10)

4. $\arccos x = \arcsin \sqrt{1-x^2} \quad [0 \le x \le 1]$
   $= \pi - \arcsin \sqrt{1-x^2} \quad [-1 \le x \le 0]$ NV 48 (6)

5. $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x} \quad [0 < x \le 1]$
   $= \pi + \arctan \frac{\sqrt{1-x^2}}{x} \quad [-1 \le x < 0]$ NV 48 (8)

6. $\arccos x = \operatorname{arccot} \frac{x}{\sqrt{1-x^2}} \quad [-1 \le x < 1]$ NV 46 (4)

7. $\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}$ NV 6 (3)

8. $\arctan x = \arccos \frac{1}{\sqrt{1+x^2}} \quad [x \ge 0]$
   $= -\arccos \frac{1}{\sqrt{1+x^2}} \quad [x \le 0]$ NV 48 (7)

9. $\arctan x = \operatorname{arccot} \frac{1}{x} \quad [x > 0]$
   $= -\operatorname{arccot} \frac{1}{x} - \pi \quad [x < 0]$ NV 49 (9)

10.[11] $\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x > 0]$
   $= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} \quad [x < 0]$ NV 49 (11)

11. $\operatorname{arccot} x = \arccos \frac{x}{\sqrt{1+x^2}}$ NV 46 (4)

12. $$\begin{aligned} \operatorname{arccot} x &= \arctan \frac{1}{x} && [x > 0] \\ &= \pi + \arctan \frac{1}{x} && [x < 0] \end{aligned}$$ NV 49 (12)

1.625

1. $$\begin{aligned} \arcsin x + \arcsin y &= \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) && \left[xy \le 0 \text{ or } x^2+y^2 \le 1\right] \\ &= \pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) && \left[x > 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right] \\ &= -\pi - \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) && \left[x < 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right] \end{aligned}$$ NV 54(1), GI I (880)

2. $$\begin{aligned} \arcsin x + \arcsin y &= \arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) && [x \ge 0, \quad y \ge 0] \\ &= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right) && [x < 0, \quad y < 0] \end{aligned}$$ NV 55

3. $$\begin{aligned} \arcsin x + \arcsin y &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} && \left[xy \le 0 \text{ or } x^2+y^2 < 1\right] \\ &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi && \left[x > 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right] \\ &= \arctan \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi && \left[x < 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right] \end{aligned}$$ NV 56

4. $$\begin{aligned} \arcsin x - \arcsin y &= \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) && \left[xy \ge 0 \text{ or } x^2+y^2 \le 1\right] \\ &= \pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) && \left[x > 0, \quad y < 0 \text{ and } x^2+y^2 > 1\right] \\ &= -\pi - \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) && \left[x < 0, \quad y > 0 \text{ and } x^2+y^2 > 1\right] \end{aligned}$$ NV 55(2)

5. $$\begin{aligned} \arcsin x - \arcsin y &= \arccos\left(x\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) && [xy > y] \\ &= -\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} + xy\right) && [x < y] \end{aligned}$$ NV 56

6. $$\begin{aligned} \arccos x + \arccos y &= \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) && [x + y \ge 0] \\ &= 2\pi - \arccos\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) && [x + y < 0] \end{aligned}$$ NV 57 (3)

7.[11] $$\begin{aligned} \arccos x - \arccos y &= -\arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) && [x \ge y] \\ &= \arccos\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) && [x < y] \end{aligned}$$ NV 57 (4)

8. $$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy} \qquad [xy < 1]$$
$$= \pi + \arctan \frac{x+y}{1-xy} \qquad [x > 0, \quad xy > 1]$$
$$= -\pi + \arctan \frac{x+y}{1-xy} \qquad [x < 0, \quad xy > 1]$$
NV 59(5), GI I (879)

9. $$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy} \qquad [xy > -1]$$
$$= \pi + \arctan \frac{x-y}{1+xy} \qquad [x > 0, \quad xy < -1]$$
$$= -\pi + \arctan \frac{x-y}{1+xy} \qquad [x < 0, \quad xy < -1]$$
NV 59(6)

1.626

1. $$2 \arcsin x = \arcsin\left(2x\sqrt{1-x^2}\right) \qquad \left[|x| \le \frac{1}{\sqrt{2}}\right]$$
$$= \pi - \arcsin\left(2x\sqrt{1-x^2}\right) \qquad \left[\frac{1}{\sqrt{2}} < x \le 1\right]$$
$$= -\pi - \arcsin\left(2x\sqrt{1-x^2}\right) \qquad \left[-1 \le x < -\frac{1}{\sqrt{2}}\right]$$
NV 61 (7)

2. $$2 \arccos x = \arccos\left(2x^2 - 1\right) \qquad [0 \le x \le 1]$$
$$= 2\pi - \arccos\left(2x^2 - 1\right) \qquad [-1 \le x < 0]$$
NV 61 (8)

3. $$2 \arctan x = \arctan \frac{2x}{1-x^2} \qquad [|x| < 1]$$
$$= \arctan \frac{2x}{1-x^2} + \pi \qquad [x > 1]$$
$$= \arctan \frac{2x}{1-x^2} - \pi \qquad [x < -1]$$
NV 61 (9)

1.627

1. $$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2} \qquad [x > 0]$$
$$= -\frac{\pi}{2} \qquad [x < 0]$$
GI I (878)

2. $$\arctan x + \arctan \frac{1-x}{1+x} = \frac{\pi}{4} \qquad [x > -1]$$
$$= -\frac{3}{4}\pi \qquad [x < -1]$$
NV 62, GI I (881)

1.628

1. $$\arcsin\frac{2x}{1+x^2} = -\pi - 2\arctan x \qquad [x \le -1]$$
$$= 2\arctan x \qquad [-1 \le x \le 1]$$
$$= \pi - 2\arctan x \qquad [x \ge 1]$$
NV 65

2. $$\arccos\frac{1-x^2}{1+x^2} = 2\arctan x \qquad [x \ge 0]$$
$$= -2\arctan x \qquad [x \le 0]$$
NV 66

1.629 $$\frac{2x-1}{2} - \frac{1}{\pi}\arctan\left(\tan\frac{2x-1}{2}\pi\right) = E(x)$$ GI (886)

1.631 Relations between the inverse hyperbolic functions.

1. $$\operatorname{arcsinh} x = \operatorname{arccosh}\sqrt{x^2+1} = \operatorname{arctanh}\frac{x}{\sqrt{x^2+1}}$$ JA

2. $$\operatorname{arccosh} x = \operatorname{arcsinh}\sqrt{x^2-1} = \operatorname{arctanh}\frac{\sqrt{x^2-1}}{x}$$ JA

3. $$\operatorname{arctanh} x = \operatorname{arcsinh}\frac{x}{\sqrt{1-x^2}} = \operatorname{arccosh}\frac{1}{\sqrt{1-x^2}} = \operatorname{arccoth}\frac{1}{x}$$ JA

4. $$\operatorname{arcsinh} x \pm \operatorname{arcsinh} y = \operatorname{arcsinh}\left(x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right)$$ JA

5. $$\operatorname{arccosh} x \pm \operatorname{arccosh} y = \operatorname{arccosh}\left(xy \pm \sqrt{\left(x^2-1\right)\left(y^2-1\right)}\right)$$ JA

6. $$\operatorname{arctanh} x \pm \operatorname{arctanh} y = \operatorname{arctanh}\frac{x \pm y}{1 \pm xy}$$ JA

1.64 Series representations

1.641

1. $$\arcsin x = \frac{\pi}{2} - \arccos x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \dots$$
$$= \sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}\left(k!\right)^2(2k+1)}x^{2k+1} = x\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right)$$
$$\left[x^2 \le 1\right]$$ FI II 479

2. $$\operatorname{arcsinh} x = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \dots;$$
$$= \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}\left(k!\right)^2(2k+1)}x^{2k+1}$$
$$= x\,F\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-x^2\right)$$
$$\left[x^2 \le 1\right]$$ FI II 480

1.642

1. $$\operatorname{arcsinh} x = \ln 2x + \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} + \dots$$
$$= \ln 2x + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2k)!x^{-2k}}{2^{2k}\left(k!\right)^2 2k} \qquad [x \geq 1]$$
AD (6480.2)a

2. $$\operatorname{arccosh} x = \ln 2x - \sum_{k=1}^{\infty}\frac{(2k)!x^{-2k}}{2^{2k}\left(k!\right)^2 2k} \qquad [x \geq 1]$$ AD (6480.3)a

1.643

1. $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$$
$$\left[x^2 \leq 1\right]$$ FI II 479

2. $$\operatorname{arctanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1} \qquad \left[x^2 < 1\right]$$ AD (6480.4)

1.644

1. $$\arctan x = \frac{x}{\sqrt{1+x^2}}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}\left(k!\right)^2(2k+1)}\left(\frac{x^2}{1+x^2}\right)^k$$
$$= \frac{x}{\sqrt{1+x^2}}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{x^2}{1+x^2}\right) \qquad \left[x^2 < \infty\right]$$
AD (641.3)

2. $$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)x^{2k+1}}$$ AD (641.4)

1.645

1. $$\operatorname{arcsec} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2\cdot 3x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5x^5} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{(2k)!x^{-(2k+1)}}{\left(k!\right)^2 2^{2k}(2k+1)}$$
$$= \frac{\pi}{2} - \frac{1}{x}F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{1}{x^2}\right) \qquad \left[x^2 > 1\right]$$
AD (641.5)

2. $$(\arcsin x)^2 = \sum_{k=0}^{\infty}\frac{2^{2k}\left(k!\right)^2 x^{2k+2}}{(2k+1)!(k+1)} \qquad \left[x^2 \leq 1\right]$$ AD (642.2), GI III (152)a

3. $$(\arcsin x)^3 = x^3 + \frac{3!}{5!}3^2\left(1+\frac{1}{3^2}\right)x^5 + \frac{3!}{7!}3^2\cdot 5^2\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)x^7 + \dots$$
$$\left[x^2 \leq 1\right]$$
BR* 188, AD (642.2), GI III (153)a

1.646

1. $$\operatorname{arcsinh}\frac{1}{x} = \operatorname{arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}(k!)^2(2k+1)} x^{-2k-1}$$
$\left[x^2 \geq 1\right]$ AD (6480.5)

2. $$\operatorname{arccosh}\frac{1}{x} = \operatorname{arcsech} x = \ln\frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2\, 2k} x^{2k}$$
$[0 < x \leq 1]$ AD (6480.6)

3. $$\operatorname{arcsinh}\frac{1}{x} = \operatorname{arcosech} x = \ln\frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k)!}{2^{2k}(k!)^2\, 2k} x^{2k}$$
$[0 < x \leq 1]$ AD (6480.7)a

4. $$\operatorname{arctanh}\frac{1}{x} = \operatorname{arccoth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1}$$
$\left[x^2 > 1\right]$ AD (6480.8)

1.647

1. $$\sum_{k=1}^{\infty} \frac{\tanh(2k-1)(\pi/2)}{(2k-1)^{4n+3}} = \frac{\pi^{4n+3}}{2}\left(2\sum_{j=1}^{n} \frac{(-1)^{j-1}\left(2^{2j}-1\right)\left(2^{4n-2j+4}-1\right)B^*_{2j-1}B^*_{4n-2j+3}}{(2j)!(4n-2j+4)!} + \frac{(-1)^n\left(2^{2n+2}-1\right)^2 {B^*_{2n+1}}^2}{[(2n+2)!]^2}\right)$$
$n = 0, 1, 2, \ldots,$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\operatorname{sech}(2k-1)(\pi/2)}{(2k-1)^{4n+1}} = \frac{\pi^{4n+1}}{2^{4n+3}}\left(2\sum_{j=1}^{n-1} \frac{(-1)^j B^*_{2j}B^*_{4n-2j}}{(2j)!(4n-2j)!} + \frac{2B^*_{4n}}{(4n)!} + \frac{(-1)^n {B^*_{2n}}^2}{[(2n)]!^2}\right),$$
$n = 1, 2, \ldots$

(The summation term on the right is to be omitted for $n = 1$.) (See page xxxiii for the definition of B^*_r.)

2 Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign (=) means that the functions on the left and right of this symbol differ by a constant. For example (see **201** 15), we write

$$\int \frac{dx}{1+x^2} = \arctan x = -\arctan x$$

although

$$\arctan x = -\arctan x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example, $\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{1+x^2}\right|$). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula **2.02** 8 for $n = -1$ or formula **2.02** 15 for $a = b$). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right-hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters x, y, t, $\ldots$ denote independent variables; f, g, φ, $\ldots$ denote functions of x, y, t, $\ldots$; f', g', φ', $\ldots$, f'', g'', φ'', $\ldots$ denote their first, second, etc., derivatives; a, b, m, p, $\ldots$ denote constants, by which we generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is made, provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas **2.148** 4 and **2.424** 6, we make no remark since it is clear from the form of these formulas themselves that n must be a natural number (that is, a positive integer).

2.01 The basic integrals

1. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1)$$

2. $$\int \frac{dx}{x} = \ln x$$

3. $$\int e^x\,dx = e^x$$

4. $$\int a^x\,dx = \frac{a^x}{\ln a}$$

5. $$\int \sin x\,dx = -\cos x$$

6.[11] $$\int \cos x\,dx = \sin x$$

7. $$\int \frac{dx}{\sin^2 x} = -\cot x$$

8.[11] $$\int \frac{dx}{\cos^2 x} = \tan x$$

9. $$\int \frac{\sin x}{\cos^2 x}\,dx = \sec x$$

10. $$\int \frac{\cos x}{\sin^2 x}\,dx = -\operatorname{cosec} x$$

11. $$\int \tan x\,dx = -\ln\cos x$$

12. $$\int \cot x\,dx = \ln\sin x$$

13. $$\int \frac{dx}{\sin x} = \ln\tan\frac{x}{2}$$

14. $$\int \frac{dx}{\cos x} = \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \ln\left(\sec x + \tan x\right)$$

15. $$\int \frac{dx}{1+x^2} = \arctan x = \frac{\pi}{2} - \operatorname{arccot} x$$

16. $$\int \frac{dx}{1-x^2} = \operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

17. $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x = -\arccos x$$

18. $$\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

19. $$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

20. $$\int \sinh x\,dx = \cosh x$$

21. $$\int \cosh x\,dx = \sinh x$$

22.[11] $$\int \frac{dx}{\sinh^2 x} = -\coth x$$

23. $$\int \frac{dx}{\cosh^2 x} = \tanh x$$

24. $$\int \tanh x\,dx = \ln\cosh x$$

25. $$\int \coth x\,dx = \ln\sinh x$$

26. $$\int \frac{dx}{\sinh x} = \ln\tanh\frac{x}{2}$$

2.02 General formulas

1. $\int af\,dx = a\int f\,dx$

2. $\int [af \pm b\varphi \pm c\psi \pm \ldots]\,dx = a\int f\,dx \pm b\int \varphi\,dx \pm c\int \psi\,dx \pm \ldots$

3. $\frac{d}{dx}\int f\,dx = f$

4. $\int f'\,dx = f$

5. $\int f'\varphi\,dx = f\varphi - \int f\varphi'\,dx$ [integration by parts]

6. $\int f^{(n+1)}\varphi\,dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \ldots + (-1)^n \varphi^{(n)} f + (-1)^{n+1}\int \varphi^{(n+1)} f\,dx$

7. $\int f(x)\,dx = \int f[\varphi(y)]\varphi'(y)\,dy$ $[x = \varphi(y)]$ [change of variable]

8.[11] $\int (f)^n f'\,dx = \frac{(f)^{n+1}}{n+1}$ $[n \neq -1]$

For $n = -1$

$\int \frac{f'\,dx}{f} = \ln f$

9. $\int (af+b)^n f'\,dx = \frac{(af+b)^{n+1}}{a(n+1)}$

10. $\int \frac{f'\,dx}{\sqrt{af+b}} = \frac{2\sqrt{af+b}}{a}$

11. $\int \frac{f'\varphi - \varphi' f}{\varphi^2}\,dx = \frac{f}{\varphi}$

12. $\int \frac{f'\varphi - \varphi' f}{f\varphi}\,dx = \ln\frac{f}{\varphi}$

13. $\int \frac{dx}{f\,(f \pm \varphi)} = \pm\int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi\,(f \pm \varphi)}$

14. $\int \frac{f'\,dx}{\sqrt{f^2+a}} = \ln\left(f + \sqrt{f^2+a}\right)$

15. $\int \frac{f\,dx}{(f+a)(f+b)} = \frac{a}{a-b}\int \frac{dx}{(f+a)} - \frac{b}{a-b}\int \frac{dx}{(f+b)}$

For $a = b$

$\int \frac{f\,dx}{(f+a)^2} = \int \frac{dx}{f+a} - a\int \frac{dx}{(f+a)^2}$

16. $\int \frac{f\,dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi\,dx}{(f+\varphi)^n}$

17. $\int \frac{f'\,dx}{p^2+q^2f^2} = \frac{1}{pq}\arctan\frac{qf}{p}$

18. $$\int \frac{f'\,dx}{q^2f^2-p^2} = \frac{1}{2pq}\ln\frac{qf-p}{qf+p}$$

19. $$\int \frac{f\,dx}{1-f} = -x + \int \frac{dx}{1-f}$$

20. $$\int \frac{f^2\,dx}{f^2-a^2} = \frac{1}{2}\int\frac{f\,dx}{f-a} + \frac{1}{2}\int\frac{f\,dx}{f+a}$$

21. $$\int \frac{f'\,dx}{\sqrt{a^2-f^2}} = \arcsin\frac{f}{a}$$

22. $$\int \frac{f'\,dx}{af^2+bf} = \frac{1}{b}\ln\frac{f}{af+b}$$

23. $$\int \frac{f'\,dx}{f\sqrt{f^2-a^2}} = \frac{1}{a}\,\mathrm{arcsec}\,\frac{f}{a}$$

24. $$\int \frac{(f'\varphi - f\varphi')\,dx}{f^2+\varphi^2} = \arctan\frac{f}{\varphi}$$

25. $$\int \frac{(f'\varphi - f\varphi')\,dx}{f^2-\varphi^2} = \frac{1}{2}\ln\frac{f-\varphi}{f+\varphi}$$

2.1 Rational Functions

2.10 General integration rules

2.101 To integrate an arbitrary rational function $\frac{F(x)}{f(x)}$, where $F(x)$ and $f(x)$ are polynomials with no common factors, we first need to separate out the integral part $E(x)$ [where $E(x)$ is a polynomial], if there is an integral part, and then to integrate separately the integral part and the remainder; thus: $\int \frac{F(x)\,dx}{f(x)} = \int E(x)\,dx + \int \frac{\varphi(x)}{f(x)}\,dx.$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions.*

2.102 If a, b, c, …, m are roots of the equation $f(x)=0$ and if α, β, γ, …, μ are their corresponding multiplicities, so that $f(x) = (x-a)^\alpha(x-b)^\beta\dots(x-m)^\mu$, then $\frac{\varphi(x)}{f(x)}$ can be decomposed into the following partial fractions:

$$\begin{aligned}\frac{\varphi(x)}{f(x)} = {} & \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \dots + \frac{A_1}{x-a} + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \dots + \frac{B_1}{x-b} + \dots \\ & + \frac{M_\mu}{(x-m)^\mu} + \frac{M_{\mu-1}}{(x-m)^{\mu-1}} + \dots + \frac{M_1}{x-m},\end{aligned}$$

where the numerators of the individual fractions are determined by the following formulas:

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \qquad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \qquad \dots, \qquad M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$\psi_1(x) = \frac{\varphi(x)(x-a)^\alpha}{f(x)}, \qquad \psi_2(x) = \frac{\varphi(x)(x-b)^\beta}{f(x)}, \qquad \dots, \qquad \psi_m(x) = \frac{\varphi(x)(x-m)^\mu}{f(x)}$$

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If $a, b, \ldots, m$ are simple roots, that is, if $\alpha = \beta = \ldots = \mu$=1, then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \qquad B = \frac{\varphi(b)}{f'(b)}, \qquad \ldots, \qquad M = \frac{\varphi(m)}{f'(m)}.$$

If some of the roots of the equation $f(x) = 0$ are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1 x + N_1}{x^2 + 2Bx + C} + \frac{M_2 x + N_2}{\left(x^2 + 2Bx + C\right)^2} + \ldots + \frac{M_p x + N_p}{\left(x^2 + 2Bx + C\right)^p}.$$

2.103 Thus, the integration of a proper rational fraction $\frac{\varphi(x)}{f(x)}$ reduces to integrals of the form $\int \frac{g\,dx}{(x-a)^\alpha}$ or $\int \frac{Mx+N}{\left(A+2Bx+Cx^2\right)^p}\,dx$. Fractions of the first form yield rational functions for $\alpha > 1$ and logarithms for $\alpha = 1$. Fractions of the second form yield rational functions and logarithms or arctangents:

1. $$\int \frac{g\,dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}$$

2. $$\int \frac{g\,dx}{x-a} = g \int \frac{d(x-a)}{x-a} = g \ln|x-a|$$

3. $$\int \frac{Mx+N}{\left(A+2Bx+Cx^2\right)^p}\,dx = \frac{NB - MA + (NC - MB)x}{2(p-1)\left(AC - B^2\right)\left(A+2Bx+Cx^2\right)^{p-1}} + \frac{(2p-3)(NC-MB)}{2(p-1)\left(AC-B^2\right)} \int \frac{dx}{\left(A+2Bx+Cx^2\right)^{p-1}}$$

4. $$\int \frac{dx}{A+2Bx+Cx^2} = \frac{1}{\sqrt{AC-B^2}} \arctan \frac{Cx+B}{\sqrt{Ac-B^2}} \qquad \text{for } \left[AC > B^2\right]$$
$$= \frac{1}{2\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| \qquad \text{for } \left[AC < B^2\right]$$

5. $$\int \frac{(Mx+N)\,dx}{A+2Bx+Cx^2}$$
$$= \frac{M}{2C} \ln\left|A+2Bx+Cx^2\right| + \frac{NC-MB}{C\sqrt{AC-B^2}} \arctan \frac{Cx+B}{\sqrt{AC-B^2}} \qquad \text{for } \left[AC > B^2\right]$$
$$= \frac{M}{2C} \ln\left|A+2Bx+Cx^2\right| + \frac{NC-MB}{2C\sqrt{B^2-AC}} \ln \left| \frac{Cx+B-\sqrt{B^2-AC}}{Cx+B+\sqrt{B^2-AC}} \right| \qquad \text{for } \left[AC < B^2\right]$$

The Ostrogradskiy–Hermite method

2.104 By means of the Ostrogradskiy-Hermite method, we can find the rational part of $\int \frac{\varphi(x)}{f(x)}\,dx$ without finding the roots of the equation $f(x) = 0$ and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)}\,dx = \frac{M}{D} + \int \frac{N\,dx}{Q}$$ FI II 49

Here, M, N, D, and Q are rational functions of x. Specifically, D is the greatest common divisor of the function $f(x)$ and its derivative $f'(x)$; $Q = \frac{f(x)}{D}$; M is a polynomial of degree no higher than $m-1$, where m is the degree of the polynomial D; N is a polynomial of degree no higher than $n-1$, where n is the degree of the polynomial Q. The coefficients of the polynomials M and N are determined by equating the coefficients of like powers of x in the following identity:

$$\varphi(x) = M'Q - M\left(T - Q'\right) + ND$$

where $T = \frac{f'(x)}{D}$ and M' and Q' are the derivatives of the polynomials M and Q.

2.11–2.13 Forms containing the binomial $a + bx^k$

2.110 Reduction formulas for $z_k = a + bx^k$ and an explicit expression for the general case.

1. $$\begin{aligned}\int x^n z_k^m\,dx &= \frac{x^{n+1} z_k^m}{km+n+1} + \frac{amk}{km+n+1}\int x^n z_k^{m-1}\,dx \\ &= \frac{x^{n+1}}{m+1}\sum_{s=0}^{p} \frac{(ak)^s (m+1)m(m-1)\dots(m-s+1) z_k^{m-s}}{[mk+n+1][(m-1)k+n+1]\dots[(m-s)k+n+1]} \\ &\quad + \frac{(ak)^{p+1} m(m-1)\dots(m-p+1)(m-p)}{[mk+n+1][(m-1)k+n+1]\dots[(m-p)k+n+1]}\int x^n z_k^{m-p-1}\,dx\end{aligned}$$ LA 126(4)

2. $$\int x^n z_k^m\,dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)}\int x^n z_k^{m+1}\,dx$$ LA 126 (6)

3. $$\int x^n z_k^m\,dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1}\int x^{n+k} z_k^{m-1}\,dx$$

4. $$\int x^n z_k^m\,dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)}\int x^{n-k} z_k^{m+1}\,dx$$ LA 125 (2)

5. $$\int x^n z_k^m\,dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)}\int x^{n-k} z_k^m\,dx$$ LA 126 (3)

6. $$\int x^n z_k^m\,dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)}\int x^{n+k} z_k^m\,dx$$ LA 126 (5)

7.* $$\int x^n \left(nx^b + c\right)^k dx = \frac{n^k}{b}\sum_{i=0}^{k} \frac{(-1)^i\, k!\,\Gamma\left(\frac{a+1}{b}\right)\left(n^b + \frac{c}{n}\right)^{k-i}}{(k-i)!\,\Gamma\left(\frac{a+1}{b} + i + 1\right)} x^{a+1+ib}$$

$[a,\ b,\ k \ge 0$ are all integers$]$

8.* $$\int x^n z_k^m\,dx = \frac{b^m}{k}\sum_{i=0}^{m} \frac{(-1)^i m!\,J!\left(x^k + \frac{a}{b}\right)^{m-i} x^{k(J+i+1)}}{(m-i)!(J+i+1)!}$$

$$J = \frac{n+1}{k} - 1 \qquad [a, b, k, m, n \text{ real},\quad k \ne 0,\quad m \ge 0 \text{ an integer}]$$

Forms containing the binomial $z_1 = a + bx$

2.111

1. $$\int z_1^m \, dx = \frac{z_1^{m+1}}{b(m+1)}$$

For $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1$$

2. $$\int \frac{x^n \, dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} \, dx}{z_1^m}$$

For $n = m - 1$, we may use the formula

3.[8] $$\int \frac{x^{m-1} \, dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b} \int \frac{x^{m-2} \, dx}{z_1^{m-1}}$$

For $m = 1$

$$\int \frac{x^n \, dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2 x^{n-2}}{(n-2)b^3} - \ldots + (-1)^{n-1} \frac{a^{n-1} x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1$$

4. $$\int \frac{x^n \, dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{k a^{k-1} x^{n-k}}{(n-k) b^{k+1}} + (-1)^{n-1} \frac{a^n}{b^{n+1} z_1} + (-1)^{n+1} \frac{n a^{n-1}}{b^{n+1}} \ln z_1$$

5. $$\int \frac{x \, dx}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1$$

6. $$\int \frac{x^2 \, dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1$$

2.113

1. $$\int \frac{dx}{z_1^2} = -\frac{1}{b z_1}$$

2. $$\int \frac{x \, dx}{z_1^2} = -\frac{x}{b z_1} + \frac{1}{b^2} \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2} \ln z_1$$

3. $$\int \frac{x^2 \, dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3} \ln z_1$$

2.114

1. $$\int \frac{dx}{z_1^3} = -\frac{1}{2b z_1^2}$$

2. $$\int \frac{x \, dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2}\right] \frac{1}{z_1^2}$$

3. $$\int \frac{x^2 \, dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right] \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1$$

4.[6] $$\int \frac{x^3 \, dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2} x^2 - 2\frac{a^2}{b^3} x - \frac{5}{2} \frac{a^3}{b^4}\right] \frac{1}{z_1^2} - 3\frac{a}{b^4} \ln z_1$$

2.115

1. $$\int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}$$

2. $$\int \frac{x\,dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right]\frac{1}{z_1^3}$$

3. $$\int \frac{x^2\,dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right]\frac{1}{z_1^3}$$

4. $$\int \frac{x^3\,dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right]\frac{1}{z_1^3} + \frac{1}{b^4}\ln z_1$$

2.116

1. $$\int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}$$

2. $$\int \frac{x\,dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right]\frac{1}{z_1^4}$$

3. $$\int \frac{x^2\,dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right]\frac{1}{z_1^4}$$

4. $$\int \frac{x^3\,dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right]\frac{1}{z_1^4}$$

2.117

1. $$\int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)}\int \frac{dx}{x^{n-1}z_1^m}$$

2. $$\int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}$$

3. $$\int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a}\int \frac{dx}{xz_1^{m-1}}$$

4. $$\int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1}\frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n}\ln\frac{z_1}{x}$$

2.118

1. $$\int \frac{dx}{xz_1} = -\frac{1}{a}\ln\frac{z_1}{x},$$

2. $$\int \frac{dx}{x^2 z_1} = -\frac{1}{ax} + \frac{b}{a^2}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1} = -\frac{1}{2ax^2} + \frac{b}{a^2x} - \frac{b^2}{a^3}\ln\frac{z_1}{x}$$

2.119

1. $$\int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^2} = -\left[\frac{1}{ax} + \frac{2b}{a^2}\right]\frac{1}{z_1} + \frac{2b}{a^3}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^2} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3}\right]\frac{1}{z_1} - \frac{3b^2}{a^4}\ln\frac{z_1}{x}$$

2.121

1. $$\int \frac{dx}{x z_1^3} = \left[\frac{3}{2a} + \frac{bx}{a^2}\right]\frac{1}{z_1^2} - \frac{1}{a^3}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^3} = -\left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3}\right]\frac{1}{z_1^2} + \frac{3b}{a^4}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4}\right]\frac{1}{z_1^2} - \frac{6b^2}{a^5}\ln\frac{z_1}{x}$$

2.122

1. $$\int \frac{dx}{x z_1^4} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3}\right]\frac{1}{z_1^3} - \frac{1}{a^4}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^4} = -\left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4}\right]\frac{1}{z_1^3} + \frac{4b}{a^5}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^4} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5}\right]\frac{1}{z_1^3} - \frac{10b^2}{a^6}\ln\frac{z_1}{x}$$

2.123

1.[11] $$\int \frac{dx}{x z_1^5} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4}\right]\frac{1}{z_1^4} - \frac{1}{a^5}\ln\frac{z_1}{x}$$

2. $$\int \frac{dx}{x^2 z_1^5} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2x}{3a^3} - \frac{35b^3x^2}{2a^4} - \frac{5b^4x^3}{a^5}\right]\frac{1}{z_1^4} + \frac{5b}{a^6}\ln\frac{z_1}{x}$$

3. $$\int \frac{dx}{x^3 z_1^5} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2x} + \frac{125b^2}{4a^3} + \frac{65b^3x}{a^4} + \frac{105b^4x^2}{2a^5} + \frac{15b^5x^3}{a^6}\right]\frac{1}{z_1^4} - \frac{15b^2}{a^7}\ln\frac{z_1}{x}$$

2.124 Forms containing the binomial $z_2 = a + bx^2$.

1. $$\int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}}\arctan x\sqrt{\frac{b}{a}} \qquad \text{if } [ab > 0] \qquad (\text{see also } \mathbf{2.141}\ 2)$$
$$= \frac{1}{2i\sqrt{ab}}\ln\frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \qquad \text{if } [ab < 0] \qquad (\text{see also } \mathbf{2.143}\ 2 \text{ and } \mathbf{2.143}3)$$

2. $$\int \frac{x\,dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}}$$ (see also **2.145** 2, **2.145** 6, and **2.18**)

Forms containing the binomial $z_3 = a + bx^3$

Notation: $\alpha = \sqrt[3]{\frac{a}{b}}$

2.125

1. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3}\,dx}{z_3^m}$$

2. $$\int \frac{x^n\,dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)} \int \frac{x^n\,dx}{z_3^{m-1}}$$ LA 133 (1)

2.126

1. $$\int \frac{dx}{z_3} = \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{x\sqrt{3}}{2\alpha - x}\right\}$$
$$= \frac{\alpha}{3a}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} + \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$
(see also **2.141** 3 and **2.143**)

2. $$\int \frac{x\,dx}{z_3} = -\frac{1}{3b\alpha}\left\{\frac{1}{2}\ln\frac{(x+\alpha)^2}{x^2-\alpha x+\alpha^2} - \sqrt{3}\arctan\frac{2x-\alpha}{\alpha\sqrt{3}}\right\}$$
(see also **2.145** 3. and **2.145** 7)

3. $$\int \frac{x^2\,dx}{z_3} = \frac{1}{3b}\ln\left(1 + x^3\alpha^{-3}\right) = \frac{1}{3b}\ln z_3$$

4. $$\int \frac{x^3\,dx}{z_3} = \frac{x}{b} - \frac{a}{b}\int\frac{dx}{z_3}$$ (see **2.126** 1)

5. $$\int \frac{x^4\,dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

2.127

1. $$\int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a}\int\frac{dx}{z_3}$$ (see **2.126** 1)

2. $$\int \frac{x\,dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a}\int\frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{x^2\,dx}{z_3^2} = -\frac{1}{3bz_3}$$

4. $$\int \frac{x^3\,dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b}\int\frac{dx}{z_3}$$ (see **2.126** 1)

2.128

1. $$\int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)}\int\frac{dx}{x^{n-3}z_3^m}$$

2. $$\int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)}\int\frac{dx}{x^n z_3^{m-1}}$$ LA 133 (2)

2.129

1. $$\int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}$$

2. $$\int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3}$$ (see **2.126** 1)

2.131

1. $$\int \frac{dx}{xz_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}$$

2. $$\int \frac{dx}{x^2 z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2}\right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x\,dx}{z_3}$$ (see **2.126** 2)

3. $$\int \frac{dx}{x^3 z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2}\right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3}$$ (see **2.126** 1)

Forms containing the binomial $z_4 = a + bx^4$

Notation: $\alpha = \sqrt[4]{\frac{a}{b}} \qquad \alpha' = \sqrt[4]{-\frac{a}{b}}$

2.132

1.[8] $$\int \frac{dx}{z_4} = \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x\sqrt{2} + \alpha^2}{x^2 - \alpha x\sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x \sqrt{2}}{\alpha^2 - x^2} \right\}$$ for $ab > 0$ (see also **2.141** 4)

$$= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2 \arctan \frac{x}{\alpha'} \right\}$$ for $ab < 0$ (see also **2.143** 5)

2. $$\int \frac{x\,dx}{z_4} = \frac{1}{2\sqrt{ab}} \arctan x^2 \sqrt{\frac{b}{a}}$$ for $ab > 0$ (see also **2.145** 4)

$$= \frac{1}{4i\sqrt{ab}} \ln \frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}}$$ for $ab < 0$ (see also **2.145** 8)

3. $$\int \frac{x^2\,dx}{z_4} = \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x\sqrt{2} + \alpha^2}{x^2 + \alpha x\sqrt{2} + \alpha^2} + 2 \arctan \frac{\alpha x \sqrt{2}}{\alpha^2 - x^2} \right\}$$ for $ab > 0$

$$= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2 \arctan \frac{x}{\alpha'} \right\}$$ for $ab < 0$

4. $$\int \frac{x^3\,dx}{z_4} = \frac{1}{4b} \ln z_4$$

2.133

1. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n\,dx}{z_4^{m-1}}$$ LA 134 (1)

2. $$\int \frac{x^n\,dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4}\,dx}{z_4^m}$$

2.134

1. $$\int \frac{dx}{z_4^2} = \frac{x}{4az_4} + \frac{3}{4a}\int \frac{dx}{z_4}$$ (see **2.132** 1)

2. $$\int \frac{x\,dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a}\int \frac{x\,dx}{z_4}$$ (see **2.132** 2)

3. $$\int \frac{x^2\,dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a}\int \frac{x^2\,dx}{z_4}$$ (see **2.132** 3)

4. $$\int \frac{x^3\,dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}$$

2.135 $$\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a}\int \frac{dx}{x^{n-4}z_4^m}$$

For $n = 1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a}\int \frac{dx}{xz_4^{m-1}} - \frac{b}{a}\int \frac{dx}{x^{-3}z_4^m}$$

2.136

1. $$\int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a}\ln\frac{x^4}{z_4}$$

2. $$\int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x^2\,dx}{z_4}$$ (see **2.132** 3)

2.14 Forms containing the binomial $1 \pm x^n$

2.141

1. $$\int \frac{dx}{1+x} = \ln(1+x)$$

2.[11] $$\int \frac{dx}{1+x^2} = \arctan x = -\arctan\left(\frac{1}{x}\right)$$ (see also **2.124** 1)

3. $$\int \frac{dx}{1+x^3} = \frac{1}{3}\ln\frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2-x}$$ (see also **2.126** 1)

4. $$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2}$$

(see also **2.132** 1)

2.142 $$\int \frac{dx}{1+x^n} = -\frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} P_k\cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n}\sum_{k=0}^{\frac{n}{2}-1} Q_k\sin\left(\frac{2k+1}{n}\pi\right)$$

for n a positive even number

TI (43)a

$$= \frac{1}{n}\ln(1+x) - \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k\cos\left(\frac{2k+1}{n}\pi\right) + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k\sin\left(\frac{2k+1}{n}\pi\right)$$

for n a positive odd number

TI (45)

where

$$P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\left(\frac{2k+1}{n}\pi\right) + 1\right)$$
$$Q_k = \arctan\frac{x\sin\left(\frac{2k+1}{n}\pi\right)}{1 - x\cos\left(\frac{2k+1}{n}\pi\right)} = \arctan\frac{x - \cos\left(\frac{2k+1}{n}\pi\right)}{\sin\left(\frac{2k+1}{n}\pi\right)}$$

2.143

1. $$\int \frac{dx}{1-x} = -\ln(1-x)$$

2. $$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} = \operatorname{arctanh} x \qquad [-1 < x < 1]$$ (see also **2.141** 1)

3. $$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\frac{x-1}{x+1} = -\operatorname{arccoth} x \qquad [x > 1, \quad x < -1]$$

4. $$\int \frac{dx}{1-x^3} = \frac{1}{3}\ln\frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{2+x}$$ (see also **2.126** 1)

5. $$\int \frac{dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x}{1-x} + \frac{1}{2}\arctan x = \frac{1}{2}\left(\operatorname{arctanh} x + \arctan x\right)$$ (see also **2.132** 1)

2.144

1. $$\int \frac{dx}{1-x^n} = \frac{1}{n}\ln\frac{1+x}{1-x} - \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} P_k\cos\frac{2k}{n}\pi + \frac{2}{n}\sum_{k=1}^{\frac{n}{2}-1} Q_k\sin\frac{2k}{n}\pi$$
for n a positive even number TI (47)

where $P_k = \frac{1}{2}\ln\left(x^2 + 2x\cos\frac{2k+1}{n}\pi + 1\right)$, $Q_k = \arctan\frac{x + \cos\frac{2k+1}{n}\pi}{\sin\frac{2k+1}{n}\pi}$

2. $$\int \frac{dx}{1-x^n} = -\frac{1}{n}\ln(1-x) + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} P_k\cos\frac{2k+1}{n}\pi + \frac{2}{n}\sum_{k=0}^{\frac{n-3}{2}} Q_k\sin\frac{2k+1}{n}\pi$$
for n a positive odd number TI (49)

where $P_k = \frac{1}{2}\ln\left(x^2 - 2x\cos\frac{2k}{n}\pi + 1\right)$, $Q_k = \arctan\frac{x - \cos\frac{2k}{n}\pi}{\sin\frac{2k}{n}\pi}$

2.145

1. $$\int \frac{x\,dx}{1+x} = x - \ln(1+x)$$

2. $$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right)$$

3. $$\int \frac{x\,dx}{1+x^3} = -\frac{1}{6}\ln\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$ (see also **2.126** 2)

4. $$\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\arctan x^2$$

5. $$\int \frac{x\,dx}{1-x} = -\ln(1-x) - x$$

6. $$\int \frac{x\,dx}{1-x^2} = -\frac{1}{2}\ln\left(1-x^2\right)$$

7. $$\int \frac{x\,dx}{1-x^3} = -\frac{1}{6}\ln\frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$ (see also **2.126** 2)

8. $$\int \frac{x\,dx}{1-x^4} = \frac{1}{4}\ln\frac{1+x^2}{1-x^2}$$ (see also **2.132** 2)

2.146 For m and n natural numbers.

1. $$\int \frac{x^{m-1}\,dx}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n}\ln\left\{1-2x\cos\frac{2k-1}{2n}\pi+x^2\right\} + \frac{1}{n}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n}\arctan\frac{x-\cos\frac{2k-1}{2n}\pi}{\sin\frac{2k-1}{2n}\pi}$$
$[m < 2n]$ TI (44)a

2. $$\int \frac{x^{m-1}\,dx}{1+x^{2n+1}} = (-1)^{m+1}\frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left\{1-2x\cos\frac{2k-1}{2n+1}\pi+x^2\right\} + \frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\frac{x-\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}$$
$[m \le 2n]$ TI (46)a

3.[11] $$\int \frac{x^{m-1}\,dx}{1-x^{2n}} = \frac{1}{2n}\left\{(-1)^{m+1}\ln(1+x) - \ln(1-x)\right\} - \frac{1}{2n}\sum_{k=1}^{n-1}\cos\frac{km\pi}{n}\ln\left(1-2x\cos\frac{k\pi}{n}+x^2\right) + \frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{km\pi}{n}\arctan\left(\frac{x-\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right)$$
$[m < 2n]$ TI (48)

4. $$\int \frac{x^{m-1}\,dx}{1-x^{2n+1}} = -\frac{1}{2n+1}\ln(1-x) + (-1)^{m+1}\frac{1}{2n+1}\sum_{k=1}^{n}\cos\frac{m\pi(2k-1)}{2n+1}\ln\left(1+2x\cos\frac{2k-1}{2n+1}\pi+x^2\right) + (-1)^{m+1}\frac{2}{2n+1}\sum_{k=1}^{n}\sin\frac{m\pi(2k-1)}{2n+1}\arctan\frac{x+\cos\frac{2k-1}{2n+1}\pi}{\sin\frac{2k-1}{2n+1}\pi}$$
$[m \le 2n]$ TI (50)

2.147

1. $$\int \frac{x^m\,dx}{1-x^{2n}} = \frac{1}{2}\int\frac{x^m\,dx}{1-x^n} + \frac{1}{2}\int\frac{x^m\,dx}{1+x^n}$$

2. $$\int \frac{x^m\,dx}{\left(1+x^2\right)^n} = -\frac{1}{2n-m-1}\cdot\frac{x^{m-1}}{\left(1+x^2\right)^{n-1}} + \frac{m-1}{2n-m-1}\int\frac{x^{m-2}\,dx}{\left(1+x^2\right)^n}$$ LA 139 (28)

3. $$\int \frac{x^m}{1+x^2}\,dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2}\,dx$$

4. $$\int \frac{x^m\,dx}{\left(1-x^2\right)^n} = \frac{1}{2n-m-1}\frac{x^{m-1}}{\left(1-x^2\right)^{n-1}} - \frac{m-1}{2n-m-1}\int \frac{x^{m-2}\,dx}{\left(1-x^2\right)^n}$$
$$= \frac{1}{2n-2}\frac{x^{m-1}}{\left(1-x^2\right)^{n-1}} - \frac{m-1}{2n-2}\int \frac{x^{m-2}\,dx}{\left(1-x^2\right)^{n-1}}$$
LA 139 (33)

5. $$\int \frac{x^m\,dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2}\,dx}{1-x^2}$$

2.148

1. $$\int \frac{dx}{x^m\left(1+x^2\right)^n} = -\frac{1}{m-1}\frac{1}{x^{m-1}\left(1+x^2\right)^{n-1}} - \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}\left(1+x^2\right)^n}$$ LA 139 (29)

For $m = 1$
$$\int \frac{dx}{x\left(1+x^2\right)^n} = \frac{1}{2n-2}\frac{1}{\left(1+x^2\right)^{n-1}} + \int \frac{dx}{x\left(1+x^2\right)^{n-1}}$$ LA 139 (31)

For $m = 1$ and $n = 1$
$$\int \frac{dx}{x\left(1+x^2\right)} = \ln \frac{x}{\sqrt{1+x^2}}$$

2. $$\int \frac{dx}{x^m\left(1+x^2\right)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2}\left(1+x^2\right)}$$

3. $$\int \frac{dx}{\left(1+x^2\right)^n} = \frac{1}{2n-2}\frac{x}{\left(1+x^2\right)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{\left(1+x^2\right)^{n-1}}$$ FI II 40

4. $$\int \frac{dx}{\left(1+x^2\right)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\cdots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)\left(1+x^2\right)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!}\arctan x$$
TI (91)

2.149

1. $$\int \frac{dx}{x^m\left(1-x^2\right)^n} = -\frac{1}{(m-1)x^{m-1}\left(1-x^2\right)^{n-1}} + \frac{2n+m-3}{m-1}\int \frac{dx}{x^{m-2}\left(1-x^2\right)^n}$$ LA 139 (34)

For $m = 1$
$$\int \frac{dx}{x\left(1-x^2\right)^n} = \frac{1}{2(n-1)\left(1-x^2\right)^{n-1}} + \int \frac{dx}{x\left(1-x^2\right)^{n-1}}$$ LA 139 (36)

For $m = 1$ and $n = 1$
$$\int \frac{dx}{x\left(1-x^2\right)} = \ln \frac{x}{\sqrt{1-x^2}}$$

2. $$\int \frac{dx}{\left(1-x^2\right)^n} = \frac{1}{2n-2}\frac{x}{\left(1-x^2\right)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{\left(1-x^2\right)^{n-1}}$$ LA 139 (35)

3. $$\int \frac{dx}{\left(1-x^2\right)^n} = \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)(2n-5)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)\left(1-x^2\right)^{n-k}} + \frac{(2n-3)!!}{2^n\cdot(n-1)!}\ln\frac{1+x}{1-x}$$
TI (91)

2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

Notation: $z = a + bx; \quad t = \alpha + \beta x; \quad \Delta = a\beta - \alpha b$

2.151 $$\int z^n t^m\,dx = \frac{z^{n+1}t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b}\int z^n t^{m-1}\,dx$$

2.152

1. $$\int \frac{z}{t}\,dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2}\ln t$$

2. $$\int \frac{t}{z}\,dx = \frac{\beta x}{b} - \frac{\Delta}{b^2}\ln z$$

2.153 $$\begin{aligned}\int \frac{t^m\,dx}{z^n} &= \frac{1}{(m-n+1)b}\frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b}\int \frac{t^{m-1}\,dx}{z^n} \\ &= \frac{1}{(n-1)\Delta}\frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta}\int \frac{t^m\,dx}{z^{n-1}} \\ &= -\frac{1}{(n-1)b}\frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b}\int \frac{t^{m-1}}{z^{n-1}}\,dx\end{aligned}$$

2.154 $$\int \frac{dx}{zt} = \frac{1}{\Delta}\ln\frac{t}{z}$$

2.155 $$\begin{aligned}\int \frac{dx}{z^n t^m} &= -\frac{1}{(m-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta}\int \frac{dx}{t^{m-1}z^n} \\ &= \frac{1}{(n-1)\Delta}\frac{1}{t^{m-1}z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta}\int \frac{dx}{t^m z^{n-1}}\end{aligned}$$

2.156 $$\int \frac{x\,dx}{zt} = \frac{1}{\Delta}\left(\frac{a}{b}\ln z - \frac{\alpha}{\beta}\ln t\right)$$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160 Reduction formulas for $R_k = a + bx^k + cx^{2k}$.

1. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma}\int x^{m+k-1}R_k^n\,dx - \frac{(m+2k+2kn)c}{ma}\int x^{m+2k-1}R_k^n\,dx$$

2. $$\int x^{m-1}R_k^n\,dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m}\int x^{m+k-1}R_k^{n-1}\,dx - \frac{2ckn}{m}\int x^{m+2k-1}R_k^{n-1}\,dx$$

3. $$\begin{aligned}\int x^{m-1}R_k^n\,dx &= \frac{x^{m-2k}R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c}\int x^{m-2k-1}R_k^n\,dx - \frac{(m-k+kn)b}{(m+2kn)c}\int x^{m-k-1}R_k^n\,dx \\ &= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn}\int x^{m-1}R_k^{n-1}\,dx + \frac{bkn}{m+2kn}\int x^{m+k-1}R_k^{n-1}\,dx\end{aligned}$$

2.161 Forms containing the trinomial $R_2 = a + bx^2 + cx^4$.

Notation: $f = \dfrac{b}{2} - \dfrac{1}{2}\sqrt{b^2 - 4ac}, \quad g = \dfrac{b}{2} + \dfrac{1}{2}\sqrt{b^2 - 4ac},$

$$h = \sqrt{b^2 - 4ac}, \quad q = \sqrt[4]{\frac{a}{c}}, \quad l = 2a(n-1)\left(b^2 - 4ac\right), \quad \cos\alpha = -\frac{b}{2\sqrt{ac}}$$

1. $$\int \frac{dx}{R_2}$$
$$= \frac{c}{h}\left\{\int \frac{dx}{cx^2+f} - \int \frac{dx}{cx^2+g}\right\} \qquad \left[h^2 > 0\right] \qquad \text{LA 146 (5)}$$
$$= \frac{1}{4cq^3 \sin\alpha}\left\{\sin\frac{\alpha}{2} \ln \frac{x^2 + 2qx\cos\frac{\alpha}{2} + q^2}{x^2 - 2qx\cos\frac{\alpha}{2} + q^2} + 2\cos\frac{\alpha}{2} \arctan \frac{x^2 - q^2}{2qx \sin\frac{\alpha}{2}}\right\} \qquad \left[h^2 < 0\right] \qquad \text{LA 146 (8)a}$$

2. $$\int \frac{x\,dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2+f}{cx^2+g} \qquad \left[h^2 > 0\right] \qquad \text{LA 146 (6)}$$
$$= \frac{1}{2cq^2 \sin\alpha} \arctan \frac{x^2 - q^2\cos\alpha}{q^2 \sin\alpha} \qquad \left[h^2 < 0\right] \qquad \text{LA 146 (9)a}$$

3. $$\int \frac{x^2\,dx}{R_2} = \frac{g}{h}\int \frac{dx}{cx^2+g} - \frac{f}{h}\int \frac{dx}{cx^2+f} \qquad \left[h^2 > 0\right] \qquad \text{LA 146 (7)}$$

4. $$\int \frac{dx}{R_2^2} = \frac{bcx^3 + \left(b^2 - 2ac\right)x}{lR_2} + \frac{b^2 - 6ac}{l}\int \frac{dx}{R_2} + \frac{bc}{l}\int \frac{x^2\,dx}{R_2}$$

5. $$\int \frac{dx}{R_2^n} = \frac{bcx^3 + \left(b^2 - 2ac\right)x}{lR_{n-1}^2} + \frac{(4n-7)bc}{l}\int \frac{x^2\,dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l}\int \frac{dx}{R_2^{n-1}}$$
$$[n > 1] \qquad \text{LA 146}$$

6.[9] $$\int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a}\int \frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)bc}{(m-1)a}\int \frac{dx}{x^{m-4}R_2^n}$$
LA 147 (12)a

2.17 Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of x

Notation: $R = a + bx + cx^2$; $\quad \Delta = 4ac - b^2$

2.171

1. $$\int x^{m+1} R^n\,dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)}\int x^{m-1}R^n\,dx - \frac{b(m+n+1)}{c(m+2n+2)}\int x^m R^n\,dx$$
TI (97)

2. $$\int \frac{R^n\,dx}{x^{m+1}} = -\frac{R^{n+1}}{amx^m} + \frac{b(n-m+1)}{am}\int \frac{R^n\,dx}{x^m} + \frac{c(2n-m+2)}{am}\int \frac{R^n\,dx}{x^{m-1}} \qquad \text{LA 142(3), TI (96)a}$$

3. $$\int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta}\int \frac{dx}{R^n} \qquad \text{TI (94)a}$$

4. $$\int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1}\sum_{k=0}^{n-1} \frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\cdots(n-k)\Delta^{k+1}R^{n-k}} + 2^n \frac{(2n-1)!!c^n}{n!\Delta^n}\int \frac{dx}{R}$$
TI (96)a

2.172[11] $$\int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta} - (b+2cx)}{(b+2cx) + \sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} \operatorname{arctanh} \frac{b+2cx}{\sqrt{-\Delta}} \qquad \text{for } [\Delta < 0]$$
$$= \frac{-2}{b+2cx} \qquad \text{for } [\Delta = 0,\ b \text{ and } c \text{ non-zero}]$$
$$= \frac{2}{\sqrt{\Delta}} \arctan \frac{b+2cx}{\sqrt{\Delta}} \qquad \text{for } [\Delta > 0]$$

2.173

1. $$\int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta}\int \frac{dx}{R}$$ (see **2.172**)

2. $$\int \frac{dx}{R^3} = \frac{b+2cx}{\Delta}\left\{\frac{1}{2R^2} + \frac{3c}{\Delta R}\right\} + \frac{6c^2}{\Delta^2}\int \frac{dx}{R}$$ (see **2.172**)

2.174

1. $$\int \frac{x^m\,dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c}\int \frac{x^{m-1}\,dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c}\int \frac{x^{m-2}\,dx}{R^n}$$

For $m = 2n-1$, this formula is inapplicable. Instead, we may use

2. $$\int \frac{x^{2n-1}\,dx}{R^n} = \frac{1}{c}\int \frac{x^{2n-3}\,dx}{R^{n-1}} - \frac{a}{c}\int \frac{x^{2n-3}\,dx}{R^n} - \frac{b}{c}\int \frac{x^{2n-2}\,dx}{R^n}$$

2.175

1. $$\int \frac{x\,dx}{R} = \frac{1}{2c}\ln R - \frac{b}{2c}\int \frac{dx}{R}$$ (see **2.172**)

2. $$\int \frac{x\,dx}{R^2} = -\frac{2a+bx}{\Delta R} - \frac{b}{\Delta}\int \frac{dx}{R}$$ (see **2.172**)

3. $$\int \frac{x\,dx}{R^3} = -\frac{2a+bx}{2\Delta R^2} - \frac{3b(b+2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2}\int \frac{dx}{R}$$ (see **2.172**)

4. $$\int \frac{x^2\,dx}{R} = \frac{x}{c} - \frac{b}{2c^2}\ln R + \frac{b^2-2ac}{2c^2}\int \frac{dx}{R}$$ (see **2.172**)

5. $$\int \frac{x^2\,dx}{R^2} = \frac{ab + \left(b^2-2ac\right)x}{c\Delta R} + \frac{2a}{\Delta}\int \frac{dx}{R}$$ (see **2.172**)

6. $$\int \frac{x^2\,dx}{R^3} = \frac{ab + \left(b^2-2ac\right)x}{2c\Delta R^2} + \frac{\left(2ac+b^2\right)(b+2cx)}{2c\Delta^2 R} + \frac{2ac+b^2}{\Delta^2}\int \frac{dx}{R}$$

(see **2.172**)

7. $$\int \frac{x^3\,dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^3}\ln R - \frac{b\left(b^2-3ac\right)}{2c^3}\int \frac{dx}{R}$$

(see **2.172**)

8. $$\int \frac{x^3\,dx}{R^2} = \frac{1}{2c^2}\ln R + \frac{a\left(2ac-b^2\right) + b\left(3ac-b^2\right)x}{c^2\Delta R} - \frac{b\left(6ac-b^2\right)}{2c^2\Delta}\int \frac{dx}{R}$$

(see **2.172**)

9. $$\int \frac{x^3\,dx}{R^3} = -\left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta}\right)\frac{1}{2R^2} - \frac{3ab}{2c\Delta}\int \frac{dx}{R^2}$$ (see **2.173** 1)

2.176 $$\int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)}\int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)}\int \frac{dx}{x^{m-2}R^n}$$

2.177

1. $$\int \frac{dx}{xR} = \frac{1}{2a}\ln \frac{x^2}{R} - \frac{b}{2a}\int \frac{dx}{R}$$ (see **2.172**)

2. $$\int \frac{dx}{xR^2} = \frac{1}{2a^2} \ln \frac{x^2}{R} + \frac{1}{2aR} \left\{ 1 - \frac{b(b+2cx)}{\Delta} \right\} - \frac{b}{2a^2} \left(1 + \frac{2ac}{\Delta} \right) \int \frac{dx}{R}$$

(see **2.172**)

3. $$\int \frac{dx}{xR^3} = \frac{1}{4aR^2} + \frac{1}{2a^2R} + \frac{1}{2a^3} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R^3} - \frac{b}{2a^2} \int \frac{dx}{R^2} - \frac{b}{2a^3} \int \frac{dx}{R}$$

(see **2.172**, **2.173**)

4. $$\int \frac{dx}{x^2R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{R}$$ (see **2.172**)

5. $$\int \frac{dx}{x^2R^2} = -\frac{b}{a^3} \ln \frac{x^2}{R} - \frac{a+bx}{a^2xR} + \frac{\left(b^2 - 3ac\right)(b+2cx)}{a^2 \Delta R} - \frac{1}{\Delta} \left(\frac{b^4}{a^3} - \frac{6b^2c}{a^2} + \frac{6c^2}{a} \right) \int \frac{dx}{R}$$

(see **2.172**)

6. $$\int \frac{dx}{x^2R^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3}$$ (see **2.173** and **2.177** 3)

7. $$\int \frac{dx}{x^3R} = -\frac{ac - b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b\left(3ac - b^2\right)}{2a^3} \int \frac{dx}{R}$$

(see **2.172**)

8. $$\int \frac{dx}{x^3R^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x} \right) \frac{1}{R} + \left(\frac{3b^2}{a^2} - \frac{2c}{a} \right) \int \frac{dx}{xR^2} + \frac{9bc}{2a^2} \int \frac{dx}{R^2}$$

(see **2.173** 1 and **2.177** 2)

9. $$\int \frac{dx}{x^3R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2x} \right) \frac{1}{R^2} + \left(\frac{6b^2}{a^2} - \frac{3c}{a} \right) \int \frac{dx}{xR^3} + \frac{10bc}{a^2} \int \frac{dx}{R^3}$$

(see **2.173** 2 and **2.177** 3)

2.18 Forms containing the quadratic trinomial $a + bx + cx^2$ and the binomial $\alpha + \beta x$

Notation: $R = a + bx + cx^2$; $z = \alpha + \beta x$; $A = a\beta^2 - \alpha b\beta + c\alpha^2$;
$B = b\beta - 2c\alpha$; $\Delta = 4ac - b^2$

1. $$\int z^m R^n \, dx = \frac{\beta z^{m-1} R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c} \int z^{m-1} R^n \, dx - \frac{(m-1)A}{(m+2n+1)c} \int z^{m-2} R^n \, dx$$

2. $$\int \frac{R^n \, dx}{z^m} = -\frac{1}{(m-2n-1)\beta} \frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2} \int \frac{R^{n-1} \, dx}{z^m} - \frac{nB}{(m-2n-1)\beta^2} \int \frac{R^{n-1} \, dx}{z^{m-1}};$$ LA 184 (4)a

$$= \frac{-\beta}{(m-1)A} \frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A} \int \frac{R^n \, dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A} \int \frac{R^n \, dx}{z^{m-2}}$$ LA 148 (5)

$$= -\frac{1}{(m-1)\beta} \frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2} \int \frac{R^{n-1} \, dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2} \int \frac{R^{n-1} \, dx}{z^{m-2}}$$ LA 418 (6)

3. $$\int \frac{z^m\,dx}{R^n} = \frac{\beta}{(m-2n+1)c}\frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c}\int\frac{z^{m-1}\,dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c}\int\frac{z^{m-2}\,dx}{R^n}$$ LA 147 (1)

$$= \frac{b+2cx}{(n-1)\Delta}\frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta}\int\frac{z^m\,dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta}\int\frac{z^{m-1}\,dx}{R^{n-1}}$$ LA 148 (3)

4.[3] $$\int\frac{dx}{z^mR^n} = -\frac{\beta}{(m-1)A}\frac{1}{z^{m-1}R^{n-1}} - \frac{(m+n-2)B}{(m-1)A}\int\frac{dx}{z^{m-1}R^n} - \frac{(m+2n-3)c}{(m-1)A}\int\frac{dx}{z^{m-2}R^n}$$ LA 148 (7)

$$= \frac{\beta}{2(n-1)A}\frac{1}{z^{m-1}R^{n-1}} - \frac{B}{2A}\int\frac{dx}{z^{m-1}R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A}\int\frac{dx}{z^mR^{n-1}}$$ LA 148 (8)

For $m = 1$ and $n = 1$

$$\int\frac{dx}{zR} = \frac{\beta}{2A}\ln\frac{z^2}{R} - \frac{B}{2A}\int\frac{dx}{R}$$

For $A = 0$

$$\int\frac{dx}{z^mR^n} = -\frac{\beta}{(m+n-1)B}\frac{1}{z^mR^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B}\int\frac{dx}{z^{m-1}R^n}$$ LA 148 (9)

2.2 Algebraic Functions

2.20 Introduction

2.201 The integrals $\int R\left(x, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^r, \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^s, \ldots\right) dx$, where $r, s, \ldots$ are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x+\beta}{\gamma x+\delta} = t^m,$$ FI II 57

where m is the common denominator of the fractions $r, s, \ldots$.

2.202 Integrals of the form $\int x^m\left(a+bx^n\right)^p\,dx$,* where m, n, and p are rational numbers, can be expressed in terms of elementary functions only in the following cases:

(a) When p is an integer; then, this integral takes the form of a sum of the integrals shown in **2.201**;

(b) When $\frac{m+1}{n}$ is an integer: by means of the substitution $x^n = z$, this integral can be transformed to the form $\frac{1}{n}\int(a+bz)^p z^{\frac{m+1}{n}-1}\,dz$, which we considered in **2.201**;

(c) When $\frac{m+1}{n} + p$ is an integer; by means of the same substitution $x^n = z$, this integral can be reduced to an integral of the form $\frac{1}{n}\int\left(\frac{a+bz}{z}\right)^p z^{\frac{m+1}{n}+p-1}\,dz$, considered in **2.201**;

For reduction formulas for integrals of binomial differentials, see **2.110**.

*Translator: The authors term such integrals "integrals of binomial differentials."

2.21 Forms containing the binomial $a + bx^k$ and $\sqrt{x}$

Notation: $z_1 = a + bx$.

2.211 $\displaystyle\int \frac{dx}{z_1\sqrt{x}} = \frac{2}{\sqrt{ab}} \arctan\sqrt{\frac{bx}{a}}$ $[ab > 0]$

$\displaystyle = \frac{1}{i\sqrt{ab}} \ln \frac{a - bx + 2i\sqrt{xab}}{z_1}$ $[ab < 0]$

2.212 $\displaystyle\int \frac{x^m\sqrt{x}}{z_1}\,dx = 2\sqrt{x}\sum_{k=0}^{m} \frac{(-1)^k a^k x^{m-k}}{(2m-2k+1)b^{k+1}} + (-1)^{m+1}\frac{a^{m+1}}{b^{m+1}}\int\frac{dx}{z_1\sqrt{x}}$

(see **2.211**)

2.213

1. $\displaystyle\int \frac{\sqrt{x}\,dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
2. $\displaystyle\int \frac{x\sqrt{x}\,dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right)2\sqrt{x} + \frac{a^2}{b^2}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
3. $\displaystyle\int \frac{x^2\sqrt{x}\,dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3}\right)2\sqrt{x} - \frac{a^3}{b^3}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
4. $\displaystyle\int \frac{dx}{z_1^2\sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
5. $\displaystyle\int \frac{\sqrt{x}\,dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
6. $\displaystyle\int \frac{x\sqrt{x}\,dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b}\int\frac{\sqrt{x}\,dx}{z_1^2}$ (see **2.213** 5)
7. $\displaystyle\int \frac{x^2\sqrt{x}\,dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right)\frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2}\int\frac{\sqrt{x}\,dx}{z_1^2}$ (see **2.213** 5)
8. $\displaystyle\int \frac{dx}{z_1^3\sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2z_1}\right)\sqrt{x} + \frac{3}{8a^2}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
9. $\displaystyle\int \frac{\sqrt{x}\,dx}{z_1^3} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1}\right)\sqrt{x} + \frac{1}{8ab}\int\frac{dx}{z_1\sqrt{x}}$ (see **2.211**)
10. $\displaystyle\int \frac{x\sqrt{x}\,dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b}\int\frac{\sqrt{x}\,dx}{z_1^3}$ (see **2.213** 9)
11. $\displaystyle\int \frac{x^2\sqrt{x}\,dx}{z_1^3} = \left(\frac{x^2}{b} + \frac{5ax}{b^2}\right)\frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2}\int\frac{\sqrt{x}\,dx}{z_1^3}$ (see **2.213** 9)

Notation: $z_2 = a + bx^2$, $\alpha = \sqrt[4]{\frac{a}{b}}$, $\alpha' = \sqrt[4]{-\frac{a}{b}}$.

2.214 $\displaystyle\int \frac{dx}{z_2\sqrt{x}} = \frac{1}{b\alpha^3\sqrt{2}}\left[\ln\frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2 - x}\right]$ $\left[\frac{a}{b} > 0\right]$

$\displaystyle = \frac{1}{2b\alpha'^3}\left(\ln\frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2\arctan\frac{\sqrt{x}}{\alpha'}\right)$ $\left[\frac{a}{b} < 0\right]$

2.215 $$\int \frac{\sqrt{x}\,dx}{z_2} = \frac{1}{b\alpha\sqrt{2}}\left[-\ln\frac{x+\alpha\sqrt{2x}+\alpha^2}{\sqrt{z_2}} + \arctan\frac{\alpha\sqrt{2x}}{\alpha^2-x}\right] \qquad \left[\frac{a}{b}>0\right]$$
$$= \frac{1}{2b\alpha'}\left[\ln\frac{\alpha'-\sqrt{x}}{\alpha'+\sqrt{x}} + 2\arctan\frac{\sqrt{x}}{\alpha'}\right] \qquad \left[\frac{a}{b}<0\right]$$

2.216

1. $$\int \frac{x\sqrt{x}\,dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

2. $$\int \frac{x^2\sqrt{x}\,dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

3. $$\int \frac{dx}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

4. $$\int \frac{\sqrt{x}\,dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

5. $$\int \frac{x\sqrt{x}\,dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

6. $$\int \frac{x^2\sqrt{x}\,dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

7. $$\int \frac{dx}{z_2^3\sqrt{x}} = \left(\frac{1}{4az_2^2} + \frac{7}{16a^2z_2}\right)\sqrt{x} + \frac{21}{32a^2}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

8. $$\int \frac{\sqrt{x}\,dx}{z_2^3} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2z_2}\right)x\sqrt{x} + \frac{5}{32a^2}\int\frac{\sqrt{x}\,dx}{z_2}$$ (see **2.215**)

9. $$\int \frac{x\sqrt{x}\,dx}{z_2^3} = \frac{\left(bx^2-3a\right)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab}\int\frac{dx}{z_2\sqrt{x}}$$ (see **2.214**)

10. $$\int \frac{x^2\sqrt{x}\,dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b}\int\frac{\sqrt{x}\,dx}{z_2^3}$$ (see **2.216** 8)

2.22–2.23 Forms containing $\sqrt[n]{(a+bx)^k}$

Notation: $z = a + bx$.

2.220 $$\int x^n \sqrt[l]{z^{lm+f}}\,dx = \left\{\sum_{k=0}^{n} \frac{(-1)^k\binom{n}{k} z^{n-k}a^k}{ln - lk + l(m+1) + f}\right\}\frac{l\sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}$$

The square root

2.221 $$\int x^n\sqrt{z^{2m-1}}\,dx = \left\{\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}z^{n-k}a^k}{2n-2k+2m+1}\right\}\frac{2\sqrt{z^{2m+1}}}{b^{n+1}}$$

2.222

1. $$\int\frac{dx}{\sqrt{z}} = \frac{2}{b}\sqrt{z}$$

2. $$\int \frac{x\,dx}{\sqrt{z}} = \left(\frac{1}{3}z - a\right)\frac{2\sqrt{z}}{b^2}$$

3. $$\int \frac{x^2\,dx}{\sqrt{z}} = \left(\frac{1}{5}z^2 - \frac{2}{3}az + a^2\right)\frac{2\sqrt{z}}{b^3}$$

2.223

1. $$\int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}$$

2. $$\int \frac{x\,dx}{\sqrt{z^3}} = (z+a)\frac{2}{b^2\sqrt{z}}$$

3. $$\int \frac{x^2\,dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2\right)\frac{2}{b^3\sqrt{z}}$$

2.224

1. $$\int \frac{z^m\,dx}{x^n\sqrt{z}} = -\frac{z^m\sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)}\frac{b}{a}\int \frac{z^m\,dx}{x^{n-1}\sqrt{z}}$$

2. $$\int \frac{z^m\,dx}{x^n\sqrt{z}} = -z^m\sqrt{z}\left\{\frac{1}{(n-1)ax^{n-1}} + \sum_{k=1}^{n-2}\frac{(2m-2n+3)(2m-2n+5)\dots(2m-2n+2k+1)}{2^k(n-1)(n-2)\dots(n-k-1)x^{n-k-1}}\frac{b^k}{a^{k+1}}\right\} + \frac{(2m-2n+3)(2m-2n+5)\dots(2m-3)(2m-1)}{2^{n-1}(n-1)!x}\frac{b^{n-1}}{a^{n-1}}\int \frac{z^m\,dx}{x\sqrt{z}}$$

For $n = 1$

3. $$\int \frac{z^m}{x\sqrt{z}}\,dx = \frac{2z^m}{(2m-1)\sqrt{z}} + a\int \frac{z^{m-1}}{x\sqrt{z}}\,dx$$

4. $$\int \frac{z^m}{x\sqrt{z}}\,dx = \sum_{k=1}^{m}\frac{2a^{m-k}z^k}{(2k-1)\sqrt{z}} + a^m\int \frac{dx}{x\sqrt{z}}$$

5.[6] $$\int \frac{dx}{x\sqrt{z}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{z}-\sqrt{a}}{\sqrt{z}+\sqrt{a}}\right| \qquad [a > 0]$$
$$= \frac{2}{\sqrt{-a}}\arctan\frac{\sqrt{z}}{\sqrt{-a}} \qquad [a < 0]$$

2.225

1. $$\int \frac{\sqrt{z}\,dx}{x} = 2\sqrt{z} + a\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{\sqrt{z}\,dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

3. $$\int \frac{\sqrt{z}\,dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.226

1. $$\int \frac{\sqrt{z^3}\,dx}{x} = \left(\frac{z}{3}+a\right) 2\sqrt{z} + a^2 \int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{\sqrt{z^3}\,dx}{x^2} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a}\int \frac{\sqrt{z^3}\,dx}{x}$$ (see **2.226** 1)

3. $$\int \frac{\sqrt{z^3}\,dx}{x^3} = -\left(\frac{1}{2ax^2}+\frac{b}{4a^2x}\right)\sqrt{z^5} + \frac{3b^2}{8a^2}\int \frac{\sqrt{z^3}\,dx}{x}$$ (see **2.226** 1)

2.227 $$\int \frac{dx}{xz^m\sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1)a^{m-k}z^k\sqrt{z}} + \frac{1}{a^m}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.228

1. $$\int \frac{dx}{x^2\sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{dx}{x^3\sqrt{z}} = \left(-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right)\sqrt{z} + \frac{3b^2}{8a^2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2.229

1. $$\int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}} + \frac{1}{a}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

2. $$\int \frac{dx}{x^2\sqrt{z^3}} = \left(-\frac{1}{ax}-\frac{3b}{a^2}\right)\frac{1}{\sqrt{z}} - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

3. $$\int \frac{dx}{x^3\sqrt{z^3}} = \left(-\frac{1}{2ax^2}+\frac{5b}{4a^2x}+\frac{15b^2}{4a^3}\right)\frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3}\int \frac{dx}{x\sqrt{z}}$$ (see **2.224** 4)

Cube root

2.231

1. $$\int \sqrt[3]{z^{3m+1}}x^n\,dx = \left\{\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k}a^k}{3n-3k+3(m+1)+1}\right\} \frac{3\sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}$$

2. $$\int \frac{x^n\,dx}{\sqrt[3]{z^{3m+2}}} = \left\{\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k}a^k}{3n-3k-3(m-1)-2}\right\} \frac{3}{b^{n+1}\sqrt[3]{z^{3(m-1)+2}}}$$

3. $$\int \sqrt[3]{z^{3m+2}}x^n\,dx = \left\{\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k}a^k}{3n-3k+3(m+1)+2}\right\} \frac{3\sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}$$

4. $$\int \frac{x^n\,dx}{\sqrt[3]{z^{3m+1}}} = \left\{\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} z^{n-k}a^k}{3n-3k-3(m-1)-1}\right\} \frac{3}{b^{n+1}\sqrt[3]{z^{3(m-1)+1}}}$$

5. $$\int \frac{z^n\,dx}{x^m \sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1)ax^{m-1}} + \frac{3n-3m+4}{3(m-1)}\frac{b}{a}\int \frac{z^n\,dx}{x^{m-1}\sqrt[3]{z^2}}$$

For $m = 1$

$$\int \frac{z^n\,dx}{x\sqrt[3]{z^2}} = \frac{3z^n}{(3n-2)\sqrt[3]{z^2}} + a\int \frac{z^{n-1}\,dx}{x\sqrt[3]{z^2}}$$

6. $$\int \frac{dx}{xz^n\sqrt[3]{z^2}} = \frac{3\sqrt[3]{z}}{(3n-1)az^n} + \frac{1}{a}\int \frac{\sqrt[3]{z}\,dx}{xz^n}$$

2.232 $$\int \frac{dx}{x\sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}}\left\{\frac{3}{2}\ln\frac{\sqrt[3]{z}-\sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3}\arctan\frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z}+2\sqrt[3]{a}}\right\}$$

2.233

1. $$\int \frac{\sqrt[3]{z}\,dx}{x} = 3\sqrt[3]{z} + a\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

2. $$\int \frac{\sqrt[3]{z}\,dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

3. $$\int \frac{\sqrt[3]{z}\,dx}{x^3} = \left(-\frac{1}{2ax^2} + \frac{b}{3a^2x}\right)z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

4. $$\int \frac{dx}{x^2\sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

5. $$\int \frac{dx}{x^3\sqrt[3]{z^2}} = \left[-\frac{1}{2ax^2} + \frac{5b}{6a^2x}\right]\sqrt[3]{z} + \frac{5b^2}{9a^2}\int \frac{dx}{x\sqrt[3]{z^2}}$$ (see **2.232**)

2.234

1. $$\int \frac{z^n\,dx}{x^m\sqrt[3]{z^2}} = -\frac{z^n\sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)}\frac{b}{a}\int \frac{z^n\,dx}{x^{m-1}\sqrt[3]{z}}$$

For $m = 1$:

2. $$\int \frac{z^n\,dx}{x\sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a\int \frac{z^{n-1}\,dx}{x\sqrt[3]{z}}$$

3. $$\int \frac{dx}{xz^n\sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a}\int \frac{\sqrt[3]{z^2}\,dx}{xz^n}$$

2.235 $$\int \frac{dx}{x\sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}}\left\{\frac{3}{2}\ln\frac{\sqrt[3]{z}-\sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3}\arctan\frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z}+2\sqrt[3]{a}}\right\}$$

2.236

1. $$\int \frac{\sqrt[3]{z^2}\,dx}{x} = \frac{3}{2}\sqrt[3]{z^2} + a\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

2. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a}\sqrt[3]{z^2} + \frac{2b}{3}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

3. $$\int \frac{\sqrt[3]{z^2}\,dx}{x^3} = \left[-\frac{1}{2ax^2} + \frac{b}{6a^2x}\right] z^{5/3} - \frac{b^2}{6a^2}\sqrt[3]{z^2} - \frac{b^2}{9a}\int \frac{dx}{x\sqrt[3]{z}}$$

(see **2.235**)

4. $$\int \frac{dx}{x^2\sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

5. $$\int \frac{dx}{x^3\sqrt[3]{z}} = \left[-\frac{1}{2ax^2} + \frac{2b}{3a^2x}\right]\sqrt[3]{z} + \frac{2b^2}{9a^2}\int \frac{dx}{x\sqrt[3]{z}}$$ (see **2.235**)

2.24 Forms containing $\sqrt{a+bx}$ and the binomial $\alpha + \beta x$

Notation: $z = a + bx, \quad t = \alpha + \beta\text{x}, \quad \Delta = a\beta - b\alpha.$

2.241

1. $$\int \frac{z^m t^n\,dx}{\sqrt{z}} = \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1}\sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta}\int \frac{z^{m-1}t^n\,dx}{\sqrt{z}}$$ LA 176 (1)

2. $$\int \frac{t^n z^m\,dx}{\sqrt{z}} = 2\sqrt{z^{2m+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{\alpha^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p+2m+1}$$

2.242

1.[11] $$\int \frac{t\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z}}{b} + \beta\left(\frac{z}{3} - a\right)\frac{2\sqrt{z}}{b^2}$$

2. $$\int \frac{t^2\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z}}{b} + 2\alpha\beta\left(\frac{z}{3} - a\right)\frac{2\sqrt{z}}{b^2} + \beta^2\left(\frac{z^2}{5} - \frac{2}{3}za + a^2\right)\frac{2\sqrt{z}}{b^3}$$

3. $$\int \frac{t^3\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z}}{b} + 3\alpha^2\beta\left(\frac{z}{3} - \alpha\right)\frac{2\sqrt{z}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{5} - \frac{2}{3}za + a^2\right)\frac{2\sqrt{z}}{b^3}$$
$$+\beta^\alpha\left(\frac{z^3}{7} - \frac{3z^2a}{5} + za^2 - a^3\right)\frac{2\sqrt{z}}{b^4}$$

4. $$\int \frac{tz\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^3}}{3b} + \beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2}$$

5. $$\int \frac{t^2z\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^3}}{3b} + 2\alpha\beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2} + \beta^2\left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3}\right)\frac{2\sqrt{z^3}}{b^3}$$

6. $$\int \frac{t^3z\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^3}}{3b} + 3\alpha^2\beta\left(\frac{z}{5} - \frac{a}{3}\right)\frac{2\sqrt{z^3}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{7} - \frac{2za}{5} + \frac{a^2}{3}\right)\frac{2\sqrt{z^3}}{b^3}$$
$$+\beta^3\left(\frac{z^3}{9} - \frac{3z^2a}{7} + \frac{3za^2}{5} - \frac{a^3}{3}\right)\frac{2\sqrt{z^3}}{b^4}$$

7. $$\int \frac{tz^2\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^5}}{5b} + \beta\left(\frac{z}{7} - \frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2}$$

8. $$\int \frac{t^2z^2\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^5}}{5b} + 2\alpha\beta\left(\frac{z}{7} - \frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2} + \beta^2\left(\frac{z^2}{9} - \frac{2za}{7} + \frac{a^2}{5}\right)\frac{2\sqrt{z^5}}{b^3}$$

9. $$\int \frac{t^3 z^2\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^5}}{5b} + 3\alpha^2\beta\left(\frac{z}{7}-\frac{a}{5}\right)\frac{2\sqrt{z^5}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{9}-\frac{2za}{7}+\frac{a^2}{5}\right)\frac{2\sqrt{z^5}}{b^3} + \beta^3\left(\frac{z^3}{11}-\frac{3z^2a}{9}+\frac{3za^2}{7}-\frac{a^3}{5}\right)\frac{2\sqrt{z^5}}{b^4}$$

10. $$\int \frac{tz^3\,dx}{\sqrt{z}} = \frac{2\alpha\sqrt{z^7}}{7b} + \beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2}$$

11. $$\int \frac{t^2 z^3\,dx}{\sqrt{z}} = \frac{2\alpha^2\sqrt{z^7}}{7b} + 2\alpha\beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2} + \beta^2\left(\frac{z^2}{11}-\frac{2za}{9}+\frac{a^2}{7}\right)\frac{2\sqrt{z^7}}{b^3}$$

12. $$\int \frac{t^3 z^3\,dx}{\sqrt{z}} = \frac{2\alpha^3\sqrt{z^7}}{7b} + 3\alpha^2\beta\left(\frac{z}{9}-\frac{a}{7}\right)\frac{2\sqrt{z^7}}{b^2} + 3\alpha\beta^2\left(\frac{z^2}{11}-\frac{2za}{9}+\frac{a^2}{7}\right)\frac{2\sqrt{z^7}}{b^3} + \beta^3\left(\frac{z^3}{13}-\frac{3z^2a}{11}+\frac{3za^2}{9}-\frac{a^3}{7}\right)\frac{2\sqrt{z^7}}{b^4}$$

2.243

1. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{t^{n+1}}{z^m}\sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta}\int\frac{t^n\,dx}{z^{m-1}\sqrt{z}} = -\frac{2}{(2m-1)b}\frac{t^n}{z^m}\sqrt{z} + \frac{2n\beta}{(2m-1)b}\int\frac{t^{n-1}\,dx}{z^{m-1}\sqrt{z}}$$

LA 176 (2)

2. $$\int \frac{t^n\,dx}{z^m\sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}}\sum_{k=0}^{n}\binom{n}{k}\frac{a^{n-k}\beta^k}{b^{k+1}}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\frac{z^{k-p}a^p}{2k-2p-2m+1}$$

2.244

1. $$\int \frac{t\,dx}{z\sqrt{z}} = -\frac{2a}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2\sqrt{z}}$$

2. $$\int \frac{t^2\,dx}{z\sqrt{z}} = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2\sqrt{z}} + \frac{2\beta^2\left(\frac{z^2}{3}-2za-a^2\right)}{b^3\sqrt{z}}$$

3. $$\int \frac{t^3\,dx}{z\sqrt{z}} = -\frac{2\alpha^3}{b\sqrt{z}} + \frac{6\alpha^2\beta(z+a)}{b^2\sqrt{z}} + \frac{6\alpha\beta^2\left(\frac{z^2}{3}-2za-a^2\right)}{b^3\sqrt{z}} + \frac{2\beta^3\left(\frac{z^3}{5}-z^2a+3za^2+a^3\right)}{b^4\sqrt{z}}$$

4. $$\int \frac{t\,dx}{z^2\sqrt{z}} = -\frac{2a}{3b\sqrt{z^3}} - \frac{2\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}}$$

5. $$\int \frac{t^2\,dx}{z^2\sqrt{z}} = -\frac{2a^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{2\beta^2\left(z^2+2az-\frac{a^2}{3}\right)}{b^3\sqrt{z^3}}$$

6. $$\int \frac{t^3\,dx}{z^2\sqrt{z}} = -\frac{2\alpha^3}{3b\sqrt{z^3}} - \frac{6\alpha^2\beta\left(z-\frac{a}{3}\right)}{b^2\sqrt{z^3}} + \frac{6\alpha\beta^2\left(z^2+2za-\frac{a^2}{3}\right)}{b^3\sqrt{z^3}} + \frac{2\beta^3\left(\frac{z^3}{3}-3z^2a-3za^2+\frac{a^3}{3}\right)}{b^4\sqrt{z^3}}$$

7. $$\int \frac{t\,dx}{z^3\sqrt{z}} = -\frac{2\alpha}{5b\sqrt{z^5}} - \frac{2\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}}$$

8. $$\int \frac{t^2\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{2\beta^2\left(z^2-\frac{2za}{3}+\frac{a^2}{5}\right)}{b^3\sqrt{z^5}}$$

9. $$\int \frac{t^3\,dx}{z^3\sqrt{z}} = -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta\left(\frac{z}{3}-\frac{a}{5}\right)}{b^2\sqrt{z^5}} - \frac{6\alpha\beta^2\left(z^2-\frac{2za}{3}+\frac{a^2}{5}\right)}{b^3\sqrt{z^5}} + \frac{2\beta^3\left(z^3+3z^2a-za^2+\frac{a^3}{5}\right)}{b^4\sqrt{z^5}}$$

2.245

1. $$\int \frac{z^m\,dx}{t^n\sqrt{z}} = -\frac{2}{(2n-2m-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta}\int\frac{z^{m-1}\,dx}{t^n\sqrt{z}}$$ LA 176 (3)
$$= -\frac{1}{(n-1)\beta}\frac{z^{m-1}}{t^{n-1}}\sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta}\int\frac{z^{m-1}}{t^{n-1}\sqrt{z}}\,dx$$
$$= -\frac{1}{(n-1)\Delta}\frac{z^m}{t^{n-1}}\sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta}\int\frac{z^m\,dx}{t^{n-1}\sqrt{z}}$$

2. $$\int \frac{z^m\,dz}{t^n\sqrt{z}} = -z^m\sqrt{z}\left[\frac{1}{(n-1)\Delta}\frac{1}{t^{n-1}} + \sum_{k=2}^{n-1}\frac{(2n-2m-3)(2n-2m-5)\dots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k}\frac{1}{t^{n-k}}\right\}\Bigg] - \frac{(2n-2m-3)(2n-2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}\cdot(n-1)!\Delta^n}\int\frac{z^m\,dx}{t\sqrt{z}}$$

For $n=1$

3. $$\int \frac{z^m\,dx}{t\sqrt{z}} = \frac{2}{(2m-1)\beta}\frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta}\int\frac{z^{m-1}\,dx}{t\sqrt{z}}$$

4. $$\int \frac{z^m\,dx}{t\sqrt{z}} = 2\sum_{k=0}^{m-1}\frac{\Delta^k}{(2m-2k-1)\beta^{k+1}}\frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m}\int\frac{dx}{t\sqrt{z}}$$

2.246 $$\int\frac{dx}{t\sqrt{z}}\ \frac{1}{\sqrt{\beta\Delta}}\ln\frac{\beta\sqrt{z}-\sqrt{\beta\Delta}}{\beta\sqrt{z}+\sqrt{\beta\Delta}}$$ $[\beta\Delta > 0]$
$$= \frac{2}{\sqrt{-\beta\Delta}}\arctan\frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}}$$ $[\beta\Delta < 0]$
$$= -\frac{2\sqrt{z}}{bt}$$ $[\Delta = 0]$

2.247 $$\int\frac{dx}{tz^m\sqrt{z}} = \frac{2}{z^{m-1}\sqrt{z}} + \sum_{k=1}^{m}\frac{\beta^{k-1}z^k}{\Delta^k(2m-2k+1)} + \frac{\beta^m}{\Delta^m}\int\frac{dx}{t\sqrt{z}}$$

(see **2.246**)

2.248

1. $$\int\frac{dx}{tz\sqrt{z}} = \frac{2}{\Delta\sqrt{z}} + \frac{\beta}{\Delta}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

2. $$\int \frac{dx}{tz^2\sqrt{z}} = \frac{2}{3\Delta z\sqrt{z}} + \frac{2\beta}{\Delta^2\sqrt{z}} + \frac{\beta^2}{\Delta^2}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

3. $$\int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

4. $$\int \frac{dx}{t^2\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

5. $$\int \frac{dx}{t^2 z\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

6. $$\int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

7. $$\int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

8. $$\int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

9. $$\int \frac{dx}{t^3 z\sqrt{z}} = -\frac{1}{2\Delta t^2\sqrt{z}} + \frac{5b}{4\Delta^2 t\sqrt{z}} + \frac{15b^2}{4\Delta^3\sqrt{z}} + \frac{15b^2\beta}{8\Delta^3}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

10. $$\int \frac{dx}{t^3 z^2\sqrt{z}} = -\frac{1}{2\Delta t^2 z\sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 tz\sqrt{z}} + \frac{35b^2}{12\Delta^2 z\sqrt{z}} + \frac{35b^2\beta}{4\Delta^4\sqrt{z}} + \frac{35b^2\beta^2}{8\Delta^4}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

11. $$\int \frac{dx}{t^3 z^3\sqrt{z}} = -\frac{1}{2\Delta t^2 z^2\sqrt{z}} + \frac{9b}{4\Delta^2 tz^2\sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2\sqrt{z}} + \frac{21b^2\beta}{4\Delta^4 z\sqrt{z}} + \frac{63b^2\beta^2}{4\Delta^5\sqrt{z}} + \frac{63b^2\beta^3}{8\Delta^5}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

12. $$\int \frac{z\,dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

13. $$\int \frac{z^2\,dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

14. $$\int \frac{z^3\,dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

15. $$\int \frac{z\,dx}{t^2\sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

16. $$\int \frac{z^2\,dx}{t^2\sqrt{z}} = -\frac{z^2\sqrt{z}}{\Delta t} + \frac{bz\sqrt{z}}{\beta\Delta} + \frac{3b\sqrt{z}}{\beta^2} + \frac{3b\Delta}{2\beta^2}\int\frac{dx}{t\sqrt{z}}$$ (see **2.246**)

17. $$\int \frac{z^3\,dx}{t^2\sqrt{z}} = -\frac{z^3\sqrt{z}}{\Delta t} + \frac{bz^2\sqrt{z}}{\beta\Delta} + \frac{5bz\sqrt{z}}{3\beta^2} + \frac{5b\Delta\sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

18.[3] $$\int \frac{z\,dx}{t^3\sqrt{z}} = -\frac{z\sqrt{z}}{2\Delta t^2} + \frac{bz\sqrt{z}}{4\Delta^2 t} - \frac{b^2\sqrt{z}}{4\beta\Delta^2} + \frac{b^2}{8\beta\Delta}\int \frac{dx}{t\sqrt{z}}$$ (see **2.246**)

19. $$\int \frac{z^2\,dx}{t^3\sqrt{z}} = -\frac{z^2\sqrt{z}}{2\Delta t^2} + \frac{bz^2\sqrt{z}}{4\Delta^2 t} + \frac{b^2 z\sqrt{z}}{4\beta\Delta^2} + \frac{3b^2\sqrt{z}}{4\beta^2\Delta} + \frac{3b^2}{8\beta^2}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

20. $$\int \frac{z^3\,dx}{t^3\sqrt{z}} = -\frac{z^3\sqrt{z}}{2\Delta t^2} + \frac{3bz^3\sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2\sqrt{z}}{4\beta\Delta^2} + \frac{5b^2 z\sqrt{z}}{4\beta^2\Delta} + \frac{15b^2\sqrt{z}}{4\beta^3} + \frac{15b^2\Delta}{8\beta^3}\int \frac{dx}{t\sqrt{z}}$$

(see **2.246**)

2.249

1. $$\int \frac{dx}{z^m t^n\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{\sqrt{z}}{t^{n-1}z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta}\int \frac{dx}{t^n z^{m-1}\sqrt{z}}$$ LA 177 (4)
$$= -\frac{1}{(n-1)\Delta}\frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta}\int \frac{dx}{t^{n-1}z^m\sqrt{z}}$$

2. $$\int \frac{dx}{z^m t^n\sqrt{z}} = \frac{\sqrt{z}}{z^m}\left[\frac{-1}{(n-1)\Delta}\frac{1}{t^{n-1}} + \sum_{k=2}^{n-1}(-1)^k\frac{(2n+2m-3)(2n+2m-5)\dots(2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k}\cdot\frac{1}{t^{n-k}}\right] + (-1)^{n-1}\frac{(2n+2m-3)(2n+2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}}\int\frac{dx}{tz^m\sqrt{z}}$$

For $n=1$
$$\int \frac{dx}{z^m t\sqrt{z}} = \frac{2}{(2m-1)\Delta}\frac{1}{z^{m-1}\sqrt{z}} + \frac{\beta}{\Delta}\int \frac{dx}{tz^{m-1}\sqrt{z}}$$

2.25 Forms containing $\sqrt{a+bx+cx^2}$

Integration techniques

2.251 It is possible to rationalize the integrand in integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right)dx$ by using one or more of the following three substitutions, known as the "Euler substitutions":

1. $\sqrt{a+bx+cx^2} = xt \pm \sqrt{a}$ for $a > 0$;
2. $\sqrt{a+bx+cx^2} = t \pm x\sqrt{c}$ for $c > 0$;
3. $\sqrt{c\,(x-x_1)\,(x-x_2)} = t\,(x-x_1)$ when x_1 and x_2 are real roots of the equation $a+bx+cx^2=0$.

2.252 Besides the Euler substitutions, there is also the following method of calculating integrals of the form $\int R\left(x, \sqrt{a+bx+cx^2}\right) dx$. By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of x and an expression of the form $\frac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, where $P_1(x)$ and $P_2(x)$ are both polynomials. By separating the integral portion of the rational function $\frac{P_1(x)}{P_2(x)}$ from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals, each of which is in one of the following three forms:

1. $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$, where $P(x)$ is a polynomial of some degree r;
2. $\int \frac{dx}{(x+p)^k\sqrt{a+bx+cx^2}}$;
3. $\int \frac{(Mx+N)\,dx}{(a+\beta x+x^2)^m\,\sqrt{c\,(a_1+b_1x+x^2)}}$, $\qquad \left(a_1 = \frac{a}{c}, \quad b_1 = \frac{b}{c}\right)$.

In more detail:

1. $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}} = Q(x)\sqrt{a+bx+cx^2} + \lambda\int \frac{dx}{\sqrt{a+bx+cx^2}}$, where $Q(x)$ is a polynomial of degree $(r-1)$. Its coefficients, and also the number λ, can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)\left(a+bx+cx^2\right) + \frac{1}{2}Q(x)(b+2cx) + \lambda \qquad \text{LI II 77}$$

Integrals of the form $\int \frac{P(x)\,dx}{\sqrt{a+bx+cx^2}}$ (where $r \le 3$) can also be calculated by use of formulas **2.26**.

2. Integrals of the form $\int \frac{P(x)\,dx}{(x+p)^k\sqrt{a+bx+cx^2}}$, where the degree n of the polynomial $P(x)$ is lower than k can, by means of the substitution $t = \frac{1}{x+p}$, be reduced to an integral of the form $\int \frac{P(t)\,dt}{\sqrt{a+\beta t+\gamma t^2}}$. (See also **2.281**).

3. Integrals of the form $\int \frac{(Mx+N)\,dx}{(\alpha+\beta x+x^2)^m\,\sqrt{c\,(a_1+b_1x+x^2)}}$ can be calculated by the following procedure:

   - If $b_1 \ne \beta$, by using the substitution

$$x = \frac{a_1-\alpha}{\beta b_1} + \frac{t-1}{t+1}\frac{\sqrt{(a_1-\alpha)^2 - (\alpha b_1 - a_1\beta)(\beta - b_1)}}{\beta - b_1}$$

we can reduce this integral to an integral of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c\,(t^2+q)}}$, where $P(t)$ is a polynomial of degree no higher than $2m-1$. The integral $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{t^2+q}}$ can be reduced to the sum of integrals of the forms $\int \frac{t\,dt}{(t^2+p)^k\sqrt{t^2+q}}$ and $\int \frac{dt}{(t^2+p)^k\sqrt{t^2+q}}$.

- If $b_1=\beta$, we can reduce it to integrals of the form $\int \frac{P(t)\,dt}{(t^2+p)^m\sqrt{c\,(t^2+q)}}$ by means of the substitution $t=x+\frac{b_1}{2}$.

The integral $\int \frac{t\,dt}{(t^2+p)^k\sqrt{c\,(t^2+q)}}$ can be evaluated by means of the substitution $t^2+q=u^2$.

The integral $\int \frac{dt}{(t^2+p)^k\sqrt{c\,(t^2+q)}}$ can be evaluated by means of the substitution $\frac{t}{\sqrt{t^2+q}}=v$ (see also **2.283**). FI II 78-82

2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x

Notation: $R=a+bx+cx^2, \quad \Delta=4ac-b^2$

For simplified formulas for the case $b=0$, see **2.27**.

2.260

1. $$\int x^m\sqrt{R^{2n+1}}\,dx=\frac{x^{m-1}\sqrt{R^{2n+3}}}{(m+2n+2)c}-\frac{(2m+2n+1)b}{2(m+2n+2)c}\int x^{m-1}\sqrt{R^{2n+1}}\,dx -\frac{(m-1)a}{(m+2n+2)c}\int x^{m-2}\sqrt{R^{2n+1}}\,dx$$ TI (192)a

2. $$\int\sqrt{R^{2n+1}}\,dx=\frac{2cx+b}{4(n+1)c}\sqrt{R^{2n+1}}+\frac{2n+1}{8(n+1)}\frac{\Delta}{c}\int\sqrt{R^{2n-1}}\,dx$$ TI (188)

3. $$\int\sqrt{R^{2n+1}}\,dx=\frac{(2cx+b)\sqrt{R}}{4(n+1)c}\left\{R^n+\sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)}\left(\frac{\Delta}{c}\right)^{k+1}R^{n-k-1}\right\} +\frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{\Delta}{c}\right)^{n+1}\int\frac{dx}{\sqrt{R}}$$ TI (190)

2.261[11] For $n=-1$

$$\int\frac{dx}{\sqrt{R}}=\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{cR}+2cx+b}{\sqrt{\Delta}}\right) \qquad [c>0]$$ TI (127)

$$=\frac{1}{\sqrt{c}}\operatorname{arcsinh}\left(\frac{2cx+b}{\sqrt{\Delta}}\right) \qquad [c>0, \quad \Delta>0]$$ DW

$$=\frac{1}{\sqrt{c}}\ln(2cx+b) \qquad [c>0, \quad \Delta=0]$$ DW

$$=\frac{-1}{\sqrt{-c}}\arcsin\left(\frac{2cx+b}{\sqrt{-\Delta}}\right) \qquad [c<0, \quad \Delta<0]$$ TI (128)

2.262

1. $$\int \sqrt{R}\,dx = \frac{(2cx+b)\sqrt{R}}{4c} + \frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2. $$\int x\sqrt{R}\,dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx+b)b}{8c^2}\sqrt{R} - \frac{b\Delta}{16c^2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

3. $$\int x^2\sqrt{R}\,dx = \left(\frac{x}{4c} - \frac{5b}{24c^2}\right)\sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{(2cx+b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

4. $$\int x^3\sqrt{R}\,dx = \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2}\right)\sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{(2cx+b)\sqrt{R}}{4c} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2}\right)\frac{\Delta}{8c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

5. $$\int \sqrt{R^3}\,dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2}\right)(2cx+b)\sqrt{R} + \frac{3\Delta^2}{128c^2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

6. $$\int x\sqrt{R^3}\,dx = \frac{\sqrt{R^5}}{5c} - (2cx+b)\left(\frac{b}{16c^2}\sqrt{R^3} + \frac{3\Delta b}{128c^3}\sqrt{R}\right) - \frac{3\Delta^2 b}{256c^3}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

7. $$\int x^2\sqrt{R^3}\,dx = \left(\frac{x}{6c} - \frac{7b}{60c^2}\right)\sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) + \left(\frac{7b^2}{4c} - a\right)\frac{\Delta^2}{256c^3}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

8. $$\int x^3\sqrt{R^3}\,dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2}\right)\sqrt{R^5} - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2}\right)\left(2x + \frac{b}{c}\right)\left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c}\sqrt{R}\right) - \left(\frac{3b^2}{4c} - a\right)\frac{3\Delta^2 b}{512c^4}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2.263

1. $$\int \frac{x^m\,dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c}\int \frac{x^{m-1}\,dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c}\int \frac{x^{m-2}\,dx}{\sqrt{R^{2n+1}}}$$ TI (193)a

For $m = 2n$

2. $$\int \frac{x^{2n}\,dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c}\int \frac{x^{2n-1}}{\sqrt{R^{2n+1}}}\,dx + \frac{1}{c}\int \frac{x^{2n-2}}{\sqrt{R^{2n-1}}}\,dx$$ TI (194)a

3. $$\int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta}\int \frac{dx}{\sqrt{R^{2n-1}}}$$ TI (189)

4. $$\int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}}\left\{1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\frac{c^k}{\Delta^k}R^k\right\}$$
$[n \geq 1]$. TI (191)

2.264

1. $$\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

2. $$\int \frac{x\,dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

3. $$\int \frac{x^2\,dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{R} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right)\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

4. $$\int \frac{x^3\,dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^2}\right)\int \frac{dx}{\sqrt{R}}$$
(see **2.261**)

5. $$\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx+b)}{\Delta\sqrt{R}}$$

6. $$\int \frac{x\,dx}{\sqrt{R^3}} = -\frac{2(2a+bx)}{\Delta\sqrt{R}}$$

7. $$\int \frac{x^2\,dx}{\sqrt{R^3}} = -\frac{\left(\Delta - b^2\right)x - 2ab}{c\Delta\sqrt{R}} + \frac{1}{c}\int \frac{dx}{\sqrt{R}}$$ (see **2.261**)

8. $$\int \frac{x^3\,dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b\left(10ac - 3b^2\right)x + a\left(8ac - 3b^2\right)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2}\int \frac{dx}{\sqrt{R}}$$
(see **2.261**)

2.265 $$\int \frac{\sqrt{R^{2n+1}}}{x^m}\,dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-1}}\,dx + \frac{(2n-m+4)c}{(m-1)a}\int \frac{\sqrt{R^{2n+1}}}{x^{m-2}}\,dx$$
TI (195)

For $m = 1$
$$\int \frac{\sqrt{R^{2n+1}}}{x}\,dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2}\int \sqrt{R^{2n-1}}\,dx + a\int \frac{\sqrt{R^{2n-1}}}{x}\,dx$$ TI (198)

For $a = 0$
$$\int \frac{\sqrt{\left(bx+cx^2\right)^{2n+1}}}{x^m}\,dx = \frac{2\sqrt{\left(bx+cx^2\right)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b}\int \frac{\sqrt{\left(bx+cx^2\right)^{2n+1}}}{x^{m-1}}$$ LA 169 (3)

For $m = 0$ see **2.260** 2 and **2.260** 3.
For $n = -1$ and $m = 1$:

2.266[8]
$$\int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \qquad [a>0] \qquad \text{TI (137)}$$
$$= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \qquad [a<0, \quad \Delta<0] \qquad \text{TI (138)}$$
$$= \frac{1}{\sqrt{-a}} \arctan \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \qquad [a<0] \qquad \text{LA 178 (6)a}$$
$$= -\frac{1}{\sqrt{a}} \operatorname{arcsinh} \frac{2a+bx}{x\sqrt{\Delta}} \qquad [a>0, \quad \Delta>0] \qquad \text{DW}$$
$$= -\frac{1}{\sqrt{a}} \operatorname{arctanh} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \qquad [a>0]$$
$$= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} \qquad [a>0, \quad \Delta=0]$$
$$= -\frac{2\sqrt{bx+cx^2}}{bx} \qquad [a=0, \quad b\neq 0] \qquad \text{LA 170 (16)}$$
$$= \frac{1}{\sqrt{a}} \operatorname{arccosh}\left(\frac{2a+bx}{x\sqrt{-\Delta}}\right) \qquad [a>0, \Delta<0]$$

2.267

1. $$\int \frac{\sqrt{R}\,dx}{x} = \sqrt{R} + a\int \frac{dx}{x\sqrt{R}} + \frac{b}{2}\int \frac{dx}{\sqrt{R}}$$ (see **2.261** and **2.266**)

2. $$\int \frac{\sqrt{R}\,dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{R}} + c\int \frac{dx}{\sqrt{R}}$$ (see **2.261** and **2.266**)

For $a=0$
$$\int \frac{\sqrt{bx+cx^2}}{x^2}\,dx = -\frac{2\sqrt{bx+cx^2}}{x} + c\int \frac{dx}{\sqrt{bx+cx^2}}$$ (see **2.261**)

3. $$\int \frac{\sqrt{R}\,dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right)\sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right)\int \frac{dx}{x\sqrt{R}}$$

(see **2.266**)

For $a=0$
$$\int \frac{\sqrt{bx+cx^2}}{x^3}\,dx = -\frac{2\sqrt{\left(bx+cx^2\right)^3}}{3bx^3}$$

4. $$\int \frac{\sqrt{R^3}}{x}\,dx = \frac{\sqrt{R^3}}{3} + \frac{2bcx+b^2+8ac}{8c}\sqrt{R} + a^2\int \frac{dx}{x\sqrt{R}} + \frac{b\left(12ac-b^2\right)}{16c}\int \frac{dx}{\sqrt{R}}$$

(see **2.261** and **2.266**)

5. $$\int \frac{\sqrt{R^3}}{x^2}\,dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx+b}{a}\sqrt{R^3} + \frac{3}{4}(2cx+3b)\sqrt{R} + \frac{3}{2}ab\int \frac{dx}{x\sqrt{R}} + \frac{3\left(4ac+b^2\right)}{8}\int \frac{dx}{\sqrt{R}}$$

(see **2.261** and **2.266**)

For $a=0$
$$\int \frac{\sqrt{\left(bx+cx^2\right)^3}}{x^2} = \frac{\sqrt{\left(bx+cx^2\right)^3}}{2x} + \frac{3b}{4}\sqrt{bx+cx^2} + \frac{3b^2}{8}\int \frac{dx}{\sqrt{bx+cx^2}}$$

(see **2.261**)

6. $$\int \frac{\sqrt{R^3}}{x^3}\,dx = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right)\sqrt{R^5} + \frac{bcx + 2ac + b^2}{4a^2}\sqrt{R^3} + \frac{3\left(bcx + 2ac + b^2\right)}{4a}\sqrt{R}$$
$$+\frac{3}{8}\left(4ac + b^2\right)\int \frac{dx}{x\sqrt{R}} + \frac{3}{2}bc\int \frac{dx}{\sqrt{R}}$$
(see **2.261** and **2.266**)

For $a = 0$
$$\int \frac{\sqrt{\left(bx + cx^2\right)^3}}{x^3}\,dx = \left(c - \frac{2b}{x}\right)\sqrt{bx + cx^2} + \frac{3bc}{2}\int \frac{dx}{\sqrt{bx + cx^2}}$$
(see **2.261**)

2.268 $$\int \frac{dx}{x^m\sqrt{R^{2n+1}}} = -\frac{1}{(m-1)ax^{m-1}\sqrt{R^{2n-1}}}$$
$$-\frac{(2n+2m-3)b}{2(m-1)a}\int \frac{dx}{x^{m-1}\sqrt{R^{2n+1}}} - \frac{(2n+m-2)c}{(m-1)a}\int \frac{dx}{x^{m-2}\sqrt{R^{2n+1}}}$$
TI (196)

For $m = 1$
$$\int \frac{dx}{x\sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a\sqrt{R^{2n-1}}} - \frac{b}{2a}\int \frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a}\int \frac{dx}{x\sqrt{R^{2n-1}}}$$ TI (199)

For $a = 0$
$$\int \frac{dx}{x^m\sqrt{\left(bx + cx^2\right)^{2n+1}}} = -\frac{2}{(2n+2m-1)bx^m\sqrt{\left(bx + cx^2\right)^{2n-1}}}$$
$$-\frac{(4n+2m-2)c}{(2n+2m-1)b}\int \frac{dx}{x^{m-1}\sqrt{\left(bx + cx^2\right)^{2n+1}}}$$
(cf. **2.265**)

2.269

1. $$\int \frac{dx}{x\sqrt{R}}$$ (see **2.266**)

2. $$\int \frac{dx}{x^2\sqrt{R}} = -\frac{\sqrt{R}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{R}}$$ (see **2.266**)

For $a = 0$
$$\int \frac{dx}{x^2\sqrt{bx + cx^2}} = \frac{2}{3}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x}\right)\sqrt{bx + cx^2}$$

3. $$\int \frac{dx}{x^3\sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right)\sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right)\int \frac{dx}{x\sqrt{R}}$$
(see **2.266**)

For $a = 0$
$$\int \frac{dx}{x^3\sqrt{bx + cx^2}} = \frac{2}{5}\left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x}\right)\sqrt{bx + cx^2}$$

4. $$\int \frac{dx}{x\sqrt{R^3}} = -\frac{2\left(bcx - 2ac + b^2\right)}{a\Delta\sqrt{R}} + \frac{1}{a}\int \frac{dx}{x\sqrt{R}}$$ (see **2.266**)

For $a = 0$

$$\int \frac{dx}{x\sqrt{\left(bx+cx^2\right)^3}} = \frac{2}{3}\left(-\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3}\right)\frac{1}{\sqrt{bx+cx^2}}$$

5.[11] $$\int \frac{dx}{x^2\sqrt{R^3}} = -\frac{A}{\sqrt{R}} - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{R}}$$

$$\text{where } A = \left(-\frac{1}{ax} - \frac{b\left(10ac-3b^2\right)}{a^2\Delta} - \frac{c\left(8ac-3b^2\right)x}{a^2\Delta}\right)$$ (see **2.266**)

For $a=0$

$$\int \frac{dx}{x^2\sqrt{\left(bx+cx^2\right)^3}} = \frac{2}{5}\left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4}\right)\frac{1}{\sqrt{bx+cx^2}}$$

6. $$\int \frac{dx}{x^3\sqrt{R^3}}$$

$$= \left(-\frac{1}{ax^2} + \frac{5b}{2a^2x} - \frac{15b^4-62acb^2+24a^2c^2}{2a^3\Delta} - \frac{bc\left(15b^2-52ac\right)x}{2a^3\Delta}\right)\frac{1}{2\sqrt{R}} + \frac{15b^2-12ac}{8a^3}\int \frac{dx}{x\sqrt{R}}$$

(see **2.266**)

For $a=0$

$$\int \frac{dx}{x^3\sqrt{\left(bx+cx^2\right)^3}} = \frac{2}{7}\left(-\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5}\right)\frac{1}{\sqrt{bx+cx^2}}$$

2.27 Forms containing $\sqrt{a+cx^2}$ and integral powers of x

Notation: $u=\sqrt{a+cx^2}$.

$$I_1 = \frac{1}{\sqrt{c}}\ln\left(x\sqrt{c}+u\right) \qquad [c>0]$$

$$= \frac{1}{\sqrt{-c}}\arcsin x\sqrt{-\frac{c}{a}} \qquad [c<0 \text{ and } a>0]$$

$$I_2 = \frac{1}{2\sqrt{a}}\ln\frac{u-\sqrt{a}}{u+\sqrt{a}} \qquad [a>0 \text{ and } c>0]$$

$$= \frac{1}{2\sqrt{a}}\ln\frac{\sqrt{a}-u}{\sqrt{a}+u} \qquad [a>0 \text{ and } c>0]$$

$$= \frac{1}{\sqrt{-a}}\operatorname{arcsec} x\sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}}\arccos\frac{1}{x}\sqrt{-\frac{a}{c}} \qquad [a<0 \text{ and } c>0]$$

2.271

1. $$\int u^5\,dx = \frac{1}{6}xu^5 + \frac{5}{24}axu^3 + \frac{5}{16}a^2xu + \frac{5}{16}a^3I_1$$ DW

2. $$\int u^3\,dx = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2I_1$$ DW

3. $$\int u\,dx = \frac{1}{2}xu + \frac{1}{2}aI_1$$ DW

4. $$\int \frac{dx}{u} = I_1$$ DW

5. $$\int \frac{dx}{u^3} = \frac{1}{a}\frac{x}{u}$$ DW

6. $$\int \frac{dx}{u^{2n+1}} = \frac{1}{a^n}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}\binom{n-1}{k}\frac{c^k x^{2k+1}}{u^{2k+1}}$$

7. $$\int \frac{x\,dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}}$$ DW

2.272

1. $$\int x^2u^3\,dx = \frac{1}{6}\frac{xu^5}{c} - \frac{1}{24}\frac{axu^3}{c} - \frac{1}{16}\frac{a^2xu}{c} - \frac{1}{16}\frac{a^3}{c}I_1$$ DW

2. $$\int x^2u\,dx = \frac{1}{4}\frac{xu^3}{c} - \frac{1}{8}\frac{axu}{c} - \frac{1}{8}\frac{a^2}{c}I_1$$ DW

3. $$\int \frac{x^2}{u}\,dx = \frac{1}{2}\frac{xu}{c} - \frac{1}{2}\frac{a}{c}I_1$$ DW

4. $$\int \frac{x^2}{u^3}\,dx = -\frac{x}{cu} + \frac{1}{c}I_1$$ DW

5. $$\int \frac{x^2}{u^5}\,dx = \frac{1}{3}\frac{x^3}{au^3}$$ DW

6. $$\int \frac{x^2\,dx}{u^{2n+1}} = \frac{1}{a^{n-1}}\sum_{k=0}^{n-2}\frac{(-1)^k}{2k+3}\binom{n-2}{k}\frac{c^k x^{2k+3}}{u^{2k+3}}$$

7. $$\int \frac{x^3\,dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2u^{2n-3}} + \frac{a}{(2n-1)c^2u^{2n-1}}$$ DW

2.273

1. $$\int x^4u^3\,dx = \frac{1}{8}\frac{x^3u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2xu^3}{64c^2} + \frac{3a^3xu}{128c^2} + \frac{3a^4}{128c^2}I_1$$ DW

2. $$\int x^4u\,dx = \frac{1}{6}\frac{x^3u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2xu}{16c^2} + \frac{a^3}{16c^2}I_1$$ DW

3. $$\int \frac{x^4}{u}\,dx = \frac{1}{4}\frac{x^3u}{c} - \frac{3}{8}\frac{axu}{c^2} + \frac{3}{8}\frac{a^2}{c^2}I_1$$ DW

4. $$\int \frac{x^4}{u^3}\,dx = \frac{1}{2}\frac{xu}{c^2} + \frac{ax}{c^2u} - \frac{3}{2}\frac{a}{c^2}I_1$$ DW

5. $$\int \frac{x^4}{u^5}\,dx = -\frac{x}{c^2u} - \frac{1}{3}\frac{x^3}{cu^3} + \frac{1}{c^2}I_1$$ DW

6. $$\int \frac{x^4}{u^7}\,dx = \frac{1}{5}\frac{x^5}{au^5}$$ DW

7. $$\int \frac{x^4\,dx}{u^{2n+1}} = \frac{1}{a^{n-2}}\sum_{k=0}^{n-3}\frac{(-1)^k}{2k+5}\binom{n-3}{k}\frac{c^k x^{2k+5}}{u^{2k+5}}$$

8. $$\int \frac{x^5\,dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3u^{2n-5}} + \frac{2a}{(2n-3)c^{u2n-3}} - \frac{a^2}{(2n-1)c^3u^{2n-1}}$$ DW

2.274

1. $$\int x^6u^3\,dx = \frac{1}{10}\frac{x^5u^5}{c} - \frac{ax^3u^5}{16c^2} + \frac{a^2xu^5}{32c^3} - \frac{a^3xu^3}{128c^3} - \frac{3a^4xu}{256c^3} - \frac{3}{256}\frac{a^5}{c^3}I_1$$

2. $$\int x^6u\,dx = \frac{1}{8}\frac{x^5u^3}{c} - \frac{5}{48}\frac{ax^3u^3}{c^2} + \frac{5a^2xu^3}{64c^3} - \frac{5a^3xu}{128c^3} - \frac{5}{128}\frac{a^4}{c^3}I_1$$

3. $$\int \frac{x^6}{u}\,dx = \frac{1}{6}\frac{x^5u}{c} - \frac{5}{24}\frac{ax^3u}{c^2} + \frac{5}{16}\frac{a^2xu}{c^3} - \frac{5}{16}\frac{a^3}{c^3}I_1$$ DW

4. $$\int \frac{x^6}{u^3}\,dx = \frac{1}{4}\frac{x^5}{cu} - \frac{5}{8}\frac{ax^3}{c^2u} - \frac{15}{8}\frac{a^2x}{c^3u} + \frac{15}{8}\frac{a^2}{c^3}I_1$$ DW

5. $$\int \frac{x^6}{u^5}\,dx = \frac{1}{2}\frac{x^5}{cu^3} + \frac{10}{3}\frac{ax^3}{c^2u^3} + \frac{5}{2}\frac{a^2x}{c^3u^3} - \frac{5}{2}\frac{a}{c^3}I_1$$ DW

6. $$\int \frac{x^6}{u^7}\,dx = -\frac{23}{15}\frac{x^5}{cu^5} - \frac{7}{3}\frac{ax^3}{c^2u^5} - \frac{a^2x}{c^3u^5} + \frac{1}{c^3}I_1$$ DW

7. $$\int \frac{x^6}{u^9}\,dx = \frac{1}{7}\frac{x^7}{au^7}$$ DW

8. $$\int \frac{x^6\,dx}{u^{2n+1}} = \frac{1}{a^{n-3}}\sum_{k=0}^{n-4}\frac{(-1)^k}{2k+7}\binom{n-4}{k}\frac{c^kx^{2k+7}}{u^{2k+7}}$$

9. $$\int \frac{x^7\,dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4u^{2n-7}} + \frac{3a}{(2n-5)c^4u^{2n-5}} - \frac{3a^2}{(2n-3)c^4u^{2n-3}} + \frac{a^3}{(2n-1)c^4u^{2n-1}}$$ DW

2.275

1. $$\int \frac{u^5}{x}\,dx = \frac{u^5}{5} + \frac{1}{3}au^3 + a^2u + a^3I_2$$ DW

2. $$\int \frac{u^3}{x}\,dx = \frac{u^3}{3} + au + a^2I_2$$ DW

3. $$\int \frac{u}{x}\,dx = u + aI_2$$ DW

4. $$\int \frac{dx}{xu} = I_2$$ DW

5. $$\int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n}I_2 + \sum_{k=0}^{n-1}\frac{1}{(2k+1)a^{n-k}u^{2k+1}}$$

6. $$\int \frac{u^5}{x^2}\,dx = -\frac{u^5}{x} + \frac{5}{4}cxu^3 + \frac{15}{8}acxu + \frac{15}{8}a^2I_1$$ DW

7. $$\int \frac{u^3}{x^2}\,dx = -\frac{u^3}{x} + \frac{3}{2}cxu + \frac{3}{2}aI_1$$ DW

8. $$\int \frac{u}{x^2}\,dx = -\frac{u}{x} + cI_1$$ DW

9. $$\int \frac{dx}{x^2u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u}\right)^{2k-1} \right\}$$

2.276

1. $$\int \frac{u^5}{x^3}\,dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2cI_2$$ DW

2. $$\int \frac{u^3}{x^3}\,dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2$$ DW

3. $$\int \frac{u}{x^3}\,dx = -\frac{u}{2x^2} + \frac{c}{2}I_2$$ DW

4. $$\int \frac{dx}{x^3u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2$$ DW

5. $$\int \frac{dx}{x^3u^3} = -\frac{1}{2ax^2u} - \frac{3c}{2a^2u} - \frac{3c}{2a^2}I_2$$ DW

6. $$\int \frac{dx}{x^3u^5} = -\frac{1}{2ax^2u^3} - \frac{5}{6}\frac{c}{a^2u^3} - \frac{5}{2}\frac{c}{a^3u} - \frac{5}{2}\frac{c}{a^3}I_2$$ DW

7. $$\int \frac{u^5}{x^4}\,dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2xu}{2} + \frac{5}{2}acI_1$$ DW

8. $$\int \frac{u^3}{x^4}\,dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + cI_1$$ DW

9. $$\int \frac{u}{x^4}\,dx = -\frac{u^3}{3ax^3}$$ DW

10. $$\int \frac{dx}{x^4u^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{u^3}{3x^3} + (n+1)\frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left(\frac{x}{u}\right)^{2k-3} \right\}$$

2.277

1. $$\int \frac{u^3}{x^5}\,dx = -\frac{u^3}{4x^4} - \frac{3}{8}\frac{cu^3}{ax^2} + \frac{3}{8}\frac{c^2u}{a} + \frac{3}{8}c^2I_2$$ DW

2. $$\int \frac{u}{x^5}\,dx = -\frac{u}{4x^4} - \frac{1}{8}\frac{cu}{ax^2} - \frac{1}{8}\frac{c^2}{a}I_2$$ DW

3. $$\int \frac{dx}{x^5u} = -\frac{u}{4ax^4} + \frac{3}{8}\frac{cu}{a^2x^2} + \frac{3}{8}\frac{c^2}{a^2}I_2$$ DW

4. $$\int \frac{dx}{x^5u^3} = -\frac{1}{4ax^4u} + \frac{5}{8}\frac{c}{a^2x^2u} + \frac{15}{8}\frac{c^2}{a^3u} + \frac{15}{8}\frac{c^2}{a^3}I_2$$ DW

2.278

1. $$\int \frac{u^3}{x^6}\,dx = -\frac{u^5}{5ax^5}$$ DW

2. $$\int \frac{u}{x^6}\,dx = -\frac{u^3}{5ax^5} + \frac{2}{15}\frac{cu^3}{a^2x^3}$$ DW

3. $$\int \frac{dx}{x^6 u} = \frac{1}{a^3}\left(-\frac{u^5}{5x^5} + \frac{2}{3}\frac{cu^3}{x^3} - \frac{c^2 u}{x}\right)$$ DW

4. $$\int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}}\left\{-\frac{u^5}{5x^5} + \frac{1}{3}\binom{n+2}{1}\frac{cu^3}{x^3} - \binom{n+2}{2}\frac{c^2 u}{x} + \sum_{k=3}^{n+2}\frac{(-1)^k}{2k-5}\binom{n+2}{k}c^k\left(\frac{x}{u}\right)^{2k-5}\right\}$$

2.28 Forms containing $\sqrt{a+bx+cx^2}$ and first- and second-degree polynomials

Notation: $R = a + bx + cx^2$

See also 2.252

2.281[3] $$\int \frac{dx}{(x+p)^n\sqrt{R}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c + (b-2pc)t + (a - bp + cp^2)\,t^2}} \qquad \left[t = \frac{1}{x+p} > 0\right]$$

2.282

1.[3] $$\int \frac{\sqrt{R}\,dx}{x+p} = c\int \frac{x\,dx}{\sqrt{R}} + (b - cp)\int \frac{dx}{\sqrt{R}} + \left(a - bp + cp^2\right)\int \frac{dx}{(x+p)\sqrt{R}} \qquad [x+p>0]$$

2. $$\int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}$$

3. $$\int \frac{\sqrt{R}\,dx}{(x+p)(x+q)} = \frac{1}{q-p}\int \frac{\sqrt{R}\,dx}{x+p} + \frac{1}{p-q}\int \frac{\sqrt{R}\,dx}{x+q}$$

4. $$\int \frac{(x+p)\sqrt{R}\,dx}{x+q} = \int \sqrt{R}\,dx + (p-q)\int \frac{\sqrt{R}\,dx}{x+q}$$

5. $$\int \frac{(rx+s)\,dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p}\int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q}\int \frac{dx}{(x+q)\sqrt{R}}$$

2.283 $$\int \frac{(Ax+B)\,dx}{(p+R)^n\sqrt{R}} = \frac{A}{c}\int \frac{du}{(p+u^2)^n} + \frac{2Bc - Ab}{2c}\int \frac{\left(1 - cv^2\right)^{n-1}\,dv}{\left[p + a - \frac{b^2}{4c} - cpv^2\right]^n},$$

where $u = \sqrt{R}$ and $v = \dfrac{b+2cx}{2c\sqrt{R}}$.

2.284 $$\int \frac{Ax+B}{(p+R)\sqrt{R}}\,dx = \frac{A}{c}I_1 + \frac{2Bc - Ab}{\sqrt{c^2 p\left[b^2 - 4(a+p)c\right]}}I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}}\arctan\sqrt{\frac{R}{p}} \qquad [p>0]$$

$$= \frac{1}{2\sqrt{-p}}\ln\frac{\sqrt{-p}-\sqrt{R}}{\sqrt{-p}+\sqrt{R}} \qquad [p<0]$$

$$I_2 = \arctan\sqrt{\frac{p}{b^2-4(a+p)c}}\frac{b+2cx}{\sqrt{R}} \qquad \left[p\left\{b^2-4(a+p)c\right\}>0, \quad p<0\right]$$

$$= -\arctan\sqrt{\frac{p}{b^2-4(a+p)c}}\frac{b+2cx}{\sqrt{R}} \qquad \left[p\left\{b^2-4(a+p)c\right\}>0, \quad p>0\right]$$

$$= \frac{1}{2i}\ln\frac{\sqrt{4(a+p)c-b^2}\sqrt{R}+\sqrt{p}(b+2cx)}{\sqrt{4(a+p)c-b^2}\sqrt{R}-\sqrt{p}(b+2cx)} \qquad \left[p\left\{b^2-4(a+p)c\right\}<0, \quad p>0\right]$$

$$= \frac{1}{2i}\ln\frac{\sqrt{b^2-4(a+p)c}\sqrt{R}-\sqrt{-p}(b+2cx)}{\sqrt{b^2-4(a+p)c}\sqrt{R}+\sqrt{-p}(b+2cx)} \qquad \left[p\left\{b^2-4(a+p)c\right\}<0, \quad p<0\right]$$

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.290 Integrals of the form $\int R\left(x, \sqrt{P(x)}\right)\,dx$, where $P(x)$ is a third- or fourth-degree polynomial, can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see **8.11**). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see **3.13**, **3.17**).

2.291 Certain integrals of the form $\int R\left(x, \sqrt{P(x)}\right)\,dx$, where $P_n(x)$ is a polynomial of not more than fourth degree, can be reduced to integrals of the form $\int R\left(x, \sqrt[k]{P_n(x)}\right)\,dx$ with $k \geq 2$. Below are examples of this procedure.

1. $$\int \frac{dx}{\sqrt{1-x^6}} = -\int \frac{dz}{\sqrt{3+3z^2+z^4}} \qquad \left[x^2 = \frac{1}{1+z^2}\right]$$

2. $$\int \frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2}\int \frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}} \qquad \left[x^2 = z\right]$$

3. $$\int \left(a+2bx+cx^2+gx^3\right)^{\pm 1/3}\,dx = \frac{3}{2}\int \frac{z^2 A^{\pm\frac{1}{3}}\,dz}{B}$$
$$\left[a+2bx+cx^2=z^3, \quad A = g\left(\frac{-b+\sqrt{b^2+\left(z^3-a\right)c}}{c}\right)^3 + z^3, \quad B = \sqrt{b^2+\left(z^3-a\right)c}\right]$$

4. $$\int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}}$$
$$= -\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[x = z+\sqrt{z^2-1}\right]$$
$$= -\frac{1}{\sqrt{2}}\int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[x = z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(4z^3-3z\right) + 2b\left(2z^2-1\right) + 2cz + d$.

5. $$\int \frac{dx}{\sqrt{a+bx^2+cx^4+bx^6+ax^8}} = \frac{1}{2}\int \frac{dy}{\sqrt{y}\sqrt{a+by+cy^2+by^3+ay^4}} \qquad [x=\sqrt{y}]$$
$$= -\frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[y = z+\sqrt{z^2-1}\right]$$
$$= \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}}\int \frac{dz}{\sqrt{(z-1)p}} \qquad \left[y = z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(2z^2-1\right)+2bz+c$.

6. $$\int \frac{dx}{\sqrt{a+bx^4+cx^8}} = \frac{1}{2}\sqrt[8]{\frac{a}{c}}\int \frac{dt}{\sqrt{t}\sqrt{ab_1t^2+at^4}} \qquad \left[x=\sqrt[8]{\frac{a}{c}}\sqrt{t}\right];$$
$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad \left[t = z+\sqrt{z^2-1}\right]$$
$$= -\frac{1}{2\sqrt{2}}\sqrt[8]{\frac{a}{c}}\left\{\int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}}\right\} \qquad \left[t = z-\sqrt{z^2-1}\right]$$
where $p = 2a\left(2z^2-1\right)+b_1$; $b_1 = b\sqrt{\frac{a}{c}}$.

7. $$\int \frac{x\,dx}{\sqrt[4]{a+bx^2+cx^4}} = 2\int \frac{z^2\,dz}{\sqrt{A+Bz^4}} \qquad \left[a+bx^2+cx^4=z^4, \quad A=b^2-4ac, \quad B=4c\right]$$

8. $$\int \frac{dx}{\sqrt[4]{a+2bx^2+cx^4}} = \int \frac{\sqrt{b^2-a\left(c-z^4\right)}+b}{\left(c-z^4\right)\sqrt{b^2-a\left(c-z^4\right)}}z^2\,dz = \int R_1\left(z^4\right)z^2\,dz + \int \frac{R_2\left(z^4\right)z^2\,dz}{\sqrt{b^2-a\left(c-z^4\right)}},$$

where $R_1\left(z^4\right)$ and $R_2\left(z^4\right)$ are rational functions of z^4 and $a+2bx^2+cx^4 = x^4z^4$.

2.292 In certain cases, integrals of the form $\int R\left(x, \sqrt{P(x)}\right)dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = f_1\left(\frac{1}{k^2x}\right), \qquad f_2(x) = f_2\left(\frac{1-k^2x}{k^2(1-x)}\right), \qquad f_3(x) = f_3\left(\frac{1-x}{1-k^2x}\right),$$

hold, then

1. $$\int \frac{f_1(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_1(z)\,dz \qquad \left[z=\sqrt{x(1-x)\left(1-k^2x\right)}\right]$$

2. $$\int \frac{f_2(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_2(z)\,dz \qquad \left[z=\frac{\sqrt{x\left(1-k^2x\right)}}{\sqrt{1-x}}\right]$$

3. $$\int \frac{f_3(x)\,dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = \int R_3(z)\,dz \qquad \left[z=\frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}}\right]$$

where $R_1(z)$, $R_2(z)$, and $R_3(z)$ are rational functions of z.

2.3 The Exponential Function

2.31 Forms containing e^{ax}

2.311 $\int e^{ax}\,dx = \frac{e^{ax}}{a}$

2.312 a^x in the integrands should be replaced with $e^{x \ln a} = a^x$

2.313

1. $\int \frac{dx}{a + be^{mx}} = \frac{1}{am}\left[mx - \ln\left(a + be^{mx}\right)\right]$ PE (410)

2. $\int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln\left(1 + e^x\right)$ PE (409)

2.314 $$\int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \arctan\left(e^{mx}\sqrt{\frac{a}{b}}\right) \qquad [ab > 0]$$ PE (411)

$$= \frac{1}{2m\sqrt{-ab}} \ln\left|\frac{b + e^{mx}\sqrt{-ab}}{b - e^{mx}\sqrt{-ab}}\right| \qquad [ab < 0]$$

2.315 $$\int \frac{dx}{\sqrt{a + be^{mx}}} = \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} \qquad [a > 0]$$

$$= \frac{2}{m\sqrt{-a}} \arctan \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} \qquad [a < 0]$$

2.32 The exponential combined with rational functions of x

2.321

1. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$

2.[11] $\int x^n e^{ax}\,dx = e^{ax}\left(\sum_{k=0}^{n} \frac{(-1)^k k! \binom{n}{k}}{a^{k+1}} x^{n-k}\right)$

2.322

1. $\int x e^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$

2. $\int x^2 e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$

3. $\int x^3 e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)$

4.[10] $\int x^4 e^{ax}\,dx = e^{ax}\left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5}\right)$

2.323 $$\int P_m(x) e^{ax}\,dx = \frac{e^{ax}}{a} \sum_{k=0}^{m} (-1)^k \frac{P^{(k)}(x)}{a^k},$$

where $P_m(x)$ is a polynomial in x of degree m and $P^{(k)}(x)$ is the k^{th} derivative of $P_m(x)$ with respect to x.

2.324

1. $$\int \frac{e^{ax}\,dx}{x^m} = \frac{1}{m-1}\left[-\frac{e^{ax}}{x^{m-1}} + a\int \frac{e^{ax}\,dx}{x^{m-1}}\right]$$

2. $$\int \frac{e^{ax}}{x^n}\,dx = -e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!}\operatorname{Ei}(ax)$$

2.325

1. $$\int \frac{e^{ax}}{x}\,dx = \operatorname{Ei}(ax)$$

2. $$\int \frac{e^{ax}}{x^2}\,dx = -\frac{e^{ax}}{x} + a\operatorname{Ei}(ax)$$

3. $$\int \frac{e^{ax}}{x^3}\,dx = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2}\operatorname{Ei}(ax)$$

4.* $$\int \frac{e^{ax}}{x^4}\,dx = -\frac{e^{ax}}{3x^3} - \frac{ae^{ax}}{6x^2} - \frac{a^2e^{ax}}{6x} + \frac{a^3}{6}\operatorname{Ei}(ax)$$

5.* $$\int \frac{e^{\pm ax^n}}{x^m}\,dx = \frac{1}{m-1}\left[-\frac{e^{\pm ax^n}}{x^{m-1}} \pm na\int \frac{e^{\pm ax^n}}{x^{m-n}}\,dx\right] \qquad [m \neq 1]$$

6.* $$\int \frac{e^{ax^n}}{x^m}\,dx = \frac{(-1)^{z+1}a^z\,\Gamma(-z,-ax^n)}{n} = \frac{(-1)^{z+1}a^z}{n}\int_{-ax^n}^{\infty}\frac{e^{-t}}{t^{z+1}}\,dt$$

$$z = \frac{m-1}{n}, \quad \text{for } \Gamma(\alpha,x) \text{ see 8.350.2} \quad [n \neq 0]$$

7.* $$\int \frac{e^{ax^n}}{x}\,dx = \frac{\operatorname{Ei}(ax^n)}{n} \qquad [a \neq 0, \quad n \neq 0]$$

8.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -e^{ax^n}\frac{\sum_{k=0}^{z-1}k!\frac{a^{z-k-1}}{x^{n(k+1)}}}{nz!} + \frac{a^z\operatorname{Ei}(ax^n)}{nz!}$$

$$\left[a \neq 0, \quad z = \frac{m-1}{n} = 1,2,\dots, \quad m = 2,3,\dots\right]$$

9.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{nx^n} + \frac{a\operatorname{Ei}(ax^n)}{n} \qquad \left[a \neq 0, \quad z = \frac{m-1}{n} = 1\right]$$

10.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{2nx^{2n}} - \frac{ae^{ax^n}}{2nx^n} + \frac{a^2\operatorname{Ei}(ax^n)}{2n} \qquad \left[a \neq 0, \quad z = \frac{m-1}{n} = 2\right]$$

11.* $$\int \frac{e^{ax^n}}{x^m}\,dx = -\frac{e^{ax^n}}{3nx^{3n}} - \frac{e^{ax^n}}{6nx^{2n}} - \frac{a^2e^{ax^n}}{6nx^n} + \frac{a^3\operatorname{Ei}(ax^n)}{6n}$$

$$\left[a \neq 0, \quad z = \frac{m-1}{n} = 3\right]$$

12.* $$\int \frac{e^{ax^2}}{x^2}\,dx = -\frac{e^{ax^2}}{x} + \sqrt{a\pi}\operatorname{erfi}\left(\sqrt{a}x\right) \qquad \text{where } \operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i}$$

13.* $$\int e^{\left(ax^2+2bx+c\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{ac-b^2}{a}\right)\operatorname{erfi}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$ $[a \neq 0]$

2.326 $$\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}$$ $[a \neq 0]$

2.33

1.[8] $$\int e^{-\left(ax^2+2bx+c\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\operatorname{erf}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$ $[a \neq 0]$

2.* $$\int e^{ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\operatorname{erfi}\left(\sqrt{a}x\right)$$ where $\operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i}$ $[a \neq 0]$

3.* $$\int e^{ax^2+bx+c}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{ac-b^2}{a}\right)\operatorname{erfi}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)$$ where $\operatorname{erfi}(z) = \frac{\operatorname{erf}(iz)}{i}$ $[a \neq 0]$

4.* $$\int x^m e^{\pm ax^n}\,dx = \pm\frac{x^{m+1-n}}{na} \mp \frac{m+1-n}{na}\int x^{m-n}e^{\pm ax^n}\,dx$$ $[a \neq 0, \quad n \neq 0]$

5.* $$\int x^m e^{ax^n}\,dx = \frac{e^{ax^n}}{n}\left[(\gamma-1)!\sum_{k=0}^{\gamma-1}(-1)^{k+1-\gamma}\frac{x^{nk}}{k!a^{\gamma-k}}\right]$$ $\left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 1, 2, \ldots\right]$

6.* $$\int x^m e^{ax^n}\,dx = \frac{e^{ax^n}}{na}$$ $\left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 1\right]$

7.* $$\int x^m e^{ax^n}\,dx = \frac{e^{ax^n}}{n}\left(\frac{x^n}{a}-\frac{1}{a^2}\right)$$ $\left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 2\right]$

8.* $$\int x^m e^{ax^n}\,dx = \frac{e^{ax^n}}{n}\left(\frac{x^{2n}}{a}-\frac{2x^n}{a^2}+\frac{2}{a^3}\right)$$ $\left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 3\right]$

9.* $$\int x^m e^{ax^n}\,dx = \frac{e^{ax^n}}{n}\left(\frac{x^{3n}}{a}-\frac{3x^{2n}}{a^2}+\frac{6x^n}{a^3}-\frac{6}{a^4}\right)$$ $\left[a \neq 0, \quad \gamma = \frac{m+1}{n} = 4\right]$

10.* $$\int x^m e^{-\beta x^n}\,dx = -\frac{\Gamma\left(\gamma, \beta x^n\right)}{n\beta^\gamma}$$ for $\Gamma(\alpha, x)$ see 8.350.2
$$= -\frac{1}{n\beta^\gamma}\int_{\beta x^n}^{\infty} t^{\gamma-1}e^{-t}\,dx$$ $\left[\gamma = \frac{m+1}{n}, \quad \beta \neq 0, \quad n \neq 0\right]$

11.* $$\int x^m \exp\left(-\beta x^n\right)\,dx = -\frac{(\gamma-1)!}{n}\exp\left(-\beta x^n\right)\left[\sum_{k=0}^{\gamma-1}\frac{x^{nk}}{k!\beta^{\gamma-k}}\right]$$ $\left[\gamma = \frac{m+1}{n} = 1, 2, \ldots\right]$

12.* $$\int x^m \exp\left(-\beta x^n\right)\, dx = -\frac{\exp\left(-\beta x^n\right)}{n\beta} \qquad \left[\gamma = \frac{m+1}{n} = 1\right]$$

13.* $$\int x^m \exp\left(-\beta x^n\right)\, dx = -\frac{\exp\left(-\beta x^n\right)}{n}\left(\frac{x^n}{\beta} + \frac{1}{\beta^2}\right) \qquad \left[\gamma = \frac{m+1}{n} = 2\right]$$

14.* $$\int x^m \exp\left(-\beta x^n\right)\, dx = -\frac{\exp\left(-\beta x^n\right)}{n}\left(\frac{x^{2n}}{\beta} + \frac{2x^n}{\beta^2} + \frac{2}{\beta^3}\right)$$
$$\left[\gamma = \frac{m+1}{n} = 3\right]$$

15.* $$\int x^m \exp\left(-\beta x^n\right)\, dx = -\frac{\exp\left(-\beta x^n\right)}{n}\left(\frac{x^{3n}}{\beta} + \frac{3x^{2n}}{\beta^2} + \frac{6x^n}{\beta^3} + \frac{6}{\beta^4}\right)$$
$$\left[\gamma = \frac{m+1}{n} = 4\right]$$

16.* $$\int e^{-\beta x^n}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\operatorname{erf}\left(\sqrt{\beta}x\right) \qquad [\beta \neq 0]$$

17.* $$\int \frac{\exp\left(-\beta x^n\right)}{x^m}\, dx = -\frac{\beta^z\,\Gamma\left(-z, \beta x^n\right)}{n} = -\frac{\beta^z}{n}\int_{\beta x^n}^{\infty} \frac{e^{-t}}{t^{z+a}}\, dt$$
$$z = \frac{m-1}{n}$$

18.* $$\int \frac{\exp\left(-\beta x^n\right)}{x}\, dx = \frac{\operatorname{Ei}\left(-\beta x^n\right)}{n}$$

19.* $$\int \frac{\exp\left(-\beta x^n\right)}{x^m}\, dx = (-1)^z \frac{\exp\left(-\beta x^n\right)}{nz!}\sum_{k=0}^{z-1}(-1)^{k!}\frac{\beta^{z-k-1}}{x^{n(k+1)}} + (-1)^z\frac{\beta^z}{nz!}\operatorname{Ei}\left(-\beta x^n\right)$$
$$\left[z = \frac{m-1}{n} = 1, 2, \ldots, \quad m = 2, 3, \ldots\right]$$

20.* $$\int \frac{\exp\left(-\beta x^n\right)}{x^m}\, dx = -\frac{\exp\left(-\beta x^n\right)}{nx^n} - \frac{\beta\operatorname{Ei}\left(-\beta x^n\right)}{n} \qquad \left[z = \frac{m-1}{n} = 1\right]$$

21.* $$\int \frac{\exp\left(-\beta x^n\right)}{x^m}\, dx = -\frac{\exp\left(-\beta x^n\right)}{2nx^{2n}} + \frac{\beta\exp\left(-\beta x^n\right)}{2nx^n} + \frac{\beta^2\operatorname{Ei}\left(-\beta x^n\right)}{2n}$$
$$\left[z = \frac{m-1}{n} = 2\right]$$

22.* $$\int \frac{\exp\left(-\beta x^n\right)}{x^m}\, dx = -\frac{\exp\left(-\beta x^n\right)}{3nx^{3n}} + \frac{\beta\exp\left(-\beta x^n\right)}{6nx^{2n}} - \frac{\beta^2\exp\left(-\beta x^n\right)}{6nx^n} - \frac{\beta^3\operatorname{Ei}\left(-\beta x^n\right)}{6n}$$
$$\left[z = \frac{m-1}{n} = 3\right]$$

23.* $$\int \frac{\exp\left(-\beta x^2\right)}{x^2}\, dx = -\frac{\exp\left(-\beta x^2\right)}{x} - \sqrt{\beta\pi}\operatorname{erf}\left(\sqrt{\beta}x\right)$$

2.4 Hyperbolic Functions

2.41–2.43 Powers of $\sinh x$, $\cosh x$, $\tanh x$, and $\coth x$

2.411
$$\begin{aligned}\int \sinh^p x \cosh^q x\,dx &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sinh^p x \cosh^{q-2} x\,dx \\ &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{p+q} - \frac{p-1}{p+q}\int \sinh^{p-2} x \cosh^{q} x\,dx \\ &= \frac{\sinh^{p-1} x \cosh^{q+1} x}{q+1} - \frac{p-1}{q+1}\int \sinh^{p-2} x \cosh^{q+2} x\,dx \\ &= \frac{\sinh^{p+1} x \cosh^{q-1} x}{p+1} - \frac{q-1}{p+1}\int \sinh^{p+2} x \cosh^{q-2} x\,dx \\ &= \frac{\sinh^{p+1} x \cosh^{q+1} x}{p+1} - \frac{p+q+2}{p+1}\int \sinh^{p+2} x \cosh^{q} x\,dx \\ &= -\frac{\sinh^{p+1} x \cosh^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \sinh^{p} x \cosh^{q+2} x\,dx\end{aligned}$$

2.412

1. $$\begin{aligned}\int \sinh^p x \cosh^{2n} x\,dx = \frac{\sinh^{p+1} x}{2n+p}\Bigg[\cosh^{2n-1} x \\ + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\cosh^{2n-2k-1} x\Bigg] \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \sinh^p x\,dx\end{aligned}$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

2. $$\int \sinh^{2m} x\,dx = (-1)^m \binom{2m}{m}\frac{x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\frac{\sinh(2m-2k)x}{2m-2k}$$ TI (543)

3. $$\int \sinh^{2m+1} x\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\frac{\cosh(2m-2k+1)x}{2m-2k+1};$$ TI (544)
$$= (-1)^n\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{\cosh^{2k+1} x}{2k+1}$$ GU (351) (5)

4. $$\begin{aligned}\int \sinh^p x \cosh^{2n+1} x\,dx \\ = \frac{\sinh^{p+1} x}{2n+p+1}\left\{\cosh^{2n} x + \sum_{k=1}^{n}\frac{2^k n(n-1)\dots(n-k+1)\cosh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}\end{aligned}$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.413

1. $$\int \cosh^p x \sinh^{2n} x\,dx = \frac{\cosh^{p+1} x}{2n+p}\left[\sinh^{2n-1} x + \sum_{k=1}^{n-1}(-1)^k \frac{(2n-1)(2n-3)\dots(2n-2k+1)\sinh^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right] + (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \cosh^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: -2, -4, ..., $-2n$. If p is a natural number and $n = 0$, we have

2. $$\int \cosh^{2m} x\,dx = \binom{2m}{m}\frac{x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\frac{\sinh(2m-2k)x}{2m-2k}$$ TI (541)

3. $$\int \cosh^{2m+1} x\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\frac{\sinh(2m-2k+1)x}{2m-2k+1}$$ TI (542)

$$= \sum_{k=0}^{m}\binom{m}{k}\frac{\sinh^{2k+1} x}{2k+1}$$ GU (351) (8)

4. $$\int \cosh^p x \sinh^{2n+1} x\,dx = \frac{\cosh^{p+1} x}{2n+p+1}\left[\sinh^{2n} x + \sum_{k=1}^{n}(-1)^k \frac{2^k n(n-1)\dots(n-k+1)\sinh^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right]$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: -1, -3, ..., $-(2n+1)$.

2.414

1. $$\int \sinh ax\,dx = \frac{1}{a}\cosh ax$$

2. $$\int \sinh^2 ax\,dx = \frac{1}{4a}\sinh 2ax - \frac{x}{2}$$

3. $$\int \sinh^3 x\,dx = -\frac{3}{4}\cosh x + \frac{1}{12}\cosh 3x = \frac{1}{3}\cosh^3 x - \cosh x$$

4. $$\int \sinh^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sinh 2x + \frac{1}{32}\sinh 4x = \frac{3}{8}x - \frac{3}{8}\sinh x\cosh x + \frac{1}{4}\sinh^3 x\cosh x$$

5. $$\int \sinh^5 x\,dx = \frac{5}{8}\cosh x - \frac{5}{48}\cosh 3x + \frac{1}{80}\cosh 5x$$

$$= \frac{4}{5}\cosh x + \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\cosh^3 x$$

6. $$\int \sinh^6 x\,dx = -\frac{5}{16}x + \frac{15}{64}\sinh 2x - \frac{3}{64}\sinh 4x + \frac{1}{192}\sinh 6x$$

$$= -\frac{5}{16}x + \frac{1}{6}\sinh^5 x\cosh x - \frac{5}{24}\sinh^3 x\cosh x + \frac{5}{16}\sinh x\cosh x$$

7. $$\int \sinh^7 x\,dx = -\tfrac{35}{64}\cosh x + \tfrac{7}{64}\cosh 3x - \tfrac{7}{320}\cosh 5x + \tfrac{1}{448}\cosh 7x$$
$$= -\tfrac{24}{35}\cosh x + \tfrac{8}{35}\cosh^3 x - \tfrac{6}{35}\cosh x\sinh^4 x + \tfrac{1}{7}\cosh x\sinh^6 x$$

8. $$\int \cosh ax\,dx = \frac{1}{a}\sinh ax$$

9. $$\int \cosh^2 ax\,dx = \frac{x}{2} + \frac{1}{4a}\sinh 2ax$$

10. $$\int \cosh^3 x\,dx = \tfrac{3}{4}\sinh x + \tfrac{1}{12}\sinh 3x = \sinh x + \tfrac{1}{3}\sinh^3 x$$

11. $$\int \cosh^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sinh 2x + \tfrac{1}{32}\sinh 4x = \tfrac{3}{8}x + \tfrac{3}{8}\sinh x\cosh x + \tfrac{1}{4}\sinh x\cosh^3 x$$

12. $$\int \cosh^5 x\,dx = \tfrac{5}{8}\sinh x + \tfrac{5}{48}\sinh 3x + \tfrac{1}{80}\sinh 5x$$
$$= \tfrac{4}{5}\sinh x + \tfrac{1}{5}\cosh^4 x\sinh x + \tfrac{4}{15}\sinh^3 x$$

13. $$\int \cosh^6 x\,dx = \tfrac{5}{16}x + \tfrac{15}{64}\sinh 2x + \tfrac{3}{64}\sinh 4x + \tfrac{1}{192}\sinh 6x$$
$$= \tfrac{5}{16}x + \tfrac{5}{16}\sinh x\cosh x + \tfrac{5}{24}\sinh x\cosh^3 x + \tfrac{1}{6}\sinh x\cosh^5 x$$

14. $$\int \cosh^7 x\,dx = \tfrac{35}{64}\sinh x + \tfrac{7}{64}\sinh 3x + \tfrac{7}{320}\sinh 5x + \tfrac{1}{448}\sinh 7x$$
$$= \tfrac{24}{35}\sinh x + \tfrac{8}{35}\sinh^3 x + \tfrac{6}{35}\sinh x\cosh^4 x + \tfrac{1}{7}\sinh x\cosh^6 x$$

2.415

1. $$\int \sinh ax\cosh bx\,dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}$$

2. $$\int \sinh ax\cosh ax\,dx = \frac{1}{4a}\cosh 2ax$$

3. $$\int \sinh^2 x\cosh x\,dx = \tfrac{1}{3}\sinh^3 x$$

4. $$\int \sinh^3 x\cosh x\,dx = \tfrac{1}{4}\sinh^4 x$$

5. $$\int \sinh^4 x\cosh x\,dx = \tfrac{1}{5}\sinh^5 x$$

6. $$\int \sinh x\cosh^2 x\,dx = \tfrac{1}{3}\cosh^3 x$$

7. $$\int \sinh^2 x\cosh^2 x\,dx = -\frac{x}{8} + \frac{1}{32}\sinh 4x$$

8. $$\int \sinh^3 x\cosh^2 x\,dx = \tfrac{1}{5}\left(\sinh^2 x - \tfrac{2}{3}\right)\cosh^3 x$$

9. $$\int \sinh^4 x\cosh^2 x\,dx = \frac{x}{16} - \frac{1}{64}\sinh 2x - \frac{1}{64}\sinh 4x + \frac{1}{192}\sinh 6x$$

10. $$\int \sinh x \cosh^3 x \, dx = \tfrac{1}{4} \cosh^4 x$$

11. $$\int \sinh^2 x \cosh^3 x \, dx = \tfrac{1}{5} \left(\cosh^2 x + \tfrac{2}{3}\right) \sinh^3 x$$

12. $$\int \sinh^3 x \cosh^3 x \, dx = -\tfrac{3}{64} \cosh 2x + \tfrac{1}{192} \cosh 6x = \tfrac{1}{48} \cosh^3 2x - \tfrac{1}{16} \cosh 2x$$
$$= \frac{\sinh^6 x}{6} + \frac{\sinh^4 x}{4} = \frac{\cosh^6 x}{6} - \frac{\cosh^4 x}{4}$$

13. $$\int \sinh^4 x \cosh^3 x \, dx = \tfrac{1}{7} \sinh^3 x \left(\cosh^4 x - \tfrac{3}{5} \cosh^2 x - \tfrac{2}{5}\right) = \tfrac{1}{7} \left(\cosh^2 x + \tfrac{2}{5}\right) \sinh^5 x$$

14. $$\int \sinh x \cosh^4 x \, dx = \tfrac{1}{5} \cosh^5 x$$

15. $$\int \sinh^2 x \cosh^4 x \, dx = -\frac{x}{16} - \frac{1}{64} \sinh 2x + \frac{1}{64} \sinh 4x + \frac{1}{192} \sinh 6x$$

16. $$\int \sinh^3 x \cosh^4 x \, dx = \tfrac{1}{7} \cosh^3 x \left(\sinh^4 x + \tfrac{3}{5} \sinh^2 x - \tfrac{2}{5}\right) = \tfrac{1}{7} \left(\sinh^2 x - \tfrac{2}{5}\right) \cosh^5 x$$

17. $$\int \sinh^4 x \cosh^4 x \, dx = \frac{3x}{128} - \frac{1}{128} \sinh 4x + \frac{1}{1024} \sinh 8x$$

2.416

1.[10] $$\int \frac{\sinh^p x}{\cosh^{2n} x} \, dx = \frac{\sinh^{p+1} x}{2n-1} \left[\operatorname{sech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \sec h^{2n-2k-1} x \right] + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \sinh^p x \, dx$$

This formula is applicable for arbitrary real p. For $\int \sinh^p x \, dx$, where p is a natural number, see **2.412** 2 and **2.412** 3. For $n = 0$ and p a negative integer, we have for this integral:

2. $$\int \frac{dx}{\sinh^{2m} x} = \frac{\cosh x}{2m-1} \left[-\operatorname{cosech}^{2m-1} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{cosec} h^{2m-2k-1} x \right]$$

3. $$\int \frac{dx}{\sinh^{2m+1} x} = \frac{\cosh x}{2m} \left[-\operatorname{cosech}^{2m} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \operatorname{cosec} h^{2m-2k} x \right] + (-1)^m \frac{(2m-1)!!}{(2m)!!} \ln \tanh \frac{x}{2}$$

2.417

1. $$\int \frac{\sinh^p x}{\cosh^{2n+1} x}\,dx = \frac{\sinh^{p+1} x}{2n}\left[\operatorname{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \sec h^{2n-2k} x\right] + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\sinh^p x}{\cosh x}\,dx$$

This formula is applicable for arbitrary real p. For $n = 0$ and p integral, we have

2. $$\int \frac{\sinh^{2m+1} x}{\cosh x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\sinh^{2k} x + (-1)^m \ln\cosh x = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\binom{m}{k}\cosh^{2k} x + (-1)^m \ln\cosh x \qquad [m \geq 1]$$

3. $$\int \frac{\sinh^{2m} x}{\cosh x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k-1}\sinh^{2k-1} x + (-1)^m \arctan(\sinh x) \qquad [m \geq 1]$$

4. $$\int \frac{dx}{\sinh^{2m+1} x \cosh x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln\tanh x$$

5. $$\int \frac{dx}{\sinh^{2m} x \cosh x} = \sum_{k=1}^{m} \frac{(-1)^k \operatorname{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \arctan\sinh x$$

2.418

1. $$\int \frac{\cosh^p x}{\sinh^{2n} x}\,dx = -\frac{\cosh^{p+1} x}{2n-1}\left[\operatorname{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec} h^{2n-2k-1} x\right] + \frac{(-1)^n(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \cosh^p x\,dx$$

This formula is applicable for arbitrary real p. For the integral $\int \cosh^p x\,dx$, where p is a natural number, see **2.413** 2 and **2.413** 3. If p is a negative integer, we have for this integral:

2. $$\int \frac{dx}{\cosh^{2m} x} = \frac{\sinh x}{2m-1}\left\{\operatorname{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k(m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)}\operatorname{sech}^{2m-2k-1} x\right\}$$

3. $$\int \frac{dx}{\cosh^{2m+1} x} = \frac{\sinh x}{2m}\left\{\operatorname{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m-1)(m-2)\dots(m-k)}\operatorname{sech}^{2m-2k} x\right\} + \frac{(2m-1)!!}{(2m)!!}\arctan\sinh x$$

2.419

1. $$\int \frac{\cosh^p x}{\sinh^{2n+1} x}\,dx = -\frac{\cosh^{p+1} x}{2n}\left[\operatorname{cosech}^{2n} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k (n-1)(n-2)\dots(n-k)} \operatorname{cosec} h^{2n-2k} x\right] + \frac{(-1)^n (2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\cosh^p x}{\sinh x}\,dx$$

This formula is applicable for arbitrary real p. For $n = 0$ and p an integer

2. $$\int \frac{\cosh^{2m} x}{\sinh x}\,dx = \sum_{k=1}^{m} \frac{\cosh^{2k-1} x}{2k-1} + \ln\tanh\frac{x}{2}$$

3. $$\int \frac{\cosh^{2m+1} x}{\sinh x}\,dx = \sum_{k=1}^{m} \frac{\cosh^{2k} x}{2k} + \ln\sinh x = \sum_{k=1}^{m} \binom{m}{k} \frac{\sinh^{2k} x}{2k} + \ln\sinh x$$

4. $$\int \frac{dx}{\sinh x \cosh^{2m} x} = \sum_{k=1}^{m} \frac{\operatorname{sech}^{2m-2k+1} x}{2m-2k+1} + \ln\tanh\frac{x}{2}$$

5. $$\int \frac{dx}{\sinh x \cosh^{2m+1} x} = \sum_{k=1}^{m} \frac{\operatorname{sech}^{2m-2k+2} x}{2m-2k+2} + \ln\tanh x$$

2.421 In formulas **2.421** 1 and **2.421** 2, $s = 1$ for m odd and $m < 2n+1$; in all other cases, $s = 0$. GI (351)(11, 13)

1.[10] $$\int \frac{\sinh^{2n+1} x}{\cosh^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} (-1)^{n+k} \binom{n}{k} \frac{\cosh^{2k-m+1} x}{2k-m+1} + s(-1)^{n+\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln\cosh x$$

2. $$\int \frac{\cosh^{2n+1} x}{\sinh^m x}\,dx = \sum_{\substack{k=0\\ k\neq\frac{m-1}{2}}}^{n} \binom{n}{k} \frac{\sinh^{2k-m+1} x}{2k-m+1} + s\binom{n}{\frac{m-1}{2}} \ln\sinh x$$

2.422

1. $$\int \frac{dx}{\sinh^{2m} x \cosh^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1} \binom{m+n-1}{k} \tanh^{2k-2m+1} x$$

2. $$\int \frac{dx}{\sinh^{2m+1} x \cosh^{2n+1} x} = \sum_{\substack{k=0\\ k\neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \tanh^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln\tanh x$$

GI (351)(15)

2.423

1. $$\int \frac{dx}{\sinh x} = \ln\tanh\frac{x}{2} = \frac{1}{2}\ln\frac{\cosh x - 1}{\cosh x + 1}$$

2. $$\int \frac{dx}{\sinh^2 x} = -\coth x$$

3. $$\int \frac{dx}{\sinh^3 x} = -\frac{\cosh x}{2\sinh^2 x} - \frac{1}{2}\ln\tanh\frac{x}{2}$$

4. $$\int \frac{dx}{\sinh^4 x} = -\frac{\cosh x}{3\sinh^3 x} + \frac{2}{3}\coth x = -\frac{1}{3}\coth^3 x + \coth x$$

5. $$\int \frac{dx}{\sinh^5 x} = -\frac{\cosh x}{4\sinh^4 x} + \frac{3}{8}\frac{\cosh x}{\sinh^2 x} + \frac{3}{8}\ln\tanh\frac{x}{2}$$

6. $$\int \frac{dx}{\sinh^6 x} = -\frac{\cosh x}{5\sinh^5 x} + \frac{4}{15}\coth^3 x - \frac{4}{5}\coth x$$
$$= -\frac{1}{5}\coth^5 x + \frac{2}{3}\coth^3 x - \coth x$$

7. $$\int \frac{dx}{\sinh^7 x} = -\frac{\cosh x}{6\sinh^2 x}\left(\frac{1}{\sinh^4 x} - \frac{5}{4\sinh^2 x} + \frac{15}{8}\right) - \frac{5}{16}\ln\tanh\frac{x}{2}$$

8. $$\int \frac{dx}{\sinh^8 x} = \coth x - \coth^3 x + \frac{3}{5}\coth^5 x - \frac{1}{7}\coth^7 x$$

9. $$\int \frac{dx}{\cosh x} = \arctan(\sinh x)$$
$$= \arcsin(\tanh x)$$
$$= 2\arctan(e^x)$$
$$= \operatorname{gd} x$$

10. $$\int \frac{dx}{\cosh^2 x} = \tanh x$$

11. $$\int \frac{dx}{\cosh^3 x} = \frac{\sinh x}{2\cosh^2 x} + \frac{1}{2}\arctan(\sinh x)$$

12. $$\int \frac{dx}{\cosh^4 x} = \frac{\sinh x}{3\cosh^3 x} + \frac{2}{3}\tanh x$$
$$= -\frac{1}{3}\tanh^3 x + \tanh x$$

13. $$\int \frac{dx}{\cosh^5 x} = \frac{\sinh x}{4\cosh^4 x} + \frac{3}{8}\frac{\sinh x}{\cosh^2 x} + \frac{3}{8}\arctan(\sinh x)$$

14. $$\int \frac{dx}{\cosh^6 x} = \frac{\sinh x}{5\cosh^5 x} - \frac{4}{15}\tanh^3 x + \frac{4}{5}\tanh x$$
$$= \frac{1}{5}\tanh^5 x - \frac{2}{3}\tanh^3 x + \tanh x$$

15. $$\int \frac{dx}{\cosh^7 x} = \frac{\sinh x}{6\cosh^2 x}\left(\frac{1}{\cosh^4 x} + \frac{5}{4\cosh^2 x} + \frac{15}{8}\right) + \frac{5}{16}\arctan(\sinh x)$$

16. $$\int \frac{dx}{\cosh^8 x} = -\frac{1}{7}\tanh^7 x + \frac{3}{5}\tanh^5 x - \tanh^3 x + \tanh x$$

17. $$\int \frac{\sinh x}{\cosh x}\,dx = \ln\cosh x$$

18. $$\int \frac{\sinh^2 x}{\cosh x}\, dx = \sinh x - \arctan(\sinh x)$$

19. $$\int \frac{\sinh^3 x}{\cosh x}\, dx = \frac{1}{2}\sinh^2 x - \ln\cosh x = \frac{1}{2}\cosh^2 x - \ln\cosh x$$

20. $$\int \frac{\sinh^4 x}{\cosh x}\, dx = \frac{1}{3}\sinh^3 x - \sinh x + \arctan(\sinh x)$$

21. $$\int \frac{\sinh x}{\cosh^2 x}\, dx = -\frac{1}{\cosh x}$$

22. $$\int \frac{\sinh^2 x}{\cosh^2 x}\, dx = x - \tanh x$$

23. $$\int \frac{\sinh^3 x}{\cosh^2 x}\, dx = \cosh x + \frac{1}{\cosh x}$$

24. $$\int \frac{\sinh^4 x}{\cosh^2 x}\, dx = -\frac{3}{2}x + \frac{1}{4}\sinh 2x + \tanh x$$

25. $$\int \frac{\sinh x}{\cosh^3 x}\, dx = -\frac{1}{2\cosh^2 x} = \frac{1}{2}\tanh^2 x$$

26. $$\int \frac{\sinh^2 x}{\cosh^3 x}\, dx = -\frac{\sinh x}{2\cosh^2 x} + \frac{1}{2}\arctan(\sinh x)$$

27. $$\int \frac{\sinh^3 x}{\cosh^3 x}\, dx = -\frac{1}{2}\tanh^2 x + \ln\cosh x = \frac{1}{2\cosh^2 x} + \ln\cosh x$$

28. $$\int \frac{\sinh^4 x}{\cosh^3 x}\, dx = \frac{\sinh x}{2\cosh x} + \sinh x - \frac{3}{2}\arctan(\sinh x)$$

29. $$\int \frac{\sinh x}{\cosh^4 x}\, dx = -\frac{1}{3\cosh^3 x}$$

30. $$\int \frac{\sinh^2 x}{\cosh^4 x}\, dx = \frac{1}{3}\tanh^3 x$$

31. $$\int \frac{\sinh^3 x}{\cosh^4 x}\, dx = -\frac{1}{\cosh x} + \frac{1}{3\cosh^3 x}$$

32. $$\int \frac{\sinh^4 x}{\cosh^4 x}\, dx = -\frac{1}{3}\tanh^3 x - \tanh x + x$$

33. $$\int \frac{\cosh x}{\sinh x}\, dx = \ln\sinh x$$

34. $$\int \frac{\cosh^2 x}{\sinh x}\, dx = \cosh x + \ln\tanh\frac{x}{2}$$

35. $$\int \frac{\cosh^3 x}{\sinh x}\,dx = \frac{1}{2}\cosh^2 x + \ln\sinh x$$

36. $$\int \frac{\cosh^4 x}{\sinh x}\,dx = \frac{1}{3}\cosh^3 x + \cosh x + \ln\tanh\frac{x}{2}$$

37. $$\int \frac{\cosh x}{\sinh^2 x}\,dx = -\frac{1}{\sinh x}$$

38. $$\int \frac{\cosh^2 x}{\sinh^2 x}\,dx = x - \coth x$$

39. $$\int \frac{\cosh^3 x}{\sinh^2 x}\,dx = \sinh x - \frac{1}{\sinh x}$$

40. $$\int \frac{\cosh^4 x}{\sinh^2 x}\,dx = \frac{3}{2}x + \frac{1}{4}\sinh 2x - \coth x$$

41. $$\int \frac{\cosh x}{\sinh^3 x}\,dx = -\frac{1}{2\sinh^2 x} = -\frac{1}{2}\coth^2 x$$

42. $$\int \frac{\cosh^2 x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \ln\tanh\frac{x}{2}$$

43. $$\int \frac{\cosh^3 x}{\sinh^3 x}\,dx = -\frac{1}{2\sinh^2 x} + \ln\sinh x = -\frac{1}{2}\coth^2 x + \ln\sinh x$$

44. $$\int \frac{\cosh^4 x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \cosh x + \frac{3}{2}\ln\tanh\frac{x}{2}$$

45. $$\int \frac{\cosh x}{\sinh^4 x}\,dx = -\frac{1}{3\sinh^3 x}$$

46. $$\int \frac{\cosh^2 x}{\sinh^4 x}\,dx = -\frac{1}{3}\coth^3 x$$

47. $$\int \frac{\cosh^3 x}{\sinh^4 x}\,dx = -\frac{1}{\sinh x} - \frac{1}{3\sinh^3 x}$$

48. $$\int \frac{\cosh^4 x}{\sinh^4 x}\,dx = -\frac{1}{3}\coth^3 x - \coth x + x$$

49. $$\int \frac{dx}{\sinh x\cosh x} = \ln\tanh x$$

50. $$\int \frac{dx}{\sinh x\cosh^2 x} = \frac{1}{\cosh x} + \ln\tanh\frac{x}{2}$$

51. $$\int \frac{dx}{\sinh x\cosh^3 x} = \frac{1}{2\cosh^2 x} + \ln\tanh x = -\frac{1}{2}\tanh^2 x + \ln\tanh x$$

52. $$\int \frac{dx}{\sinh x \cosh^4 x} = \frac{1}{\cosh x} + \frac{1}{3\cosh^3 x} + \ln\tanh\frac{x}{2}$$

53. $$\int \frac{dx}{\sinh^2 x \cosh x} = -\frac{1}{\sinh x} - \arctan\sinh x$$

54. $$\int \frac{dx}{\sinh^2 x \cosh^2 x} = -2\coth 2x$$

55. $$\int \frac{dx}{\sinh^2 x \cosh^3 x} = -\frac{\sinh x}{2\cosh^2 x} - \frac{1}{\sinh x} - \frac{3}{2}\arctan\sinh x$$

56. $$\int \frac{dx}{\sinh^2 x \cosh^4 x} = \frac{1}{3\sinh x \cosh^3 x} - \frac{8}{3}\coth 2x$$

57. $$\int \frac{dx}{\sinh^3 x \cosh x} = -\frac{1}{2\sinh^2 x} - \ln\tanh x$$
$$= -\frac{1}{2}\coth^2 x + \ln\coth x$$

58. $$\int \frac{dx}{\sinh^3 x \cosh^2 x} = -\frac{1}{\cosh x} - \frac{\cosh x}{2\sinh^2 x} - \frac{3}{2}\ln\tanh\frac{x}{2}$$

59. $$\int \frac{dx}{\sinh^3 x \cosh^3 x} = -\frac{2\cosh 2x}{\sinh^2 2x} - 2\ln\tanh x$$
$$= \frac{1}{2}\tanh^2 x - \frac{1}{2}\coth^2 x - 2\ln\tanh x$$

60. $$\int \frac{dx}{\sinh^3 x \cosh^4 x} = -\frac{2}{\cosh x} - \frac{1}{3\cosh^2 x} - \frac{\cosh x}{2\sinh^2 x} - \frac{5}{2}\ln\tanh\frac{x}{2}$$

61. $$\int \frac{dx}{\sinh^4 x \cosh x} = \frac{1}{\sinh x} - \frac{1}{3\sinh^3 x} + \arctan\sinh x$$

62. $$\int \frac{dx}{\sinh^4 x \cosh^2 x} = -\frac{1}{3\cosh x \sinh^3 x} + \frac{8}{3}\coth 2x$$

63. $$\int \frac{dx}{\sinh^4 x \cosh^3 x} = \frac{2}{\sinh x} - \frac{1}{3\sinh^3 x} + \frac{\sinh x}{2\cosh^2 x} + \frac{5}{2}\arctan\sinh x$$

64. $$\int \frac{dx}{\sinh^4 x \cosh^4 x} = 8\coth 2x - \frac{8}{3}\coth^3 2x$$

2.424

1. $$\int \tanh^p x\,dx = -\frac{\tanh^{p-1} x}{p-1} + \int \tanh^{p-2} x\,dx \qquad [p \neq 1]$$

2. $$\int \tanh^{2n+1} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k}\binom{n}{k}\frac{1}{\cosh^{2k} x} + \ln\cosh x$$
$$= -\sum_{k=1}^{n} \frac{\tanh^{2n-2k+2} x}{2n-2k+2} + \ln\cosh x$$

3. $$\int \tanh^{2n} x\,dx = -\sum_{k=1}^{n} \frac{\tanh^{2n-2k+1} x}{2n-2k+1} + x$$ GU (351)(12)

4. $$\int \coth^p x\,dx = -\frac{\coth^{p-1} x}{p-1} + \int \coth^{p-2} x\,dx \qquad [p \neq 1]$$

5. $$\int \coth^{2n+1} x\,dx = -\sum_{k=1}^{n} \frac{1}{2n}\binom{n}{k}\frac{1}{\sinh^{2k} x} + \ln\sinh x$$
$$= -\sum_{k=1}^{n} \frac{\coth^{2n-2k+2} x}{2n-2k+2} + \ln\sinh x$$

6. $$\int \coth^{2n} x\,dx = -\sum_{k=1}^{n} \frac{\coth^{2n-2k+1} x}{2n-2k+1} + x$$ GU (351)(14)

For formulas containing powers of $\tanh x$ and $\coth x$ equal to $n = 1, 2, 3, 4$, see **2.423** 17, **2.423** 22, **2.423** 27, **2.423** 32, **2.423** 33, **2.423** 38, **2.423** 43, **2.423** 48.

Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

2.425

1. $$\int \sinh(ax+b)\sinh(cx+d)\,dx = \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] - \frac{1}{2(a-c)}\sinh[(a-c)x+b-d]$$ $\left[a^2 \neq c^2\right]$ GU (352)(2a)

2. $$\int \sinh(ax+b)\cosh(cx+d)\,dx = \frac{1}{2(a+c)}\cosh[(a+c)x+b+d] + \frac{1}{2(a-c)}\cosh[(a-c)x+b-d]$$ $\left[a^2 \neq c^2\right]$ GU (352)(2c)

3. $$\int \cosh(ax+b)\cosh(cx+d)\,dx = \frac{1}{2(a+c)}\sinh[(a+c)x+b+d] + \frac{1}{2(a-c)}\sinh[(a-c)x+b-d]$$ $\left[a^2 \neq c^2\right]$ GU (352)(2b)

When $a = c$:

4. $$\int \sinh(ax+b)\sinh(ax+d)\,dx = -\frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$$ GU (352)(3a)

5. $$\int \sinh(ax+b)\cosh(ax+d)\,dx = \frac{x}{2}\sinh(b-d) + \frac{1}{4a}\cosh(2ax+b+d)$$ GU (352)(3c)

6. $$\int \cosh(ax+b)\cosh(ax+d)\,dx = \frac{x}{2}\cosh(b-d) + \frac{1}{4a}\sinh(2ax+b+d)$$ GU (352)(3b)

2.426

1. $$\int \sinh ax \sinh bx \sinh cx\,dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} - \frac{\cosh(a-b+c)x}{4(a-b+c)} - \frac{\cosh(a+b-c)x}{4(a+b-c)}$$ GU (352)(4a)

2. $$\int \sinh ax \sinh bx \cosh cx\, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} - \frac{\sinh(-a+b+c)x}{4(-a+b+c)} - \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4b)

3. $$\int \sinh ax \cosh bx \cosh cx\, dx = \frac{\cosh(a+b+c)x}{4(a+b+c)} - \frac{\cosh(-a+b+c)x}{4(-a+b+c)} + \frac{\cosh(a-b+c)x}{4(a-b+c)} + \frac{\cosh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4c)

4. $$\int \cosh ax \cosh bx \cosh cx\, dx = \frac{\sinh(a+b+c)x}{4(a+b+c)} + \frac{\sinh(-a+b+c)x}{4(-a+b+c)} + \frac{\sinh(a-b+c)x}{4(a-b+c)} + \frac{\sinh(a+b-c)x}{4(a+b-c)}$$

GU (352)(4d)

2.427

1. $$\int \sinh^p x \sinh ax\, dx = \frac{1}{p+a}\left\{\sinh px \cosh ax - p\int \sinh^{p-1} x \cosh(a-1)x\, dx\right\}$$

2. $$\int \sinh^p x \sinh(2n+1)x\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k+1)x - \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k)x\right] + \frac{\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p+1-2n)} \int \sinh^{p-2n} x \sinh x\, dx$$

[p is not a negative integer]

3. $$\int \sinh^p x \sinh 2nx\, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \cosh(2n-2k)x - \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \sinh(2n-2k-1)x\right]$$

[p is not a negative integer] GU (352)(5)a

2.428

1. $$\int \sinh^p x \cosh ax\, dx = \frac{1}{p+a}\left\{\sinh^p x \sinh ax - p\int \sinh^{p-1} x \sinh(a-1)x\, dx\right\}$$

2. $$\int \sinh^p x \cosh(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \times \left\{ \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k+1)x - \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k)x \right] + \frac{\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p+1-2n)} \int \sinh^{p-2n} x \cosh x\,dx \right\}$$

[p is not a negative integer]

3. $$\int \sinh^p x \cosh 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sinh^{p-2k} x \sinh(2n-2k)x - \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\,\Gamma(p-2k)} \sinh^{p-2k-1} x \cosh(2n-2k-1)x \right] + \frac{\Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\,\Gamma(p+1-2n)} \int \sinh^{p-2n} x\,dx \right\}$$

[p is not a negative integer] GU (352)(6)a

2.429

1. $$\int \cosh^p x \sinh ax\,dx = \frac{1}{p+a} \left\{ \cosh^p x \cosh ax + p \int \cosh^{p-1} x \sinh(a-1)x\,dx \right\}$$

2. $$\int \cosh^p x \sinh(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)} \int \cosh^{p-n} x \sinh(n+1)x\,dx \right]$$

[p is not a negative integer]

3. $$\int \cosh^p x \sinh 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left[\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cosh^{p-k} x \cosh(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cosh^{p-n} x \sinh nx\,dx \right]$$

[p is not a negative integer] GU (352)(7)a

2.431

1. $$\int \cosh^p x \cosh ax\,dx = \frac{1}{p+a}\left\{\cosh^p x \sinh ax + p\int \cosh^{p-1} x \cosh(a-1)x\,dx\right\}$$

2. $$\int \cosh^p x \cosh(2n+1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left[\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)}\cosh^{p-k} x \sinh(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)}\int \cosh^{p-n} x \cosh(n+1)x\,dx\right]$$

[p is not a negative integer]

3. $$\int \cosh^p x \cosh 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left[\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)}\cosh^{p-k} x \sinh(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)}\cosh^{p-n} x \cosh nx\,dx\right]$$

[p is not a negative integer] GU (352)(8)a

2.432

1. $$\int \sinh(n+1)x \sinh^{n-1} x\,dx = \frac{1}{n}\sinh^n x \sinh nx$$

2. $$\int \sinh(n+1)x \cosh^{n-1} x\,dx = \frac{1}{n}\cosh^n x \cosh nx$$

3. $$\int \cosh(n+1)x \sinh^{n-1} x\,dx = \frac{1}{n}\sinh^n x \cosh nx$$

4. $$\int \cosh(n+1)x \cosh^{n-1} x\,dx = \frac{1}{n}\cosh^n x \sinh nx$$

2.433

1. $$\int \frac{\sinh(2n+1)x}{\sinh x}\,dx = 2\sum_{k=0}^{n-1}\frac{\sinh(2n-2k)x}{2n-2k} + x$$

2. $$\int \frac{\sinh 2nx}{\sinh x}\,dx = 2\sum_{k=0}^{n-1}\frac{\sinh(2n-2k-1)x}{2n-2k-1}$$ GU (352)(5d)

3. $$\int \frac{\cosh(2n+1)x}{\sinh x}\,dx = 2\sum_{k=0}^{n-1}\frac{\cosh(2n-2k)x}{2n-2k} + \ln\sinh x$$

4. $$\int \frac{\cosh 2nx}{\sinh x}\,dx = 2\sum_{k=0}^{n-1}\frac{\cosh(2n-2k-1)x}{2n-2k-1} + \ln\tanh\frac{x}{2}$$ GU (352)(6d)

5. $$\int \frac{\sinh(2n+1)x}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k\frac{\cosh(2n-2k)x}{2n-2k} + (-1)^n\ln\cosh x$$

6. $$\int \frac{\sinh 2nx}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\cosh(2n-2k-1)x}{2n-2k-1}$$ GU (352)(7d)

7. $$\int \frac{\cosh(2n+1)x}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\sinh(2n-2k)x}{2n-2k} + (-1)^n x$$

8. $$\int \frac{\cosh 2nx}{\cosh x}\,dx = 2\sum_{k=0}^{n-1}(-1)^k \frac{\sinh(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\tanh x)$$ GU (352)(8d)

9. $$\int \frac{\sinh 2x}{\sinh^n x}\,dx = -\frac{2}{(n-2)\sinh^{n-2} x}$$

For $n=2$:

10. $$\int \frac{\sinh 2x}{\sinh^2 x}\,dx = 2\ln\sinh x$$

11. $$\int \frac{\sinh 2x\,dx}{\cosh^n x} = \frac{2}{(2-n)\cosh^{n-2} x}$$

For $n=2$:

12. $$\int \frac{\sinh 2x}{\cosh^2 x}\,dx = 2\ln\cosh x$$

13. $$\int \frac{\cosh 2x}{\sinh x}\,dx = 2\cosh x + \ln\tanh\frac{x}{2}$$

14. $$\int \frac{\cosh 2x}{\sinh^2 x}\,dx = -\coth x + 2x$$

15. $$\int \frac{\cosh 2x}{\sinh^3 x}\,dx = -\frac{\cosh x}{2\sinh^2 x} + \frac{3}{2}\ln\tanh\frac{x}{2}$$

16. $$\int \frac{\cosh 2x}{\cosh x}\,dx = 2\sinh x - \arcsin(\tanh x)$$

17. $$\int \frac{\cosh 2x}{\cosh^2 x}\,dx = -\tanh x + 2x$$

18. $$\int \frac{\cosh 2x}{\cosh^3 x}\,dx = -\frac{\sinh x}{2\cosh^2 x} + \frac{3}{2}\arcsin(\tanh x)$$

19. $$\int \frac{\sinh 3x}{\sinh x}\,dx = x + \sinh 2x$$

20. $$\int \frac{\sinh 3x}{\sinh^2 x}\,dx = 3\ln\tanh\frac{x}{2} + 4\cosh x$$

21. $$\int \frac{\sinh 3x}{\sinh^3 x}\,dx = -3\coth x + 4x$$

22. $$\int \frac{\sinh 3x}{\cosh^n x}\,dx = \frac{4}{(3-n)\cosh^{n-3} x} - \frac{1}{(1-n)\cosh^{n-1} x}$$

For $n=1$ and $n=3$:

23. $$\int \frac{\sinh 3x}{\cosh x}\,dx = 2\sinh^2 x - \ln\cosh x$$

24. $$\int \frac{\sinh 3x}{\cosh^3 x}\,dx = \frac{1}{2\cosh^2 x} + 4\ln\cosh x$$

25. $$\int \frac{\cosh 3x}{\sinh^n x}\,dx = \frac{4}{(3-n)\sinh^{n-3} x} + \frac{1}{(1-n)\sinh^{n-1} x}$$

For $n = 1$ and $n = 3$:

26. $$\int \frac{\cosh 3x}{\sinh x}\,dx = 2\sinh^2 x + \ln\sinh x$$

27. $$\int \frac{\cosh 3x}{\sinh^3 x}\,dx = -\frac{1}{2\sinh^2 x} + 4\ln\sinh x$$

28. $$\int \frac{\cosh 3x}{\cosh x}\,dx = \sinh 2x - x$$

29. $$\int \frac{\cosh 3x}{\cosh^2 x}\,dx = 4\sinh x - 3\arcsin(\tanh x)$$

30. $$\int \frac{\cosh 3x}{\cosh^3 x}\,dx = 4x - 3\tanh x$$

2.44–2.45 Rational functions of hyperbolic functions

2.441

1. $$\int \frac{A + B\sinh x}{(a + b\sinh x)^n}\,dx = \frac{aB - bA}{(n-1)\left(a^2+b^2\right)} \cdot \frac{\cosh x}{(a + b\sinh x)^{n-1}} + \frac{1}{(n-1)\left(a^2+b^2\right)} \int \frac{(n-1)(aA + bB) + (n-2)(aB - bA)\sinh x}{(a + b\sinh x)^{n-1}}\,dx$$

For $n = 1$:

2. $$\int \frac{A + B\sinh x}{a + b\sinh x}\,dx = \frac{B}{b}x - \frac{aB - bA}{b}\int \frac{dx}{a + b\sinh x}$$ (see **2.441** 3)

3. $$\int \frac{dx}{a + b\sinh x} = \frac{1}{\sqrt{a^2+b^2}} \ln \frac{a\tanh\frac{x}{2} - b + \sqrt{a^2+b^2}}{a\tanh\frac{x}{2} - b - \sqrt{a^2+b^2}} = \frac{2}{\sqrt{a^2+b^2}} \operatorname{arctanh} \frac{a\tanh\frac{x}{2} - b}{\sqrt{a^2+b^2}}$$

2.442

1. $$\int \frac{A + B\cosh x}{(a + b\sinh x)^n}\,dx = -\frac{B}{(n-1)b\,(a + b\sinh x)^{n-1}} + A\int \frac{dx}{(a + b\sinh x)^n}$$

For $n = 1$:

2. $$\int \frac{A + B\cosh x}{a + b\sinh x}\,dx = \frac{B}{b}\ln(a + b\sinh x) + A\int \frac{dx}{a + b\sinh x}$$

(see **2.441** 3)

2.443

1. $$\int \frac{A + B\cosh x}{(a + b\cosh x)^n}\,dx = \frac{aB - bA}{(n-1)\left(a^2 - b^2\right)} \cdot \frac{\sinh x}{(a + b\cosh x)^{n-1}} + \frac{1}{(n-1)\left(a^2 - b^2\right)} \int \frac{(n-1)(aA - bB) + (n-2)(aB - bA)\cosh x}{(a + b\cosh x)^{n-1}}\,dx$$

For $n = 1$:

2. $$\int \frac{A + B\cosh x}{a + b\cosh x}\,dx = \frac{B}{b}x - \frac{aB - bA}{b} \int \frac{dx}{a + b\cosh x} \qquad \text{(see } \mathbf{2.443}\ 3\text{)}$$

3. $$\int \frac{dx}{a + b\cosh x} = \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a\cosh x}{a + b\cosh x} \qquad \left[b^2 > a^2, \quad x < 0\right]$$
$$= -\frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a\cosh x}{a + b\cosh x} \qquad \left[b^2 > a^2, \quad x > 0\right]$$
$$= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}\tanh \frac{x}{2}}{a + b - \sqrt{a^2 - b^2}\tanh \frac{x}{2}} \qquad \left[a^2 > b^2\right]$$

2.444

1. $$\int \frac{dx}{\cosh a + \cosh x} = \operatorname{cosech} a \left[\ln\cosh \frac{x + a}{2} - \ln\cosh \frac{x - a}{2}\right] = 2\operatorname{cosech} a \operatorname{arctanh}\left(\tanh \frac{x}{2} \tanh \frac{a}{2}\right)$$

2.[11] $$\int \frac{dx}{\cos a + \cosh x} = 2\operatorname{cosec} a \arctan\left(\tanh \frac{x}{2} \tan \frac{a}{2}\right)$$

2.445

1. $$\int \frac{B\sinh x}{(a + b\cosh x)^n}\,dx = -\frac{B}{(n-1)b(a + b\cosh x)^{n-1}} \qquad [n \neq 1]$$

For $n = 1$:

2. $$\int \frac{B\sinh x}{a + b\cosh x}\,dx = \frac{B}{b}\ln(a + b\cosh x) \qquad \text{(see } \mathbf{2.443}\ 3\text{)}$$

In evaluating definite integrals by use of formulas **2.441–2.443** and **2.445**, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{arcsinh}\left(-\frac{a}{b}\right)$$

in formulas **2.441** or **2.442** or over the points

$$x = \operatorname{arccosh}\left(-\frac{a}{b}\right)$$

in formulas **2.443** or **2.445**. Formulas **2.443** are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

2.446

1. $$\int \frac{A + B\cosh x}{(\varepsilon + \cosh x)^n}\,dx = \frac{B\sinh x}{(1-n)(\varepsilon + \cosh x)^n} + \left(\varepsilon A + \frac{n}{n-1}B\right)\frac{(n-1)!}{(2n-1)!!}\sinh x \sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!} \times \frac{\varepsilon^h}{(\varepsilon + \cosh x)^{n-k}} \qquad [\varepsilon = \pm 1, \quad n > 1]$$

For $n = 1$:

2. $$\int \frac{A + B\cosh x}{\varepsilon + \cosh x}\,dx = Bx + (\varepsilon A - B)\frac{\cosh x - \varepsilon}{\sinh x} \qquad [\varepsilon = \pm 1]$$

2.447

1. $$\int \frac{\sinh x\,dx}{a\cosh x + b\sinh x} = \frac{a\ln\cosh\left(x + \operatorname{arctanh}\frac{b}{a}\right) bx}{a^2 - b^2} \qquad [a > |b|]$$
$$= \frac{bx - a\ln\sinh\left(x + \operatorname{arctanh}\frac{a}{b}\right)}{b^2 - a^2} \qquad [b > |a|]$$ MZ 215

For $a = b = 1$:

2. $$\int \frac{\sinh x\,dx}{\cosh x + \sinh x} = \frac{x}{2} + \frac{1}{4}e^{-2x}$$

For $a = -b = 1$:

3. $$\int \frac{\sinh x\,dx}{\cosh x - \sinh x} = -\frac{x}{2} + \frac{1}{4}e^{2x}$$ MZ 215

2.448

1. $$\int \frac{\cosh x\,dx}{a\cosh x + b\sinh x} = \frac{ax - b\ln\cosh\left(x + \operatorname{arctanh}\frac{b}{a}\right)}{a^2 - b^2} \qquad [a > |b|]$$
$$= \frac{-ax + b\ln\sinh\left(x + \operatorname{arctanh}\frac{a}{b}\right)}{b^2 - a^2} \qquad [b > |a|]$$

For $a = b = 1$:

2. $$\int \frac{\cosh x\,dx}{\cosh x + \sinh x} = \frac{x}{2} - \frac{1}{4}e^{-2x}$$

For $a = -b = 1$:

3. $$\int \frac{\cosh x\,dx}{\cosh x - \sinh x} = \frac{x}{2} + \frac{1}{4}e^{2x}$$ MZ 214, 215

2.449

1.[6] $$\int \frac{dx}{(a\cosh x + b\sinh x)^n} = \frac{1}{\sqrt{(a^2 - b^2)^n}}\int \frac{dx}{\sinh^n\left(x + \operatorname{arctanh}\frac{b}{a}\right)} \qquad [a > |b|]$$
$$= \frac{1}{\sqrt{(b^2 - a^2)^n}}\int \frac{dx}{\cosh^n\left(x + \operatorname{arctanh}\frac{a}{b}\right)} \qquad [b > |a|]$$

For $n = 1$:

2. $$\int \frac{dx}{a\cosh x + b\sinh x} = \frac{1}{\sqrt{a^2-b^2}} \arctan\left|\sinh\left(x + \operatorname{arctanh}\frac{b}{a}\right)\right| \qquad [a > |b|]$$
$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\tanh\frac{x + \operatorname{arctanh}\frac{a}{b}}{2}\right| \qquad [b > |a|]$$

For $a = b = 1$:

3. $$\int \frac{ax}{\cosh x + \sinh x} = -e^{-x} = \sinh x - \cosh x$$

For $a = -b = 1$:

4. $$\int \frac{dx}{\cosh x - \sinh x} = e^{x} = \sinh x + \cosh x$$ MZ 214

2.451

1. $$\int \frac{A + B\cosh x + C\sinh x}{(a + b\cosh x + c\sinh x)^n}\,dx$$
$$= \frac{Bc - Cb + (Ac - Ca)\cosh x + (Ab - Ba)\sinh x}{(1-n)\left(a^2 - b^2 + c^2\right)(a + b\cosh x + c\sinh x)^{n-1}} + \frac{1}{(n-1)\left(a^2-b^2+c^2\right)}$$
$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba)\cosh x - (n-2)(Ac - Ca)\sinh x}{(a + b\cosh x + c\sinh x)^{n-1}}\,dx$$
$$\left[a^2 + c^2 \neq b^2\right]$$
$$= \frac{Bc - Cb - Ca\cosh x - Ba\sinh x}{(n-1)a\,(a + b\cosh x + c\sinh x)^n} + \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2}\right](c\cosh x + b\sinh x)\,\frac{(n-1)!}{(2n-1)!!}$$
$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k}\,\frac{1}{(a + b\cosh x + c\sinh x)^{n-k}}$$
$$\left[a^2 + c^2 = b^2\right]$$

2. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x + c\sinh x}\,dx = \frac{Cb - Bc}{b^2 - c^2}\ln(a + b\cosh x + c\sinh x)$$
$$+ \frac{Bb - Cc}{b^2 - c^2}x + \left(A - a\frac{Bb - Cc}{b^2 - c^2}\right)\int \frac{dx}{a + b\cosh x + c\sinh x}$$
$$\left[b^2 \neq c^2\right]$$ (see **2.451** 4)

3. $$\int \frac{A + B\cosh x + C\sinh x}{a + b\cosh x \pm b\sinh x}\,dx = \frac{C \mp B}{2a}(\cosh x \mp \sinh x) + \left[\frac{A}{a} - \frac{(B \mp C)\,b}{2a^2}\right]x$$
$$+ \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)\,b}{2a^2}\right]\ln(a + b\cosh x \pm b\sinh x)$$
$$[ab \neq 0]$$

4. $$\int \frac{dx}{a + b\cosh x + c\sinh x}$$
$$= \frac{2}{\sqrt{b^2 - a^2 - c^2}} \arctan \frac{(b-a)\tanh\frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} \qquad \left[b^2 > a^2 + c^2 \text{ and } a \neq b\right]$$
$$= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b)\tanh\frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b)\tanh\frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} \qquad \left[b^2 < a^2 + c^2 \text{ and } a \neq b\right]$$
$$= \frac{1}{c} \ln\left(a + c\tanh\frac{x}{2}\right) \qquad [a = b \text{ and } c \neq 0]$$
$$= \frac{2}{(a-b)\tanh\frac{x}{2} + c} \qquad \left[b^2 = a^2 + c^2\right]$$

GU (351)(18)

2.452

1. $$\int \frac{A + B\cosh x + C\sinh x}{(a_1 + b_1\cosh x + c_1\sinh x)(a_2 + b_2\cosh x + c_2\sinh x)}\, dx$$
$$= A_0 \ln \frac{a_1 + b_1\cosh x + c_1\sinh x}{a_2 + b_2\cosh x + c_2\sinh x} + A_1 \int \frac{dx}{a_1 + b_1\cosh x + c_1\sinh x} + A_2 \int \frac{dx}{a_2 + b_2\cosh x + c_2\sinh x}$$

where GU (351)(19)

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2} \qquad A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} b_1 & c_1 \\ B & C \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ C & A \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2\right].$$

2. $$\int \frac{A\cosh^2 x + 2B\sinh x\cosh x + C\sinh^2 x}{a\cosh^2 x + 2b\sinh x\cosh x + c\sinh^2 x}\, dx$$
$$= \frac{1}{4b^2 - (a+c)^2} \Big\{ [4Bb - (A+C)(a+c)]x$$
$$+ [(A+C)b - \mathrm{B}(a+c)] \ln\left(a\cosh^2 x + 2b\sinh x\cosh x + c\sinh^2 x\right)$$
$$+ \left[2(A-C)b^2 - 2Bb(a-c) + (Ca - Ac)(a+c)\right] f(x) \Big\}$$

where GU (351)(24)

$$f(x) = \frac{1}{2\sqrt{b^2-ac}} \ln \frac{c\tanh x + b - \sqrt{b^2-ac}}{c\tanh x + b + \sqrt{b^2-ac}} \qquad [b^2 > ac]$$

$$= \frac{1}{\sqrt{ac-b^2}} \arctan \frac{c\tanh x + b}{\sqrt{ac-b^2}} \qquad [b^2 < ac]$$

$$= -\frac{1}{c\tanh x + b} \qquad [b^2 = ac]$$

2.453

1. $$\int \frac{(A + B\sinh x)\ dx}{\sinh x\,(a + b\sinh x)} = \frac{1}{a}\left[A\ln\left|\tanh\frac{x}{2}\right| + (aB - bA)\int\frac{dx}{a + b\sinh x}\right]$$

(see **2.441** 3)

2. $$\int \frac{(A + B\sinh x)\ dx}{\sinh x\,(a + b\cosh x)} = \frac{A}{a^2-b^2}\left(a\ln\left|\tanh\frac{x}{2}\right| + b\ln\left|\frac{a + b\cosh x}{\sinh x}\right|\right) + B\int\frac{dx}{a + b\cosh x}$$

(see **2.443** 3)

For $a^2 = b^2 = 1$:

3. $$\int \frac{(A + B\sinh x)\ dx}{\sinh x\,(1 + \cosh x)} = \frac{A}{2}\left(\ln\left|\tanh\frac{x}{2}\right| - \frac{1}{2}\tanh^2\frac{x}{2}\right) + B\tanh\frac{x}{2}$$

4. $$\int \frac{(A + B\sinh x)\ dx}{\sinh x\,(1 - \cosh x)} = \frac{A}{2}\left(-\ln\left|\coth\frac{x}{2}\right| + \frac{1}{2}\coth^2\frac{x}{2}\right) + B\coth\frac{x}{2}$$

2.454

1. $$\int \frac{(A + B\sinh x)\ dx}{\cosh x\,(a + b\sinh x)} = \frac{1}{a^2+b^2}\left[(Aa + Bb)\arctan(\sinh x) + (Ab - Ba)\ln\left|\frac{a + b\sinh x}{\cosh x}\right|\right]$$

2. $$\int \frac{(A + B\cosh x)\ dx}{\sinh x\,(a + b\sinh x)} = \frac{1}{a}\left(A\ln\left|\tanh\frac{x}{2}\right| + B\ln\left|\frac{\sinh x}{a + b\sinh x}\right| - Ab\int\frac{dx}{a + b\sinh x}\right)$$

(see **2.441** 3)

2.455

1. $$\int \frac{(A + B\cosh x)\ dx}{\sinh x\,(a + b\cosh x)} = \frac{1}{a^2-b^2}\left[(Aa + Bb)\ln\left|\tanh\frac{x}{2}\right| + (Ab - Ba)\ln\left|\frac{a + b\cosh x}{\sinh x}\right|\right]$$

For $a^2 = b^2 = 1$:

2. $$\int \frac{(A + B\cosh x)\ dx}{\sinh x\,(1 + \cosh x)} = \frac{A+B}{2}\ln\left|\tanh\frac{x}{2}\right| - \frac{A-B}{4}\tanh^2\frac{x}{2}$$

3. $$\int \frac{(A + B\cosh x)\ dx}{\sinh x\,(1 - \cosh x)} = \frac{A+B}{4}\coth^2\frac{x}{2} - \frac{A-B}{2}\ln\coth\frac{x}{2}$$

2.456 $$\int \frac{(A + B\cosh x)\ dx}{\cosh x\,(a + b\sinh x)} = \frac{A}{a^2+b^2}\left[a\arctan(\sinh x) + b\ln\left|\frac{a + b\sinh x}{\cosh x}\right|\right] + B\int\frac{dx}{a + b\sinh x}$$

(see **2.441** 3)

2.457

1. $$\int \frac{(A + B\cosh x)\,dx}{\cosh x\,(a + b\cosh x)} = \frac{1}{a}\left[A \arctan\sinh x - (Ab - Ba)\int \frac{dx}{a + b\cosh x}\right]$$

(see **2.443** 3)

2.458

1. $$\int \frac{dx}{a + b\sinh^2 x}$$
$$= \frac{1}{\sqrt{a(b-a)}} \arctan\left(\sqrt{\frac{b}{a} - 1}\tanh x\right) \qquad \left[\frac{b}{a} > 1\right]$$
$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arctanh}\left(\sqrt{1 - \frac{b}{a}}\tanh x\right) \qquad \left[0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \sinh^2 x < -\frac{a}{b}\right]$$
$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arccoth}\left(\sqrt{1 - \frac{b}{a}}\tanh x\right) \qquad \left[\frac{b}{a} < 0 \text{ and } \sinh^2 x > -\frac{a}{b}\right]$$

MZ 195

2. $$\int \frac{dx}{a + b\cosh^2 x}$$
$$= \frac{1}{\sqrt{-a(a+b)}} \arctan\left(\sqrt{-\left(1 + \frac{b}{a}\right)}\coth x\right) \qquad \left[\frac{b}{a} < -1\right]$$
$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arctanh}\left(\sqrt{1 + \frac{b}{a}}\coth x\right) \qquad \left[-1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x > -\frac{a}{b}\right]$$
$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arccoth}\left(\sqrt{1 + \frac{b}{a}}\coth x\right) \qquad \left[\frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x < -\frac{a}{b}\right]$$

MZ 202

For $a^2 = b^2 = 1$:

3. $$\int \frac{dx}{1 + \sinh^2 x} = \tanh x$$

4. $$\int \frac{dx}{1 - \sinh^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctanh}\left(\sqrt{2}\tanh x\right) \qquad \left[\sinh^2 x < 1\right]$$
$$= \frac{1}{\sqrt{2}} \operatorname{arccoth}\left(\sqrt{2}\tanh x\right) \qquad \left[\sinh^2 x > 1\right]$$

5. $$\int \frac{dx}{1 + \cosh^2 x} = \frac{1}{\sqrt{2}} \operatorname{arccoth}\left(\sqrt{2}\coth x\right)$$

6. $$\int \frac{dx}{1 - \cosh^2 x} = \coth x$$

2.459

1. $$\int \frac{dx}{\left(a + b\sinh^2 x\right)^2} = \frac{1}{2a(b-a)}\left[\frac{b\sinh x\cosh x}{a + b\sinh^2 x} + (b - 2a)\int \frac{dx}{a + b\sinh^2 x}\right]$$

(see **2.458** 1) MZ 196

2. $$\int \frac{dx}{\left(a+b\cosh^2 x\right)^2} = \frac{1}{2a(a+b)}\left[-\frac{b\sinh x\cosh x}{a+b\cosh^2 x} + (2a+b)\int\frac{dx}{a+b\cosh^2 x}\right]$$

(see **2.458** 2) MZ 203

3. $$\int \frac{dx}{\left(a+b\sinh^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\tanh x\right) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\tanh x}{1+p^2\tanh^2 x} + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\tanh^2 x\right)\frac{2p\tanh x}{\left(1+p^2\tanh^2 x\right)^2}\right]$$

$$\left[p^2 = \frac{b}{a} - 1 > 0\right]$$

$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\tanh x\right) + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\tanh x}{1-q^2\tanh^2 x} + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\tanh^2 x\right)\frac{2q\tanh x}{\left(1-q^2\tanh^2 x\right)^2}\right]$$

$$\left[q^2 = 1 - \frac{b}{a} > 0\right]$$

MZ 196

4. $$\int \frac{dx}{\left(a+b\cosh^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3-\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\coth x\right) + \left(3-\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\coth x}{1+p^2\coth^2 x} + \left(1+\frac{2}{p^2}-\frac{1}{p^2}\coth^2 x\right)\frac{2p\coth x}{\left(1+p^2\coth^2 x\right)^2}\right]$$

$$\left[p^2 = -1 - \frac{b}{a} > 0\right]$$

$$= \frac{1}{8qa^3}\left[\left(3+\frac{2}{q^2}+\frac{3}{q^4}\right)\varphi(x)^* + \left(3+\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\coth x}{1-q^2\coth^2 x} + \left(1-\frac{2}{q^2}+\frac{1}{q^2}\coth^2 x\right)\frac{2q\coth x}{\left(1-q^2\coth^2 x\right)^2}\right]$$

$$\left[q^2 = 1 + \frac{b}{a} > 0\right]$$

2.46 Algebraic functions of hyperbolic functions

2.461

1. $$\int \sqrt{\tanh x}\,dx = \operatorname{arctanh}\sqrt{\tanh x} - \arctan\sqrt{\tanh x}$$ MZ 221

*In 2.459.4, if $\frac{b}{a} < 0$ and $\cosh^2 x > -\frac{a}{b}$, then $\varphi(x) = \operatorname{arctanh}\left(q\coth x\right)$. If $\frac{b}{a} < 0$, but $\cosh^2 x < -\frac{a}{b}$, or if $\frac{b}{a} > 0$, then $\varphi(x) = \operatorname{arccoth}\left(q\coth x\right)$.

2. $$\int \sqrt{\coth x}\, dx = \operatorname{arccoth} \sqrt{\coth x} - \arctan \sqrt{\coth x}$$ MZ 222

2.462

1. $$\int \frac{\sinh x\, dx}{\sqrt{a^2 + \sinh^2 x}} = \operatorname{arcsinh} \frac{\cosh x}{\sqrt{a^2 - 1}} = \ln\left(\cosh x + \sqrt{a^2 + \sinh^2 x}\right) \qquad [a^2 > 1]$$
$$= \operatorname{arccosh} \frac{\cosh x}{\sqrt{1 - a^2}} = \ln\left(\cosh x + \sqrt{a^2 + \sinh^2 x}\right) \qquad [a^2 < 1]$$
$$= \ln \cosh x \qquad [a^2 = 1]$$

2. $$\int \frac{\sinh x\, dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\cosh x}{\sqrt{a^2 + 1}} \qquad [\sinh^2 x < a^2]$$

3. $$\int \frac{\sinh x\, dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{\sqrt{a^2 + 1}} = \ln\left(\cosh x + \sqrt{\sinh^2 x - a^2}\right)$$
$$[\sinh^2 x > a^2]$$ MZ 199

4. $$\int \frac{\cosh x\, dx}{\sqrt{a^2 + \sinh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{a} = \ln\left(\sinh x + \sqrt{a^2 + \sinh^2 x}\right)$$

5. $$\int \frac{\cosh x\, dx}{\sqrt{a^2 - \sinh^2 x}} = \arcsin \frac{\sinh x}{a} \qquad [\sinh^2 x < a^2]$$

6. $$\int \frac{\cosh x\, dx}{\sqrt{\sinh^2 x - a^2}} = \operatorname{arccosh} \frac{\sinh x}{a} = \ln\left(\sinh x + \sqrt{\sinh^2 x - a^2}\right)$$
$$[\sinh^2 x > a^2]$$

7. $$\int \frac{\sinh x\, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\cosh x}{a} = \ln\left(\cosh x + \sqrt{a^2 + \cosh^2 x}\right)$$

8. $$\int \frac{\sinh x\, dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\cosh x}{a} \qquad [\cosh^2 x < a^2]$$

9. $$\int \frac{\sinh x\, dx}{\sqrt{\cosh^2 x - a^2}} = \operatorname{arccosh} \frac{\cosh x}{a} = \ln\left(\cosh x + \sqrt{\cosh^2 x - a^2}\right)$$
$$[\cosh^2 x > a^2]$$ MZ 215, 216

10. $$\int \frac{\cosh x\, dx}{\sqrt{a^2 + \cosh^2 x}} = \operatorname{arcsinh} \frac{\sinh x}{\sqrt{a^2 + 1}} = \ln\left(\sinh x + \sqrt{a^2 + \cosh^2 x}\right)$$

11. $$\int \frac{\cosh x\, dx}{\sqrt{a^2 - \cosh^2 x}} = \arcsin \frac{\sinh x}{\sqrt{a^2 - 1}} \qquad [\cosh^2 x < a^2]$$

12. $$\int \frac{\cosh x\, dx}{\sqrt{\cosh^2 x - a^2}} = \operatorname{arccosh} \frac{\sinh x}{\sqrt{a^2 - 1}} \qquad [a^2 > 1]$$
$$= \ln \sinh x \qquad [a^2 = 1]$$

MZ 206

13. $$\int \frac{\coth x\,dx}{\sqrt{a+b\sinh x}} = 2\sqrt{a}\,\operatorname{arccoth}\sqrt{1+\frac{b}{a}\sinh x} \qquad [b\sinh x>0, \quad a>0]$$
$$= 2\sqrt{a}\,\operatorname{arctanh}\sqrt{1+\frac{b}{a}\sinh x} \qquad [b\sinh x<0, \quad a>0]$$
$$= 2\sqrt{-a}\,\operatorname{arctanh}\sqrt{-\left(1+\frac{b}{a}\sinh x\right)} \qquad a<0$$

14. $$\int \frac{\tanh x\,dx}{\sqrt{a+b\cosh x}} = 2\sqrt{a}\,\operatorname{arccoth}\sqrt{1+\frac{b}{a}\cosh x} \qquad [b\cosh x>0, \quad a>0]$$
$$= 2\sqrt{a}\,\operatorname{arctanh}\sqrt{1+\frac{b}{a}\cosh x} \qquad [b\cosh x<0, \quad a>0]$$
$$= 2\sqrt{-a}\,\operatorname{arctanh}\sqrt{-\left(1+\frac{b}{a}\cosh x\right)} \qquad [a<0]$$

MZ 220, 221

2.463

1. $$\int \frac{\sinh x\sqrt{a+b\cosh x}}{p+q\cosh x}\,dx$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arccoth}\sqrt{\frac{q\,(a+b\cosh x)}{aq-bp}} \qquad \left[b\cosh x>0, \quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\cosh x)}{aq-bp}} \qquad \left[b\cosh x<0, \quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{bp-aq}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\cosh x)}{bp-aq}} \qquad \left[\frac{aq-bp}{q}<0\right]$$

MZ 220

2. $$\int \frac{\cosh x\sqrt{a+b\sinh x}}{p+q\sinh x}\,dx$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arccoth}\sqrt{\frac{q\,(a+b\sinh x)}{aq-bp}} \qquad \left[b\sinh x>0, \quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{aq-bp}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\sinh x)}{aq-bp}} \qquad \left[b\sinh x<0, \quad \frac{aq-bp}{q}>0\right]$$
$$= 2\sqrt{\frac{bp-aq}{q}}\,\operatorname{arctanh}\sqrt{\frac{q\,(a+b\sinh x)}{bp-aq}} \qquad \left[\frac{aq-bp}{q}<0\right]$$

MZ 221

2.464

1. $$\int \frac{dx}{\sqrt{k^2+k'^2\cosh^2 x}} = \int \frac{dx}{\sqrt{1+k'^2\sinh^2 x}} = F\left(\arcsin\left(\tanh x\right), k\right)$$
$[x>0]$ BY (295.00)(295.10)

2. $$\int \frac{dx}{\sqrt{\cosh^2 x - k^2}} = \int \frac{dx}{\sqrt{\sinh^2 x + k'^2}} = F\left(\arcsin\left(\frac{1}{\cosh x}\right), k\right)$$
$[x>0]$ BY (295.40)(295.30)

3. $$\int \frac{dx}{\sqrt{1-k'^2\cosh^2 x}} = F\left(\arcsin\left(\frac{\tanh x}{k}\right), k\right) \qquad \left[0 < x < \operatorname{arccosh}\frac{1}{k'}\right]$$ BY (295.20)

Notation: In **2.464** 4–**2.464** 8, we set $\alpha = \arccos\frac{1-\sinh 2ax}{1+\sinh 2ax}$, $r = \frac{1}{\sqrt{2}}$ $[ax > 0]$

4. $$\int \frac{dx}{\sqrt{\sinh 2ax}} = \frac{1}{2a} F(\alpha, r)$$ BY (296.50)

5. $$\int \sqrt{\sinh 2ax}\, dx = \frac{1}{2a}\left[F(\alpha,r) - 2E(\alpha,r)\right] + \frac{1}{a}\frac{\sqrt{\sinh 2ax\left(1+\sinh^2 2ax\right)}}{1+\sinh 2ax}$$ BY (296.53)

6. $$\int \frac{\cosh^2 2ax\, dx}{(1+\sinh 2ax)^2\sqrt{\sinh 2ax}} = \frac{1}{2a} E(\alpha, r)$$ BY (296.51)

7. $$\int \frac{(1-\sinh 2ax)^2\, dx}{(1+\sinh 2ax)^2\sqrt{\sinh 2ax}} = \frac{1}{2a}\left[2E(\alpha,r) - F(\alpha,r)\right]$$ BY (296.55)

8. $$\int \frac{\sqrt{\sinh 2ax}\, dx}{(1+\sinh 2ax)^2} = \frac{1}{4a}\left[F(\alpha,r) - E(\alpha,r)\right]$$ BY (296.54)

Notation: In **2.464** 9–**2.464** 15, we set $\alpha = \arcsin\sqrt{\frac{\cosh 2ax - 1}{\cosh 2ax}}$, $r = \frac{1}{\sqrt{2}}$ $[x \neq 0]$:

9. $$\int \frac{dx}{\sqrt{\cosh 2ax}} = \frac{1}{a\sqrt{2}} F(\alpha, r)$$ BY (296.00)

10. $$\int \sqrt{\cosh 2ax}\, dx = \frac{1}{a\sqrt{2}}\left[F(\alpha,r) - 2E(\alpha,r)\right] + \frac{\sinh 2ax}{a\sqrt{\cosh 2ax}}$$ BY (296.03)

11. $$\int \frac{dx}{\sqrt{\cosh^3 2ax}} = \frac{1}{a\sqrt{2}}\left[2E(\alpha,r) - F(\alpha,r)\right]$$ BY (296.04)

12. $$\int \frac{dx}{\sqrt{\cosh^5 2ax}} = \frac{1}{3\sqrt{2}a} F(\alpha,r) + \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}$$ BY (296.04)

13. $$\int \frac{\sinh^2 2ax\, dx}{\sqrt{\cosh 2ax}} = -\frac{\sqrt{2}}{3a} F(\alpha,r) + \frac{1}{3a}\sinh 2ax\sqrt{\cosh 2ax}$$ BY (296.07)

14. $$\int \frac{\tanh^2 2ax\, dx}{\sqrt{\cosh 2ax}} = \frac{\sqrt{2}}{3a} F(\alpha,r) - \frac{\tanh 2ax}{3a\sqrt{\cosh 2ax}}$$ BY (296.05)

15. $$\int \frac{\sqrt{\cosh 2ax}\, dx}{p^2 + (1-p^2)\cosh 2ax} = \frac{1}{a\sqrt{2}}\Pi\left(\alpha, p^2, r\right)$$ BY (296.02)

Notation: In **2.464** 16–**2.464** 20, we set:

$$\alpha = \arccos\frac{\sqrt{a^2+b^2} - a - b\sinh x}{\sqrt{a^2+b^2} + a + b\sinh x},$$

$$r = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}} \qquad \left[a > 0, \quad b > 0, \quad x > -\operatorname{arcsinh}\frac{a}{b}\right]$$

16. $$\int \frac{dx}{\sqrt{a+b\sinh x}} = \frac{1}{\sqrt[4]{a^2+b^2}} F(\alpha, r)$$ BY (298.00)

17. $$\int \sqrt{a+b\sinh x}\,dx = \sqrt[4]{a^2+b^2}\,[F(\alpha,r) - 2\,E(\alpha,r)] + \frac{2b\cosh x\sqrt{a+b\sinh x}}{\sqrt{a^2+b^2}+a+b\sinh x}$$ BY (298.02)

18. $$\int \frac{\sqrt{a+b\sinh x}}{\cosh^2 x}\,dx = \sqrt[4]{a^2+b^2}\,E(\alpha,r) - \frac{\sqrt{a^2+b^2}-a}{2\sqrt[4]{a^2+b^2}}\,F(\alpha,r) - \frac{a+\sqrt{a^2+b^2}}{b}\cdot\frac{\sqrt{a^2+b^2}-a-b\sinh x}{\sqrt{a^2+b^2}+a+b\sinh x}\cdot\frac{\sqrt{a+b\sinh x}}{\cosh x}$$ BY (298.03)

19. $$\int \frac{\cosh^2 x\,dx}{\left[\sqrt{a^2+b^2}+a+b\sinh x\right]^2\sqrt{a+b\sinh x}} = \frac{1}{b^2\sqrt[4]{a^2+b^2}}\,E(\alpha,r)$$ BY (298.01)

20. $$\int \frac{\sqrt{a+b\sinh x}\,dx}{\left[\sqrt{a^2+b^2}-a-b\sinh x\right]^2} = -\frac{1}{\sqrt[4]{a^2+b^2}\left(\sqrt{a^2+b^2}-a\right)}\,E(\alpha,r) + \frac{b}{\sqrt{a^2+b^2}-a}\cdot\frac{\cosh x\sqrt{a+b\sinh x}}{a^2+b^2-(a+b\sinh x)^2}$$ BY (298.04)

Notation: In **2.464** 21–**2.464** 31, we set $\alpha = \arcsin\left(\tanh\frac{x}{2}\right)$, $r=\sqrt{\frac{a-b}{a+b}}$ $[0<b<a, x>0]$:

21. $$\int \frac{dx}{\sqrt{a+b\cosh x}} = \frac{2}{\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.25)

22. $$\int \sqrt{a+b\cosh x}\,dx = 2\sqrt{a+b}\,[F(\alpha,r)-E(\alpha,r)] + 2\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.29)

23. $$\int \frac{\cosh x\,dx}{\sqrt{a+b\cosh x}} = \frac{2}{\sqrt{a+b}}\,F(\alpha,r) - \frac{2\sqrt{a+b}}{b}\,E(\alpha,r) + \frac{2}{b}\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.33)

24. $$\int \frac{\tanh^2\frac{x}{2}}{\sqrt{a+b\cosh x}}\,dx = \frac{2\sqrt{a+b}}{a-b}\,[F(\alpha,r)-E(\alpha,r)]$$ BY (297.28)

25.[11] $$\int \frac{\tanh^4\frac{x}{2}}{\sqrt{a+b\cosh x}}\,dx = \frac{2\sqrt{a+b}}{3(a-b)^2}\,[(3a+b)\,F(\alpha,r) - 4a\,E(\alpha,r)] + \frac{2}{3(a-b)}\,\frac{\sinh\frac{x}{2}\sqrt{a+b\cosh x}}{\cosh^3\dfrac{x}{2}}$$ BY (297.28)

26. $$\int \frac{\cosh x - 1}{\sqrt{a+b\cosh x}}\,dx = \frac{2}{b}\left[\left(\tanh\frac{x}{2}\right)\sqrt{a+b\cosh x} - \sqrt{a+b}\,E(\alpha,r)\right]$$ BY (297.31)

27. $$\int \frac{(\cosh x-1)^2}{\sqrt{a+b\cosh x}}\,dx = \frac{4\sqrt{a+b}}{3b^2}\,[(a+3b)\,E(\alpha,r) - b\,F(\alpha,r)] + \frac{4}{3b^2}\left[b\cosh^2\frac{x}{2} - (a+3b)\right]\tanh\frac{x}{2}\sqrt{a+b\cosh x}$$ BY (297.31)

28. $$\int \frac{\sqrt{a+b\cosh x}}{\cosh x+1}\,dx = \sqrt{a+b}\,E(\alpha,r)$$ BY (297.26)

29. $$\int \frac{dx}{(\cosh x+1)\sqrt{a+b\cosh x}} = \frac{\sqrt{a+b}}{a-b}\,E(\alpha,r) - \frac{2b}{(a-b)\sqrt{a+b}}\,F(\alpha,r)$$ BY (297.30)

30. $$\int \frac{dx}{(\cosh x + 1)^2 \sqrt{a + b\cosh x}} = \frac{1}{3(a-b)^2\sqrt{a+b}} \left[b(5b-a)\, F(\alpha, r) + (a-3b)(a+b)\, E(\alpha, r) \right] + \frac{1}{6(a-b)} \cdot \frac{\sinh \frac{x}{2}}{\cosh^3 \frac{x}{2}} \sqrt{a + b\cosh x}$$ 297.30)

31. $$\int \frac{(1 + \cosh x)\ dx}{[1 + p^2 + (1 - p^2)\cosh x] \sqrt{a + b\cosh x}} = \frac{2}{\sqrt{a+b}} \Pi\left(\alpha, p^2, r\right)$$ BY (297.27)

Notation: In **2.464** 32–**2.464** 40, we set:

$$\alpha = \arcsin \sqrt{\frac{a - b\cosh x}{a - b}}$$
$$r = \sqrt{\frac{a-b}{a+b}} \qquad \left[0 < b < a, \quad 0 < x < \operatorname{arccosh} \frac{a}{b}\right]$$

32. $$\int \frac{dx}{\sqrt{a - b\cosh x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r)$$ BY (297.50)

33. $$\int \sqrt{a - b\cosh x}\, dx = 2\sqrt{a+b}\, [F(\alpha, r) - E(\alpha, r)]$$ BY (297.54)

34. $$\int \frac{\cosh x\, dx}{\sqrt{a - b\cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r) - \frac{2}{\sqrt{a+b}} F(\alpha, r)$$ BY (297.56)

35. $$\int \frac{\cosh^2 x\, dx}{\sqrt{a - b\cosh x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha, r) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) + \frac{2}{3b} \sinh x \sqrt{a - b\cosh x}$$ BY (297.56)

36. $$\int \frac{(1 + \cosh x)\ dx}{\sqrt{a - b\cosh x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r)$$ BY (297.51)

37. $$\int \frac{dx}{\cosh x \sqrt{a - b\cosh x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\alpha, \frac{a-b}{a}, r\right)$$ BY (297.57)

38. $$\int \frac{dx}{(1 + \cosh x)\sqrt{a - b\cosh x}} = \frac{1}{\sqrt{a+b}} E(\alpha, r) - \frac{1}{a+b} \tanh \frac{x}{2} \sqrt{a - b\cosh x}$$ BY (297.58)

39. $$\int \frac{dx}{(1 + \cosh x)^2 \sqrt{a - b\cosh x}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+3b)\, E(\alpha, r) - b\, F(\alpha, r)] - \frac{1}{3(a+b)^2} \frac{\tanh \frac{x}{2} \sqrt{a - b\cosh x}}{\cosh x + 1} [2a + 4b + (a+3b)\cosh x]$$ BY (297.58)

40. $$\int \frac{dx}{(a - b - ap^2 + bp^2\cosh x)\sqrt{a - b\cosh x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi\left(\alpha, p^2, r\right)$$ BY (297.52)

Notation: In **2.464** 41 –**2.464** 47, we set:

$$\alpha = \arcsin \sqrt{\frac{b\,(\cosh x - 1)}{b\cosh x - a}},$$
$$r = \sqrt{\frac{a+b}{2b}} \qquad [0 < a < b, x > 0]$$

41. $$\int \frac{dx}{\sqrt{b\cosh - a}} = \sqrt{\frac{2}{b}}\, F(\alpha, r)$$ BY (297.00)

42. $$\int \sqrt{b\cosh x - a}\, dx = (b-a)\sqrt{\frac{2}{b}}\, F(\alpha, r) - 2\sqrt{2b}\, E(\alpha, r) + \frac{2b\sinh x}{\sqrt{b\cosh x - a}}$$ BY (297.05)

43. $$\int \frac{dx}{\sqrt{(b\cosh x - a)^3}} = \frac{1}{b^2 - a^2}\cdot\sqrt{\frac{2}{b}}\,[2b\,E(\alpha, r) - (b-a)\,F(\alpha, r)]$$ BY (297.06)

44. $$\int \frac{dx}{\sqrt{(b\cosh x - a)^5}} = \frac{1}{3\,(b^2 - a^2)^2}\sqrt{\frac{2}{b}}\,[(b-3a)(b-a)\,F(\alpha, r) + 8ab\,E(\alpha, r)] + \frac{2b}{3\,(b^2-a^2)}\cdot\frac{\sinh x}{\sqrt{(b\cosh x - a)^3}}$$ BY (297.06)

45. $$\int \frac{\cosh x\, dx}{\sqrt{b\cosh x - a}} = \sqrt{\frac{2}{b}}\,[F(\alpha, r) - 2\,E(\alpha, r)] + \frac{2\sinh x}{\sqrt{b\cosh x - a}}$$ BY (297.03)

46. $$\int \frac{(\cosh x + 1)\, dx}{\sqrt{(b\cosh x - a)^3}} = \frac{2}{b-a}\sqrt{\frac{2}{b}}\, E(\alpha, r)$$ BY (297.01)

47. $$\int \frac{\sqrt{b\cosh x - a}\, dx}{p^2 b - a + b\,(1-p^2)\cosh x} = \sqrt{\frac{2}{b}}\,\Pi\left(\alpha, p^2, r\right)$$ BY (297.02)

Notation: In **2.464** 48–**2.464** 55, we set $\alpha = \arcsin\sqrt{\frac{b\cosh x - a}{b\,(\cosh x - 1)}}$ and $r = \sqrt{\frac{2b}{a+b}}$ for $\left[0 < b < a, x > \operatorname{arccosh}\frac{a}{b}\right]$:

48. $$\int \frac{dx}{\sqrt{b\cosh x - a}} = \frac{2}{\sqrt{a+b}}\, F(\alpha, r)$$ BY (297.75)

49. $$\int \sqrt{b\cosh x - a}\, dx = -2\sqrt{a+b}\, E(\alpha, r) + 2\coth\frac{x}{2}\sqrt{b\cosh x - a}$$ BY (297.79)

50. $$\int \frac{\coth^2\frac{x}{2}\, dx}{\sqrt{b\cosh x - a}} = \frac{2\sqrt{a+b}}{a-b}\, E(\alpha, r)$$ BY (297.76)

51. $$\int \frac{\sqrt{b\cosh x - a}}{\cosh x - 1}\, dx = \sqrt{a+b}\,[F(\alpha, r) - E(\alpha, r)]$$ BY (297.77)

52. $$\int \frac{dx}{(\cosh x - 1)\sqrt{b\cosh x - a}} = \frac{\sqrt{a+b}}{a-b}\, E(\alpha, r) - \frac{1}{\sqrt{a+b}}\, F(\alpha, r)$$ BY (297.78)

53. $$\int \frac{dx}{(\cosh x - 1)^2\sqrt{b\cosh x - a}} = \frac{1}{3(a-b)^2\sqrt{a+b}}\Big[(a-2b)(a-b)\,F(\alpha, r) + (3a-b)(a+b)\,E(\alpha, r)\Big] + \frac{a+b}{6b(a-b)}\cdot\frac{\cosh\frac{x}{2}}{\sinh^3\frac{x}{2}}\sqrt{b\cosh x - a}$$ BY (297.78)

54. $$\int \frac{dx}{(\cosh x + 1)\sqrt{b\cosh x - a}} = \frac{1}{\sqrt{a+b}}\,[F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b\cosh x - a}}{(a+b)\sinh x}$$ BY (297.80)

55. $$\int \frac{dx}{(\cosh x+1)^2\sqrt{b\cosh x-a}} = \frac{1}{3\sqrt{(a+b)^3}}\left[(a+b)\,F(\alpha,r) - (a+3b)\,E(\alpha,r)\right] + \frac{\sqrt{b\cosh x-a}}{3(a+b)\sinh x}\left(2\frac{a+3b}{a+b}-\tanh^2\frac{x}{2}\right)$$ BY (297.80)

Notation: In **2.464** 56–**2.464** 60, we set

$$\alpha = \arccos\frac{\sqrt[4]{b^2-a^2}}{\sqrt{a\sinh x+b\cosh x}},$$
$$r=\frac{1}{\sqrt{2}} \qquad \left[0<a<b, \quad -\operatorname{arcsinh}\frac{a}{\sqrt{b^2-a^2}}<x\right]$$

56. $$\int\frac{dx}{\sqrt{a\sinh x+b\cosh x}} = \sqrt[4]{\frac{4}{b^2-a^2}}\,F(\alpha,r)$$ BY (299.00)

57. $$\int\sqrt{a\sinh x+b\cosh x}\,dx = \sqrt[4]{4\left(b^2-a^2\right)}\left[F(\alpha,r)-2\,E(\alpha,r)\right]+\frac{2\left(a\cosh x+b\sinh x\right)}{\sqrt{a\sinh x+b\cosh x}}$$ BY (299.02)

58. $$\int\frac{dx}{\sqrt{\left(a\sinh x+b\cosh x\right)^3}} = \sqrt[4]{\frac{4}{\left(b^2-a^2\right)^3}}\left[2\,E(\alpha,r)-F(\alpha,r)\right]$$ BY (299.03)

59. $$\int\frac{dx}{\sqrt{\left(a\sinh x+b\cosh x\right)^5}} = \frac{1}{3}\sqrt[4]{\frac{4}{\left(b^2-a^2\right)^5}}\,F(\alpha,r)+\frac{2}{3\left(b^2-a^2\right)}\cdot\frac{a\cosh x+b\sinh x}{\sqrt{\left(a\sinh x+b\cosh x\right)^3}}$$ BY (299.03)

60. $$\int\frac{\left(\sqrt{b^2-a^2}+a\sinh x+b\cosh x\right)\,dx}{\sqrt{\left(a\sinh x+b\cosh x\right)^3}} = 2\sqrt[4]{\frac{4}{b^2-a^2}}\,E(\alpha,r)$$ BY (299.01)

2.47 Combinations of hyperbolic functions and powers

2.471

1. $$\int x^r\sinh^p x\cosh^q x\,dx$$
$$= \frac{1}{(p+q)^2}\left[(p+q)x^r\sinh^{p-1}x\cosh^{q-1}x - rx^{r-1}\sinh^p x\cosh^q x + r(r+1)\int x^{r-2}\sinh^p x\cosh^q x\,dx + rp\int x^{r-1}\sinh^{p-1}x\cosh^{q-1}x\,dx + (q-1)(p+q)\int x^r\sinh^p x\cosh^{q-2}x\,dx\right]$$
$$= \frac{1}{(p+q)^2}\left[(p+q)x^r\sinh^{p-1}x\cosh^{q+1}x - rx^{r-1}\sinh^p x\cosh^q x + r(r-1)\int x^{r-2}\sinh^p x\cosh^q x\,dx - rq\int x^{r-1}\sinh^{p-1}x\cosh^{q-1}x\,dx - (p-1)(p+q)\int x^r\sinh^{p-2}x\cosh^q x\,dx\right]$$
GU (353)(1)

2. $$\int x^n \sinh^{2m} x\,dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cosh(2m-2k)x\,dx$$

3. $$\int x^n \sinh^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int x^n \sinh(2m-2k+1)x\,dx$$

4. $$\int x^n \cosh^{2m} x\,dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cosh(2m-2k)x\,dx$$

5. $$\int x^n \cosh^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \int x^n \cosh(2m-2k+1)x\,dx$$

2.472

1. $$\int x^n \sinh x\,dx = x^n \cosh x - n \int x^{n-1} \cosh x\,dx$$
$$= x^n \cosh x - nx^{n-1} \sinh x + n(n-1) \int x^{n-2} \sinh x\,dx$$

2. $$\int x^n \cosh x\,dx = x^n \sinh x - n \int x^{n-1} \sinh x\,dx$$
$$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1) \int x^{n-2} \cosh x\,dx$$

3. $$\int x^{2n} \sinh x\,dx = (2n)! \left\{ \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \cosh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!} \sinh x \right\}$$

4. $$\int x^{2n+1} \sinh x\,dx = (2n+1)! \sum_{k=0}^{n} \left\{ \frac{x^{2k+1}}{(2k+1)!} \cosh x - \frac{x^{2k}}{(2k)!} \sinh x \right\}$$

5.[11] $$\int x^{2n} \cosh x\,dx = (2n)! \left\{ \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \sinh x - \sum_{k=1}^{n} \frac{x^{2k-1}}{(2k-1)!} \cosh x \right\}$$

6. $$\int x^{2n+1} \cosh x\,dx = (2n+1)! \sum_{k=0}^{n} \left\{ \frac{x^{2k+1}}{(2k+1)!} \sinh x - \frac{x^{2k}}{(2k)!} \cosh x \right\}$$

7. $$\int x \sinh x\,dx = x \cosh x - \sinh x$$

8. $$\int x^2 \sinh x\,dx = \left(x^2+2\right) \cosh x - 2x \sinh x$$

9. $$\int x \cosh x\,dx = x \sinh x - \cosh x$$

10. $$\int x^2 \cosh x\,dx = \left(x^2+2\right) \sinh x - 2x \cosh x$$

2.473 **Notation**: $z_1 = a + bx$

1. $$\int z_1 \sinh kx\,dx = \frac{1}{k} z_1 \cosh kx - \frac{b}{k^2} \sinh kx$$

2. $$\int z_1 \cosh kx\,dx = \frac{1}{k} z_1 \sinh kx - \frac{b}{k^2}\cosh kx$$

3. $$\int z_1^2 \sinh kx\,dx = \frac{1}{k}\left(z_1^2 + \frac{2b^2}{k^2}\right)\cosh kx - \frac{2bz_1}{k^2}\sinh kx$$

4. $$\int z_1^2 \cosh kx\,dx = \frac{1}{k}\left(z_1^2 + \frac{2b^2}{k^2}\right)\sinh kx - \frac{2bz_1}{k^2}\cosh kx$$

5. $$\int z_1^3 \sinh kx\,dx = \frac{z_1}{k}\left(z_1^2 + \frac{6b^2}{k^2}\right)\cosh kx - \frac{3b}{k^2}\left(z_1^2 + \frac{2b^2}{k^2}\right)\sinh kx$$

6. $$\int z_1^3 \cosh kx\,dx = \frac{z_1}{k}\left(z_1^2 + \frac{6b^2}{k^2}\right)\sinh kx - \frac{3b}{k^2}\left(z_1^3 + \frac{2b^2}{k^2}\right)\cosh kx$$

7. $$\int z_1^4 \sinh kx\,dx = \frac{1}{k}\left(z_1^4 + \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cosh kx - \frac{4bz_1}{k^2}\left(z_1^2 + \frac{6b^2}{k^2}\right)\sinh kx$$

8. $$\int z_1^4 \cosh kx\,dx = \frac{1}{k}\left(z_1^4 + \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sinh kx - \frac{4bz_1}{k^2}\left(z_1^2 + \frac{6b^2}{k^2}\right)\cosh kx$$

9. $$\int z_1^5 \sinh kx\,dx = \frac{z_1}{k}\left(z_1^4 + \frac{20b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\cosh kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2}z_1^2 + 24\frac{b^4}{k^4}\right)\sinh kx$$

10. $$\int z_1^5 \cosh kx\,dx = \frac{z_1}{k}\left(z_1^4 + 20\frac{b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\sinh kx - \frac{5b}{k^2}\left(z_1^4 + 12\frac{b^2}{k^2}z_1^2 + 24\frac{b^4}{k^4}\right)\cosh kx$$

11. $$\int z_1^6 \sinh kx\,dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2}z_1^4 + 360\frac{b^4}{k^4}z_1^2 + 720\frac{b^6}{k^6}\right)\cosh kx - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2}z_1^2 + 120\frac{b^4}{k^4}\right)\sinh kx$$

12. $$\int z_1^6 \cosh kx\,dx = \frac{1}{k}\left(z_1^6 + 30\frac{b^2}{k^2}z_1^4 + 360\frac{b^4}{k^4}z_1^2 + 720\frac{b^6}{k^6}\right)\sinh kx - \frac{6bz_1}{k^2}\left(z_1^4 + 20\frac{b^2}{k^2}z_1 + 120\frac{b^4}{k^4}\right)\cosh kx$$

2.474

1. $$\int x^n \sinh^2 x\,dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{\lfloor n/2 \rfloor}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!}\sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cosh 2x\right\}$$ GU (353)(2b)

2. $$\int x^n \cosh^2 x\,dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\sum_{k=0}^{\lfloor n/2 \rfloor}\left\{\frac{x^{n-2k}}{2^{2k}(n-2k)!}\sinh 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cosh 2x\right\}$$ GU (353)(3e)

3. $$\int x \sinh^2 x\,dx = \frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x - \frac{x^2}{4}$$

4. $$\int x^2 \sinh^2 x\,dx = \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\sinh 2x - \frac{x}{4}\cosh 2x - \frac{x^3}{6}$$ MZ 257

5. $$\int x\cosh^2 x\,dx = \frac{x}{4}\sinh 2x - \frac{1}{8}\cosh 2x + \frac{x^2}{4}$$

6. $$\int x^2\cosh^2 x\,dx = \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\sinh 2x - \frac{x}{4}\cosh 2x + \frac{x^3}{6}$$ MZ 261

7. $$\int x^n \sinh^3 x\,dx$$
$$= \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\cosh 3x}{3^{2k+1}} - 3\cosh x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\sinh 3x}{3^{2k+2}} - 3\sinh x\right)\right\}$$
GU (353)(2f)

8. $$\int x^n \cosh^3 x\,dx$$
$$= \frac{n!}{4}\sum_{k=0}^{\lfloor n/2\rfloor}\left\{\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\sinh 3x}{3^{2k+1}} + 3\sinh x\right) - \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\cosh 3x}{3^{2k+2}} + 3\cosh x\right)\right\}$$
GU (353)(3f)

9. $$\int x\sinh^3 x\,dx = \frac{3}{4}\sinh x - \frac{1}{36}\sinh 3x - \frac{3}{4}x\cosh x - \frac{x}{12}\cosh 3x$$

10. $$\int x^2\sinh^3 x\,dx = -\left(\frac{3x^2}{4} + \frac{3}{2}\right)\cosh x + \left(\frac{x^2}{12} + \frac{1}{54}\right)\cosh 3x + \frac{3x}{2}\sinh x - \frac{x}{18}\sinh 3x.$$ MZ 257

11. $$\int x\cosh^3 x\,dx = -\frac{3}{4}\cosh x - \frac{1}{36}\cosh 3x + \frac{3}{4}x\sinh x + \frac{x}{12}\sinh 3x$$

12. $$\int x^2\cosh^3 x\,dx = \left(\frac{3}{4}x^2 + \frac{3}{2}\right)\sinh x + \left(\frac{x^2}{12} + \frac{1}{54}\right)\sinh 3x - \frac{3}{2}x\cosh x - \frac{x}{18}\cosh 3x$$ MZ 262

2.475

1. $$\int \frac{\sinh^q x}{x^p}\,dx = -\frac{(p-2)\sinh^q x + qx\sinh^{q-1} x\cosh x}{(p-1)(p-2)x^{p-1}}$$
$$+\frac{q(q-1)}{(p-1)(p-2)}\int \frac{\sinh^{q-2} x}{x^{p-2}}\,dx + \frac{q^2}{(p-1)(p-2)}\int \frac{\sinh^q x}{x^{p-2}}\,dx \qquad [p>2]$$
GU (353)(6a)

2. $$\int \frac{\cosh^q x}{x^p}\,dx = -\frac{(p-2)\cosh^q x + qx\cosh^{q-1} x\sinh x}{(p-1)(p-2)x^{p-1}}$$
$$-\frac{q(q-1)}{(p-1)(p-2)}\int \frac{\cosh^{q-2} x}{x^{p-2}}\,dx + \frac{q^2}{(p-1)(p-2)}\int \frac{\cosh^q x}{x^{p-2}}\,dx \qquad [p>2]$$
GU (353)(7a)

3. $$\int \frac{\sinh x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\cosh x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\sinh x\right\} + \frac{1}{(2n-1)!}\operatorname{chi}(x)$$
GU (353)(6b)

4. $$\int \frac{\sinh x}{x^{2n+1}}\,dx = -\frac{1}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\cosh x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\sinh x\right\} + \frac{1}{(2n)!}\operatorname{shi}(x)$$ GU (353)(6b)

5. $$\int \frac{\cosh x}{x^{2n}}\,dx = -\frac{1}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(2k+1)!}{x^{2k+1}}\sinh x + \sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\cosh x\right\} + \frac{1}{(2n-1)!}\operatorname{shi}(x)$$ GU (353)(7b)

6. $$\int \frac{\cosh x}{x^{2n+1}}\,dx = -\frac{1}{(2n)!x}\left\{\sum_{k=0}^{n-1}\frac{(2k)!}{x^{2k}}\sinh x + \sum_{k=0}^{n-1}\frac{(2k+1)!}{x^{2k+1}}\cosh x\right\} + \frac{1}{(2n)!}\operatorname{chi}(x)$$ GU (353)(7b)

7. $$\int \frac{\sinh^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\operatorname{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}}\binom{2m}{m}\ln x$$ GU (353)(6c)

8. $$\int \frac{\sinh^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\operatorname{shi}(2m-2k+1)x$$ GU (353)(6d)

9. $$\int \frac{\cosh^{2m} x}{x}\,dx = \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\operatorname{chi}(2m-2k)x + \frac{1}{2^{2m}}\binom{2m}{m}\ln x$$ GU (353)(7c)

10. $$\int \frac{\cosh^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\operatorname{chi}(2m-2k+1)x$$ GU (353)(7c)

11. $$\int \frac{\sinh^{2m} x}{x^2}\,dx = \frac{(-1)^{m-1}}{2^{2m}x}\binom{2m}{m} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+1}\binom{2m}{k}\left\{\frac{\cosh(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x\right\}$$

12. $$\int \frac{\sinh^{2m+1} x}{x^2}\,dx = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+1}\binom{2m+1}{k} \times \left\{\frac{\sinh(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{chi}(2m-2k+1)x\right\}$$

13. $$\int \frac{\cosh^{2m} x}{x^2}\,dx = -\frac{1}{2^{2m}x}\binom{2m}{m} - \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\left\{\frac{\cosh(2m-2k)x}{x} - (2m-2k)\operatorname{shi}(2m-2k)x\right\}$$

14. $$\int \frac{\cosh^{2m+1} x}{x^2}\,dx = -\frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\left\{\frac{\cosh(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{shi}(2m-2k+1)x\right\}$$

2.476

1. $$\int \frac{\sinh kx}{a+bx}\,dx = \frac{1}{b}\left[\cosh\frac{ka}{b}\operatorname{shi}(u) - \sinh\frac{ka}{b}\operatorname{chi}(u)\right] = \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\operatorname{Ei}(u) - \exp\left(\frac{ka}{b}\right)\operatorname{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

2. $$\int \frac{\cosh kx}{a+bx}\,dx = \frac{1}{b}\left[\cosh\frac{ka}{b}\operatorname{chi}(u) - \sinh\frac{ka}{b}\operatorname{shi}(u)\right]$$
$$= \frac{1}{2b}\left[\exp\left(-\frac{ka}{b}\right)\operatorname{Ei}(u) + \exp\left(\frac{ka}{b}\right)\operatorname{Ei}(-u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

3. $$\int \frac{\sinh kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\sinh kx}{a+bx} + \frac{k}{b}\int\frac{\cosh kx}{a+bx}\,dx$$ (see **2.476** 2)

4. $$\int \frac{\cosh kx}{(a+bx)^2}\,dx = -\frac{1}{b}\cdot\frac{\cosh kx}{a+bx} + \frac{k}{b}\int\frac{\sinh kx}{a+bx}\,dx$$ (see **2.476** 1)

5. $$\int \frac{\sinh kx}{(a+bx)^3}\,dx = -\frac{\sinh kx}{2b(a+bx)^2} - \frac{k\cosh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int\frac{\sinh kx}{a+bx}\,dx$$ (see **2.476** 1)

6. $$\int \frac{\cosh kx}{(a+bx)^3}\,dx = -\frac{\cosh kx}{2b(a+bx)^2} - \frac{k\sinh kx}{2b^2(a+bx)} + \frac{k^2}{2b^2}\int\frac{\cosh kx}{a+bx}\,dx$$ (see **2.476** 2)

7. $$\int \frac{\sinh kx}{(a+bx)^4}\,dx = -\frac{\sinh kx}{3b(a+bx)^3} - \frac{k\cosh kx}{6b^2(a+bx)^2} - \frac{k^2\sinh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int\frac{\cosh kx}{a+bx}\,dx$$ (see **2.476** 2)

8. $$\int \frac{\cosh kx}{(a+bx)^4}\,dx = -\frac{\cosh kx}{3b(a+bx)^3} - \frac{k\sinh kx}{6b^2(a+bx)^2} - \frac{k^2\cosh kx}{6b^3(a+bx)} + \frac{k^3}{6b^3}\int\frac{\sinh kx}{a+bx}\,dx$$ (see **2.476** 1)

9. $$\int \frac{\sinh kx}{(a+bx)^5}\,dx = -\frac{\sinh kx}{4b(a+bx)^4} - \frac{k\cosh kx}{12b^2(a+bx)^3} - \frac{k^2\sinh kx}{24b^3(a+bx)^2}$$
$$-\frac{k^3\cosh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int\frac{\sinh kx}{a+bx}\,dx$$ (see **2.476** 1)

10. $$\int \frac{\cosh kx}{(a+bx)^5}\,dx = -\frac{\cosh kx}{4b(a+bx)^4} - \frac{k\sinh kx}{12b^2(a+bx)^3} - \frac{k^2\cosh kx}{24b^3(a+bx)^2}$$
$$-\frac{k^3\sinh kx}{24b^4(a+bx)} + \frac{k^4}{24b^4}\int\frac{\cosh kx}{a+bx}\,dx$$ (see **2.476** 2)

11. $$\int \frac{\sinh kx}{(a+bx)^6}\,dx = -\frac{\sinh kx}{5b(a+bx)^5} - \frac{k\cosh kx}{20b^2(a+bx)^4} - \frac{k^2\sinh kx}{60b^3(a+bx)^3} - \frac{k^3\cosh kx}{120b^4(a+bx)^2}$$
$$-\frac{k^4\sinh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int\frac{\cosh kx}{a+bx}\,dx$$ (see **2.476** 2)

12. $$\int \frac{\cosh kx}{(a+bx)^6}\,dx = -\frac{\cosh kx}{5b(a+bx)^5} - \frac{k\sinh kx}{20b^2(a+bx)^4} - \frac{k^2\cosh kx}{60b^3(a+bx)^3} - \frac{k^3\sinh kx}{120b^4(a+bx)^2}$$
$$-\frac{k^4\cosh kx}{120b^5(a+bx)} + \frac{k^5}{120b^5}\int\frac{\sinh kx}{a+bx}\,dx$$ (see **2.476** 1)

2.477

1. $$\int \frac{x^p\,dx}{\sinh^q x} = \frac{-px^{p-1}\sinh x - (q-2)x^p\cosh x}{(q-1)(q-2)\sinh^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}}{\sinh^{q-2} x}\,dx - \frac{q-2}{q-1}\int \frac{x^p\,dx}{\sinh^{q-2} x}$$
$[q > 2]$ GU (353)(8a)

2. $$\int \frac{x^p\,dx}{\cosh^q x} = \frac{px^{p-1}\cosh x + (q-2)x^p\sinh x}{(q-1)(q-2)\cosh^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\cosh^{q-2} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\cosh^{q-2} x}$$
$[q > 2]$ GU (353)(10a)

3. $$\int \frac{x^n}{\sinh x}\,dx = \sum_{k=0}^{\infty} \frac{\left(2-2^{2k}\right)B_{2k}}{(n+2k)(2k)!}x^{n+2k}$$
$[|x| < \pi, \quad n > 0]$ GU(353)(8b)

4. $$\int \frac{x^n}{\cosh x}\,dx = \sum_{k=0}^{\infty} \frac{E_{2k}x^{n+2k+1}}{(n+2k+1)(2k)!}$$
$\left[|x| < \frac{\pi}{2}, \quad n \geq 0\right]$ GU (353)(10b)

5. $$\int \frac{dx}{x^n \sinh x} = -\left[1+(-1)^n\right]\frac{2^{n-1}-1}{n!}B_n \ln x + \sum_{\substack{k=0\\ k\neq \frac{n}{2}}}^{\infty} \frac{2-2^{2k}}{(2k-n)(2k)!}B_{2k}x^{2k-n}$$
$[|x| < \pi, \quad n \geq 1]$ GU (353)(9b)

6.[11] $$\int \frac{dx}{x^n \cosh x} = \sum_{\substack{k=0\\ k\neq \frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!}x^{2k-n+1} + \frac{1}{2}\left[1+(-1)^n\right] + \frac{E_{n-1}}{(n-1)!}\ln x$$
$\left[|x| < \frac{\pi}{2}\right]$ GU (353)(11b)

7. $$\int \frac{x^n}{\sinh^2 x}\,dx = -x^n \coth x + n\sum_{k=0}^{\infty} \frac{2^{2k}B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1}$$
$[n > 1, \quad |x| < \pi]$ GU (353)(8c)

8. $$\int \frac{x^n}{\cosh^2 x}\,dx = x^n \tanh x - n\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(n+2k-1)(2k)!}x^{n+2k-1}$$
$\left[n > 1, \quad |x| < \frac{\pi}{2}\right]$ GU (353)(10c)

9. $$\int \frac{dx}{x^n \sinh^2 x} = -\frac{\coth x}{x^n} - \left[1-(-1)^n\right]\frac{2^n n}{(n+1)!}B_{n+1}\ln x - \frac{n}{x^{n+1}}\sum_{\substack{k=0\\ k\neq \frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!}(2x)^{2^k}$$
$[|x| < \pi]$ GU (353)(9c)

10. $$\int \frac{dx}{x^n \cosh^2 x} = \frac{\tanh x}{x^n} + [1-(-1)^n] - \frac{2n\left(2^{n+1}-1\right)n}{(n+1)!} B_{n+1} \ln x + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{\left(2^{2k}-1\right) B_{2k}}{(2k-n-1)(2k)!} (2x)^{2^k}$$
$\left[|x| < \frac{\pi}{2}\right]$ GU (353)(11c)

11. $$\int \frac{x}{\sinh^{2n} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \left\{ \frac{x \cosh x}{\sinh^{2n-2k+1} x} + \frac{1}{(2n-2k)\sinh^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\, dx}{\sinh^2 x}$$
(see **2.477** 17) GU (353)(8e)

12. $$\int \frac{x}{\sinh^{2n-1} x}\, dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \left\{ \frac{x \cosh x}{\sinh^{2n-2k} x} + \frac{1}{(2n-2k-1)\sinh^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\, dx}{\sinh x}$$
(see **2.477** 15) GU (353)(8e)

13. $$\int \frac{x}{\cosh^{2n} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \left\{ \frac{x \sinh x}{\cosh^{2n-2k+1} x} + \frac{1}{(2n-2k)\cosh^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x\, dx}{\cosh^2 x}$$
(see **2.477** 18) GU (353)(10e)

14. $$\int \frac{x}{\cosh^{2n-1} x}\, dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \left\{ \frac{x \sinh x}{\cosh^{2n-2k} x} + \frac{1}{(2n-2k-1)\cosh^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x\, dx}{\cosh x}$$
(see **2.477** 16) GU (353)(10e)

15. $$\int \frac{x\, dx}{\sinh x} = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1}$$
$|x| < \pi$ GU (353)(8b)a

16. $$\int \frac{x\, dx}{\cosh x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!}$$
$|x| < \frac{\pi}{2}$ GU (353)(10b)a

17. $$\int \frac{x\, dx}{\sinh^2 x} = -x \coth x + \ln \sinh x$$
MZ 257

18. $$\int \frac{x\, dx}{\cosh^2 x} = x \tanh x - \ln \cosh x$$
MZ 262

19. $$\int \frac{x\, dx}{\sinh^3 x} = -\frac{x \cosh x}{2\sinh^2 x} - \frac{1}{2\sinh x} - \frac{1}{2} \int \frac{x\, dx}{\sinh x}$$
(see **2.477** 15) MZ 257

20. $$\int \frac{x\,dx}{\cosh^3 x} = \frac{x \sinh x}{2\cosh^2 x} + \frac{1}{2\cosh x} + \frac{1}{2}\int \frac{x\,dx}{\cosh x}$$ (see **2.477** 16) MZ 262

21. $$\int \frac{x\,dx}{\sinh^4 x} = -\frac{x\cosh x}{3\sinh^3 x} - \frac{1}{6\sinh^2 x} + \frac{2}{3}x\coth x - \frac{2}{3}\ln\sinh x$$ MZ 258

22. $$\int \frac{x\,dx}{\cosh^4 x} = \frac{x\sinh x}{3\cosh^3 x} + \frac{1}{6\cosh^2 x} + \frac{2}{3}x\tanh x - \frac{2}{3}\ln\cosh x$$ MZ 262

23. $$\int \frac{x\,dx}{\sinh^5 x} = -\frac{x\cosh x}{4\sinh^4 x} - \frac{1}{12\sinh^3 x} + \frac{3x\cosh x}{8\sinh^2 x} + \frac{3}{8\sinh x} + \frac{3}{8}\int \frac{x\,dx}{\sinh x}$$ (see **2.477** 15) MZ 258

24. $$\int \frac{x\,dx}{\cosh^5 x} = \frac{x\sinh x}{4\cosh^4 x} + \frac{1}{12\cosh^3 x} + \frac{3x\sinh x}{8\cosh^2 x} + \frac{3}{8\cosh x} + \frac{3}{8}\int \frac{x\,dx}{\cosh x}$$ (see **2.477** 16) MZ 262

2.478

1. $$\int \frac{x^n \cosh x\,dx}{(a+b\sinh x)^m} = -\frac{x^n}{(m-1)b\,(a+b\sinh x)^{m-1}} + \frac{n}{(m-1)b}\int \frac{x^{n-1}\,dx}{(a+b\sinh x)^{m-1}}$$ $[m \neq 1]$ MZ 263

2. $$\int \frac{x^n \sinh x\,dx}{(a+b\cosh x)^m} = -\frac{x^n}{(m-1)b\,(a+b\cosh x)^{m-1}} + \frac{n}{(m-1)b}\int \frac{x^{n-1}\,dx}{(a+b\cosh x)^{m-1}}$$ $[m \neq 1]$ MZ 263

3. $$\int \frac{x\,dx}{1+\cosh x} = x\tanh\frac{x}{2} - 2\ln\cosh\frac{x}{2}$$

4. $$\int \frac{x\,dx}{1-\cosh x} = x\coth\frac{x}{2} - 2\ln\sinh\frac{x}{2}$$

5. $$\int \frac{x\sinh x\,dx}{(1+\cosh x)^2} = -\frac{x}{1+\cosh x} + \tanh\frac{x}{2}$$

6. $$\int \frac{x\sinh x\,dx}{(1-\cosh x)^2} = \frac{x}{1-\cosh x} - \coth\frac{x}{2}$$ MZ 262-264

7. $$\int \frac{x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2\sin 2t}\left[L(u+t) - L(u-t) - 2\,L(t)\right]$$ $[u = \arctan(\tanh x \cot t), \quad t \neq \pm n\pi]$ LO III 402

8. $$\int \frac{x\cosh x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2\sin t}\left[L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{v+t}{2}\right) + L\left(\frac{v-t}{2}\right) - 2\,L\left(\frac{t}{2}\right) - 2\,L\left(\frac{\pi-t}{2}\right)\right]$$ $\left[u = 2\arctan\left(\tanh\frac{x}{2}\cdot\cot\frac{t}{2}\right), \quad v = 2\arctan\left(\coth\frac{x}{2}\cdot\cot\frac{t}{2}\right); \quad t \neq \pm n\pi\right]$ LO III 403

2.479

1. $$\int x^p \frac{\sinh^{2m} x}{\cosh^n x}\,dx = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} \int \frac{x^p\,dx}{\cosh^{n-2k} x}$$ (see **4.477** 2)

2. $$\int x^p \frac{\sinh^{2m+1} x}{\cosh^n x}\,dx = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} \int x^p \frac{\sinh x}{\cosh^{n-2k} x}\,dx$$ $[n > 1]$ (see **2.479** 3)

3. $$\int x^p \frac{\sinh x}{\cosh^n x}\,dx = -\frac{x^p}{(n-1)\cosh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}\,dx}{\cosh^{n-1} x}$$ $[n > 1]$ (see **2.477** 2) GU (353)(12)

4. $$\int x^p \frac{\cosh^{2m} x}{\sinh^n x}\,dx = \sum_{k=0}^{m} \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x}$$ (see **2.477** 1)

5. $$\int x^p \frac{\cosh^{2m+1} x}{\sinh^n x}\,dx = \sum_{k=0}^{m} \binom{m}{k} \int \frac{x^p \cosh x}{\sinh^{n-2k} x}\,dx$$ (see **2.479** 6)

6. $$\int x^p \frac{\cosh x}{\sinh^n x}\,dx = -\frac{x^p}{(n-1)\sinh^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1}\,dx}{\sinh^{n-1} x}$$ $[n > 1]$ (see **2.477** 1) GU (353)(13c)

7. $$\int x^p \tanh x\,dx = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right) B_{2k}}{(2k+p)(2k)!} x^{p+2k}$$ $\left[p > -1, \quad |x| < \frac{\pi}{2}\right]$ GU (353)(12d)

8. $$\int x^p \coth x\,dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k}$$ $[p \geq +1, \quad |x| < \pi]$ GU (353)(13d)

9. $$\int \frac{x \cosh x}{\sinh^2 x}\,dx = \ln \tanh \frac{x}{2} - \frac{x}{\sinh x}$$

10. $$\int \frac{x \sinh x}{\cosh^2 x}\,dx = -\frac{x}{\cosh x} + \arctan(\sinh x)$$ MZ 263

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.481

1. $$\int e^{ax} \sinh(bx+c)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a \sinh(bx+c) - b\cosh(bx+c)\right]$$ $\left[a^2 \neq b^2\right]$

2. $$\int e^{ax} \cosh(bx+c)\,dx = \frac{e^{ax}}{a^2-b^2}\left[a \cosh(bx+c) - b\sinh(bx+c)\right]$$ $\left[a^2 \neq b^2\right]$

For $a^2 = b^2$:

3. $$\int e^{ax}\sinh(ax+c)\,dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

4. $$\int e^{-ax}\sinh(ax+c)\,dx = \frac{1}{2}xe^{c} + \frac{1}{4a}e^{-(2ax+c)}$$

5. $$\int e^{ax}\cosh(ax+c)\,dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}$$

6. $$\int e^{-ax}\cosh(ax+c)\,dx = \frac{1}{2}xe^{c} - \frac{1}{4a}e^{-(2ax+c)}$$ MZ 275-277

2.482

1. $$\int x^p e^{ax}\sinh bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx - \int x^p e^{(a-b)x}\,dx\right\}$$ $[a^2 \neq b^2]$

2. $$\int x^p e^{ax}\cosh bx\,dx = \frac{1}{2}\left\{\int x^p e^{(a+b)x}\,dx + \int x^p e^{(a-b)x}\,dx\right\}$$ $[a^2 \neq b^2]$

For $a^2 = b^2$:

3. $$\int x^p e^{ax}\sinh ax\,dx = \frac{1}{2}\int x^p e^{2ax}\,dx - \frac{x^{p+1}}{2(p+1)}$$ (see **2.321**)

4. $$\int x^p e^{-ax}\sinh ax\,dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2}\int x^p e^{-2ax}\,dx$$ (see **2.321**)

5. $$\int x^p e^{ax}\cosh ax\,dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2}\int x^p e^{2ax}\,dx$$ (see **2.321**) MZ 276, 278

2.483

1. $$\int xe^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\sinh bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\cosh bx\right]$$ $[a^2 \neq b^2]$

2. $$\int xe^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left[\left(ax - \frac{a^2+b^2}{a^2-b^2}\right)\cosh bx - \left(bx - \frac{2ab}{a^2-b^2}\right)\sinh bx\right]$$ $[a^2 \neq b^2]$

3. $$\int x^2 e^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2-b^2}\left\{\left[ax^2 - \frac{2\left(a^2+b^2\right)}{a^2-b^2}x + \frac{2a\left(a^2+3b^2\right)}{\left(a^2-b^2\right)^2}\right]\sinh bx - \left[bx^2 - \frac{4ab}{a^2-b^2}x + \frac{2b\left(3a^2+b^2\right)}{\left(a^2-b^2\right)^2}\right]\cosh x\right\}$$ $[a^2 \neq b^2]$

4. $$\int x^2 e^{ax} \cosh bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ax^2 - \frac{2\left(a^2 + b^2\right)}{a^2 - b^2} x + \frac{2a\left(a^2 + 3b^2\right)}{\left(a^2 - b^2\right)^2} \right] \cosh bx - \left[bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b\left(3a^2 + b^2\right)}{\left(a^2 - b^2\right)^2} \right] \sinh x \right\} \qquad \left[a^2 \neq b^2\right]$$

For $a^2 = b^2$:

5. $$\int x e^{ax} \sinh ax \, dx = \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) - \frac{x^2}{4}$$

6. $$\int x e^{-ax} \sinh ax \, dx = \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right) + \frac{x^2}{4}$$ MZ 276, 278

7. $$\int x e^{ax} \cosh ax \, dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right)$$

8. $$\int x e^{-ax} \cosh ax \, dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right)$$

9. $$\int x^2 e^{ax} \sinh ax \, dx = \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right) - \frac{x^3}{6}$$

10. $$\int x^2 e^{-ax} \sinh ax \, dx = \frac{e^{-2ax}}{4a} \left(x^2 + \frac{x}{a} + \frac{1}{2a^2} \right) + \frac{x^3}{6}$$

11. $$\int x^2 e^{ax} \cosh ax \, dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right)$$

2.484

1. $$\int e^{ax} \sinh bx \frac{dx}{x} = \frac{1}{2} \left\{ \mathrm{Ei}[(a+b)x] - \mathrm{Ei}[(a-b)x] \right\} \qquad \left[a^2 \neq b^2\right]$$

2. $$\int e^{ax} \cosh bx \frac{dx}{x} = \frac{1}{2} \left\{ \mathrm{Ei}[(a+b)x] + \mathrm{Ei}[(a-b)x] \right\} \qquad \left[a^2 \neq b^2\right]$$

3. $$\int e^{ax} \sinh bx \frac{dx}{x^2} = -\frac{e^{ax} \sinh bx}{2x} + \frac{1}{2} \left\{ (a+b) \, \mathrm{Ei}[(a+b)x] - (a-b) \, \mathrm{Ei}[(a-b)x] \right\} \qquad \left[a^2 \neq b^2\right]$$

4. $$\int e^{ax} \cosh bx \frac{dx}{x^2} = -\frac{e^{ax} \cosh bx}{2x} + \frac{1}{2} \left\{ (a+b) \, \mathrm{Ei}[(a+b)x] + (a-b) \, \mathrm{Ei}[(a-b)x] \right\} \qquad \left[a^2 \neq b^2\right]$$

For $a^2 = b^2$:

5. $$\int e^{ax} \sinh ax \frac{dx}{x} = \frac{1}{2} \left[\mathrm{Ei}(2ax) - \ln x \right]$$

6. $$\int e^{-ax} \sinh ax \frac{dx}{x} = \frac{1}{2} \left[\ln x - \mathrm{Ei}(-2ax) \right]$$

7. $$\int e^{ax} \cosh ax \frac{dx}{x} = \frac{1}{2} \left[\ln x + \mathrm{Ei}(2ax) \right]$$

8. $$\int e^{ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x}\left(e^{2ax} - 1\right) + a \operatorname{Ei}(2ax)$$

9. $$\int e^{-ax} \sinh ax \frac{dx}{x^2} = -\frac{1}{2x}\left(1 - e^{-2ax}\right) + a \operatorname{Ei}(-2ax)$$

10. $$\int e^{ax} \cosh ax \frac{dx}{x^2} = -\frac{1}{2x}\left(e^{2ax} + 1\right) + a \operatorname{Ei}(2ax)$$ MZ 276, 278

2.5–2.6 Trigonometric Functions

2.50 Introduction

2.501 Integrals of the form $\int R(\sin x, \cos x)\, dx$ can always be reduced to integrals of rational functions by means of the substitution $t = \tan \frac{x}{2}$.

2.502 If $R(\sin x, \cos x)$ satisfies the relation

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

it is convenient to make the substitution $t = \cos x$.

2.503 If this function satisfies the relation

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

it is convenient to make the substitution $t = \sin x$.

2.504 If this function satisfies the relation

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

it is convenient to make the substitution $t = \tan x$.

2.51–2.52 Powers of trigonometric functions

2.510
$$\begin{aligned}
\int \sin^p x \cos^q x\, dx &= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x\, dx \\
&= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^{q} x\, dx \\
&= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^{q} x\, dx \\
&= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x\, dx \\
&= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^{p} x \cos^{q-2} x\, dx \\
&= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^{p} x \cos^{q+2} x\, dx \\
&= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left\{ \sin^2 x - \frac{q-1}{p+q-2} \right\} \\
&\quad + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x\, dx
\end{aligned}$$

FI II 89, TI 214

2.511

1. $$\int \sin^p x \cos^{2n} x\,dx$$
$$= \frac{\sin^{p+1} x}{2n+p}\left\{\cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\}$$
$$+\frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \sin^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have:

2. $$\int \sin^{2l} x\,dx$$
$$= -\frac{\cos x}{2l}\left\{\sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)}\sin^{2l-2k-1} x\right\}$$
$$+\frac{(2l-1)!!}{2^l l!}x$$
(see also **2.513** 1)

TI (232)

3. $$\int \sin^{2l+1} x\,dx = -\frac{\cos x}{2l+1}\left\{\sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)}\sin^{2l-2k-2} x\right\}$$
(see also **2.513** 2) TI (233)

4. $$\int \sin^p x \cos^{2n+1} x\,dx = \frac{\sin^{p+1} x}{2n+p+1}\left\{\cos^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}$$

This formula is applicable for arbitrary real p, except for the negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.512

1. $$\int \cos^p x \sin^{2n} x\,dx$$
$$= -\frac{\cos^{p+1} x}{2n+p}\left\{\sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)\sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)}\right\}$$
$$+\frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)}\int \cos^p x\,dx$$

This formula is applicable for arbitrary real p, except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

2. $$\int \cos^{2l} x\,dx = \frac{\sin x}{2l}\left\{\cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)}\cos^{2l-2k-1} x\right\}$$
$$+\frac{(2l-1)!!}{2^l l!}x$$
(see also **2.513** 3) TI (230)

3. $$\int \cos^{2l+1} x\,dx = \frac{\sin x}{2l+1}\left\{\cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \cos^{2l-2k-2} x\right\}$$

(see also **2.513** 4) TI (231)

4. $$\int \cos^p x \sin^{2n+1} x\,dx$$
$$= -\frac{\cos^{p+1} x}{2n+p+1}\left\{\sin^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\dots(n-k+1)\sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)}\right\}$$

This formula is applicable for arbitrary real p, except for the following negative odd integers: -1, -3, …, $-(2n+1)$.

2.513

1. $$\int \sin^{2n} x\,dx = \frac{1}{2^{2n}}\binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$ (see also **2.511** 2)

TI (226)

2. $$\int \sin^{2n+1} x\,dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$ (see also **2.511** 3)

TI (227)

3. $$\int \cos^{2n} x\,dx = \frac{1}{2^{2n}}\binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

(see also **2.512** 2) TI (224)

4. $$\int \cos^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1}$$

(see also **2.512** 3) TI (225)

5. $$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x$$

6. $$\int \sin^3 x\,dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x = \frac{1}{3}\cos^3 x - \cos x$$

7. $$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$
$$= -\frac{3}{8}\sin x \cos x - \frac{1}{4}\sin^3 x \cos x + \frac{3}{8}x$$

8. $$\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos 3x - \frac{1}{80}\cos 5x$$
$$= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x$$

9. $$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x$$
$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}x$$

10. $$\int \sin^7 x\,dx = -\frac{35}{64}\cos x + \frac{7}{64}\cos 3x - \frac{7}{320}\cos 5x + \frac{1}{448}\cos 7x$$
$$= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x + \frac{8}{35}\cos^3 x - \frac{24}{35}\cos x$$

11. $$\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x\cos x + \frac{1}{2}x$$

12. $$\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x$$

13. $$\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = \frac{3}{8}x + \frac{3}{8}\sin x\cos x + \frac{1}{4}\sin x\cos^3 x$$

14. $$\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x = \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x\sin x$$

15. $$\int \cos^6 x\,dx = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x$$
$$= \frac{5}{16}x + \frac{5}{16}\sin x\cos x + \frac{5}{24}\sin x\cos^3 x + \frac{1}{6}\sin x\cos^5 x$$

16. $$\int \cos^7 x\,dx = \frac{35}{64}\sin x + \frac{7}{64}\sin 3x + \frac{7}{320}\sin 5x + \frac{1}{448}\sin 7x$$
$$= \frac{24}{35}\sin x - \frac{8}{35}\sin^3 x + \frac{6}{35}\sin x\cos^4 x + \frac{1}{7}\sin x\cos^6 x$$

17. $$\int \sin x\cos^2 x\,dx = -\frac{1}{4}\left(\frac{1}{3}\cos 3x + \cos x\right) = -\frac{\cos^3 x}{3}$$

18. $$\int \sin x\cos^3 x\,dx = -\frac{\cos^4 x}{4}$$

19. $$\int \sin x\cos^4 x\,dx = -\frac{\cos^5 x}{5}$$

20. $$\int \sin^2 x\cos x\,dx = -\frac{1}{4}\left(\frac{1}{3}\sin 3x - \sin x\right) = \frac{\sin^3 x}{3}$$

21. $$\int \sin^2 x\cos^2 x\,dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right)$$

22. $$\int \sin^2 x\cos^3 x\,dx = -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right)$$
$$= \frac{\sin^3 x}{5}\left(\cos^2 x + \frac{2}{3}\right) = \frac{\sin^3 x}{5}\left(\frac{5}{3} - \sin^2 x\right)$$

23. $$\int \sin^2 x\cos^4 x\,dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x$$

24. $$\int \sin^3 x\cos x\,dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{\sin^4 x}{4}$$

25. $$\int \sin^3 x\cos^2 x\,dx = \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right)$$
$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

26. $$\int \sin^3 x \cos^3 x \, dx = \frac{1}{32}\left(\frac{1}{6}\cos 6x - \frac{3}{2}\cos 2x\right)$$

27. $$\int \sin^3 x \cos^4 x \, dx = \frac{1}{7}\cos^3 x \left(-\frac{2}{5} - \frac{3}{5}\sin^2 x + \sin^4 x\right)$$

28. $$\int \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5}$$

29. $$\int \sin^4 x \cos^2 x \, dx = \frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x$$

30. $$\int \sin^4 x \cos^3 x \, dx = \frac{1}{7}\sin^3 x \left(\frac{2}{5} + \frac{3}{5}\cos^2 x - \cos^4 x\right)$$

31. $$\int \sin^4 x \cos^4 x \, dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x$$

2.514 $$\int \frac{\sin^p x}{\cos^{2n} x}\, dx = \frac{\sin^{p+1} x}{2n-1}\left\{\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\sec^{2n-2k-1} x\right\} + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \sin^p x\, dx$$

This formula is applicable for arbitrary real p. For $\int \sin^p x\, dx$, where p is a natural number, see **2.511** 2, 3 and **2.513** 1, 2. If $n = 0$ and p is a negative integer, we have for this integral:

2.515

1. $$\int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1}\left\{\operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)}\operatorname{cosec}^{2l-2k-1} x\right\}$$
TI (242)

2. $$\int \frac{dx}{\sin^{2l+1} x} = -\frac{\cos x}{2l}\left\{\operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{28^k(l-1)(l-2)\dots(l-k)}\operatorname{cosec}^{2l-2k} x\right\} + \frac{(2l-1)!!}{2^l l!}\ln\tan\frac{x}{2}$$
TI (243)

2.516

1. $$\int \frac{\sin^p x\, dx}{\cos^{2n+1} x} = \frac{\sin^{p+1} x}{2n}\left\{\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\cdots(2n-p-2k+1)}{2^k(n-1)(n-2)\cdots(n-k)}\sec^{2n-2k} x\right\} + \frac{(2n-p-1)(2n-p-3)\cdots(3-p)(1-p)}{2^n n!}\int \frac{\sin^p x}{\cos x}\, dx$$

This formula is applicable for arbitrary real p. For $n = 0$ and p a natural number, we have

2. $$\int \frac{\sin^{2l+1} x\, dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k} x}{2k} - \ln\cos x$$

3. $$\int \frac{\sin^{2l} x\,dx}{\cos x} = -\sum_{k=1}^{l} \frac{\sin^{2k-1} x}{2k-1} + \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

2.517

1. $$\int \frac{dx}{\sin^{2m+1} x \cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+2)\sin^{2m-2k+2} x} + \ln\tan x$$

2. $$\int \frac{dx}{\sin^{2m} x \cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+1)\sin^{2m-2k+1} x} + \ln\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

2.518

1. $$\int \frac{\sin^p x}{\cos^2 x}\,dx = \frac{\sin^{p-1} x}{\cos x} - (p-1)\int \sin^{p-2} x\,dx$$

2. $$\int \frac{\cos^p x\,dx}{\sin^{2n} x} = \frac{\cos^{p+1} x}{2n-1}\left\{\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x\right\} + \frac{(2n-p-2)(2n-p-4)\dots(2-p)(-p)}{(2n-1)!!}\int \cos^p x\,dx$$

This formula is applicable for arbitrary real p. For $\int \cos^p x\,dx$ where p is a natural number, see **2.512** 2, 3 and **2.513** 3, 4. If $n = 0$ and p is a negative integer, we have for this integral:

2.519

1. $$\int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1}\left\{\sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)} \sec^{2l-2k-1} x\right\}$$ TI (240)

2. $$\int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l}\left\{\sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sec^{2l-2k} x\right\} + \frac{(2l-1)!!}{2^l l!}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$ TI (241)

2.521

1. $$\int \frac{\cos^p x\,dx}{\sin^{2n+1} x} = -\frac{\cos^{p+1} x}{2n}\left\{\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \operatorname{cosec}^{2n-2k} x\right\} + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n \cdot n!}\int \frac{\cos^p x}{\sin x}\,dx$$

This formula is applicable for arbitrary real p. For $n = 0$ and p a natural number, we have

2. $$\int \frac{\cos^{2l+1} x\,dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k} x}{2k} + \ln \sin x$$

3. $$\int \frac{\cos^{2l} x\,dx}{\sin x} = \sum_{k=1}^{l} \frac{\cos^{2k-1} x}{2k-1} + \ln \tan \frac{x}{2}$$

2.522

1. $$\int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+2)\cos^{2m-2k+2} x} + \ln \tan x$$

2. $$\int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+1)\cos^{2m-2k+1} x} + \ln \tan \frac{x}{2}$$ GW (331)(15)

2.523 $$\int \frac{\cos^m x}{\sin^2 x}\,dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1)\int \cos^{m-2} x\,dx$$

2.524 In formulas **2.524** 1 and **2.524** 2, $s = 1$ for m odd and $m < 2n+1$; in other cases, $s = 0$.

1. $$\int \frac{\sin^{2n+1} x}{\cos^m x}\,dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^{n} (-1)^{k+1} \binom{n}{k} \frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cos x$$

GU (331)(11d)

2. $$\int \frac{\cos^{2n+1} x}{\sin^m x}\,dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^{n} (-1)^{k} \binom{n}{k} \frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \sin x$$

2.525

1. $$\int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\tan^{2k-2m+1} x}{2k-2m+1}$$ TI (267)

2. $$\int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\tan^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln \tan x$$

TI (268), GU (331)(15f)

2.526

1. $$\int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

2. $$\int \frac{dx}{\sin^2 x} = -\cot x$$

3. $$\int \frac{dx}{\sin^3 x} = -\frac{1}{2}\frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \tan \frac{x}{2}$$

4. $$\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3}\cot x = -\frac{1}{3}\cot^3 x - \cot x$$

5. $$\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8} \ln \tan \frac{x}{2}$$

6. $$\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \cot^3 x - \frac{4}{5} \cot x$$
$$= -\frac{1}{5} \cot^5 x - \frac{2}{3} \cot^3 x - \cot x$$

7. $$\int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6 \sin^2 x} \left(\frac{1}{\sin^4 x} + \frac{5}{4 \sin^2 x} + \frac{15}{8} \right) + \frac{5}{16} \ln \tan \frac{x}{2}$$

8. $$\int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7} \cot^7 x + \frac{3}{5} \cot^5 x + \cot^3 x + \cot x \right)$$

9. $$\int \frac{dx}{\cos x} = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln \cot \left(\frac{\pi}{4} - \frac{x}{2} \right) = \ln \sqrt{\frac{1 + \sin x}{1 - \sin x}}$$

10. $$\int \frac{dx}{\cos^2 x} = \tan x$$

11. $$\int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

12. $$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \tan x = \frac{1}{3} \tan^3 x + \tan x$$

13. $$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

14. $$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \tan^3 x + \frac{4}{5} \tan x = \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x$$

15. $$\int \frac{dx}{\cos^7 x} = \frac{\sin x}{6 \cos^6 x} + \frac{5 \sin x}{24 \cos^4 x} + \frac{5 \sin x}{16 \cos^2 x} + \frac{5}{16} \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

16. $$\int \frac{dx}{\cos^8 x} = \frac{1}{7} \tan^7 x + \frac{3}{5} \tan^5 x + \tan^3 x + \tan x$$

17. $$\int \frac{\sin x}{\cos x} \, dx = -\ln \cos x$$

18. $$\int \frac{\sin^2 x}{\cos x} \, dx = -\sin x + \ln \tan \left(\frac{\pi}{4} = \frac{x}{2} \right)$$

19. $$\int \frac{\sin^3 x}{\cos x} \, dx = -\frac{\sin^2 x}{2} - \ln \cos x = \frac{1}{2} \cos^2 x - \ln \cos x$$

20. $$\int \frac{\sin^4 x}{\cos x} \, dx = -\frac{1}{3} \sin^3 x - \sin x + \ln \tan \left(\frac{x}{2} + \frac{\pi}{4} \right)$$

21. $$\int \frac{\sin^2 x \, dx}{\cos^2 x} = \frac{1}{\cos x}$$

22. $$\int \frac{\sin^2 x \, dx}{\cos^2 x} = \tan x - x$$

23. $$\int \frac{\sin^3 x \, dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}$$

24. $$\int \frac{\sin^4 x \, dx}{\cos^2 x} = \tan x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x$$

25. $$\int \frac{\sin x\,dx}{\cos^3 x} = \frac{1}{2\cos^2 x} = \frac{1}{2}\tan^2 x$$

26. $$\int \frac{\sin^2 x\,dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

27. $$\int \frac{\sin^3 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \ln\cos x$$

28. $$\int \frac{\sin^4 x\,dx}{\cos^3 x} = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

29. $$\int \frac{\sin x\,dx}{\cos^4 x} = \frac{1}{3\cos^3 x}$$

30. $$\int \frac{\sin^2 x\,dx}{\cos^4 x} = \frac{1}{3}\tan^3 x$$

31. $$\int \frac{\sin^3 x\,dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x}$$

32. $$\int \frac{\sin^4 x\,dx}{\cos^4 x} = \frac{1}{3}\tan^3 x - \tan x + x$$

33. $$\int \frac{\cos x\,dx}{\sin x} = \ln\sin x$$

34. $$\int \frac{\cos^2 x\,dx}{\sin x} = \cos x + \ln\tan\frac{x}{2}$$

35. $$\int \frac{\cos^3 x\,dx}{\sin x} = \frac{\cos^2 x}{2} + \ln\sin x$$

36. $$\int \frac{\cos^4 x\,dx}{\sin x} = \frac{1}{3}\cos^3 x + \cos x + \ln\tan\left(\frac{x}{2}\right)$$

37. $$\int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x}$$

38. $$\int \frac{\cos^2 x}{\sin^2 x}\,dx = -\cot x - x$$

39. $$\int \frac{\cos^3 x}{\sin^2 x}\,dx = -\sin x - \frac{1}{\sin x}$$

40. $$\int \frac{\cos^4 x}{\sin^2 x}\,dx = -\cot x - \frac{1}{2}\sin x\cos x - \frac{3}{2}x$$

41. $$\int \frac{\cos x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x}$$

42. $$\int \frac{\cos^2 x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\tan\frac{x}{2}$$

43. $$\int \frac{\cos^3 x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - \ln\sin x$$

44. $$\int \frac{\cos^4 x}{\sin^3 x}\,dx = -\frac{1}{2}\frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2}\ln\tan\frac{x}{2}$$

45. $$\int \frac{\cos x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x}$$

46. $$\int \frac{\cos^2 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x$$

47. $$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}$$

48. $$\int \frac{\cos^4 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x + \cot x + x$$

49. $$\int \frac{dx}{\sin x\cos x} = \ln\tan x$$

50. $$\int \frac{dx}{\sin x\cos^2 x} = \frac{1}{\cos x} + \ln\tan\frac{x}{2}$$

51. $$\int \frac{dx}{\sin x\cos^3 x} = \frac{1}{2\cos^2 x} + \ln\tan x$$

52. $$\int \frac{dx}{\sin x\cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln\tan\frac{x}{2}$$

53. $$\int \frac{dx}{\sin^2 x\cos x} = \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) - \operatorname{cosec} x$$

54. $$\int \frac{dx}{\sin^2 x\cos^2 x} = -2\cot 2x$$

55. $$\int \frac{dx}{\sin^2 x\cos^3 x} = \left(\frac{1}{2\cos^2 x} - \frac{3}{2}\right)\frac{1}{\sin x} + \frac{3}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

56. $$\int \frac{dx}{\sin^2 x\cos^4 x} = \frac{1}{3\sin x\cos^3 x} - \frac{8}{3}\cot 2x$$

57. $$\int \frac{dx}{\sin^3 x\cos x} = -\frac{1}{2\sin^2 x} + \ln\tan x$$

58. $$\int \frac{dx}{\sin^3 x\cos^2 x} = -\frac{1}{\cos x}\left(\frac{1}{2\sin^2 x} - \frac{3}{2}\right) + \frac{3}{2}\ln\tan\frac{x}{2}$$

59. $$\int \frac{dx}{\sin^3 x\cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2\ln\tan x$$

60. $$\int \frac{dx}{\sin^3 x\cos^4 x} = \frac{2}{\cos x} + \frac{1}{3\cos^3 x} - \frac{\cos x}{2\sin^2 x} + \frac{5}{2}\ln\tan\frac{x}{2}$$

61. $$\int \frac{dx}{\sin^4 x\cos x} = -\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

62. $$\int \frac{dx}{\sin^4 x\cos^2 x} = -\frac{1}{3\cos x\sin^3 x} - \frac{8}{3}\cot 2x$$

63. $$\int \frac{dx}{\sin^4 x\cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

64. $$\int \frac{dx}{\sin^4 x\cos^4 x} = -8\cot 2x - \frac{8}{3}\cot^3 2x$$

2.527

1. $$\int \tan^p x\,dx = \frac{\tan^{p-1} x}{p-1} - \int \tan^{p-2} x\,dx \qquad [p \neq 1]$$

2. $$\int \tan^{2n+1} x\,dx = \sum_{k=1}^{n} (-1)^{n+k} \binom{n}{k} \frac{1}{2k\cos^{2k} x} - (-1)^n \ln\cos x$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k-1} \tan^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln\cos x$$

3. $$\int \tan^{2n} x\,dx = \sum_{k=1}^{n} (-1)^{k-1} \frac{\tan^{2n-2k+1} x}{2n-2k+1} + (-1)^n x$$ GU (331)(12)

4. $$\int \cot^p x\,dx = -\frac{\cot^{p-1} x}{p-1} - \int \cot^{p-2} x\,dx \qquad [p \neq 1]$$

5. $$\int \cot^{2n+1} x\,dx = \sum_{k=1}^{n} (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k\sin^{2k} x} + (-1)^n \ln\sin x$$
$$= \sum_{k=1}^{n} (-1)^k \frac{\cot^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln\sin x$$

6. $$\int \cot^{2n} x\,dx = \sum_{k=1}^{n} (-1)^k \frac{\cot^{2n-2k+1} x}{2n-2k+1} + (-1)^n x$$ GU (331)(14)

For special formulas for $p = 1, 2, 3, 4$, see **2.526** 17, **2.526** 33, **2.526** 22, **2.526** 38, **2.526** 27, **2.526** 43, **2.526** 32, and **2.526** 48.

2.53–2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.531

1. $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$$

2. $$\int \cos(ax+b)\,dx = -\frac{1}{a}\sin(ax+b)$$

2.532

1. $$\int \sin(ax+b)\sin(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)}$$
$$[a^2 \neq c^2]$$

2.[8] $$\int \sin(ax+b)\cos(cx+d)\,dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)}$$
$$[a^2 \neq c^2]$$

3. $$\int \cos(ax+b)\cos(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)}$$
$$\left[a^2 \neq c^2\right]$$

For $c = a$:

4. $$\int \sin(ax+b)\sin(ax+d)\,dx = \frac{x}{2}\cos(b-d) - \frac{\sin(2ax+b+d)}{4a}$$

5. $$\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d)}{4a}$$

6. $$\int \cos(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\cos(b-d) + \frac{\sin(2ax+b+d)}{4a}$$ GU (332)(3)

2.533

1.[8] $$\int \sin ax \cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \qquad \left[a^2 \neq b^2\right]$$

2.[8] $$\int \sin ax \sin bx \sin cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right\}$$ PE (376)

3. $$\int \sin ax \cos bx \cos cx\,dx = -\frac{1}{4}\left\{\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right\}$$ PE (378)

4. $$\int \cos ax \sin bx \sin cx\,dx = \frac{1}{4}\left\{\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right\}$$ PE (379)

5. $$\int \cos ax \cos bx \cos cx\,dx = \frac{1}{4}\left\{\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c}\right\}$$ PE (377)

2.534

1. $$\int \frac{\cos px + i \sin px}{\sin nx}\,dx = -2\int \frac{z^{p+n-1}}{1-z^{2n}}\,dz \qquad [z = \cos x + i \sin x]$$ Pe (374)

2. $$\int \frac{\cos px + i \sin px}{\cos nx}\,dx = -2i\int \frac{z^{p+n-1}}{1-z^{2n}}\,dz \qquad [z = \cos x + i \sin x]$$ Pe (373)

2.535

1. $$\int \sin^p x \sin ax\,dx = \frac{1}{p+a}\left\{-\sin^p x \cos ax + p\int \sin^{p-1} x \cos(a-1)x\,dx\right\}$$ GU (332)(5a)

2. $$\int \sin^p x \sin(2n+1)x\,dx$$
$$= (2n+1)\left\{\int \sin^{p+1} x\,dx + \sum_{k=1}^{n}(-1)^k \frac{\left[(2n+1)^2-1^2\right]\left[(2n+1)^2-3^2\right]\ldots\left[(2n+1)^2-(2k-1)^2\right]}{(2k+1)!}\right.$$
$$\left.\times\int \sin^{2k+p+1} x\,dx\right\}$$
TI (299)
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^{k-1}\,\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)}\sin^{p-2k} x \cos(2n-2k+1)x\right.\right.$$
$$\left.+ (-1)^k \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)}\sin^{p-2k-1} x \sin(2n-2k)x\right]$$
$$\left.+ \frac{(-1)^n\,\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p-2n+1)}\int \sin^{p-2n+1} x\,dx\right\}$$
GU (332)(5c)

3. $$\int \sin^p x \sin 2nx\,dx = 2n\left\{\frac{\sin^{p+2} x}{p+2} + \sum_{k=1}^{n-1}(-1)^k \frac{\left(4n^2-2^2\right)\left(4n^2-4^2\right)\dots\left[4n^2-(2k)^2\right]}{(2k+1)!(2k+p+2)} \sin^{2k+p+2} x\right\}$$

TI (303)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{\sum_{k=0}^{n-1} \frac{(-1)^{k-1}\,\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k)x - \frac{(-1)^k\,\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\,\Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k-1)x\right\}$$

[p is not equal to $-2, -4, \dots, -2n$]

GU (332)(5c)

2.536

1. $$\int \sin^p x \cos ax\,dx = \frac{1}{p+1}\left\{\sin^p x \sin ax - p\int \sin^{p-1} x \sin(a-1)x\,dx\right\}$$ GU (332)(6a)

2. $$\int \sin^p x \cos(2n+1)x\,dx$$

$$= \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^{n}(-1)^k \frac{\left[(2n+1)^2-1^2\right]\left[(2n+1)^2-3^2\right]\dots\left[(2n+1)^2-(2k-1)^2\right]}{(2k)!(2k+p+1)} \times \sin^{2k+p+1} x$$

TI (301)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^k\,\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\,\Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k+1)x + \frac{(-1)^k\,\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\,\Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k)x\right] + \frac{(-1)^n\,\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\,\Gamma(p-2n+1)}\int \sin^{p-2n} x \cos x\,dx\right\}$$

[p is not equal to $-3, -5, \dots, -(2n+1)$]

GU (332)(6c)

3. $$\int \sin^p x \cos 2nx\, dx$$
$$= \int \sin^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2 \cdot \left(4n^2 - 2^2\right) \dots \left[4n^2 - (2k-2)^2\right]}{(2k)!} \int \sin^{2k+p} x\, dx$$
TI (300)
$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k\, \Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\, \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k)x \right.\right.$$
$$\left. + \frac{(-1)^k\, \Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\, \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k-1)x \right]$$
$$\left. + \frac{(-1)^n\, \Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\, \Gamma(p-2n+1)} \int \sin^{p-2n} x\, dx \right\}$$
GU (332)(6c)

2.537

1. $$\int \cos^p x \sin ax\, dx = \frac{1}{p+a} \left\{ -\cos^p x \cos ax + p \int \cos^{p-1} x \sin(a-1)x\, dx \right\}$$ GU (332)(7a)

2. $$\int \cos^p x \sin(2n+1)x\, dx$$
$$= (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} \right.$$
$$\left. + \sum_{k=1}^{n} (-1)^k \frac{\left[(2n+1)^2 - 1^2\right]\left[(2n+1)^2 - 3^2\right] \dots \left[(2n+1)^2 - (2k-1)^2\right]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x \right\}$$
TI (295)
$$= \frac{\Gamma(p+1)}{\Gamma\left(\dfrac{p+3}{2}+n\right)} \left\{ -\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{2k+1}\, \Gamma(p-2k+1)} \cos^{p-k} x \cos(2n-k+1)x \right.$$
$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\, \Gamma(p-n+1)} \int \cos^{p-n} x \sin(n+1)x\, dx \right\}$$
[p is not equal to $-3, -5, \dots, -(2n+1)$]
GU (332)(7b)a

3. $$\int \cos^p x \sin 2nx\, dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{\left(4n^2-2^2\right)\left(4n^2-4^2\right)\dots\left[4n^2-(2k)^2\right]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\}$$

TI (297)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ -\sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \cos(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \sin nx\, dx \right\}$$

[p is not equal to $-2, -4, \dots, -2n$]

GU (332)(7b)a

2.538

1. $$\int \cos^p x \cos ax\, dx = \frac{1}{p+a} \left\{ \cos^p x \sin ax + p \int \cos^{p-1} x \cos(a-1)x\, dx \right\}$$ GU (332)(8a)

2. $$\int \cos^p x \cos(2n+1)x\, dx$$

$$= (-1)^n (2n+1) \left\{ \int \cos^{p+1} x\, dx + \sum_{k=1}^{n} (-1)^k \frac{\left[(2n+1)^2-1^2\right]\left[(2n+1)^2-3^2\right]\dots\left[(2n+1)^2-(2k-1)^2\right]}{(2k+1)!} \times \int \cos^{2k+p+1} x\, dx \right\}$$

TI (293)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x\, dx \right\}$$

GU (332)(8b)a

3. $$\int \cos^p x \cos 2nx\,dx = (-1)^n \left\{ \int \cos^p x\,dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2\left[4n^2-2^2\right]\dots\left[4n^2-(2k-2)^2\right]}{(2k)!} \int \cos^{2k+p} x\,dx \right\}$$

TI (294)

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\,\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\,\Gamma(p-n+1)} \int \cos^{p-n} x \cos nx\,dx \right\}$$

GU (332)(8b)a

2.539

1. $$\int \frac{\sin(2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\sin 2kx}{2k} + x$$

2. $$\int \frac{\sin 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$$ GU (332)(5e)

3. $$\int \frac{\cos(2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos 2kx}{2k} + \ln\sin x$$

4. $$\int \frac{\cos 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos(2k-1)x}{2k-1} + \ln\tan\frac{x}{2}$$ GI (332)(6e)

5. $$\int \frac{\sin(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1}\ln\cos x$$

6. $$\int \frac{\sin 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1}$$ GU (332)(7d)

7. $$\int \frac{\cos(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^n x$$

8. $$\int \frac{\cos 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$ GU (332)(8d)

2.541

1. $$\int \sin(n+1)x \sin^{n-1} x\,dx = \frac{1}{n}\sin^n x \sin nx$$ BI (71)(1)a

2. $$\int \sin(n+1)x \cos^{n-1} x\,dx = -\frac{1}{n}\cos^n x \cos nx$$ BI (71)(2)a

3. $$\int \cos(n+1)x \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \cos nx$$ BI (71)(3)a

4. $$\int \cos(n+1)x \cos^{n-1} x\, dx = \frac{1}{n} \cos^n x \sin nx$$ BI (71)(4)a

5. $$\int \sin\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \cos n\left(\frac{\pi}{2} - x\right)$$ BI (71)(5)a

6. $$\int \cos\left[(n+1)\left(\frac{\pi}{2} - x\right)\right] \sin^{n-1} x\, dx = -\frac{1}{n} \sin^n x \sin n\left(\frac{\pi}{2} - x\right)$$ BI (71)(6)a

2.542

1. $$\int \frac{\sin 2x}{\sin^n x}\, dx = -\frac{2}{(n-2)\sin^{n-2} x}$$

For $n = 2$:

2. $$\int \frac{\sin 2x}{\sin^2 x}\, dx = 2 \ln \sin x$$

2.543

1. $$\int \frac{\sin 2x\, dx}{\cos^n x} = \frac{2}{(n-2)\cos^{n-2} x}$$

For $n = 2$:

2. $$\int \frac{\sin 2x}{\cos^2 x}\, dx = -2 \ln \cos x$$

2.544

1. $$\int \frac{\cos 2x\, dx}{\sin x} = 2\cos x + \ln \tan \frac{x}{2}$$

2. $$\int \frac{\cos 2x\, dx}{\sin^2 x} = -\cot x - 2x$$

3. $$\int \frac{\cos 2x\, dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} - \frac{3}{2} \ln \tan \frac{x}{2}$$

4. $$\int \frac{\cos 2x\, dx}{\cos x} = 2\sin x - \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

5. $$\int \frac{\cos 2x\, dx}{\cos^2 x} = 2x - \tan x$$

6. $$\int \frac{\cos 2x\, dx}{\cos^3 x} = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2} \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

7. $$\int \frac{\sin 3x\, dx}{\sin x} = x + \sin 2x$$

8. $$\int \frac{\sin 3x}{\sin^2 x}\, dx = 3 \ln \tan \frac{x}{2} + 4\cos x$$

9. $$\int \frac{\sin 3x}{\sin^3 x}\, dx = -3\cot x - 4x$$

2.545

1. $$\int \frac{\sin 3x}{\cos^n x}\,dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}$$

For $n = 1$ and $n = 3$:

2. $$\int \frac{\sin 3x}{\cos x}\,dx = 2\sin^2 x + \ln\cos x$$

3. $$\int \frac{\sin 3x}{\cos^3 x}\,dx = -\frac{1}{2\cos^2 x} - 4\ln\cos x$$

2.546

1. $$\int \frac{\cos 3x}{\sin^n x}\,dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}$$

For $n = 1$ and $n = 3$:

2. $$\int \frac{\cos 3x}{\sin x}\,dx = -2\sin^2 x + \ln\sin x$$

3. $$\int \frac{\cos 3x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - 4\ln\sin x$$

2.547

1. $$\int \frac{\sin nx}{\cos^p x}\,dx = 2\int \frac{\sin(n-1)x\,dx}{\cos^{p-1} x} - \int \frac{\sin(n-2)x\,dx}{\cos^p x}$$

2. $$\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x$$

3. $$\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

4. $$\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\tan x$$

2.548

1. $$\int \frac{\sin^m x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\sum_{k=0}^{2n}(-1)^{n+k}\cos^m\left[\frac{2k+1}{2(2n+1)}\pi\right]\ln\frac{\sin\left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2}\right]}{\sin\left[\frac{k+n+1}{(2n+1)}\pi - \frac{x}{2}\right]}$$

$[m$ a natural number $\leq 2n]$ TI (378)

2. $$\int \frac{\sin^{2m} x\,dx}{\sin 2nx} = \frac{(-1)^n}{2n}\left\{\ln\cos x + \sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left(\cos^2 x - \sin^2\frac{k\pi}{2n}\right)\right\}$$

$[m$ a natural number $\leq n]$ TI (379)

3. $$\int \frac{\sin^{2m+1} x}{\sin 2nx}\,dx = \frac{(-1)^n}{2n}\left\{\ln\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left[\tan\left(\frac{n+k}{4n}\pi - \frac{x}{2}\right)\tan\left(\frac{n-k}{4n}\pi - \frac{x}{2}\right)\right]\right\}$$

$[m$ a natural number $< n]$

4. $$\int \frac{\sin^{2m} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + \sum_{k=1}^{n} (-1)^k \right.$$
$$\left. \times\ \cos^{2m} \frac{k\pi}{2n+1} \ln\left[\tan\left(\frac{2n+2k+1}{4(2n+1)}\pi - \frac{x}{2}\right)\tan\left(\frac{2n-2k+1}{2(2n+1)}\pi - \frac{x}{2}\right)\right]\right\}$$

[m a natural number $\le n$] TI (381)

5. $$\int \frac{\sin^{2m+1} x\,dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln\cos x + \sum_{k=1}^{n} (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln\left(\cos^2 x - \sin^2 \frac{k\pi}{2n+1}\right)\right\}$$

[m a natural number $\le n$] TI (382)a

6. $$\int \frac{\sin^m x\,dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left[\frac{2k+1}{4n}\pi\right] \ln \frac{\sin\left[\dfrac{2k-2n+1}{8n}\pi + \dfrac{x}{2}\right]}{\sin\left[\dfrac{2k+2n+1}{8n}\pi - \dfrac{x}{2}\right]}$$

[m a natural number $< 2n$] TI (377)

7. $$\int \frac{\cos^{2m+1} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln\sin x + \sum_{k=1}^{n} (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln\left(\sin^2 x - \sin^2 \frac{k\pi}{2n+1}\right)\right\}$$

[m a natural number $\le n$] TI (376)

8. $$\int \frac{\cos^{2m} x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln\tan\frac{x}{2} \right.$$
$$\left. + \sum_{k=1}^{n} (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln\left[\tan\left(\frac{x}{2} + \frac{k\pi}{4n+2}\right)\tan\left(\frac{x}{2} - \frac{k\pi}{4n+2}\right)\right]\right\}$$

[m a natural number $\le n$] TI (375)

9. $$\int \frac{\cos^{2m+1} x}{\sin 2nx}\,dx = \frac{1}{2n} \left\{ \ln\tan\frac{x}{2} + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln\left[\tan\left(\frac{x}{2} + \frac{k\pi}{4}\right)\tan\left(\frac{x}{2} - \frac{k\pi}{4n}\right)\right]\right\}$$

[m a natural number $< n$] TI (374)

10. $$\int \frac{\cos^{2m} x}{\sin 2nx}\,dx = \frac{1}{2n} \left\{ \ln\sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln\left(\sin^2 x - \sin^2 \frac{k\pi}{2n}\right)\right\}$$

[m a natural number $\le n$] TI (373)

11. $$\int \frac{\cos^m x}{\cos nx}\,dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n}\pi \ln \frac{\sin\left[\dfrac{2k+1}{4n}\pi + \dfrac{x}{2}\right]}{\sin\left[\dfrac{2k+1}{4n}\pi - \dfrac{x}{2}\right]}$$

[m is a natural number $\le n$] TI (372)

2.549

1. $$\int \sin x^2\,dx = \sqrt{\frac{\pi}{2}}\,S(x)$$

2. $$\int \cos x^2\,dx = \sqrt{\frac{\pi}{2}}\,C(x)$$

3.[11] $$\int \sin\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{ac-b^2}{a}\,S\left(\frac{ax+b}{\sqrt{a}}\right)+\sin\frac{ac-b^2}{a}\,C\left(\frac{ax+b}{\sqrt{a}}\right)\right\}$$

$[a>0]$

4.[11] $$\int \cos\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{ac-b^2}{a}\,C\left(\frac{ax+b}{\sqrt{a}}\right)-\sin\frac{ac-b^2}{a}\,S\left(\frac{ax+b}{\sqrt{a}}\right)\right\}$$

$[a>0]$

5. $$\int \sin\ln x\,dx = \frac{x}{2}\left(\sin\ln x-\cos\ln x\right)$$ PE (444)

6. $$\int \cos\ln x\,dx = \frac{x}{2}\left(\sin\ln x+\cos\ln x\right)$$ PE (445)

2.55–2.56 Rational functions of the sine and cosine

2.551

1. $$\int \frac{A+B\sin x}{(a+b\sin x)^n}\,dx = \frac{1}{(n-1)\left(a^2-b^2\right)}\left[\frac{(Ab-aB)\cos x}{(a+b\sin x)^{n-1}} + \int\frac{(Aa-Bb)(n-1)+(aB-bA)(n-2)\sin x}{(a+b\sin x)^{n-1}}\,dx\right]$$

TI (358)a

For $n=1$:

2. $$\int \frac{A+B\sin x}{a+b\sin x}\,dx = \frac{B}{b}x+\frac{Ab-aB}{b}\int\frac{dx}{a+b\sin x}$$ (see **2.551** 3) TI (342)

3. $$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} \qquad [a^2>b^2]$$

$$= \frac{1}{\sqrt{b^2-a^2}}\ln\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \qquad [a^2<b^2]$$

2.552

1. $$\int \frac{A+B\cos x}{(a+b\sin x)^n}\,dx = -\frac{B}{(n-1)b\,(a+b\sin x)^{n-1}}+A\int\frac{dx}{(a+b\sin x)^n}$$

(see **2.552** 3) TI (361)

For $n=1$:

2. $$\int \frac{A+B\cos x}{a+b\sin x}\,dx = \frac{B}{b}\ln\left(a+b\sin x\right)+A\int\frac{dx}{a+b\sin x}$$

(see **2.551** 3) TI (344)

3. $$\int \frac{dx}{(a+b\sin x)^n} = \frac{1}{(n-1)\left(a^2-b^2\right)}\left[\frac{b\cos x}{(a+b\sin x)^{n-1}} + \int \frac{(n-1)a-(n-2)b\sin x}{(a+b\sin x)^{n-1}}\,dx\right]$$ (see **2.551** 1)

TI (359)

2.553

1. $$\int \frac{A+B\sin x}{(a+b\cos x)^n}\,dx = \frac{B}{(n-1)b\,(a+b\cos x)^{n-1}} + A\int \frac{dx}{(a+b\cos x)^n}$$ (see **2.554** 3) TI (355)

For $n=1$:

2. $$\int \frac{A+B\sin x}{a+b\cos x}\,dx = -\frac{B}{b}\ln(a+b\cos x) + A\int \frac{dx}{a+b\cos x}$$ (see **2.553** 3*) TI (343)

3.* $$\int \frac{dx}{a+b\cos x}$$
$$= \frac{2}{\sqrt{a^2-b^2}}\arctan\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right) \qquad \left[a^2>b^2\right]$$
$$= \frac{2}{\sqrt{a^2-b^2}}\ln\left|\frac{(b-a)\tan\left(\frac{x}{2}\right)+\sqrt{b^2-b^a}}{(b-a)\tan\left(\frac{x}{2}\right)-\sqrt{b^2-b^a}}\right| \qquad \left[b^2>a^2\right]$$
$$= \frac{2}{\sqrt{b^2-a^2}}\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right) \qquad \left[b^2>a^2, \quad \left|(b-a)\tan\left(\frac{x}{2}\right)\right| < \sqrt{b^2-a^2}\right]$$
$$= \frac{2}{\sqrt{b^2-a^2}}\operatorname{arccoth}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{b^2-a^2}}\right) \qquad \left[b^2>a^2, \quad \left|(b-a)\tan\left(\frac{x}{2}\right)\right| > \sqrt{b^2-a^2}\right]$$
(compare with **2.551** 3)

2.554

1. $$\int \frac{A+B\cos x}{(a+b\cos x)^n}\,dx = \frac{1}{(n-1)\left(a^2-b^2\right)}\left[\frac{(aB-Ab)\sin x}{(a+b\cos x)^{n-1}} + \int \frac{(Aa-bB)(n-1)+(n-2)(aB-bA)\cos x}{(a+b\cos x)^{n-1}}\,dx\right]$$

TI (353)

For $n = 1$:

2. $$\int \frac{A + B\cos x}{a + b\cos x}\,dx = \frac{B}{b}x + \frac{Ab - aB}{b}\int \frac{dx}{a + b\cos x}$$ (see **2.553** 3) TI (341)

3. $$\int \frac{dx}{(a + b\cos x)^n} = -\frac{1}{(n-1)\left(a^2 - b^2\right)}\left\{\frac{b\sin x}{(a + b\cos x)^{n-1}} - \int \frac{(n-1)a - (n-2)b\cos x}{(a + b\cos x)^{n-1}}\,dx\right\}$$ (see **2.554** 1) TI (354)

In integrating the functions in formulas **2.551** 3 and **2.553** 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points $x = \arcsin\left(-\frac{a}{b}\right)$ in formula **2.551** 3 or over the points $x = \arccos\left(-\frac{a}{b}\right)$ in formula **2.553** 3.

2.555 Formulas **2.551** 3 and **2.553** 3 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

1. $$\int \frac{A + B\sin x}{(1 \pm \sin x)^n}\,dx = -\frac{1}{2^{n-1}}\left\{2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1} \pm (A \mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\tan^{2k+1}\left(\frac{\pi}{4} \mp \frac{x}{2}\right)}{2k+1}\right.$$ TI (361)a

2. $$\int \frac{A + B\cos x}{(1 \pm \cos x)^n}\,dx = \frac{1}{2^{n-1}}\left\{2B\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1} \pm (A \mp B)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\tan^{2k+1}\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]}{2k+1}\right\}$$ TI (356)

For $n = 1$:

3.[11] $$\int \frac{A + B\sin x}{1 \pm \sin x}\,dx = \pm Bx + (B \mp A)\tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right)$$ TI (250)

4. $$\int \frac{A + B\cos x}{1 \pm \cos x}\,dx = \pm Bx \pm (A \mp B)\tan\left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$ TI (248)

2.556

1. $$\int \frac{\left(1 - a^2\right)\,dx}{1 - 2a\cos x + a^2} = 2\arctan\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right)$$ $[0 < a < 1, \quad |x| < \pi]$ FI II 93

2. $$\int \frac{(1 - a\cos x)\,dx}{1 - 2a\cos x + a^2} = \frac{x}{2} + \arctan\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right)$$ $[0 < a < 1, \quad |x| < \pi]$ FI II 93

2.557

1. $$\int \frac{dx}{(a\cos x + b\sin x)^n} = \frac{1}{\sqrt{\left(a^2 + b^2\right)^n}}\int \frac{dx}{\sin^n\left(x + \arctan\frac{a}{b}\right)}$$ (see **2.515**) MZ 173a

2.[6] $$\int \frac{\sin x\,dx}{a\sin x + b\cos x} = \frac{ax - b\ln\sin\left(x + \arctan\frac{b}{a}\right)}{a^2+b^2}$$

3. $$\int \frac{\cos x\,dx}{a\cos x + b\sin x} = \frac{ax + b\ln\sin\left(x + \arctan\frac{a}{b}\right)}{a^2+b^2}$$ MZ 174a

4. $$\int \frac{dx}{a\cos x + b\sin x} = \frac{\ln\tan\left[\frac{1}{2}\left(x + \arctan\frac{a}{b}\right)\right]}{\sqrt{a^2+b^2}}$$

5. $$\int \frac{dx}{(a\cos x + b\sin x)^2} = -\frac{\cot\left(x + \arctan\frac{a}{b}\right)}{a^2+b^2} = +\frac{1}{a^2+b^2}\cdot\frac{a\sin x - b\cos x}{a\cos x + b\sin x}$$ MZ 174a

2.558

1. $$\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}\,dx$$
$$= \frac{(Bc - Cb) + (Ac - Ca)\cos x - (Ab - Ba)\sin x}{(n-1)\left(a^2-b^2-c^2\right)(a + b\cos x + c\sin x)^{n-1}} + \frac{1}{(n-1)\left(a^2-b^2-c^2\right)}$$
$$\times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)\left[(Ab - Ba)\cos x - (Ac - Ca)\sin x\right]}{(a + b\cos x + c\sin x)^{n-1}}\,dx$$
$$\left[n \neq 1, \quad a^2 \neq b^2 + c^2\right]$$
$$= \frac{Cb - Bc + Ca\cos x - Ba\sin x}{(n-1)a\,(a + b\cos x + c\sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right)(-c\cos x + b\sin x)$$
$$\times \frac{(n-1)!}{(2n-1)!!}\sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!\,a^k}\cdot\frac{1}{(a + b\cos x + c\sin x)^{n-k}}$$
$$\left[n \neq 1, \quad a^2 = b^2 + c^2\right]$$

For $n = 1$:

2.[11] $$\int \frac{A + B\cos x + C\sin x}{a + b\cos x + c\sin x}\,dx = \frac{Bc - Cb}{b^2+c^2}\ln(a + b\cos x + c\sin x) + \frac{Bb + Cc}{b^2+c^2}x$$
$$+ \left(A - \frac{Bb + Cc}{b^2+c^2}a\right)\int \frac{dx}{a + b\cos x + c\sin x}$$ (see **2.558** 4)

GU (331)(18)

3. $$\int \frac{dx}{(a + b\cos x + c\sin x)^n} = \int \frac{d(x-\alpha)}{[a + r\cos(x-\alpha)]^n},$$
where $b = r\cos\alpha,\ c = r\sin\alpha$ (see **2.554** 3)

4. $$\int \frac{dx}{a + b\cos x + c\sin x}$$
$$= \frac{2}{\sqrt{a^2-b^2-c^2}}\arctan\frac{(a-b)\tan\frac{x}{2} + c}{\sqrt{a^2-b^2-c^2}} \qquad \left[a^2 > b^2 + c^2\right]$$ TI (253), FI II 94
$$= \frac{1}{\sqrt{b^2+c^2-a^2}}\ln\frac{(a-b)\tan\frac{x}{2} + c - \sqrt{b^2+c^2-a^2}}{(a-b)\tan\frac{x}{2} + c + \sqrt{b^2+c^2-a^2}} \qquad \left[a^2 < b^2 + c^2\right]$$ TI (253)a
$$= \frac{1}{c}\ln\left(a + c\cdot\tan\frac{x}{2}\right) \qquad [a = b]$$
$$= \frac{-2}{c + (a-b)\tan\frac{x}{2}} \qquad \left[a^2 = b^2 + c^2\right]$$ TI (253)a

2.559

1. $$\int \frac{dx}{[a(1+\cos x)+c\sin x]^2} = \frac{1}{c^3}\left[\frac{c(a\sin x - c\cos x)}{a(1+\cos x)+c\sin x} - a\ln\left(a + c\tan\frac{x}{2}\right)\right]$$

2. $$\int \frac{A + B\cos x + C\sin x}{(a_1+b_1\cos x+c_1\sin x)(a_2+b_2\cos x+c_2\sin x)}\,dx$$
$$= A_0\ln\frac{a_1+b_1\cos x+c_1\sin x}{a_2+b_2\cos x+c+2\sin x} + A_1\int\frac{dx}{a_1+b_1\cos x+c_1\sin x} + A_2\int\frac{dx}{a_2+b_2\cos x+c_2\sin x}$$
(see **2.558 4**) GU (331)(19)

where

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad A_1 = \frac{\begin{vmatrix} \begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix} & \begin{vmatrix} A & C \\ a_1 & c_1 \end{vmatrix} & \begin{vmatrix} B & A \\ b_1 & a_1 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2\right]$$

3. $$\int \frac{A\cos^2 x + 2B\sin x\cos x + C\sin^2 x}{a\cos^2 x + 2b\sin x\cos x + c\sin^2 x}\,dx$$
$$= \frac{1}{4b^2+(a-c)^2}\Big\{[4Bb + (A-C)(a-c)]x + [(A-C)b - \mathrm{B}(a-c)]$$
$$\times \ln\left(a\cos^2 x + 2b\sin x\cos x + c\sin^2 x\right)$$
$$+ \left[2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)\right] f(x)\Big\}$$

where GU (331)(24)

$$f(x) = \frac{1}{2\sqrt{b^2-ac}}\ln\frac{c\tan x + b - \sqrt{b^2-ac}}{c\tan x + b + \sqrt{b^2-ac}} \qquad \left[b^2 > ac\right]$$
$$= \frac{1}{\sqrt{ac-b^2}}\arctan\frac{c\tan x + b}{\sqrt{ac-b^2}} \qquad \left[b^2 < ac\right]$$
$$= -\frac{1}{c\tan x + b} \qquad \left[b^2 = ac\right]$$

2.561

1. $$\int \frac{(A + B\sin x)\,dx}{\sin x\,(a + b\sin x)} = \frac{A}{a}\ln\tan\frac{x}{2} + \frac{Ba - Ab}{a}\int\frac{dx}{a + b\sin x}$$
(see **2.551** 3) TI (348)

2. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(a+b\cos x)} = \frac{A}{a^2-b^2}\left\{a\ln\tan\frac{x}{2} + b\ln\frac{a+b\cos x}{\sin x}\right\} + B\int\frac{dx}{a+b\cos x}$$ (see **2.553** 3) TI (349)

For $a^2 = b^2\,(=1)$:

3. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(a+b\cos x)} = \frac{A}{2}\left\{\ln\tan\frac{x}{2} + \frac{1}{1+\cos x}\right\} + B\tan\frac{x}{2}$$

4. $$\int \frac{(A+B\sin x)\,dx}{\sin x\,(1-\cos x)} = \frac{A}{2}\left\{\ln\tan\frac{x}{2} - \frac{1}{1-\cos x}\right\} - B\cot\frac{x}{2}$$

5. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(a+b\sin x)} = \frac{1}{a^2-b^2}\left\{(Aa-Bb)\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - (Ab-aB)\ln\frac{a+b\sin x}{\cos x}\right\}$$ TI (346)

For $a^2 = b^2\,(=1)$:

6. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(1\pm\sin x)} = \frac{A\pm B}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \mp \frac{A\mp B}{2\,(1\pm\sin x)}$$

7. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(a+b\cos x)} = \frac{A}{a}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + \frac{B}{a}\ln\frac{a+b\cos x}{\cos x} - \frac{Ab}{a}\int\frac{dx}{a+b\cos x}$$ (see **2.553** 3) TI (351)a

8. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(a+b\sin x)} = \frac{A}{a}\ln\tan\frac{x}{2} - \frac{B}{a}\ln\frac{a+b\sin x}{\sin x} - \frac{Ab}{a}\int\frac{dx}{a+b\sin x}$$ (see **2.551** 3) TI (352)

9. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(a+b\cos x)} = \frac{1}{a^2-b^2}\left\{(Aa-Bb)\ln\tan\frac{x}{2} + (Ab-Ba)\ln\frac{a+b\cos x}{\sin x}\right\}$$ TI (345)

For $a^2 = b^2\,(=1)$:

10. $$\int \frac{(A+B\cos x)\,dx}{\sin x\,(1\pm\cos x)} = \pm\frac{A\mp B}{2\,(1\pm\cos x)} + \frac{A\pm B}{2}\ln\tan\frac{x}{2}$$

11. $$\int \frac{(A+B\cos x)\,dx}{\cos x\,(a+b\sin x)} = \frac{A}{a^2-b^2}\left\{a\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - b\ln\frac{a+b\sin x}{\cos x}\right\} + B\int\frac{dx}{a+b\sin x}$$ (see **2.551** 3) TI (350)

For $a^2 = b^2\,(=1)$:

12. $$\int \frac{(A+B\sin x)\,dx}{\cos x\,(1\pm\sin x)} = \frac{A\pm B}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \mp \frac{A\mp B}{2\,(1\pm\sin x)}$$

13. $$\int \frac{(A+B\cos x)\,dx}{\cos x\,(a+b\cos x)} = \frac{A}{a}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + \frac{Ba-Ab}{a}\int\frac{dx}{a+b\cos x}$$ (see **2.553** 3) TI (347)

2.562

1. $$\int \frac{dx}{a + b\sin^2 x} = \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan\left(\sqrt{\frac{a+b}{a}} \tan x\right) \qquad \left[\frac{b}{a} > -1\right]$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh}\left(\sqrt{-\frac{a+b}{a}} \tan x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2 x < -\frac{a}{b}\right]$$
$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth}\left(\sqrt{-\frac{a+b}{a}} \tan x\right) \qquad \left[\frac{b}{a} < -1, \quad \sin^2 x > -\frac{a}{b}\right]$$
MZ 155

2. $$\int \frac{dx}{a + b\cos^2 x} = \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \arctan\left(\sqrt{\frac{a+b}{a}} \cot x\right) \qquad \left[\frac{b}{a} > -1\right]$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arctanh}\left(\sqrt{-\frac{a+b}{a}} \cot x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b}\right]$$
$$= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{arccoth}\left(\sqrt{-\frac{a+b}{a}} \cot x\right) \qquad \left[\frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b}\right]$$
MZ 162

3. $$\int \frac{dx}{1 + \sin^2 x} = \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2} \tan x\right)$$

4. $$\int \frac{dx}{1 - \sin^2 x} = \tan x$$

5. $$\int \frac{dx}{1 + \cos^2 x} = -\frac{1}{\sqrt{2}} \arctan\left(\sqrt{2} \cot x\right)$$

6. $$\int \frac{dx}{1 - \cos^2 x} = -\cot x$$

2.563

1. $$\int \frac{dx}{\left(a + b\sin^2 x\right)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b\sin^2 x} + \frac{b \sin x \cos x}{a + b\sin^2 x}\right]$$
(see **2.562** 1) MZ 155

2. $$\int \frac{dx}{(a + b\cos^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a + b\cos^2 x} - \frac{b \sin x \cos x}{a + b\cos^2 x}\right]$$
(see **2.562** 2) MZ 163

3. $$\int \frac{dx}{\left(a+b\sin^2 x\right)^3} = \frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\tan x\right) + \left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\tan x}{1+p^2\tan^2 x} + \left(1-\frac{2}{p^2}-\frac{1}{p^2}\tan^2 x\right)\frac{2p\tan x}{\left(1+p^2\tan^2 x\right)^2}\right]$$
$$\left[p^2 = 1+\frac{b}{a} > 0\right]$$
$$= \frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\tan x\right) + \left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\tan x}{1-q^2\tan^2 x} + \left(1+\frac{2}{q^2}+\frac{1}{q^2}\tan^2 x\right)\frac{2q\tan x}{\left(1-q^2\tan^2 x\right)^2}\right]$$
$$\left[q^2 = -1-\frac{b}{a} > 0, \quad \sin^2 x < -\frac{a}{b}; \quad \text{for } \sin^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}\left(q\tan x\right) \text{ to } \operatorname{arccoth}\left(q\tan x\right)\right]$$

MZ 156

4. $$\int \frac{dx}{\left(a+b\cos^2 x\right)^3} = -\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\arctan\left(p\cot x\right) + \left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\cot x}{1+p^2\cot^2 x} + \left(1-\frac{2}{p^2}-\frac{1}{p^2}\cot^2 x\right)\frac{2p\cot x}{\left(1+p^2\cot^2 x\right)^2}\right]$$
$$\left[p^2 = 1+\frac{b}{a} > 0\right]$$
$$= -\frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{arctanh}\left(q\cot x\right) + \left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\cot x}{1-q^2\cot^2 x} + \left(1+\frac{2}{q^2}+\frac{1}{q^2}\cot^2 x\right)\frac{2p\cot x}{\left(1-q^2\cot^2 x\right)^2}\right]$$
$$\left[q^2 = -1-\frac{b}{a} > 0, \quad \cos^2 x < -\frac{a}{b}; \quad \text{for } \cos^2 x > -\frac{a}{b}, \text{ change } \operatorname{arctanh}\left(q\cot x\right) \text{ to } \operatorname{arccoth}\left(q\cot x\right)\right]$$

MZ 163a

2.564

1. $$\int \frac{\tan x\,dx}{1+m^2\tan^2 x} = \frac{\ln\left(\cos^2 x + m^2\sin^2 x\right)}{2\left(m^2-1\right)}$$ LA 210 (10)

2. $$\int \frac{\tan\alpha - \tan x}{\tan\alpha + \tan x}\,dx = \sin 2\alpha \ln\sin(x+\alpha) - x\cos 2\alpha$$ LA 210 (11)a

3. $$\int \frac{\tan x\,dx}{a+b\tan x} = \frac{1}{a^2+b^2}\left\{bx - a\ln\left(a\cos x + b\sin x\right)\right\}$$ PE (335)

4. $$\int \frac{dx}{a+b\tan^2 x} = \frac{1}{a-b}\left[x - \sqrt{\frac{b}{a}}\arctan\left(\sqrt{\frac{b}{a}}\tan x\right)\right]$$ PE (334)

2.57 Integrals containing $\sqrt{a \pm b\sin x}$ or $\sqrt{a \pm b\cos x}$

Notation:

$$\alpha = \arcsin\sqrt{\frac{1-\sin x}{2}}, \qquad \beta = \arcsin\sqrt{\frac{b\,(1-\sin x)}{a+b}},$$

$$\gamma = \arcsin\sqrt{\frac{b\,(1-\cos x)}{a+b}}, \qquad \delta = \arcsin\sqrt{\frac{(a+b)\,(1-\cos x)}{2\,(a-b\cos x)}}, \qquad r = \sqrt{\frac{2b}{a+b}}$$

2.571

1. $$\int \frac{dx}{\sqrt{a+b\sin x}} = \frac{-2}{\sqrt{a+b}}\,F(\alpha, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
$$= -\sqrt{\frac{2}{b}}\,F\left(\beta, \frac{1}{r}\right) \qquad \left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.00, 288.50)

2. $$\int \frac{\sin x\,dx}{\sqrt{a+b\sin x}}$$
$$= \frac{2a}{b\sqrt{a+b}}\,F(\alpha, r) - \frac{2\sqrt{a+b}}{b}\,E(\alpha, r) \qquad \left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$ BY (288.03)
$$= \sqrt{\frac{2}{b}}\left\{F\left(\beta, \frac{1}{r}\right) - 2\,E\left(\beta, \frac{1}{r}\right)\right\} \qquad \left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$ BY (288.54)

3. $$\int \frac{\sin^2 x\,dx}{\sqrt{a+b\sin x}} = \frac{4a\sqrt{a+b}}{3b^2}\,E(\alpha, r) - \frac{2\left(2a^2+b^2\right)}{3b^2\sqrt{a+b}}\,F(\alpha, r) - \frac{2}{3b}\cos x\sqrt{a+b\sin x}$$
$$\left[a > b > 0, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$
$$= \sqrt{\frac{2}{b}}\left\{\frac{4a}{3b}\,E\left(\beta, \frac{1}{r}\right) - \frac{2a+b}{3b}\,F\left(\beta, \frac{1}{r}\right)\right\} - \frac{2}{3b}\cos x\sqrt{a+b\sin x}$$
$$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$
BY (288.03, 288.54)

4. $$\int \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{\sqrt{a+b}}\,F\left(\frac{x}{2}, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$
$$= \sqrt{\frac{2}{b}}\,F\left(\gamma, \frac{1}{r}\right) \qquad \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$
BY (289.00)

5. $$\int \frac{dx}{\sqrt{a-b\cos x}} = \frac{2}{\sqrt{a+b}}\,F(\delta, r) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$ BY (291.00)

6. $$\int \frac{\cos x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{b\sqrt{a+b}}\left\{(a+b)\,E\left(\frac{x}{2},r\right) - a\,F\left(\frac{x}{2},r\right)\right\}$$

$[a>b>0, \quad 0 \le x \le \pi]$ BY (289.03)

$$= \sqrt{\frac{2}{b}}\left\{2\,E\left(\gamma,\frac{1}{r}\right) - F\left(\gamma,\frac{1}{r}\right)\right\}$$

$\left[b > |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$ BY (290.04)

7.[6] $$\int \frac{\cos x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{b\sqrt{a+b}}\left\{(b-a)\,\Pi\left(\delta,r^2,r\right) + a\,F(\delta,r)\right\}$$

$[a>b>0, \quad 0 \le x \le \pi]$ BY (291.03)

8. $$\int \frac{\cos^2 x\,dx}{\sqrt{a+b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\left\{\left(2a^2+b^2\right)F\left(\frac{x}{2},r\right) - 2a(a+b)\,E\left(\frac{x}{2},r\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x}$$

$[a>b>0, \quad 0 \le x \le \pi]$ BY (289.03)

$$= \frac{1}{3b}\sqrt{\frac{2}{b}}\left\{(2a+b)\,F\left(\gamma,\frac{1}{r}\right) - 4a\,E\left(\gamma,\frac{1}{r}\right)\right\} + \frac{2}{3b}\sin x\sqrt{a+b\cos x}$$

$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$ BY (290.04)

9. $$\int \frac{\cos^2 x\,dx}{\sqrt{a-b\cos x}} = \frac{2}{3b^2\sqrt{a+b}}\left\{\left(2a^2+b^2\right)F(\delta,r) - 2a(a+b)\,E(\delta,r)\right\} + \frac{2}{3b}\sin x\frac{a+b\cos x}{\sqrt{a-b\cos x}} \quad [a>b>0,]$$

BY (291.04)a

2.572 $$\int \frac{\tan^2 x\,dx}{\sqrt{a+b\sin x}} = \frac{1}{\sqrt{a+b}}\,F(\alpha,r) + \frac{a}{(a-b)\sqrt{a+b}}\,E(\alpha,r) - \frac{b-a\sin x}{\left(a^2-b^2\right)\cos x}\sqrt{a+b\sin x}$$

$\left[0<b<a, -\frac{\pi}{2} < x < \frac{\pi}{2}\right]$

$$= \sqrt{\frac{2}{b}}\left\{\frac{2a+b}{2(a+b)}\,F\left(\beta,\frac{1}{r}\right) + \frac{ab}{a^2-b^2}\,E\left(\beta,\frac{1}{r}\right)\right\} - \frac{b-a\sin x}{\left(a^2-b^2\right)\cos x}\sqrt{a+b\sin x}$$

$\left[0<|a|<b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$

BY(288.08, 288.58)

2.573

1. $$\int \frac{1-\sin x}{1+\sin x}\cdot\frac{dx}{\sqrt{a+b\sin x}} = \frac{2}{a-b}\left\{\sqrt{a+b}\,E(\alpha,r)\right\} - \left.\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\sqrt{a+b\sin x}\right\}$$

$\left[0<b<a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$ BY (288.07)

2. $$\int \frac{1-\cos x}{1+\cos x} \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{a-b} \tan\frac{x}{2}\sqrt{a+b\cos x} - \frac{2\sqrt{a+b}}{a-b} E\left(\frac{x}{2}, r\right)$$
$[a > b > 0, \quad 0 \le x < \pi]$ BY (289.07)

2.574

1. $$\int \frac{dx}{(2-p^2+p^2\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b}\Pi\left(\alpha, p^2, r\right)$$
$\left[0 < b < a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$
BY (288.02)

2. $$\int \frac{dx}{(a+b-p^2b+p^2b\sin x)\sqrt{a+b\sin x}} = -\frac{1}{a+b}\sqrt{\frac{2}{b}}\,\Pi\left(\beta, p^2, \frac{1}{r}\right)$$
$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$
BY (288.52)

3. $$\int \frac{dx}{(2-p^2+p^2\cos x)\sqrt{a+b\cos x}} = \frac{1}{\sqrt{a+b}}\Pi\left(\frac{x}{2}, p^2, r\right)$$
$[a > b > 0, \quad 0 \le x < \pi]$ BY (289.02)

4. $$\int \frac{dx}{(a+b-p^2b+p^2b\cos x)\sqrt{a+b\cos x}} = \frac{\sqrt{2}}{(a+b)\sqrt{b}}\Pi\left(\gamma, p^2, \frac{1}{r}\right)$$
$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$
BY (290.02)

2.575

1. $$\int \frac{dx}{\sqrt{(a+b\sin x)^3}} = \frac{2b\cos x}{(a^2-b^2)\sqrt{a+b\sin x}} - \frac{2}{(a-b)\sqrt{a+b}} E(\alpha, r)$$
$\left[0 < b < a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$
BY (288.05)

$$= \sqrt{\frac{2}{b}}\left\{\frac{2b}{b^2-a^2} E\left(\beta, \frac{1}{r}\right) - \frac{1}{a+b} F\left(\beta, \frac{1}{r}\right)\right\} + \frac{2b}{b^2-a^2}\cdot\frac{\cos x}{\sqrt{a+b\sin x}}$$
$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$
BY (288.56)

2. $$\int \frac{dx}{\sqrt{(a+b\sin x)^5}} = \frac{2}{3\left(a^2-b^2\right)^2\sqrt{a+b}}\left\{\left(a^2-b^2\right)F(\alpha,r) - 4a(a+b)\,E(\alpha,r)\right\} + \frac{2b\left(5a^2-b^2+4ab\sin x\right)}{3\left(a^2-b^2\right)^2\sqrt{(a+b\sin x)^3}}\cos x$$

$$\left[0 < b < a, \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}\right]$$

BY (288.05)

$$= -\frac{1}{3\left(a^2-b^2\right)^2}\sqrt{\frac{2}{b}}\left\{(3a-b)(a-b)\,F\left(\beta,\frac{1}{r}\right) + 8ab\,E\left(\beta,\frac{1}{r}\right)\right\} + \frac{2b\left[a^2-b^2+4a\left(a+b\sin x\right)\right]}{3\left(a^2-b^2\right)^2\sqrt{(a+b\sin x)^3}}\cos x$$

$$\left[0 < |a| < b, \quad -\arcsin\frac{a}{b} < x < \frac{\pi}{2}\right]$$

BY (288.56)

3. $$\int \frac{dx}{\sqrt{(a+b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E\left(\frac{x}{2},r\right) - \frac{2b}{a^2-b^2}\cdot\frac{\sin x}{\sqrt{a+b\cos x}}$$

$$[a > b > 0, \quad 0 \le x \le \pi]$$

BY (289.05)

$$= \frac{1}{a^2-b^2}\sqrt{\frac{2}{b}}\left\{(a-b)\,F\left(\gamma,\frac{1}{r}\right) + 2b\,E\left(\gamma,\frac{1}{r}\right)\right\} + \frac{2b}{b^2-a^2}\cdot\frac{\sin x}{\sqrt{a+b\cos x}}$$

$$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$

BY (290.06)

4. $$\int \frac{dx}{\sqrt{(a-b\cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}}\,E(\delta,r) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$

(291.01)

5. $$\int \frac{dx}{\sqrt{(a+b\cos x)^5}} = \frac{2\sqrt{a+b}}{3\left(a^2-b^2\right)^2}\left\{4a\,E\left(\frac{x}{2},r\right)-(a-b)\,F\left(\frac{x}{2},r\right)\right\}$$
$$-\frac{2b}{3\left(a^2-b^2\right)^2}\cdot\frac{5a^2-b^2+4ab\cos x}{\sqrt{(a+b\cos x)^3}}\sin x$$
$[a > b > 0, \quad 0 \le x \le \pi]$
BY (289.05)
$$= \frac{1}{3\left(a^2-b^2\right)^2}\sqrt{\frac{2}{b}}\left\{(a-b)(3a-b)\,F\left(\gamma,\frac{1}{r}\right)+8ab\,E\left(\gamma,\frac{1}{r}\right)\right\}$$
$$+\frac{2b\left(5a^2-b^2+4ab\cos x\right)\sin x}{3\left(a^b-b^2\right)^2\sqrt{(a+b\cos x)^3}}$$
$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$
BY (290.06)

2.576

1. $$\int \sqrt{a+b\cos x}\,dx = 2\sqrt{a+b}\,E\left(\frac{x}{2},r\right)$$
$[a > b > 0, \quad 0 \le x \le \pi]$
BY (289.01)
$$= \sqrt{\frac{2}{b}}\left\{(a-b)\,F\left(\gamma,\frac{1}{r}\right)+2b\,E\left(\gamma,\frac{1}{r}\right)\right\}$$
$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$
BY (290.03)

2. $$\int \sqrt{a-b\cos x}\,dx = 2\sqrt{a+b}\,E(\delta,r)-\frac{2b\sin x}{\sqrt{a-b\cos x}}$$ $[a > b > 0, \quad 0 \le x \le \pi]$ BY (291.05)

2.577

1.[3] $$\int \frac{\sqrt{a-b\cos x}}{1+p\cos x}\,dx = \frac{2(a-b)}{(1+p)\sqrt{a+b}}\,\Pi\left(\delta,\frac{2ap}{(a+b)(1+p)},r\right)$$
$[a > b > 0, \quad 0 \le x \le \pi, \quad p \ne -1]$
BY (291.02)

2.[3] $$\int \sqrt{\frac{a-b\cos x}{1+p\cos x}}\,dx = \frac{2(a-b)}{\sqrt{(1+p)(a+b)}}\,\Pi\left(\delta,-r^2,\sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right)$$
$[a > b > 0, \quad 0 \le x \le \pi, \quad p \ne -1]$

2.578 $$\int \frac{\tan x\,dx}{\sqrt{a+b\tan^2 x}} = \frac{1}{\sqrt{b-a}}\arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}}\cos x\right)$$ $[b > a, \quad b > 0]$ PE (333)

2.58–2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

1. $$\int \frac{d\varphi}{\sqrt{a + b\cos\varphi + c\sin\varphi}} = 2\int \frac{d\psi}{\sqrt{a - p + 2p\cos^2\psi}} \qquad \left[\varphi = 2\psi + \alpha, \tan\alpha = \frac{c}{b}, p = \sqrt{b^2 + c^2}\right]$$

2. $$\int \frac{d\varphi}{\sqrt{a + b\cos\varphi + c\sin\varphi + d\cos^2\varphi + e\sin\varphi\cos\varphi + f\sin^2\varphi}} = 2\int \frac{dx}{\sqrt{A + Bx + Cx^2 - Dx^3 + Ex^4}}$$
$$\left[\tan\frac{\varphi}{2} = x, A = a + b + d, B = 2c + 2e, C = 2a - 2d + 4f, D = 2c - 2e, E = a - b + d\right]$$

Forms containing $\sqrt{1 - k^2\sin^2 x}$

Notation: $\Delta = \sqrt{1 - k^2\sin^2 x}$, $k' = \sqrt{1 - k^2}$

2.581

1. $$\int \sin^m x\cos^n x\Delta^r\,dx$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m-3} x\cos^{n+1} x\Delta^{r+2} + \left[(m+n-2) + (m+r-1)k^2\right]\right.$$
$$\left.\times \int \sin^{m-2} x\cos^n x\Delta^r\,dx - (m-3)\int \sin^{m-4} x\cos^n x\Delta^r\,dx\right\}$$
$$= \frac{1}{(m+n+r)k^2}\left\{\sin^{m+1} x\cos^{n-3} x\Delta^{r+2} + \left[(n+r-1)k^2 - (m+n-2){k'}^2\right]\right.$$
$$\left.\times \int \sin^m x\cos^{n-2} x\Delta^r\,dx + (n-3){k'}^2\int \sin^m x\cos^{n-4} x\Delta^r\,dx\right\}$$
$$[m + n + r \neq 0]$$

For $r = -3$ and $r = -5$:

2. $$\int \frac{\sin^m x\cos^n x}{\Delta^3}\,dx = \frac{\sin^{m-1} x\cos^{n-1} x}{k^2\Delta} - \frac{m-1}{k^2}\int \frac{\sin^{m-2} x\cos^n x}{\Delta}\,dx + \frac{n-1}{k^2}\int \frac{\sin^m x\cos^{n-2} x}{\Delta}\,dx$$

3. $$\int \frac{\sin^m x\cos^n x}{\Delta^5}\,dx = \frac{\sin^{m-1} x\cos^{n-1} x}{3k^2\Delta^3} - \frac{m-1}{3k^2}\int \frac{\sin^{m-2} x\cos^n x}{\Delta^3}\,dx + \frac{n-1}{3k^2}\int \frac{\sin^m x\cos^{n-2} x}{\Delta^3}\,dx$$

For $m = 1$ or $n = 1$:

4. $$\int \sin x\cos^n x\Delta^r\,dx = -\frac{\cos^{n-1} x\Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1){k'}^2}{(n+r+1)k^2}\int \cos^{n-2} x\sin x\Delta^r\,dx$$

5. $$\int \sin^m x\cos x\Delta^r\,dx = -\frac{\sin^{m-1} x\Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2}\int \sin^{m-2} x\cos x\Delta^r\,dx$$

For $m = 3$ or $n = 3$:

6. $$\int \sin^3 x \cos^n x \Delta^r \, dx = \frac{(n+r+1)k^2 \cos^2 x - \left[(r+2)k^2 + n + 1\right]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x \Delta^{r+2} - \frac{\left[(r+2)k^2 + n + 1\right](n-1)k'^2}{(n+r+1)(n+r+3)k^4} \int \cos^{n-2} x \sin x \Delta^r \, dx$$

7. $$\int \sin^m x \cos^3 x \Delta^r \, dx = \frac{(m+r+1)k^2 \sin^2 x - \left[(r+2)k^2 - (m+1)k'^2\right]}{(m+r+1)(m+r+3)k^4} \times \sin^{m-1} x n^{m-1} x \Delta^{r+2} + \frac{\left[(r+2)k^2 - (m-1)k'^2\right](m-1)}{(m+r+1)(m+r+3)k^4} \int \sin^{m-2} x \cos x \Delta^r \, dx$$

2.582

1. $$\int \Delta^n \, dx = \frac{n-1}{n}\left(2-k^2\right) \int \Delta^{n-2} \, dx - \frac{n-2}{n}\left(1-k^2\right) \int \Delta^{n-4} \, dx + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2}$$

LA (316)(1)a

2. $$\int \frac{dx}{\Delta^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1)k'^2 \Delta^{n-1}} + \frac{n-2}{n-1}\frac{2-k^2}{k'^2} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1}\frac{1}{k'^2} \int \frac{dx}{\Delta^{n-3}}$$ LA 317(8)a

3. $$\int \frac{\sin^n x}{\Delta} \, dx = \frac{\sin^{n-3} x}{(n-1)k^2} \cos x \cdot \Delta + \frac{n-2}{n-1}\frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} \, dx - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} \, dx$$

LA 316(1)a

4. $$\int \frac{\cos^n x}{\Delta} \, dx = \frac{\cos^{n-3} x}{(n-1)k^2} \sin x \cdot \Delta + \frac{n-2}{n-1}\frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} \, dx + \frac{n-3}{n-1}\frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} \, dx$$

LA 316(2)a

5. $$\int \frac{\tan^n x}{\Delta} \, dx = \frac{\tan^{n-3} x}{(n-1)k'^2}\frac{\Delta}{\cos^2 x} - \frac{(n-2)\left(2-k^2\right)}{(n-1)k'^2} \int \frac{\tan^{n-2} x}{\Delta} \, dx - \frac{n-3}{(n-1)k'^2} \int \frac{\tan^{n-4} x}{\Delta} \, dx$$

LA 317(3)

6. $$\int \frac{\cot^n x}{\Delta} \, dx = -\frac{\cot^{n-1} x}{n-1}\frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1}\left(2-k^2\right) \int \frac{\cot^{n-2} x}{\Delta} \, dx - \frac{n-3}{n-1} k'^2 \int \frac{\cot^{n-4} x}{\Delta} \, dx$$

LA 317(6)

2.583

1. $$\int \Delta \, dx = E(x, k)$$

2. $$\int \Delta \sin x\,dx = -\frac{\Delta\cos x}{2} - \frac{k'^2}{2k}\ln\left(k\cos x + \Delta\right)$$

3. $$\int \Delta \cos x\,dx = \frac{\Delta\sin x}{2} + \frac{1}{2k}\arcsin\left(k\sin x\right)$$

4. $$\int \Delta \sin^2 x\,dx = -\frac{\Delta}{3}\sin x\cos x + \frac{k'^2}{3k^2}\,F(x,k) + \frac{2k^2-1}{3k^2}\,E(x,k)$$

5. $$\int \Delta \sin x\cos x\,dx = -\frac{\Delta^3}{3k^2}$$

6. $$\int \Delta \cos^2 x\,dx = \frac{\Delta}{3}\sin x\cos x - \frac{k'^2}{3k^2}\,F(x,k) + \frac{k^2+1}{3k^2}\,E(x,k)$$

7. $$\int \Delta \sin^3 x\,dx = -\frac{2k^2\sin^2 x + 3k^2 - 1}{8k^2}\Delta\cos x + \frac{3k^4-2k^2-1}{8k^3}\ln\left(k\cos x + \Delta\right)$$

8. $$\int \Delta \sin^2 x\cos x\,dx = \frac{2k^2\sin^2 x - 1}{8k^2}\Delta\sin x + \frac{1}{8k^3}\arcsin\left(k\sin x\right)$$

9. $$\int \Delta \sin x\cos^2 x\,dx = -\frac{2k^2\cos^2 x + k'^2}{8k^2}\Delta\cos x + \frac{k'^4}{8k^3}\ln\left(k\cos x + \Delta\right)$$

10. $$\int \Delta \cos^3 x\,dx = \frac{2k^2\cos^2 x + 2k^2 + 1}{8k^2}\Delta\sin x + \frac{4k^2-1}{8k^3}\arcsin\left(k\sin x\right)$$

11. $$\int \Delta \sin^4 x\,dx = -\frac{3k^2\sin^2 x + 4k^2 - 1}{15k^2}\Delta\sin x\cos x - \frac{2\left(2k^4-k^2-1\right)}{15k^4}\,F(x,k) + \frac{8k^4-3k^2-2}{15k^4}\,E(x,k)$$

12. $$\int \Delta \sin^3 x\cos x\,dx = \frac{3k^4\sin^4 x - k^2\sin^2 x - 2}{15k^4}\Delta$$

13. $$\int \Delta \sin^2 x\cos^2 x\,dx = -\frac{3k^2\cos^2 x - 2k^2 + 1}{15k^2}\Delta\sin x\cos x - \frac{k'^2\left(1+k'^2\right)}{15k^4}\,F(x,k) + \frac{2\left(k^4-k^2+1\right)}{15k^4}\,E(x,k)$$

14. $$\int \Delta \sin x\cos^3 x\,dx = -\frac{3k^4\sin^4 x - k^2\left(5k^2+1\right)\sin^2 x + 5k^2 - 2}{15k^4}\Delta$$

15. $$\int \Delta \cos^4 x\,dx = \frac{3k^2\cos^2 x + 3k^2 + 1}{15k^2}\Delta\sin x\cos x + \frac{2k'^2\left(k'^2-2k^2\right)}{15k^4}\,F(x,k) + \frac{3k^4+7k^2-2}{15k^4}\,E(x,k)$$

16. $$\int \Delta \sin^5 x\,dx = \frac{-8k^4\sin^4 x - 2k^2\left(5k^2-1\right)\sin^2 x - 15k^4 + 4k^2 + 3}{48k^4}\Delta\cos x + \frac{5k^6-3k^4-k^2-1}{16k^5}\ln\left(k\cos x + \Delta\right)$$

17. $$\int \Delta \sin^4 x\cos x\,dx = \frac{8k^4\sin^4 x - 2k^2\sin^2 x - 3}{48k^4}\Delta\sin x + \frac{1}{16k^5}\arcsin\left(k\sin x\right)$$

18. $$\int \Delta \sin^3 x \cos^2 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 \left(k^2+1\right) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x + \frac{k'^4 \left(k^2+1\right)}{16k^5} \ln \left(k \cos x + \Delta\right)$$

19. $$\int \Delta \sin^2 x \cos^3 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 \left(6k^2+1\right) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x + \frac{2k^2-1}{16k^5} \arcsin \left(k \sin x\right)$$

20. $$\int \Delta \sin x \cos^4 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2 \left(7k^2+1\right) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x - \frac{k'^6}{16k^5} \ln \left(k \cos x + \Delta\right)$$

21. $$\int \Delta \cos^5 x \, dx = \frac{8k^4 \sin^4 x - 2k^2 \left(12k^2+1\right) \sin^2 x + 24k^4 + 12k^2 - 3}{48k^4} \Delta \sin x + \frac{8k^4 - 4k^2 + 1}{16k^5} \arcsin \left(k \sin x\right)$$

22. $$\int \Delta^3 \, dx = \frac{2}{3} \left(1 + k'^2\right) E(x,k) - \frac{k'^2}{3} F(x,F) + \frac{k^2}{3} \Delta \sin x \cos x$$

23. $$\int \Delta^3 \sin x \, dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3k'^4}{8k} \ln \left(k \cos x + \Delta\right)$$

24. $$\int \Delta^3 \cos x \, dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin \left(k \sin x\right)$$

25. $$\int \Delta^3 \sin^2 x \, dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x + \frac{k'^2 \left(3 - 4k^2\right)}{15k^2} F(x,k) - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x,k)$$

26. $$\int \Delta^3 \sin x \cos x \, dx = -\frac{\Delta^5}{5k^2}$$

27. $$\int \Delta^3 \cos^2 x \, dx = \frac{-3k^2 \sin^2 x + k^2 + 5}{15} \Delta \sin x \cos x - \frac{k'^2 \left(k^2 + 3\right)}{15k^2} F(x,k) - \frac{2k^4 - 7k^2 - 3}{15k^2} E(x,k)$$

28. $$\int \Delta^3 \sin^3 x \, dx = \frac{8k^4 \sin^4 x + 2k^2 \left(5k^2 - 7\right) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln \left(k \cos x + \Delta\right)$$

29. $$\int \Delta^3 \sin^2 x \cos x \, dx = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + \frac{1}{16k^3} \arcsin \left(k \sin x\right)$$

30. $$\int \Delta^3 \sin x \cos^2 x\,dx = \frac{-8k^4 \sin^4 x + 2k^2\left(k^2+7\right)\sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \times \Delta \cos x + \frac{k'^6}{16k^3} \ln\left(k\cos x + \Delta\right)$$

31. $$\int \Delta^3 \cos^3 x\,dx = \frac{8k^4 \sin^4 x - 2k^2\left(6k^2+7\right)\sin^2 x + 30k^2 + 3}{48k^2}\Delta \sin x + \frac{6k^2-1}{16k^3}\arcsin\left(k\sin x\right)$$

32. $$\int \frac{\Delta\,dx}{\sin x} = -\frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + k\ln k\left(k\cos x + \Delta\right)$$

33. $$\int \frac{\Delta\,dx}{\cos x} = \frac{k'}{2}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x} + k\arcsin\left(k\sin x\right)$$

34. $$\int \frac{\Delta\,dx}{\sin^2 x} = k'^2 F(x,k) - E(x,k) - \Delta\cot x$$

35. $$\int \frac{\Delta\,dx}{\sin x\cos x} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} + \frac{k'}{2}\ln\frac{\Delta + k'}{\Delta - k'}$$

36. $$\int \frac{\Delta\,dx}{\cos^2 x} = F(x,k) - E(x,k) + \Delta\tan x$$

37. $$\int \frac{\sin x}{\cos x}\Delta\,dx = \int \Delta\tan x\,dx = -\Delta + \frac{k'}{2}\ln\frac{\Delta + k'}{\Delta - k'}$$

38. $$\int \frac{\cos x}{\sin x}\Delta\,dx = \int \Delta\cot x\,dx = \Delta + \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}$$

39. $$\int \frac{\Delta\,dx}{\sin^3 x} = -\frac{\Delta\cos x}{2\sin^2 x} + \frac{k'^2}{4}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

40. $$\int \frac{\Delta\,dx}{\sin^2 x\cos x} = \frac{-\Delta}{\sin x} - \frac{1+k^2}{2k'}\ln\frac{\Delta - k'\sin x}{\Delta + k'\sin x}$$

41. $$\int \frac{\Delta\,dx}{\sin x\cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

42. $$\int \frac{\Delta\,dx}{\cos^3 x} = \frac{\Delta\sin x}{2\cos^2 x} + \frac{1}{4k'}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

43. $$\int \frac{\Delta\sin x\,dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k\ln\left(k\cos x + \Delta\right)$$

44. $$\int \frac{\Delta\cos x\,dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k\arcsin\left(k\sin x\right)$$

45. $$\int \frac{\Delta\sin^2 x\,dx}{\cos x} = -\frac{\Delta\sin x}{2} + \frac{2k^2-1}{2k}\arcsin\left(k\sin x\right) + \frac{k'}{2}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

46. $$\int \frac{\Delta\cos^2 x\,dx}{\sin x} = \frac{\Delta\cos x}{2} + \frac{k^2+1}{2k}\ln\left(k\cos x + \Delta\right) + \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

47. $$\int \frac{\Delta\,dx}{\sin^4 x} = \frac{1}{3}\left\{-\Delta\cot^3 x + \left(k^2-3\right)\Delta\cot x + 2k'^2 F(x,k) + \left(k^2-2\right)E(x,k)\right\}$$

48. $$\int \frac{\Delta\, dx}{\sin^3 x \cos x} = -\frac{\Delta}{2\sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

49. $$\int \frac{\Delta\, dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{k'^2} \tan x - \cot x \right) \Delta + 2\, F(x, k) - \frac{1 + k'^2}{k'^2} E(x, k)$$

50. $$\int \frac{\Delta\, dx}{\sin x \cos^3 x} = \frac{\Delta}{2\cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

51. $$\int \frac{\Delta\, dx}{\cos^4 x} = \frac{1}{3k'^2} \left\{ \left[k'^2 \tan^2 x - \left(2k^2 - 3\right) \tan x \right] \Delta + 2k'^2\, F(x, k) + \left(k^2 - 2\right) E(x, k) \right\}$$

52. $$\int \frac{\sin x}{\cos^3 x} \Delta\, dx = \frac{\Delta}{2\cos^2 x} + \frac{k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

53. $$\int \frac{\cos x}{\sin^3 x} \Delta\, dx = -\frac{\Delta}{2\sin^2 x} + \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

54. $$\int \frac{\sin^2 x}{\cos^2 x} \Delta\, dx = \int \tan^2 x \Delta\, dx = \Delta \tan x + F(x, k) - 2\, E(x, k)$$

55. $$\int \frac{\cos^2 x}{\sin^2 x} \Delta\, dx = \int \cot^2 x \Delta\, dx = -\Delta \cot x + k'^2\, F(x, k) - 2\, E(x, k)$$

56. $$\int \frac{\sin^3 x}{\cos x} \Delta\, dx = -\frac{k^2 \sin^2 x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}$$

57. $$\int \frac{\cos^3 x}{\sin x} \Delta\, dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}$$

58. $$\int \frac{\Delta\, dx}{\sin^5 x} = \frac{\left(k^2 - 3\right) \sin^2 x + 2}{8\sin^4 x} \cos x \Delta + \frac{k'^2 \left(k^2 + 3\right)}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

59. $$\int \frac{\Delta\, dx}{\sin^4 x \cos x} = -\frac{\left(3 - k^2\right) \sin^2 x + 1}{3\sin^3 x} \Delta - \frac{k'}{2} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

60. $$\int \frac{\Delta\, dx}{\sin^3 x \cos^2 x} = \frac{3\sin^2 x - 1}{2\sin^2 x \cos x} \Delta + \frac{k^2 - 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

61. $$\int \frac{\Delta\, dx}{\sin^2 x \cos^3 x} = \frac{3\sin^2 x - 2}{2\sin x \cos^2 x} \Delta - \frac{2k^2 - 3}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

62. $$\int \frac{\Delta\, dx}{\sin x \cos^4 x} = \frac{\left(2k^2 - 3\right) \sin^2 x - 3k^2 + 4}{3k'^2 \cos^3 x} \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

63. $$\int \frac{\Delta\, dx}{\cos^5 x} = \frac{\left(2k^2 - 3\right) \sin^2 x - 4k^2 + 5}{8k'^2 \cos^4 x} \sin x \Delta - \frac{4k^2 - 3}{16k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

64. $$\int \frac{\sin x}{\cos^4 x} \Delta\, dx = \frac{-\left(2k^2 + 1\right) k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta$$

65. $$\int \frac{\cos x}{\sin^4 x} \Delta\, dx = -\frac{\Delta^3}{3\sin^3 x}$$

66. $$\int \frac{\sin^2 x}{\cos^3 x} \Delta\, dx = \frac{\sin x}{2\cos^2 x} \Delta + \frac{2k^2 - 1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin\left(k \sin x\right)$$

67. $$\int \frac{\cos^2 x}{\sin^3 x} \Delta\, dx = -\frac{\cos x}{2\sin^2 x} \Delta - \frac{k^2+1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln (k \cos x + \Delta)$$

68. $$\int \frac{\sin^3 x}{\cos^2 x} \Delta\, dx = -\frac{\sin^2 x - 3}{2\cos x} \Delta - \frac{3k^2-1}{2k} \ln (k \cos x + \Delta)$$

69. $$\int \frac{\cos^3 x}{\sin^2 x} \Delta\, dx = -\frac{\sin^2 x + 2}{2\sin x} \Delta - \frac{2k^2+1}{2k} \arcsin (k \sin x)$$

70. $$\int \frac{\sin^4 x}{\cos x} \Delta\, dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \Delta \sin x + \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin (k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}$$

71. $$\int \frac{\cos^4 x}{\sin x} \Delta\, dx = \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \Delta \cos x + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln (k \cos x + \Delta)$$

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1. $$\int \frac{dx}{\Delta} = F(x, k)$$

2. $$\int \frac{\sin x\, dx}{\Delta} = \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x} = -\frac{1}{k} \ln (k \cos x + \Delta)$$

3. $$\int \frac{\cos x\, dx}{\Delta} = \frac{1}{k} \arcsin (k \sin x) = \frac{1}{k} \arctan \frac{k \sin x}{\Delta}$$

4. $$\int \frac{\sin^2 x\, dx}{\Delta} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k)$$

5. $$\int \frac{\sin x \cos x\, dx}{\Delta} = -\frac{\Delta}{k^2}$$

6. $$\int \frac{\cos^2 x\, dx}{\Delta} = \frac{1}{k^2} E(x, k) - \frac{k'^2}{k^2} F(x, k)$$

7. $$\int \frac{\sin^3 x\, dx}{\Delta} = \frac{\cos x \Delta}{2k^2} - \frac{1+k^2}{2k^3} \ln (k \cos x + \Delta)$$

8. $$\int \frac{\sin^2 x \cos x\, dx}{\Delta} = -\frac{\sin x \Delta}{2k^2} + \frac{\arcsin (k \sin x)}{2k^3}$$

9. $$\int \frac{\sin x \cos^2 x\, dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{k'^2}{2k^3} \ln (k \cos x + \Delta)$$

10. $$\int \frac{\cos^3 x\, dx}{\Delta} = \frac{\sin x \Delta}{2k^2} + \frac{2k^2 - 1}{2k^3} \arcsin (k \sin x)$$

11. $$\int \frac{\sin^4 x\, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{2+k^2}{3k^4} F(x, k) - \frac{2\left(1+k^2\right)}{3k^4} E(x, k)$$

12. $$\int \frac{\sin^3 x \cos x\, dx}{\Delta} = -\frac{1}{3k^4} \left(2 + k^2 \sin^2 x\right) \Delta$$

13. $$\int \frac{\sin^2 x \cos^2 x \, dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2-k^2}{3k^4} E(x,k) + \frac{2k^2-2}{3k^4} F(x,k)$$

14. $$\int \frac{\sin x \cos^3 x \, dx}{\Delta} = -\frac{1}{3k^4} \left(k^2 \cos^2 x - 2k'^2\right) \Delta$$

15. $$\int \frac{\cos^4 x \, dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2-2}{3k^4} E(x,k) + \frac{3k^4-5k^2+2}{3k^4} F(x,k)$$

16. $$\int \frac{\sin^5 x \, dx}{\Delta} = \frac{2k^2 \sin^2 x + 3k^2 + 3}{8k^4} \cos x \Delta - \frac{3+2k^2+3k^4}{8k^5} \ln\left(k \cos x + \Delta\right)$$

17. $$\int \frac{\sin^4 x \cos x \, dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \sin x \Delta + \frac{3}{8k^5} \arcsin\left(k \sin x\right)$$

18. $$\int \frac{\sin^3 x \cos x \, dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \cos x \Delta - \frac{k^4+2k^2-3}{8k^5} \ln\left(k \cos x + \Delta\right)$$

19. $$\int \frac{\sin^2 x \cos^3 x \, dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \sin x \Delta + \frac{4k^2-3}{8k^5} \arcsin\left(k \sin x\right)$$

20. $$\int \frac{\sin x \cos^4 x \, dx}{\Delta} = \frac{3 - 5k^2 + 2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4-6k^2+3}{8k^5} \ln\left(k \cos x + \Delta\right)$$

21. $$\int \frac{\cos^5 x \, dx}{\Delta} = \frac{2k^2 \cos^2 x + 6k^2 - 3}{8k^4} \sin x \Delta + \frac{8k^4-8k^2+3}{8k^5} \arcsin\left(k \sin x\right)$$

22. $$\int \frac{\sin^6 x \, dx}{\Delta} = \frac{3k^2 \sin^2 x + 4k^2 + 4}{15k^4} \sin x \cos x \Delta + \frac{4k^4+3k^2+8}{15k^6} F(x,k) - \frac{8k^4+7k^2+8}{15k^6} E(x,k)$$

23. $$\int \frac{\sin^5 x \cos x \, dx}{\Delta} = -\frac{3k^4 \sin^4 x + 4k^2 \sin^2 x + 8}{15k^6} \Delta$$

24. $$\int \frac{\sin^4 x \cos x \, dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{k^4+7k^2-8}{15k^6} F(x,k) - \frac{2k^4+3k^2-8}{15k^6} E(x,k)$$

25. $$\int \frac{\sin^3 x \cos^3 x \, dx}{\Delta} = \frac{3k^4 \sin^4 x - \left(5k^4 - 4k^2\right) \sin^2 x - 10k^2 + 8}{15k^6} \Delta$$

26. $$\int \frac{\sin^2 x \cos^4 x \, dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{9k^4-17k^2+8}{15k^6} F(x,k) - \frac{3k^4-13k^2+8}{15k^6} E(x,k)$$

27. $$\int \frac{\sin x \cos^5 x \, dx}{\Delta} = \frac{-3k^4 \cos^4 x + 4k^2 k'^2 \cos^2 x - 8k^4 + 16k^2 - 8}{15k^6} \Delta$$

28. $$\int \frac{\cos^6 x \, dx}{\Delta} = \frac{3k^2 \cos^2 x + 8k^2 - 4}{15k^4} \sin x \cos x \Delta + \frac{15k^6 - 34k^4 + 27k^2 - 8}{15k^6} F(x,k) + \frac{23k^4 - 23k^2 + 8}{15k^6} E(x,k)$$

29. $$\int \frac{\sin^7 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x + 10k^2\left(k^2+1\right)\sin^2 x + 15k^4 + 14k^2 + 15}{48k^6} \cos x\Delta - \frac{\left(5k^4 - 2k^2 + 5\right)\left(k^2+1\right)}{16k^7} \ln\left(k\cos x + \Delta\right)$$

30. $$\int \frac{\sin^6 x \cos x\,dx}{\Delta} = -\frac{8k^4 \sin^4 x + 10k^2 \sin^2 x + 15}{48k^6} \sin x\Delta + \frac{5}{16k^7} \arcsin\left(k\sin x\right)$$

31. $$\int \frac{\sin^5 x \cos^2 x\,dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2\left(k^2-5\right)\sin^2 x + 3k^4 + 4k^2 - 15}{48k^6} \cos x\Delta - \frac{k^6 + k^4 + 3k^2 - 5}{16k^7} \ln\left(k\cos x + \Delta\right)$$

32. $$\int \frac{\sin^4 x \cos^3 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2\left(6k^2-5\right)\sin^2 x - 18k^2 + 15}{48k^6} \sin x\Delta + \frac{6k^2 - 5}{16k^7} \arcsin\left(k\sin x\right)$$

33. $$\int \frac{\sin^3 x \cos^4 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2\left(6k^2-5\right)\sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \cos x\Delta - \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln\left(k\cos x + \Delta\right)$$

34. $$\int \frac{\sin^2 x \cos^5 x\,dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2\left(12k^2-5\right)\sin^2 x - 24k^4 + 36k^2 - 15}{48k^6} \sin x\Delta + \frac{8k^4 - 12k^2 + 5}{16k^7} \arcsin\left(k\sin x\right)$$

35. $$\int \frac{\sin x \cos^6 x\,dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2\left(13k^2-5\right)\sin^2 x - 33k^4 + 40k^2 - 15}{48k^6} \cos x\Delta + \frac{5k'^6}{16k^7} \ln\left(k\cos x + \Delta\right)$$

36. $$\int \frac{\cos^7 x\,dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2\left(18k^2-5\right)\sin^2 x + 72k^4 - 54k^2 + 15}{48k^6} \sin x\Delta + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7} \arcsin\left(k\sin x\right)$$

37. $$\int \frac{dx}{\Delta^3} = \frac{1}{k'^2} E(x,k) - \frac{k^2}{k'^2} \frac{\sin x \cos x}{\Delta}$$

38. $$\int \frac{\sin x\,dx}{\Delta^3} = -\frac{\cos x}{k'^2 \Delta}$$

39. $$\int \frac{\cos x\,dx}{\Delta^3} = \frac{\sin x}{\Delta}$$

40.[11] $$\int \frac{\sin^2 x\,dx}{\Delta^3} = \frac{1}{k'^2 k^2} E(x,k) - \frac{1}{k^2} F(x,k) - \frac{1}{k'^2} \frac{\sin x \cos x}{\Delta}$$

41. $$\int \frac{\sin x \cos x\,dx}{\Delta^3} = \frac{1}{k^2 \Delta}$$

42. $$\int \frac{\cos^2 x\,dx}{\Delta^3} = \frac{1}{k^2} F(x,k) - \frac{1}{k^2} E(x,k) + \frac{\sin x \cos x}{\Delta}$$

43. $$\int \frac{\sin^3 x\,dx}{\Delta^3} = -\frac{\cos x}{k^2 k'^2 \Delta} + \frac{1}{k^3} \ln(k\cos x + \Delta)$$

44. $$\int \frac{\sin^2 x \cos x\,dx}{\Delta^3} = \frac{\sin x}{k^2 \Delta} - \frac{1}{k^3} \arcsin(k\sin x)$$

45. $$\int \frac{\sin x \cos^2 x\,dx}{\Delta^3} = \frac{\cos x}{k^2 \Delta} - \frac{1}{k^3} \ln(k\cos x + \Delta)$$

46. $$\int \frac{\cos^3 x\,dx}{\Delta^3} = -\frac{k'^2 \sin x}{k^2 \Delta} + \frac{1}{k^3} \arcsin(k\sin x)$$

47. $$\int \frac{\sin^4 x\,dx}{\Delta^3} = \frac{k'^2 + 1}{k'^2 k^4} E(x,k) - \frac{2}{k^4} F(x,k) - \frac{\sin x \cos x}{k^2 k'^2 \Delta}$$

48. $$\int \frac{\sin^3 x \cos x\,dx}{\Delta^3} = \frac{2 - k^2 \sin^2 x}{k^4 \delta}$$

49. $$\int \frac{\sin^2 x \cos^2 x\,dx}{\Delta^3} = \frac{2 - k^2}{k^4} F(x,k) - \frac{2}{k^4} E(x,k) + \frac{\sin x \cos x}{k^2 \Delta}$$

50. $$\int \frac{\sin x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta}$$

51. $$\int \frac{\cos^4 x\,dx}{\Delta^3} = \frac{k'^2 + 1}{k^4} E(x,k) - \frac{2k'^2}{k^4} F(x,k) - \frac{k'^2 \sin x \cos x}{k^2 \Delta}$$

52.[9] $$\int \frac{\sin^5 x\,dx}{\Delta^3} = \frac{k^2 k'^2 \sin^2 x + k^2 - 3}{2k^4 k'^2 \Delta} \cos x + \frac{k^2 + 3}{2k^5} \ln(k\cos x + \Delta)$$

53. $$\int \frac{\sin^4 x \cos x\,dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k\sin x)$$

54. $$\int \frac{\sin^3 x \cos^2 x\,dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k\cos x + \Delta)$$

55. $$\int \frac{\sin^2 x \cos^3 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k\sin x)$$

56. $$\int \frac{\sin x \cos^4 x\,dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3k'^2}{2k^5} \ln(k\cos x + \Delta)$$

57. $$\int \frac{\cos^5 x\,dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4 \Delta} \sin x + \frac{4k^2 - 3}{2k^5} \arcsin(k\sin x)$$

58. $$\int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2\left(k'^2 + 1\right) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2} F(x,k) + \frac{2\left(k'^2 + 1\right)}{3k'^4} E(x,k)$$

59. $$\int \frac{\sin x\,dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x$$

60. $$\int \frac{\cos x\,dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x$$

61. $$\int \frac{\sin^2 x\, dx}{\Delta^5} = \frac{k^2+1}{3k'^4k^2} E(x,k) - \frac{1}{3k'^2k^2} F(x,k) + \frac{k^2\left(k^2+1\right)\sin^2 x - 2}{3k'^4\Delta^3} \sin x \cos x$$

62. $$\int \frac{\sin x \cos x\, dx}{\Delta^5} = \frac{1}{3k^2\Delta^3}$$

63. $$\int \frac{\cos^2 x\, dx}{\Delta^5} = \frac{1}{3k^2} F(x,k) + \frac{2k^2-1}{3k^2k'^2} E(x,k) + \frac{k^2\left(2k^2-1\right)\sin^2 x - 3k^2 + 2}{2k'^2\Delta} \sin x \cos x$$

64. $$\int \frac{\sin^3 x}{\Delta^5}\, dx = \frac{\left(3k^2-1\right)\sin^2 x - 2}{3k'^4\Delta^3} \cos x$$

65. $$\int \frac{\sin^2 x \cos x}{\Delta^5}\, dx = \frac{\sin^3 x}{3\Delta^3}$$

66. $$\int \frac{\sin x \cos^2 x}{\Delta^5}\, dx = -\frac{\cos^3 x}{3k'^2\Delta^3}$$

67. $$\int \frac{\cos^3 x\, dx}{\Delta^5} = \frac{-\left(2k^2+1\right)\sin^2 x + 3}{3\Delta^3} \sin x$$

68. $$\int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

69. $$\int \frac{dx}{\Delta \cos x} = -\frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

70. $$\int \frac{dx}{\Delta \sin^2 x} = \int \frac{1 + \cot^2 x}{\Delta}\, dx = F(x,k) - E(x,k) - \Delta \cot x$$

71. $$\int \frac{dx}{\Delta \sin x \cos x} = \int (\tan x + \cot x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

72. $$\int \frac{dx}{\Delta \cos^2 x} = \int \left(1 + \tan^2 x\right) \frac{dx}{\Delta} = F(x,k) - \frac{1}{k'^2} E(x,k) + \frac{1}{k'^2} \Delta \tan x$$

73. $$\int \frac{\sin x}{\cos x} \frac{dx}{\Delta} = \int \tan x \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

74. $$\int \frac{\cos x}{\sin x} \frac{dx}{\Delta} = \int \cot x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}$$

75. $$\int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1+k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

76. $$\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

77. $$\int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}$$

78. $$\int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} + \frac{2k^2-1}{4k'^3} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}$$

79. $$\int \frac{\sin x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x}$$

80. $$\int \frac{\cos x}{\sin^2 x} \frac{dx}{\Delta} = -\frac{\Delta}{\sin x}$$

81. $$\int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{1}{k} \arcsin (k \sin x)$$

82. $$\int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln (k \cos x + \Delta)$$

83. $$\int \frac{dx}{\Delta \sin^4 x} = \frac{1}{3} \left\{-\Delta \cot^3 x - \Delta \left(2k^2 + 3\right) \cot x + \left(k^2 + 2\right) F(x, k) - 2 \left(k^2 + 1\right) E(x, k)\right\}$$

84. $$\int \frac{dx}{\Delta \sin^3 x \cos x} = \int \left(\tan x + 2 \cot x + \cot^3 x\right) \frac{dx}{\Delta}$$
$$= -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

85. $$\int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \int \left(\tan^2 x + 2 + \cot^2 x\right) \frac{dx}{\Delta}$$
$$= \left(\frac{\tan x}{k'^2} - \cot x\right) \Delta + \frac{k^2 - 2}{k'^2} E(x, k) + 2 F(x, k)$$

86. $$\int \frac{dx}{\Delta \sin x \cos^3 x} = \int \left(\cot x + 2 \tan x + \tan^3 x\right) \frac{dx}{\Delta}$$
$$= -\frac{\Delta}{2k'^2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - 3k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$$

87. $$\int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3k'^2} \left\{\Delta \tan^3 x - \frac{5k^2 - 3}{k'^2} \Delta \tan x - \left(3k^2 - 2\right) F(x, k) + \frac{2\left(2k^2 - 1\right)}{k'^2} E(x, k)\right\}$$

88. $$\int \frac{\sin x}{\cos^3 x} \frac{dx}{\Delta} = \int \tan x \left(1 + \tan^2 x\right) \frac{dx}{\Delta} = \frac{\Delta}{2k'^2 \cos^2 x} - \frac{k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}$$

89. $$\int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}$$

90. $$\int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\tan^2 x}{\Delta} dx = \frac{\Delta}{k'^2} \tan x - \frac{1}{k'^2} E(x, k)$$

91. $$\int \frac{\cos^2 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\cot^2 x}{\Delta} dx = -\Delta \cot x - E(x, k)$$

92. $$\int \frac{\sin^3 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}$$

93. $$\int \frac{\cos^3 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta}$$

94. $$\int \frac{dx}{\Delta \sin^5 x} = -\frac{\left[3\left(1 + k^2\right) \sin^2 x + 2\right]}{8 \sin^2 x} \Delta \cos x + \frac{3k^4 + 2k^2 + 3}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}$$

95. $$\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{\left(3+2k^2\right)\sin^2 x + 1}{3\sin^3 x}\Delta - \frac{1}{2k'}\ln\frac{\Delta - k'\sin x}{\Delta + k'\sin x}$$

96. $$\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{\left(3-k^2\right)\sin^2 x - k'^2}{2k'^2\sin^2 x\cos x}\Delta + \frac{k^2+3}{4}\ln\frac{\Delta - \cos x}{\Delta + \cos x}$$

97. $$\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{\left(3-2k^2\right)\sin^2 x - 2k'^2}{2k'^2\sin x\cos^2 x}\Delta - \frac{4k^2-3}{4k'^3}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

98. $$\int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{\left(5k^2-3\right)\sin^2 x - 6k^2 + 4}{3k'^4\cos^3 x}\Delta - \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

99. $$\int \frac{dx}{\Delta \cos^5 x} = \frac{3\left(2k^2-1\right)\sin^2 x - 8k^2 + 5}{8k'^4\cos^4 x}\Delta\sin x + \frac{8k^4-8k^2+3}{16k'^5}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

100. $$\int \frac{\sin x}{\cos^4 x}\frac{dx}{\Delta} = -\frac{2k^2\cos^2 x - k'^2}{2k'^4\cos^3 x}\Delta$$

101. $$\int \frac{\cos x}{\sin^4 x}\frac{dx}{\Delta} = -\frac{2k^2\sin^2 x + 1}{3\sin^3 x}\Delta$$

102. $$\int \frac{\sin^2 x}{\cos^3 x}\frac{dx}{\Delta} = \frac{\Delta\sin x}{2k'^2\cos^2 x} - \frac{1}{4k'^3}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x}$$

103. $$\int \frac{\cos^3 x}{\sin^3 x}\frac{dx}{\Delta} = -\frac{\Delta\cos x}{2\sin^2 x} + \frac{k'^2}{4}\ln\frac{\Delta + \cos x}{\Delta - \cos x}$$

104. $$\int \frac{\sin^3 x}{\cos^2 x}\frac{dx}{\Delta} = \frac{\Delta}{k'^2\cos x} + \frac{1}{k}\ln\left(k\cos x + \Delta\right)$$

105. $$\int \frac{\cos^3 x}{\sin^2 x}\frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k}\arcsin\left(k\sin x\right)$$

106. $$\int \frac{\sin^4 x}{\cos x}\frac{dx}{\Delta} = \frac{\Delta\sin x}{2k^2} + \frac{1}{2k'}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x} - \frac{2k^2+1}{2k^3}\arcsin\left(k\sin x\right)$$

107. $$\int \frac{\cos^4 x}{\sin x}\frac{dx}{\Delta} = \frac{\Delta\cos x}{2k^2} + \frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2-1}{2k^3}\ln\left(k\cos x + \Delta\right)$$

2.585

1. $$\begin{aligned}\int \frac{(a+\sin x)^{p+3}\,dx}{\Delta} &= \frac{1}{(p+2)k^2}\left[(a+\sin x)^p\cos x\Delta \right.\\ &\quad +2(2p+3)ak^2\int\frac{(a+\sin x)^{p+2}\,dx}{\Delta} + (p+1)\left(1+k^2-6a^2k^2\right)\int\frac{(a+\sin x)^{p+1}\,dx}{\Delta}\\ &\quad -a(2p+1)\left(1+k^2-2a^2k^2\right)\int\frac{(a+b\sin x)^p\,dx}{\Delta}\\ &\quad \left. -\,p\left(1-a^2\right)\left(1-a^2k^2\right)\int\frac{(a+\sin x)^{p-1}\,dx}{\Delta}\right]\end{aligned}$$

$$\left[p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm\frac{1}{k}\right]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{a+\sin x}{\Delta}\,dx = a\,F(x,k) + \frac{1}{2k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x}$$

3. $$\int \frac{(a+\sin x)^2}{\Delta}\,dx = \frac{1+k^2a^2}{k^2}\,F(x,k) - \frac{1}{k^2}\,E(x,k) + \frac{a}{k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x}$$

4.[6] $$\int \frac{dx}{(a+\sin x)\,\Delta} = \frac{1}{a}\,\Pi\left(x, \frac{1}{a^2}, k\right) - \int \frac{\sin x\,dx}{\left(a^2-\sin^2 x\right)\Delta},$$

where

5. $$\int \frac{\sin x\,dx}{\left(a^2-\sin^2 x\right)\Delta} = \frac{-1}{2\sqrt{\left(1-a^2\right)\left(1-a^2k^2\right)}}\ln\frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2}\cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2}\cos x}$$

2.586

1. $$\int \frac{dx}{(a+\sin x)^n\,\Delta} = \frac{1}{(n-1)\left(1-a^2\right)\left(1-a^2k^2\right)}\left[-\frac{\cos x\Delta}{(a+\sin x)^{n-1}} - (2n-3)\left(1+k^2-2a^2k^2\right)a\int \frac{dx}{(a+\sin x)^{n-1}\,\Delta} - (n-2)\left(6a^2k^2-k^2-1\right)\int \frac{dx}{(a+\sin x)^{n-2}\,\Delta} - (10-4n)ak^2\int \frac{dx}{(a+\sin x)^{n-3}\,\Delta} - (n-3)k^2\int \frac{dx}{(a+\sin x)^{n-4}\,\Delta}\right]$$
$$\left[n \neq 1, \quad a \neq \pm 1, \quad a \neq \pm\frac{1}{k}\right]$$

This integral can be reduced to the integrals:

2. $$\int \frac{dx}{(a+\sin x)^2\,\Delta} = \frac{1}{\left(1-a^2\right)\left(1-a^2k^2\right)}\left[-\frac{\cos x\Delta}{a+\sin x} - a\left(1+k^2-2a^2k^2\right)\int \frac{dx}{(a+\sin x)\,\Delta} - 2ak^2\int \frac{(a+\sin x)\ dx}{\Delta} + k^2\int \frac{(a+\sin x)^2\ dx}{\Delta}\right]$$
(see **2.585** 2, 3, 4)

3. $$\int \frac{dx}{(a+\sin x)^3\,\Delta} = \frac{1}{2\left(1-a^2\right)\left(1-a^2k^2\right)}\left[-\frac{\cos x\Delta}{(a+\sin x)^2} - 3a\left(1+k^2-2a^2k^2\right)\int \frac{dx}{(a+\sin x)^2\,\Delta} - \left(6a^2k^2-k^2-1\right)\int \frac{dx}{(a+\sin x)\,\Delta} + 2ak^2\,F(x,k)\right]$$
(see **2.585** 4 and **2.586** 2)

For $a = \pm 1$, we have:

4. $$\int \frac{dx}{(1\pm\sin x)^n\,\Delta} = \frac{1}{(2n-1)k'^2}\left[\mp\frac{\cos x\Delta}{(1\pm\sin x)^n} + (n-1)\left(1-5k^2\right)\int \frac{dx}{(1\pm\sin x)^{n-1}\,\Delta} + 2(2n-3)k^2\int \frac{dx}{(1\pm\sin x)^{n-2}\,\Delta} - (n-2)k^2\int \frac{dx}{(1\pm\sin x)^{n-3}\,\Delta}\right]$$
GU (241)(6a)

This integral can be reduced to the following integrals:

5. $$\int \frac{dx}{(1 \pm \sin x)\,\Delta} = \frac{\mp \cos x \Delta}{k'^2\,(1 \pm \sin x)} + F(x,k) - \frac{1}{k'^2}\,E(x,k)$$ GU (241)(6c)

6. $$\int \frac{dx}{(1 \pm \sin x)^2\,\Delta} = \frac{1}{3k'^4}\left\{\mp \frac{k'^2 \cos x\Delta}{(1 \pm \sin x)^2} \mp \frac{\left(1 - 5k^2\right)\cos x\Delta}{1 \pm \sin x} + \left(1 - 3k^2\right)k'^2\,F(x,k) - \left(1 - 5k^2\right)E(x,k)\right\}$$ GU (241)(6b)

For $a = \pm\dfrac{1}{k}$, we have

7. $$\int \frac{dx}{(1 \pm k\sin x)^n\,\Delta} = \frac{1}{(2n-1)k'^2}\left[\pm \frac{k\cos x\Delta}{(1 \pm k\sin x)^n} + (n-1)\left(5 - k^2\right)\int \frac{dx}{(1 \pm k\sin x)^{n-1}\,\Delta} - 2(2n-3)\int \frac{dx}{(1 \pm k\sin x)^{n-2}\,\Delta} + (n-2)\int \frac{dx}{(1 \pm k\sin x)^{n-3}\,\Delta}\right]$$ GU (241)(7a)

This integral can be reduced to the following integrals:

8. $$\int \frac{dx}{(1 \pm k\sin x)\,\Delta} = \pm\frac{k\cos x\Delta}{k'^2\,(1 \pm k\sin x)} + \frac{1}{k'^2}\,E(x,k)$$ GU (241)(7b)

9. $$\int \frac{dx}{(1 \pm k\sin x)^2\,\Delta} = \frac{1}{3k'^4}\left[\pm \frac{kk'^2\cos x\Delta}{(1 \pm k\sin x)^2} \pm \frac{k\left(5 - k^2\right)\cos x\Delta}{1 \pm k\sin x} - 2k'^2\,F(x,k) + \left(5 - k^2\right)E(x,k)\right]$$ GU(241)(7c)

2.587

1. $$\int \frac{(b + \cos x)^{p+3}\;dx}{\Delta} \quad \frac{1}{(p+2)k^2}\left[(b + \cos x)^p \sin x\Delta + 2(2p+3)bk^2\int \frac{(b+\cos x)^{p+2}\;dx}{\Delta} - (p+1)\left(k'^2 - k^2 + 6b^2k^2\right)\int \frac{(b+\cos x)^{p+1}\;dx}{\Delta} + (2p+1)b\left(k'^2 - k^2 + b^2k^2\right)\int \frac{(b+\cos x)^{p}\;dx}{\Delta} + p\left(1 - b^2\right)\left(k'^2 + k^2b^2\right)\int \frac{(b+\cos x)^{p-1}\;dx}{\Delta}\right]$$ $$\left[p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k}\right]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{b + \cos x}{\Delta}\,dx = b\,F(x,k) + \frac{1}{k}\arcsin(k\sin x)$$

3. $$\int \frac{(b + \cos x)^2}{\Delta}\,dx = \frac{b^2k^2 - k'^2}{k^2}\,F(x,k) + \frac{1}{k^2}\,E(x,k) + \frac{2b}{k}\arcsin(k\sin x)$$

4. $$\int \frac{dx}{(b + \cos x)\,\Delta} = \frac{b}{b^2 - 1}\,\Pi\left(x, \frac{1}{b^2 - 1}, k\right) + \int \frac{\cos x\,dx}{\left(1 - b^2 - \sin^2 x\right)\Delta},$$

where

5. $$\int \frac{\cos x\,dx}{\left(1-b^2-\sin^2 x\right)\Delta} = \frac{1}{2\sqrt{\left(1-b^2\right)\left(k'^2+k^2b^2\right)}} \ln \frac{\sqrt{1-b^2}\Delta + k\sqrt{k'^2+k^2b^2}\sin x}{\sqrt{1-b^2}\Delta - k\sqrt{k'^2+k^2b^2}\sin x}$$

2.588

1. $$\int \frac{dx}{(b+\cos x)^n \Delta} = \frac{1}{(n-1)\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2 \sin x\Delta}{(b+\cos x)^{-1}} - (2n-3)\left(1-2k^2+2b^2k^2\right) b \int \frac{dx}{(b+\cos x)^{n-1}\Delta} - (n-2)\left(2k^2-1-6b^2k^2\right) \int \frac{dx}{(b+\cos x)^{n-2}\Delta} - (4n-10)bk^2 \int \frac{dx}{(b+\cos x)^{n-3}\Delta} + (n-3)k^2 \int \frac{dx}{(b+\cos x)^{n-4}\Delta}\right]$$
$$\left[n \neq 1, \quad b \neq \pm 1, \quad b \neq \pm \frac{ik'}{k}\right]$$

This integral can be reduced to the following integrals:

2. $$\int \frac{dx}{(b+\cos x)^2 \Delta} = \frac{1}{\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2 \sin x\Delta}{b+\cos x} - \left(1-2k^2+2b^2k^2\right) b \int \frac{dx}{(b+\cos x)\Delta} + 2bk^2 \int \frac{b+\cos x}{\Delta} dx - k^2 \int \frac{(b+\cos x)^2}{\Delta}\, dx\right]$$
(see **2.587** 2, 3, 4)

3. $$\int \frac{dx}{(b+\cos x)^3 \Delta} = \frac{1}{2\left(1-b^2\right)\left(k'^2+b^2k^2\right)} \left[\frac{-k'^2 \sin x\Delta}{(b+\cos x)^2} - 3b\left(1-2k^2+2k^2b^2\right) \int \frac{dx}{(b+\cos x)^2\Delta} - \left(2k^2-1-6b^2k^2\right) \int \frac{dx}{(b+\cos x)\Delta} - 2bk^2\, F(x,k)\right]$$
(see **2.588** 2 and **2.587** 4)

2.589

1. $$\int \frac{(c+\tan x)^{p+3}\, dx}{\Delta} = \frac{1}{(p+2)k'^2} \left[\frac{(c+\tan x)^p\,\Delta}{\cos^2 x} + 2(2n+3)ck'^2 \int \frac{(c+\tan x)^{p+2}\, dx}{\Delta} - (p+1)\left(1+k'^2+6c^2k'^2\right) \int \frac{(c+\tan x)^{p+1}\, dx}{\Delta} + (2p+1)c\left(1+k'^2+2c^2k'^2\right) \int \frac{(c+\tan x)^p\, dx}{\Delta} - p\left(1+c^2\right)\left(1+k'^2c^2\right) \int \frac{(c+\tan x)^{p-1}\, dx}{\Delta}\right]$$
$$[p \neq -2]$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

2. $$\int \frac{c+\tan x}{\Delta}\,dx = c\,F(x,k) + \frac{1}{2k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

3. $$\int \frac{(c+\tan x)^2}{\Delta}\,dx = \frac{1}{k'^2}\tan x\Delta + c^2\,F(x,k) - \frac{1}{k'^2}\,E(x,k) + \frac{c}{k'}\ln\frac{\Delta+k'}{\Delta-k'}$$

4. $$\int \frac{dx}{(c+\tan x)\,\Delta} = \frac{c}{1+c^2}\,F(x,k) + \frac{1}{c\,(1+c^2)}\,\Pi\left(x, -\frac{1+c^2}{c^2}, k\right) - \int \frac{\sin x\cos x\,dx}{\left[c^2-(1+c^2)\sin^2 x\right]\Delta},$$

where

5. $$\int \frac{\sin x\cos x\,dx}{\left[c^2-(1+c^2)\sin^2 x\right]\Delta} = \frac{1}{2\sqrt{(1+c^2)\left(1+c^2k'^2\right)}}\ln\frac{\sqrt{1+c^2k'^2}+\sqrt{1+c^2}\Delta}{\sqrt{1+c^2k'^2}-\sqrt{1+c^2}\Delta}$$

2.591

1. $$\int \frac{dx}{(c+\tan x)^n\,\Delta} = \frac{1}{(n-1)(1+c^2)\left(1+k'^2c^2\right)}\left[-\frac{\Delta}{(c+\tan x)^{n-1}\cos^2 x} + (2n-3)c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)^{n-1}\,\Delta} - (n-2)\left(1+k'^2+6c^2k'^2\right)\int\frac{dx}{(c+\tan x)^{n-2}\,\Delta} + (4n-10)ck'^2\int\frac{dx}{(c+\tan x)^{n-3}\,\Delta} - (n-3)k'^2\int\frac{dx}{(c+\tan x)^{n-4}\,\Delta}\right]$$

This integral can be reduced to the integrals:

2. $$\int \frac{dx}{(c+\tan x)^2\,\Delta} = \frac{1}{(1+c^2)\left(1+k'^2c^2\right)}\left[\frac{-\Delta}{(c+\tan x)\cos^2 x} + c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)\,\Delta} - 2ck'^2\int\frac{c+\tan x}{\Delta}dx + k'^2\int\frac{(c+\tan x)^2}{\Delta}\,dx\right]$$

(see **2.589** 2, 3, 4)

3. $$\int \frac{dx}{(c+\tan x)^3\,\Delta} = \frac{1}{2(1+c^2)\left(1+k'^2c^2\right)}\left[\frac{-\Delta}{(c+\tan x)^2\cos^2 x} + 3c\left(1+k'^2+2c^2k'^2\right)\int\frac{dx}{(c+\tan x)^2\,\Delta} - \left(1+k'^2+6c^2k'^2\right)\int\frac{dx}{(c+\tan x)\,\Delta} + 2ck'^2\,F(x,k)\right]$$

(see **2.591** 2 and **2.589** 4)

2.592

1. $P_n = \int \frac{\left(a + \sin^2 x\right)^n}{\Delta}\, dx$

The recursion formula

$$P_{n+1} = \frac{1}{(2n+3)k^2} \left\{ \left(a + \sin^2 x\right)^n \sin x \cos x \Delta + (2n+2)\left(1 + k^2 + 3ak^2\right) P_{n+1} - (2n+1)\left[1 + 2a\left(1+k^2\right) + 3a^2k^2\right] P_n + 2na(1+a)\left(1 + k^2 a\right) P_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

2. P_1 (see **2.584** 1 and **2.584** 4)

3. P_0 (see **2.584** 1)

4. $P_{-1} = \int \frac{dx}{\left(a + \sin^2 x\right)\Delta} = \frac{1}{a}\Pi\left(x, \frac{1}{a}, k\right)$

For $a = 0$

5. $\int \frac{dx}{\sin^2 x \Delta}$ (see **2.584** 70) H (124)a

6. $T_n = \int \frac{dx}{\left(h + g\sin^2 x\right)^n \Delta}$

can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2} \left\{ \frac{-g^2 \sin x \cos x \Delta}{\left(h + g\sin^2 x\right)^{n-1}} + 2(n-2)\left[g\left(1+k^2\right) + 3hk^2\right] T_{n-2} - (2n-3)\left[g^2 + 2hg\left(1+k^2\right) + 3h^2k^2\right] T_{n-1} + 2(n-1)h(g+h)\left(g + hk^2\right) T_n \right\}$$

2.593

1. $Q_n = \int \frac{\left(b + \cos^2 x\right)^n}{\Delta}\, dx$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \left\{ \left(b + \cos^2 x\right)^n \sin x \sin x \Delta - (2n+2)\left(1 - 2k^2 - 3bk^2\right) Q_{n+1} + (2n+1)\left[k'^2 + 2b\left(k'^2 - k^2\right) - 3b^2k^2\right] n\_2nb(1-b)\left(k'^2 - k^2 b\right) Q_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

2. Q_1 (see **2.584** 1 and **2.584** 6)

3. Q_0 (see **2.584** 1)

4. $$Q_{-1} = \int \frac{dx}{(b + \cos^2 x)\,\Delta} = \frac{1}{b+1}\,\Pi\left(x, -\frac{1}{b+1}, k\right)$$

For $b = 0$

5. $$\int \frac{dx}{\cos^2 x \Delta}$$ (see **2.584** 72) H (123)

2.594

1. $$R_n = \int \frac{\left(c + \tan^2 x\right)^n\, dx}{\Delta}$$

The recursion formula

$$R_{n+2} = \frac{1}{(2n+3)k'^2}\left\{\frac{\left(c + \tan^2 x\right)^n \tan x \Delta}{\cos^2 x} - (2n+2)\left(1 + k'^2 - 3ck'^2\right) R_{n+1} + (2n-1)\left[1 - 2c\left(1 + k'^2\right) + 3c^2 k'^2\right] R_n + 2nc(1-c)\left(1 - k'^2 c\right) R_{n-1}\right\}$$

reduces this integral (for n an integer) to the integrals:

2. R_1 (see **2.584** 1 and **2.584** 90)

3. R_0 (see **2.584** 1)

4. $$R_{-1} = \int \frac{dx}{\left(c + \tan^2 x\right)\Delta} = \frac{1}{c-1}\,F(x,k) + \frac{1}{c(1-c)}\,\Pi\left(x, \frac{1-c}{c}, k\right)$$

For $c = 0$, see **2.582** 5.

2.595 Integrals of the type $\int R\left(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}\right) dx$ for $p^2 > 1$.

Notation: $\alpha = \arcsin\left(p \sin x\right)$.

Basic formulas

1. $$\int \frac{dx}{\sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p}\,F\left(\alpha, \frac{1}{p}\right)$$ $\left[p^2 > 1\right]$ BY (283.00)

2. $$\int \sqrt{1 - p^2 \sin^2 x}\, dx = p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p}\,F\left(\alpha, \frac{1}{p}\right)$$ $\left[p^2 > 1\right]$ BY (283.03)

3. $$\int \frac{dx}{\left(1 - r^2 \sin^2 x\right)\sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p}\,\Pi\left(\alpha, \frac{r^2}{p^2}, \frac{1}{p}\right)$$ $\left[p^2 > 1\right]$ BY (283.02)

To evaluate integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}\right) dx$ for $p^2 > 1$, we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

(1) k with p;

(2) k'^2 with $1 - p^2$;

(3) $F(x,k)$ with $\frac{1}{p} F\left(\alpha, \frac{1}{p}\right)$;

(4) $E(x,k)$ with $p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\alpha, \frac{1}{p}\right)$.

For example (see **2.584** 15):

2.596

1.[10]
$$\int \frac{\cos^4 x\,dx}{\sqrt{1-p^2\sin^2 x}} = \frac{\sin x \cos x \sqrt{1-p^2\sin^2 x}}{3p^2} + \frac{4p^2-2}{3p^4}\left[p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\alpha, \frac{1}{p}\right)\right] + \frac{2-5p^2+3p^4}{3p^4} \cdot \frac{1}{p} F\left(\alpha, \frac{1}{p}\right)$$
$$= \frac{\sin x \cos x \sqrt{1-p^2\sin^2 x}}{3p^2} - \frac{p^2-1}{3p^3} F\left(\alpha, \frac{1}{p}\right) + \frac{4p^2-2}{3p^3} E\left(\alpha, \frac{1}{p}\right) \qquad [p^2 > 1]$$

For example (see **2.583** 36):

2.
$$\int \frac{\sqrt{1-p^2\sin^2 x}}{\cos^2 x}\,dx = \tan x \sqrt{1-p^2\sin^2 x} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \left[p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\alpha, \frac{1}{p}\right)\right]$$
$$= p\left[F\left(\alpha, \frac{1}{p}\right) - E\left(\alpha, \frac{1}{p}\right)\right] + \tan x \sqrt{1-p^2\sin^2 x} \qquad [p^2 > 1]$$

For example (see **2.584** 37):

3.
$$\int \frac{dx}{\sqrt{\left(1-p^2\sin^2 x\right)^3}} = \frac{-1}{p^2-1}\left[p\,E\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\alpha, \frac{1}{p}\right)\right] - \frac{p^2}{1-p^2} \frac{\sin x \cos x}{\sqrt{1-p^2\sin^2 x}}$$
$$= \frac{p^2}{p^2-1} \frac{\sin x \cos x}{\sqrt{1-p^2\sin^2 x}} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \frac{p}{p^2-1} E\left(\alpha, \frac{1}{p}\right) \qquad [p^2 > 1]$$

2.597 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1+p^2\sin^2 x}\right)\,dx$

Notation: $\alpha = \arcsin\left(\frac{\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}}\right)$

Basic formulas

1.
$$\int \frac{dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$
BY (282.00)

2.
$$\int \sqrt{1+p^2\sin^2 x}\,dx = \sqrt{1+p^2}\,E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2\sin^2 x}}$$
BY (282.03)

3. $$\frac{\sqrt{1+p^2\sin^2 x}\,dx}{1+(p^2-r^2p^2-r^2)\sin^2 x} = \frac{1}{\sqrt{1+p^2}}\Pi\left(\alpha, r^2, \frac{p}{\sqrt{1+p^2}}\right)$$ BY (282.02)

4. $$\int\frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}} = -\frac{1}{p}\arcsin\left(\frac{p\cos x}{\sqrt{1+p^2}}\right)$$

5. $$\int\frac{\cos x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{p}\ln\left(p\sin x+\sqrt{1+p^2\sin^2 x}\right)$$

6. $$\int\frac{dx}{\sin x\sqrt{1+p^2\sin^2 x}} = \frac{1}{2}\ln\frac{\sqrt{1+p^2\sin^2 x}-\cos x}{\sqrt{1+p^2\sin^2 x}+\cos x}$$

7. $$\int\frac{dx}{\cos x\sqrt{1+p^2\sin^2 x}} = \frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}\sin x}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}\sin x}$$

8. $$\int\frac{\tan x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{2\sqrt{1+p^2}}\ln\frac{\sqrt{1+p^2\sin^2 x}+\sqrt{1+p^2}}{\sqrt{1+p^2\sin^2 x}-\sqrt{1+p^2}}$$

9. $$\int\frac{\cot x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{2}\ln\frac{1-\sqrt{1+p^2\sin^2 x}}{1+\sqrt{1+p^2\sin^2 x}}$$

2.598 To calculate integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1+p^2\sin^2 x}\right)dx$, we may use formulas **2.583** and **2.584**, making the following modifications in them. We replace

(1) k^2 with $-p^2$;

(2) k'^2 with $1+p^2$;

(3) $F(x,k)$ with $\frac{1}{\sqrt{1+p^2}}F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$;

(4) $E(x,k)$ with $\sqrt{1+p^2}\,E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}$;

(5) $\frac{1}{k}\ln(k\cos x+\Delta)$ with $\frac{1}{p}\arcsin\frac{p\cos x}{\sqrt{1+p^2}}$;

(6) $\frac{1}{k}\arcsin(k\sin x)$ with $\frac{1}{p}\ln\left(p\sin x+\sqrt{1+p^2\sin^2 x}\right)$.

For example (see **2.584** 90):

1. $$\int\frac{\tan^2 x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{(1+p^2)}\left[\tan x\sqrt{1+p^2\sin^2 x} - \sqrt{1+p^2}\,E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + p^2\frac{\sin x\cos x}{\sqrt{1+p^2\sin^2 x}}\right]$$
$$= -\frac{1}{\sqrt{1+p^2}}E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) + \frac{\tan x}{\sqrt{1+p^2\sin^2 x}}$$

For example (see **2.584** 37):

2. $$\int \frac{dx}{\sqrt{\left(1+p^2\sin^2 x\right)^3}} = \frac{1}{\sqrt{1+p^2}}\, E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$

2.599 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{a^2\sin^2 x - 1}\right) dx \qquad \left[a^2 > 1\right]$

Notation: $\alpha = \arcsin\left(\frac{a\cos x}{\sqrt{a^2-1}}\right)$.

Basic formulas:

1. $$\int \frac{dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{1}{a}\, F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) \qquad \left[a^2 > 1\right]$$ BY (285.00)a

2. $$\int \sqrt{a^2\sin^2 x - 1}\, dx = \frac{1}{a}\, F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) - a\, E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)$$ $\left[a^2 > 1\right]$ BY (285.06)a

3. $$\int \frac{dx}{\left(1 - r^2\sin^2 x\right)\sqrt{a^2\sin^2 x - 1}} = \frac{1}{a\left(r^2-1\right)}\, \Pi\left(\alpha, \frac{r^2\left(a^2-1\right)}{a^2\left(r^2-1\right)}, \frac{\sqrt{a^2-1}}{a}\right)$$ $\left[a^2 > 1, \quad r^2 > 1\right]$ BY (285.02)a

4. $$\int \frac{\sin x\, dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{\alpha}{a} \qquad \left[a^2 > 1\right]$$

5. $$\int \frac{\cos x\, dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{a}\ln\left(a\sin x + \sqrt{a^2\sin^2 x - 1}\right) \qquad \left[a^2 > 1\right]$$

6. $$\int \frac{dx}{\sin x\sqrt{a^2\sin^2 x - 1}} = -\arctan\frac{\cos x}{\sqrt{a^2\sin^2 x - 1}} \qquad \left[a^2 > 1\right]$$

7. $$\int \frac{dx}{\cos x\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-2}}\ln\frac{\sqrt{a^2-1}\sin x + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1}\sin x - \sqrt{a^2\sin^2 x - 1}}$$ $\left[a^2 > 1\right]$

8. $$\int \frac{\tan x\, dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1} + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2\sin^2 x - 1}}$$ $\left[a^2 > 1\right]$

9. $$\int \frac{\cot x\, dx}{\sqrt{a^2\sin^2 x - 1}} = -\arcsin\left(\frac{1}{a\sin x}\right) \qquad \left[a^2 > 1\right]$$

2.611 To calculate integrals of the type $\int R\left(\sin x, \cos x, \sqrt{a^2\sin^2 x - 1}\right) dx$ for $a^2 > 1$, we may use formulas **2.583** and **2.584**. In doing so, we should follow the procedure outlined below:

(1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$F(x,k) \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta}$$

$$E(x,k) \quad \text{should be replaced with} \quad \int \Delta dx$$

$$-\frac{1}{k}\ln\left(k\cos x + \Delta\right) \quad \text{should be replaced with} \quad \int \frac{\sin x\,dx}{\Delta}$$

$$\frac{1}{k}\arcsin\left(k\sin x\right) \quad \text{should be replaced with} \quad \int \frac{\cos x\,dx}{\Delta}$$

$$\frac{1}{2}\ln\frac{\Delta - \cos x}{\Delta + \cos x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \sin x}$$

$$\frac{1}{2k'}\ln\frac{\Delta + k'\sin x}{\Delta - k'\sin x} \quad \text{should be replaced with} \quad \int \frac{dx}{\Delta \cos x}$$

$$\frac{1}{2k'}\ln\frac{\Delta + k'}{\Delta - k'} \quad \text{should be replaced with} \quad \int \frac{\tan x}{\Delta}\,dx$$

$$\frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} \quad \text{should be replaced with} \quad \int \frac{\cot x}{\Delta}\,dx$$

(2) Then, on both sides of the equations, we should replace Δ with $i\sqrt{a^2\sin^2 x - 1}$, k with a and ${k'}^2$ with $1-a^2$.

(3) Both sides of the resulting equations should be multiplied by i, as a result of which only real functions $\left(a^2 > 1\right)$ should appear on both sides of the equations.

(4) The integrals on the right sides of the equations should be replaced with their values found from formulas **2.599**.

Examples:

1. We rewrite equation **2.584** 4 in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2\sin^2 x - 1}}\,dx = \frac{1}{a^2}\int \frac{dx}{i\sqrt{a^2\sin^2 x - 1}} - \frac{1}{a^2}\int i\sqrt{a^2\sin^2 x - 1}\,dx,$$

from which we get

$$\int \frac{\sin^2 x\,dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{a^2}\left\{\int \frac{dx}{\sqrt{a^2\sin^2 x - 1}} + \int \sqrt{a^2\sin^2 x - 1}\,dx\right\} = -\frac{1}{a}E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)$$

$$\left[a^2 > 1\right]$$

2. We rewrite equation **2.584** 58 as follows:

$$\int \frac{dx}{i^5\sqrt{\left(a^2\sin^2 x - 1\right)^5}} = -\frac{2a^4\left(a^2-2\right)\sin^2 x - \left(3a^2-5\right)a^2}{3\left(1-a^2\right)^2 i^3\sqrt{\left(a^2\sin^2 x - 1\right)^3}}\sin x\cos x$$
$$-\frac{1}{3\left(1-a^2\right)}\int \frac{dx}{i\sqrt{a^2\sin^2 x - 1}} - \frac{2a^2-4}{3\left(1-a^2\right)^2}\int i\sqrt{a^2\sin^2 x - 1}\,dx$$

from which we obtain

$$\int \frac{dx}{\sqrt{\left(a^2 \sin^2 x - 1\right)^5}} = \frac{2a^4\left(a^2-2\right)\sin^2 x - \left(3a^2-5\right)a^2}{3\left(1-a^2\right)^2\sqrt{\left(a^2\sin^2 x - 1\right)^3}}\sin x\cos x + \frac{1}{3\left(1-a^2\right)^2 a}$$
$$\times\left\{\left(a^2-3\right)F\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right) - 2a^2\left(a^2-2\right)E\left(\alpha, \frac{\sqrt{a^2-1}}{a}\right)\right\}$$
$$\left[a^2>1\right]$$

3. We rewrite equation **2.584** 71 in the form

$$\int \frac{dx}{\sin x\cos x i\sqrt{a^2\sin^2 x - 1}} = \int \frac{\cot x\,dx}{i\sqrt{a^2\sin^2 x-1}} + \int \frac{\tan x\,dx}{i\sqrt{a^2\sin^2 x - 1}},$$

from which we obtain

$$\int \frac{dx}{\sin x\cos x\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}+\sqrt{a^2\sin^2 x-1}}{\sqrt{a^2-1}-\sqrt{a^2\sin^2 x - 1}} - \arcsin\left(\frac{1}{a\sin x}\right)$$
$$\left[a^2>1\right]$$

2.612 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-k^2\cos^2 x}\right)\,dx$.

To find integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-k^2\cos^2 x}\right)\,dx$, we make the substitution $x = \frac{\pi}{2} - y$, which yields

$$\int R\left(\sin x, \cos x, \sqrt{1-k^2\cos^2 x}\right)\,dx = -\int R\left(\cos y, \sin y, \sqrt{1-k^2\sin^2 y}\right)\,dy.$$

The integrals $\int R\left(\cos y, \sin y, \sqrt{1-k^2\sin^2 y}\right)\,dy$ are found from formulas **2.583** and **2.584**. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions $F(x,k)$ and $E(x,k)$ with the corresponding integrals, we obtain an expression of the form

$$-g\left(\cos y, \sin y\right) - A\int\frac{dy}{\sqrt{1-k^2\sin^2 y}} - B\int\sqrt{1-k^2\sin^2 y}\,dy$$

Returning now to the original variable x, we obtain

$$\int R\left(\sin x, \cos x, \sqrt{1-k^2\cos^2 x}\right)\,dx = -g\left(\sin x, \cos x\right) - A\int\frac{dx}{\sqrt{1-k^2\cos^2 x}} - B\int\sqrt{1-k^2\cos^2 x}\,dx$$

The integrals appearing in this expression are found from the formulas

1. $$\int \frac{dx}{\sqrt{1-k^2\cos^2 x}} = F\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\right), k\right)$$

2. $$\int \sqrt{1-k^2\cos^2 x}\,dx = E\left(\arcsin\left(\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\right), k\right) - \frac{k^2\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}$$

2.613 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-p^2\cos^2 x}\right)\,dx \quad [p>1]$.

To find integrals of the type $\int R\left(\sin x, \cos x, \sqrt{1-p^2\cos^2 x}\right)\,dx$, where $[p>1]$, we proceed as in section **2.612**. Here, we use the formulas

1. $$\int \frac{dx}{\sqrt{1-p^2\cos^2 x}} = -\frac{1}{p} F\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right) \qquad [p>1]$$

2. $$\int \sqrt{1-p^2\cos^2 x}\, dx = \frac{p^2-1}{p} F\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right) - p\, E\left(\arcsin\left(p\cos x\right), \frac{1}{p}\right)$$

2.614 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right) dx$.

To find integrals of the type $\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R\left(\sin x, \cos x, \sqrt{1+p^2\cos^2 x}\right) dx = -\int R\left(\cos y, \sin y, \sqrt{1+p^2\sin^2 y}\right) dy.$$

To calculate the integrals $-\int R\left(\cos y, \sin y, \sqrt{1+p^2\sin^2 y}\right) dy$, we need to use first what was said in **2.598** and **2.612** and then, after returning to the variable x, the formulas

1. $$\int \frac{dx}{\sqrt{1+p^2\cos^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

2. $$\int \sqrt{1+p^2\cos^2 x}\, dx = \sqrt{1+p^2}\, E\left(x, \frac{p}{\sqrt{1+p^2}}\right)$$

2.615 Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right) dx \qquad [a>1]$.

To find integrals of the type $\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R\left(\sin x, \cos x, \sqrt{a^2\cos^2 x - 1}\right) dx = -\int R\left(\cos y, \sin y, \sqrt{a^2\sin^2 y - 1}\right) dy$$

To calculate the integrals $-\int R\left(\cos y, \sin y, \sqrt{a^2\sin^2 y - 1}\right) dy$, we use what was said in **2.611** and then, after returning to the variable x, we use the formulas

1. $$\int \frac{dx}{\sqrt{a^2\cos^2 x - 1}} = \frac{1}{a} F\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a>1]$$

2. $$\int \sqrt{a^2\cos^2 x - 1}\, dx = a\, E\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) - \frac{1}{a} F\left(\arcsin\left(\frac{a\sin x}{\sqrt{a^2-1}}\right), \frac{\sqrt{a^2-1}}{a}\right) \qquad [a>1]$$

2.616[11] Integrals of the form $\int R\left(\sin x, \cos x, \sqrt{1-p^2\sin^2 x}, \sqrt{1-q^2\sin^2 x}\right)\, dx.$

Notation: $\alpha = \arcsin\left(\dfrac{\sqrt{1-p^2}\sin x}{\sqrt{1-p^2\sin^2 x}}\right)$.

1. $$\int \frac{dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}} = \frac{1}{\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.00)

2. $$\int \frac{\tan^2 x\, dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}} = \frac{\tan x\sqrt{1-q^2\sin^2 x}}{\left(1-q^2\right)\sqrt{1-p^2\sin^2 x}} - \frac{1}{\left(1-q^2\right)\sqrt{1-p^2}} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.07)

3. $$\int \frac{\tan^4 x\, dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}} = \frac{1}{3\left(1-q^2\right)^2\left(1-p^2\right)^{\frac{3}{2}}} \times \left[2\left(2-p^2-q^2\right) E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \left(1-q^2\right) F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right] + \frac{2p^2+q^2-3+\sin^2 x\left(4-3p^2-2q^2+p^2q^2\right)}{3\left(1-p^2\right)\left(1-q^2\right)^2} \frac{\sin x}{\cos^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.07)

4. $$\int \frac{\sin^2 x\, dx}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)^3}} = \frac{\sqrt{1-p^2}}{\left(1-q^2\right)\left(q^2-p^2\right)} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1}{\left(q^2-p^2\right)\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{\sin x\cos x}{\left(1-q^2\right)\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.06)

5. $$\int \frac{\cos^2 x\, dx}{\sqrt{\left(1-p^2\sin^2 x\right)^3\left(1-q^2\sin^2 x\right)}} = \frac{\sqrt{1-p^2}}{q^2-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1-q^2}{\left(q^2-p^2\right)\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$
BY (284.05)

6. $$\int \frac{\cos^4 x\,dx}{\sqrt{\left(1-p^2\sin^2 x\right)^5\left(1-q^2\sin^2 x\right)}}$$
$$= \frac{\left(1-p^2\right)^{\frac{3}{2}}}{3\left(q^2-p^2\right)^2}\left[\frac{\left(2+p^2-3q^2\right)\left(1-q^2\right)}{\left(1-p^2\right)^2} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) + 2\frac{2q^2-p^2-1}{1-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)\right] + \frac{\left(1-p^2\right)\sin x\cos x\sqrt{1-q^2\sin^2 x}}{3\left(q^2-p^2\right)\sqrt{\left(1-p^2\sin^2 x\right)^3}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$ BY (284.05)

7. $$\int \frac{dx}{1-p^2\sin^2 x}\sqrt{\frac{1-q^2\sin^2 x}{1-p^2\sin^2 x}} = \frac{1}{\sqrt{1-p^2}} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right]$$ BY (284.01)

8. $$\int \sqrt{\frac{1-p^2\sin^2 x}{\left(1-q^2\sin^2 x\right)^3}}\,dx = \frac{\sqrt{1-p^2}}{1-q^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2}\frac{\sin x\cos x}{\sqrt{\left(1-p^2\sin^2 x\right)\left(1-q^2\sin^2 x\right)}}$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right].$$ BY (284.04)

9. $$\int \frac{dx}{1+\left(p^2r^2-p^2-r^2\right)\sin^2 x}\sqrt{\frac{1-p^2\sin^2 x}{1-q^2\sin^2 x}} = \frac{1}{\sqrt{1-p^2}}\,\Pi\left(\alpha, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right)$$
$$\left[0 < p^2 < q^2 < 1, \quad 0 < x \le \frac{\pi}{2}\right].$$ BY (284.02)

2.617 **Notation**: $\alpha = \arcsin\sqrt{\frac{\sqrt{b^2+c^2}-b\sin x-c\cos x}{2\sqrt{b^2+c^2}}}, \quad r = \sqrt{\frac{2\sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}.$

1. $$\int \frac{dx}{\sqrt{a+b\sin x+c\cos x}}$$
$$= -\frac{2}{\sqrt{a+\sqrt{b^2+c^2}}} F(\alpha, r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$ BY (294.00)

$$= -\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\alpha, r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$ BY (293.00)

2. $$\int \frac{\sin x\, dx}{\sqrt{a + b\sin x + c\cos x}} = -\frac{\sqrt{2b}}{\sqrt[4]{\left(b^2+c^2\right)^3}} \left\{2\, E(\alpha, r) - F(\alpha, r)\right\} + \frac{2c}{b^2+c^2}\sqrt{a + b\sin x + c\cos x}$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.05)

3. $$\int \frac{(b\cos x - c\sin x)\, dx}{\sqrt{a + b\sin x + c\cos x}} = 2\sqrt{a + b\sin x + c\cos x}$$

4. $$\int \frac{\sqrt{b^2+c^2} + b\sin x + c\cos x}{\sqrt{a + b\sin x + c\cos x}}\, dx$$
$$= -2\sqrt{a + \sqrt{b^2+c^2}}\, E(\alpha, r) + \frac{2\left(a - \sqrt{b^2+c^2}\right)}{\sqrt{a + \sqrt{b^2+c^2}}}\, F(\alpha, r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (294.04)
$$= -2\sqrt{2}\sqrt[4]{b^2+c^2}\, E(\alpha, r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.01)

5. $$\int \sqrt{a + b\sin x + c\cos x}\, dx$$
$$= -2\sqrt{a + \sqrt{b^2+c^2}}\, E(\alpha, r)$$
$$\left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \pi \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (294.01)
$$= -2\sqrt{2}\sqrt[4]{b^2+c^2}\, E(\alpha, r) + \frac{\sqrt{2}\left(\sqrt{b^2+c^2} - a\right)}{\sqrt[4]{b^2+c^2}}\, F(\alpha, r)$$
$$\left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin\frac{b}{\sqrt{b^2+c^2}} - \arccos\left(\frac{-a}{\sqrt{b^2+c^2}}\right) \le x < \arcsin\frac{b}{\sqrt{b^2+c^2}}\right]$$
BY (293.03)

2.618 Integrals of the form $\int R\left(\sin ax, \cos ax, \sqrt{\cos 2ax}\right)\, dx = \frac{1}{a}\int R\left(\sin t, \cos t, \sqrt{1 - 2\sin^2 t}\right)\, dt$ where the substitution $t = ax$ has been used.
Notation: $\alpha = \arcsin\left(\sqrt{2}\sin ax\right)$

The integrals $\int R\left(\sin ax, \cos ax, \sqrt{\cos 2ax}\right)\, dx$ are special cases of the integrals **2.595**. for $(p = 2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

2. $$\int \frac{\cos^2 ax}{\sqrt{\cos 2ax}}\, dx = \frac{1}{a\sqrt{2}}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

3. $$\int \frac{dx}{\cos^2 ax\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\tan x}{a}\sqrt{\cos 2ax}$$
$$\left[0 < ax \le \frac{\pi}{4}\right]$$

4. $$\int \frac{dx}{\cos^4 ax\sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\left(6\cos^2 ax + 1\right)\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax}$$
$$\left[0 < x \le \frac{\pi}{4}\right]$$

5. $$\int \frac{\tan^2 ax\, dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a}\tan ax\sqrt{\cos 2ax}$$
$$\left[0 < x \le \frac{\pi}{2}\right]$$

6. $$\int \frac{\tan^4 ax\, dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a\cos^3 ax}\sqrt{\cos 2ax}$$
$$\left[0 < ax \le \frac{\pi}{4}\right]$$

7. $$\int \frac{dx}{\left(1 - 2r^2\sin^2 ax\right)\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

8. $$\int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}}$$
$$\left[0 < ax \le \frac{\pi}{4}\right]$$

9. $$\int \frac{\sin^2 ax\, dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

10. $$\int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \qquad \left[0 < ax \le \frac{\pi}{4}\right]$$

11. $$\int \sqrt{\cos 2ax}\, dx = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$
$$\left[0 < ax \le \frac{\pi}{4}\right]$$

12. $$\int \frac{\sqrt{\cos 2ax}}{\cos^2 ax}\, dx = \frac{\sqrt{2}}{a}\left\{F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\} + \frac{1}{a}\tan ax\sqrt{\cos 2ax}$$
$$\left[0 < x \le \frac{\pi}{4}\right]$$

2.619 Integrals of the form $\int R\left(\sin ax, \cos ax, \sqrt{-\cos 2ax}\right)\, dx = \frac{1}{a}\int R\left(\sin x, \cos x, \sqrt{2\sin^2 x - 1}\right)\, dx$

Notation: $\alpha = \arcsin\left(\sqrt{2}\cos ax\right)$

The integrals $\int R\left(\sin x, \cos x, \sqrt{2\sin^2 x - 1}\right)\, dx$ are special cases of the integrals **2.599** and **2.611** for $(a = 2)$. We give some formulas:

1. $$\int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

2. $$\int \frac{\cos^2 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$

3. $$\int \frac{\cos^4 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}} \left[3\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{5}{2}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] - \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}$$

4. $$\int \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{1}{a} \cot ax \sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

5. $$\int \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} = \frac{2}{3a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 6\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{3a} \frac{\cos ax}{\sin^3 ax} \left(6 \sin^2 ax + 1\right) \sqrt{-\cos 2ax}$$

6. $$\int \frac{\cot^2 ax\, dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{1}{a} \cot ax \sqrt{-\cos 2ax}$$

7. $$\int \frac{dx}{\left(1 - 2r^2 \cos^2 ax\right) \sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\, \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right)$$

8. $$\int \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}$$

9. $$\int \frac{\cos^2 ax\, dx}{\sqrt{-\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}}\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$

10. $$\int \frac{dx}{\sqrt{-\cos^5 2ax}} = -\frac{1}{3a\sqrt{2}}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3 2ax}}$$

11. $$\int \sqrt{-\cos 2ax}\, dx = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$

2.621 Integrals of the form $\int R\left(\sin ax, \cos ax, \sqrt{\sin 2ax}\right)\, dx$.

Notation: $\alpha = \arcsin \sqrt{\dfrac{2 \sin ax}{1 + \sin ax + \cos ax}}$.

1. $$\int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\, F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ BY (287.50)

2. $$\int \frac{\sin ax\, dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[\frac{1+i}{2}\, \Pi\left(\alpha, \frac{1+i}{2}, \frac{1}{\sqrt{2}}\right) + \frac{1-i}{2}\, \Pi\left(\alpha, \frac{1-i}{2}, \frac{1}{\sqrt{2}}\right) + F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$ BY (287.57)

3. $$\int \frac{\sin ax\, dx}{\left(1 + \sin ax + \cos ax\right) \sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$$ BY (287.54)

4. $$\int \frac{\sin ax\, dx}{(1 - \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{\sqrt{\tan ax} - E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\}$$
$$\left[ax \neq \frac{\pi}{2}\right]$$ BY (287.55)

5. $$\int \frac{(1 + \cos ax)\ dx}{(1 + \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ BY (287.51)

6. $$\int \frac{(1 + \cos ax)\ dx}{(1 - \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \sqrt{\tan ax}\right\}$$
$$\left[ax \neq \frac{\pi}{2}\right]$$ BY (287.56)

7. $$\int \frac{(1 - \sin ax + \cos ax)\ dx}{(1 + \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{2\, E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\}$$ BY (287.53)

8. $$\int \frac{(1 + \sin ax + \cos ax)\ dx}{[1 + \cos ax + (1 - 2r^2)\sin ax]\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right).$$ BY (287.52)

2.63–2.65 Products of trigonometric functions and powers

2.631

1. $$\begin{aligned}\int x^r \sin^p x \cos^q x\, dx = {} & \frac{1}{(p+q)^2}\Bigg[(p+q)x^r \sin^{p+1} x \cos^{q-1} x \\ & + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\, dx \\ & - rp\int x^{r-1} \sin^{p-1} x \cos^{q-1} x\, dx + (q-1)(p+q)\int x^r \sin^p x \cos^{q-2} x\, dx\Bigg] \\ = {} & \frac{1}{(p+q)^2}\Bigg[-(p+q)x^r \sin^{p-1} x \cos^{q+1} x \\ & + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\, dx \\ & + rq\int x^{r-1} \sin^{p-1} x \cos^{q-1} x\, dx + (p-1)(p+q)\int x^r \sin^{p-2} x \cos^q x\, dx\Bigg]\end{aligned}$$
GU (331)(1)

2. $$\begin{aligned}\int x^m \sin^n x\, dx = {} & \frac{x^{m-1}\sin^{n-1} x}{n^2}\{m \sin x - nx \cos x\} \\ & + \frac{n-1}{n}\int x^m \sin^{n-2} x\, dx - \frac{m(m-1)}{n^2}\int x^{m-2} \sin^n x\, dx\end{aligned}$$

3. $$\begin{aligned}\int x^m \cos^n x\, dx = {} & \frac{x^{m-1}\cos^{n-1} x}{n^2}\{m \cos x + nx \sin x\} \\ & + \frac{n-1}{n}\int x^m \cos^{n-2} x\, dx - \frac{m(m-1)}{n^2}\int x^{m-2} \cos^n x\, dx\end{aligned}$$

4. $$\int x^n \sin^{2m} x\,dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cos(2m-2k)x\,dx$$

(see **2.633** 2) TI 333

5. $$\int x^n \sin^{2m+1} x\,dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int x^n \sin(2m-2k+1)x\,dx$$

(see **2.633** 1) TI 333

6. $$\int x^n \cos^{2m} x\,dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cos(2m-2k)x\,dx$$

(see **2.633** 2) TI 333

7. $$\int x^n \cos^{2m+1} x\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \int x^n \cos(2m-2k+1)x\,dx$$

(see **2.633** 2) TI 333

2.632

1. $$\int x^{\mu-1} \sin \beta x\,dx = \frac{i}{2}\,(i\beta)^{-\mu}\,\gamma(\mu, i\beta x) - \frac{i}{2}\,(-i\beta)^{-\mu}\,\gamma\,(\mu, -i\beta x)$$

$[\operatorname{Re}\mu > -1, \quad x > 0]$ ET I 317(2)

2. $$\int x^{\mu-1} \sin ax\,dx = -\frac{1}{2a^\mu} \left\{ \exp\left[\frac{\pi i}{2}(\mu-1)\right] \Gamma(\mu, -iax) + \exp\left[\frac{\pi i}{2}(1-\mu)\right] \Gamma(\mu, iax) \right\}$$

$[\operatorname{Re}\mu < 1, \quad a > 0, \quad x > 0]$ ET I 317(3)

3. $$\int x^{\mu-1} \cos \beta x\,dx = \frac{1}{2} \left\{ (i\beta)^{-\mu}\,\gamma(\mu, i\beta x) + (-i\beta)^{-\mu}\,\gamma\,(\mu, -i\beta x) \right\}$$

$[\operatorname{Re}\mu > 0, \quad x > 0]$ ET I 319(22)

4. $$\int x^{\mu-1} \cos ax\,dx = -\frac{1}{2a^\mu} \left\{ \exp\left(i\mu\frac{\pi}{2}\right) \Gamma(\mu, -iax) + \exp\left(-i\mu\frac{\pi}{2}\right) \Gamma(\mu, iax) \right\}$$ ET I 319(23)

2.633

1. $$\int x^n \sin ax\,dx = -\sum_{k=0}^{n} k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \cos\left(ax + \frac{1}{2}k\pi\right)$$ TI (487)

2.[8] $$\int x^n \cos ax\,dx = \sum_{k=0}^{n} k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \sin\left(ax + \frac{1}{2}k\pi\right)$$ TI (486)

3. $$\int x^{2n} \sin x\,dx = (2n)! \left\{ \sum_{k=0}^{n} (-1)^{k+1} \frac{x^{2n-2k}}{(2n-2k)!} \cos x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \sin x \right\}$$

4. $$\int x^{2n+1}\sin x\,dx = (2n+1)!\left\{\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\cos x + \sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x\right\}$$

5. $$\int x^{2n}\cos x\,dx = (2n)!\left\{\sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k}}{(2n-2k)!}\sin x + \sum_{k=0}^{n-1}(-1)^{k}\frac{x^{2n-2k-1}}{(2n-2k-1)!}\cos x\right\}$$

6. $$\int x^{2n+1}\cos x\,dx = (2n+1)!\left\{\sum_{k=0}^{n}(-1)^{k}\frac{x^{2n-2k+1}}{(2n-2k+1)!}\sin x + \sum_{k=0}^{n}\frac{x^{2n-2k}}{(2n-2k)!}\cos x\right\}$$

2.634

1. $$\int P_n(x)\sin mx\,dx = -\frac{\cos mx}{m}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$$

2. $$\int P_n(x)\cos mx\,dx = \frac{\sin mx}{m}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\frac{P_n^{(2k-1)}(x)}{m^{2k-1}}$$

In formulas **2.634**, $P_n(x)$ is any n^{th}-degree polynomial, and $P_n^{(k)}(x)$ is its k^{th} derivative with respect to x.

2.635 **Notation**: $z_1 = a + bx$.

1. $$\int z_1\sin kx\,dx = -\frac{1}{k}z_1\cos kx + \frac{b}{k^2}\sin kx$$

2. $$\int z_1\cos kx\,dx = \frac{1}{k}z_1\sin kx + \frac{b}{k^2}\cos kx$$

3. $$\int z_1^2\sin kx\,dx = \frac{1}{k}\left(\frac{2b^2}{k^2} - z_1^2\right)\cos kx + \frac{2bz_1}{k^2}\sin kx$$

4. $$\int z_1^2\cos kx\,dx = \frac{1}{k}\left(z_1^2 - \frac{2b^2}{k^2}\right)\sin kx + \frac{2bz_1}{k^2}\cos kx$$

5. $$\int z_1^3\sin kx\,dx = \frac{z_1}{k}\left(\frac{6b^2}{k^2} - z_1^2\right)\cos kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\sin kx$$

6. $$\int z_1^3\cos kx\,dx = \frac{z_1}{k}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx + \frac{3b}{k^2}\left(z_1^2 - \frac{2b^2}{k^2}\right)\cos kx$$

7. $$\int z_1^4\sin kx\,dx = -\frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\sin kx$$

8. $$\int z_1^4\cos kx\,dx = \frac{1}{k}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx + \frac{4bz_1}{k^2}\left(z_1^2 - \frac{6b^2}{k^2}\right)\cos kx$$

9. $$\int z_1^5\sin kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\sin kx - \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx$$

10. $$\int z_1^5\cos kx\,dx = \frac{5b}{k^2}\left(z_1^4 - \frac{12b^2}{k^2}z_1^2 + \frac{24b^4}{k^4}\right)\cos kx + \frac{z_1}{k}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx$$

11. $$\int z_1^6 \sin kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\sin kx - \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\cos kx$$

12. $$\int z_1^6 \cos kx\,dx = \frac{6bz_1}{k^2}\left(z_1^4 - \frac{20b^2}{k^2}z_1^2 + \frac{120b^4}{k^4}\right)\cos kx + \frac{1}{k}\left(z_1^6 - \frac{30b^2}{k^2}z_1^4 + \frac{360b^4}{k^4}z_1^2 - \frac{720b^6}{k^6}\right)\sin kx$$

2.636

1. $$\int x^n \sin^2 x\,dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}$$ GU (333)(2e)

2. $$\int x^n \cos^2 x\,dx = \frac{x^{n+1}}{2(n+1)} - \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^{k+1}x^{n-2k}}{2^{2k}(n-2k)!}\sin 2x + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^{k+1}x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\cos 2x\right\}$$ GU (333)(3e)

3. $$\int x \sin^2 x\,dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x$$

4. $$\int x^2 \sin^2 x\,dx = \frac{x^3}{6} - \frac{x}{4}\cos 2x - \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x$$ MZ 241

5. $$\int x \cos^2 x\,dx = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x$$

6. $$\int x^2 \cos^2 x\,dx = \frac{x^3}{6} + \frac{x}{4}\cos 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\sin 2x$$ MZ 245

2.637

1.[11] $$\int x^n \sin^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\cos 3x}{3^{2k+1}} - 3\cos x\right) - \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\sin 3x}{3^{2k+2}} - 3\sin x\right)\right\}$$ GU(333)(2f)

2. $$\int x^n \cos^3 x\,dx = \frac{n!}{4}\left\{\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k x^{n-2k}}{(n-2k)!}\left(\frac{\sin 3x}{3^{2k+1}} + 3\sin x\right) + \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\cos 3x}{3^{2k+2}} + 3\cos x\right)\right\}$$ GU(333)(3f)

3. $$\int x \sin^3 x\,dx = \frac{3}{4}\sin x - \frac{1}{36}\sin 3x - \frac{3}{4}x\cos x + \frac{x}{12}\cos 3x$$

4. $$\int x^2 \sin^3 x\,dx = -\left(\frac{3}{4}x^2 + \frac{3}{2}\right)\cos x + \left(\frac{x^2}{12} + \frac{1}{54}\right)\cos 3x + \frac{3}{2}x\sin x - \frac{x}{18}\sin 3x$$ MZ 241

5. $$\int x \cos^3 x\,dx = \frac{3}{4}\cos x + \frac{1}{36}\cos 3x + \frac{3}{4}x\sin x + \frac{x}{12}\sin 3x$$

6. $$\int x^2 \cos^3 x\,dx = \left(\frac{3}{4}x^2 - \frac{3}{2}\right)\sin x + \left(\frac{x^2}{12} - \frac{1}{54}\right)\sin 3x + \frac{3}{2}x\cos x + \frac{x}{18}\cos 3x$$ MZ 245, 246

2.638

1. $$\int \frac{\sin^q x}{x^p}\,dx = -\frac{\sin^{q-1} x\,[(p-2)\sin x + qx\cos x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)}\int \frac{\sin^q x\,dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)}\int \frac{\sin^{q-2} x\,dx}{x^{p-2}}$$ $[p \neq 1, \quad p \neq 2]$ TI (496)

2. $$\int \frac{\cos^q x}{x^p}\,dx = -\frac{\cos^{q-1} x\,[(p-2)\cos x - qx\sin x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)}\int \frac{\cos^q x\,dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)}\int \frac{\cos^{q-2} x\,dx}{x^{p-2}}$$ $[p \neq 1, \quad p \neq 2]$ TI (495)

3.[6] $$\int \frac{\sin x\,dx}{x^p} = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1}\int \frac{\cos x\,dx}{x^{p-1}} = -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)}\int \frac{\sin x\,dx}{x^{p-2}}$$ $(p > 2)$ TI (492)

4.[6] $$\int \frac{\cos x\,dx}{x^p} = -\frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1}\int \frac{\sin x\,dx}{x^{p-1}} = -\frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)}\int \frac{\cos x\,dx}{x^{p-2}}$$ $(p > 2)$ TI (491)

2.639

1. $$\int \frac{\sin x\,dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-2}\frac{(-1)^k(2k+1)!}{x^{2k+1}}\cos x + \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\sin x\right\} + \frac{(-1)^{n+1}}{(2n-1)!}\operatorname{ci}(x)$$

GU (333)(6b)a

2. $$\int \frac{\sin x}{x^{2n+1}}\,dx = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\cos x + \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}}\sin x\right\} + \frac{(-1)^n}{(2n)!}\operatorname{si}(x)$$

GU (333)(6b)a

3. $$\int \frac{\cos x\,dx}{x^{2n}}\,dx = \frac{(-1)^{n+1}}{x(2n-1)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\cos x - \sum_{k=0}^{n-2}\frac{(-1)^k(2k+1)!}{x^{2k+1}}\sin x\right\} + \frac{(-1)^n}{(2n-1)!}\operatorname{si}(x)$$

GU (333)(7b)

4. $$\int \frac{\cos x\,dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!}\left\{\sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k+1)!}{x^{2k+1}}\cos x - \sum_{k=0}^{n-1}\frac{(-1)^{k+1}(2k)!}{x^{2k}}\sin x\right\} + \frac{(-1)^n}{(2n)!}\operatorname{ci}(x)$$

GU (333)(7b)

2.641

1. $$\int \frac{\sin kx}{a+bx}\,dx = \frac{1}{b}\left[\cos\frac{ka}{b}\operatorname{si}(u) - \sin\frac{ka}{b}\operatorname{ci}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

2. $$\int \frac{\cos kx}{a+bx}\,dx = \frac{1}{b}\left[\cos\frac{ka}{b}\operatorname{ci}(u) + \sin\frac{ka}{b}\operatorname{si}(u)\right] \qquad \left[u = \frac{k}{b}(a+bx)\right]$$

3. $$\int \frac{\sin kx}{(a+bx)^2}\,dx = -\frac{1}{b}\frac{\sin kx}{a+bx} + \frac{k}{b}\int\frac{\cos kx}{a+bx}\,dx$$ (see **2.641** 2)

4. $$\int \frac{\cos kx}{(a+bx)^2}\,dx = -\frac{1}{b}\frac{\cos kx}{a+bx} - \frac{k}{b}\int\frac{\sin kx}{a+bx}\,dx$$ (see **2.641** 1)

5. $$\int \frac{\sin kx}{(a+bx)^3}\,dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k\cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2}\int\frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

6. $$\int \frac{\cos kx}{(a+bx)^3}\,dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k \sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

7. $$\int \frac{\sin kx}{(a+bx)^4}\,dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k \cos kx}{6b^2(a+bx)^2} + \frac{k^2 \sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3} \int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

8. $$\int \frac{\cos kx}{(a+bx)^4}\,dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k \sin kx}{6b^2(a+bx)^2} + \frac{k^2 \cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

9. $$\int \frac{\sin kx}{(a+bx)^5}\,dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k \cos kx}{12b^2(a+bx)^3} + \frac{k^2 \sin kx}{24b^3(a+bx)^2} + \frac{k^3 \cos kx}{24b^4(a+bx)} \frac{k^4}{24b^4} \int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

10. $$\int \frac{\cos kx}{(a+bx)^5}\,dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k \sin kx}{12b^2(a+bx)^3} + \frac{k^2 \cos kx}{24b^3(a+bx)^2} - \frac{k^3 \sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

11. $$\int \frac{\sin kx}{(a+bx)^6}\,dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k \cos kx}{20b^2(a+bx)^4} + \frac{k^2 \sin kx}{60b^3(a+bx)^3} + \frac{k^3 \cos kx}{120b^4(a+bx)^2} - \frac{k^4 \sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cos kx}{a+bx}\,dx$$

(see **2.641** 2)

12. $$\int \frac{\cos kx}{(a+bx)^6}\,dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k \sin kx}{20b^2(a+bx)^4} + \frac{k^2 \cos kx}{60b^3(a+bx)^3} - \frac{k^3 \sin kx}{120b^4(a+bx)^2} - \frac{k^4 \cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5} \int \frac{\sin kx}{a+bx}\,dx$$

(see **2.641** 1)

2.642

1. $$\int \frac{\sin^{2m} x}{x}\,dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \operatorname{ci}[(2m-2k)x]$$

2. $$\int \frac{\sin^{2m+1} x}{x}\,dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \operatorname{si}[(2m-2k+1)x]$$

3. $$\int \frac{\cos^{2m} x}{x}\,dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \operatorname{ci}[(2m-2k)x]$$

4. $$\int \frac{\cos^{2m+1} x}{x}\,dx = \frac{1}{2^{2m}} \sum_{k=0}^{m} \binom{2m+1}{k} \operatorname{ci}[(2m-2k+1)x]$$

5. $$\int \frac{\sin^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} + \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\operatorname{si}[(2m-2k)x]\right\}$$

6. $$\int \frac{\sin^{2m+1} x}{x^2}\,dx = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+1}\binom{2m+1}{k} \times \left\{\frac{\sin(2m-2k+1)x}{x} - (2m-2k+1)\operatorname{ci}[(2m-2k+1)x]\right\}$$

7. $$\int \frac{\cos^{2m} x}{x^2}\,dx = -\binom{2m}{m}\frac{1}{2^{2m}x} - \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\binom{2m}{k}\left\{\frac{\cos(2m-2k)x}{x} + (2m-2k)\operatorname{si}[(2m-2k)x]\right\}$$

8. $$\int \frac{\cos^{2m+1} x}{x^2} = -\frac{1}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{k}\left\{\frac{\cos(2m-2k+1)x}{x} + (2m-2k+1)\operatorname{si}[(2m-2k+1)x]\right\}$$

2.643

1. $$\int \frac{x^p\,dx}{\sin^q x} = -\frac{x^{p-1}\left[p\sin x + (q-2)x\cos x\right]}{(q-1)(q-2)\sin^{q-1} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\sin^{q-2} x}$$

2. $$\int \frac{x^p\,dx}{\cos^q x} = -\frac{x^{p-1}\left[p\cos x - (q-2)x\sin x\right]}{(q-1)(q-2)\cos^{q-1} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\cos^{q-2} x}$$

3.[4] $$\int \frac{x^n}{\sin x}\,dx = \frac{x^n}{n} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2\left(2^{2k-1}-1\right)}{(n+2k)(2k)!}B_{2k}x^{n+2k}$$

$[|x| < \pi, \quad n > 0]$ TU (333)(8b)

4. $$\int \frac{dx}{x^n\sin x} = -\frac{1}{nx^n} - \left[1+(-1)^n\right](-1)^{\frac{n}{2}}\frac{2^{n-1}-1}{n!}B_n\ln x - \sum_{\substack{k=1\\k\neq\frac{n}{2}}}^{\infty}(-1)^k\frac{2\left(2^{2n}-1\right)}{(2k-n)\cdot(2k)!}B_{2k}x^{2k-n}$$

$[n > 1, \quad |x| > \pi]$ GU (333)(9b)

5.[8] $$\int \frac{x^n\,dx}{\cos x} = \sum_{k=0}^{\infty}\frac{|E_{2k}|x^{n+2k+1}}{(n+2k+1)(2k)!}$$ $\left[|x| < \frac{\pi}{2}, \quad n > 0\right]$ GU (333)(10b)

6. $$\int \frac{dx}{x^n\cos x} = \frac{1}{2}\left[1-(-1)^n\right]\frac{|E_{n-1}|}{(n-1)!}\ln x + \sum_{\substack{k=0\\k\neq\frac{n-1}{2}}}^{\infty}\frac{|E_{2k}|x^{2k-n+1}}{(2k-n+1)\cdot(2k)!}$$

$\left[|x| < \frac{\pi}{2}\right]$ GU (333)(11b)

7. $$\int \frac{x^n\,dx}{\sin^2 x} = -x^n \cot x + \frac{n}{n-1} x^{n-1} + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{n+2k-1}}{(n+2k-1)(2k)!} B_{2k}$$
$[|x| < \pi, \quad n > 1]$ GU (333)(8c)

8. $$\int \frac{dx}{x^n \sin^2 x} = -\frac{\cot x}{x^n} + \frac{n}{(n+1)x^{n+1}} - \left[1 - (-1)^n\right] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \frac{n}{2^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k}$$
$[|x| < \pi]$ GU (333)(9c)

9. $$\int \frac{x^n\,dx}{\cos^2 x} = x^n \tan x + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}\left(2^{2k}-1\right) x^{n+2k-1}}{(n+2k-1)\cdot(2k)!} B_{2k}$$
$\left[n > 1, \quad |x| < \frac{\pi}{2}\right]$ GU (333)(10c)

10. $$\int \frac{dx}{x^n \cos^2 x} = \frac{\tan x}{x^n} - \left[1 - (-1)^n\right] (-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} \left(2^{n+1}-1\right) B_{n+1} \ln x - \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k \left(2^{2k}-1\right)(2x)^{2k}}{(2k-n-1)(2k)!} B_{2k}$$
$\left[|x| < \frac{\pi}{2}\right]$ GU (333)(11c)

2.644

1. $$\int \frac{x\,dx}{\sin^{2n} x} = -\sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{\sin x + (2n-2k)x\cos x}{(2n-2k+1)(2n-2k)\sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!} \left(\ln \sin x - x \cot x\right)$$

2. $$\int \frac{x\,dx}{\sin^{2n+1} x} = -\sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{\sin x + (2n-2k-1)x\cos x}{(2n-2k)(2n-2k-1)\sin^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x\,dx}{\sin x}$$
(see **2.644** 5)

3. $$\int \frac{x\,dx}{\cos^{2n} x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{(2n-2k)x\sin x - \cos x}{(2n-2k+1)(2n-2k)\cos^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!} \left(x \tan x + \ln \cos x\right)$$

4. $$\int \frac{x\,dx}{\cos^{2n+1} x} = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{(2n-2k+1)x\sin x - \cos x}{(2n-2k)(2n-2k-1)\cos^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x\,dx}{\cos x}$$
(see **2.644** 6)

5. $$\int \frac{x\,dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2\left(2^{2k-1}-1\right)}{(2k+1)!} B_{2k} x^{2k+1}$$

6. $$\int \frac{x\,dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}$$

7. $$\int \frac{x\,dx}{\sin^2 x} = -x\cot x + \ln\sin x$$

8. $$\int \frac{x\,dx}{\cos^2 x} = x\tan x + \ln\cos x$$

9. $$\int \frac{x\,dx}{\sin^3 x} = -\frac{\sin x + x\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{x}{\sin x}\,dx$$ (see **2.644** 5)

10. $$\int \frac{x\,dx}{\cos^3 x} = \frac{x\sin x - \cos x}{2\cos^2 x} + \frac{1}{2}\int \frac{x\,dx}{\cos x}$$ (see **2.644** 6)

11. $$\int \frac{x\,dx}{\sin^4 x} = -\frac{x\cos x}{3\sin^3 x} - \frac{1}{6\sin^2 x} - \frac{2}{3}x\cot x + \frac{2}{3}\ln(\sin x)$$

12. $$\int \frac{x\,dx}{\cos^4 x} = \frac{x\sin x}{3\cos^3 x} - \frac{1}{6\cos^2 x} + \frac{2}{3}x\tan x - \frac{2}{3}\ln(\cos x)$$

13. $$\int \frac{x\,dx}{\sin^5 x} = -\frac{x\cos x}{4\sin^4 x} - \frac{1}{12\sin^3 x} - \frac{3x\cos x}{8\sin^2 x} - \frac{3}{8\sin x} + \frac{3}{8}\int \frac{x\,dx}{\sin x}$$ (see **2.644** 5)

14. $$\int \frac{x\,dx}{\cos^5 x} = \frac{x\sin x}{4\cos^4 x} - \frac{1}{12\cos^3 x} + \frac{3x\sin x}{8\cos^2 x} - \frac{3}{8\cos x} + \frac{3}{8}\int \frac{x\,dx}{\cos x}$$ (see **2.644** 6)

2.645

1. $$\int x^p \frac{\sin^{2m} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\cos^{n-2k} x}$$ (see **2.643** 2)

2. $$\int x^p \frac{\sin^{2m+1} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x}\,dx$$ (see **2.645** 3)

3. $$\int x^p \frac{\sin x\,dx}{\cos^n x} = \frac{x^p}{(n-1)\cos^{n-1} x} - \frac{p}{n-1}\int \frac{x^{p-1}}{\cos^{n-1} x}\,dx$$ $[n > 1]$ (see **2.643** 2) GU (333)(12)

4. $$\int x^p \frac{\cos^{2m} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p\,dx}{\sin^{n-2k} x}$$ (see **2.643** 1)

5. $$\int x^p \frac{\cos^{2m+1} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x}\,dx$$ (see **2.645** 6)

6. $$\int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1)\sin^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}\,dx}{\sin^{n-1} x}$$ $[n > 1]$ (see **2.643** 1) GU (333)(13)

7. $$\int \frac{x \cos x}{\sin^2 x}\, dx = -\frac{x}{\sin x} + \ln \tan \frac{x}{2}$$

8. $$\int \frac{x \sin x}{\cos^2 x}\, dx = \frac{x}{\cos x} - \ln \tan \left(\frac{x}{2} + \frac{\pi}{4}\right)$$

2.646

1. $$\int x^p \tan x\, dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k}\left(2^{2k-1} - 1\right)}{(p+2k)\cdot(2k)!} B_{2k} x^{p+2k}$$
$\left[p \geq -1, \quad |x| < \frac{\pi}{2}\right]$ GU (333)(12d)

2. $$\int x^p \cot x\, dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k}$$
$[p \geq 1, \quad |x| < \pi]$ GU (333)(13d)

3. $$\int x^p \tan^2 x\, dx = x \tan x + \ln \cos x - \frac{x^2}{2}$$

4. $$\int x \cot^2 x\, dx = -x \cot x + \ln \sin x - \frac{x^2}{2}$$

2.647

1. $$\int \frac{x^n \cos x\, dx}{(a + b \sin x)^m} = -\frac{x^n}{(m-1)b\,(a + b \sin x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1}\, dx}{(a + b \sin x)^{m-1}}$$
$[m \neq 1]$ MZ 247

2. $$\int \frac{x^n \sin x\, dx}{(a + b \cos x)^m} = \frac{x^n}{(m-1)b\,(a + b \cos x)^{m-1}} - \frac{n}{(m-1)b} \int \frac{x^{n-1}\, dx}{(a + b \cos x)^{m-1}}$$
$[m \neq 1]$ MZ 247

3. $$\int \frac{x\, dx}{1 + \sin x} = -x \tan \left(\frac{\pi}{4} - \frac{x}{2}\right) + 2 \ln \cos \left(\frac{\pi}{4} - \frac{x}{2}\right)$$ PE (329)

4. $$\int \frac{x\, dx}{1 - \sin x} = x \cot \left(\frac{\pi}{4} - \frac{x}{2}\right) + 2 \ln \sin \left(\frac{\pi}{4} - \frac{x}{2}\right)$$ PE (330)

5. $$\int \frac{x\, dx}{1 + \cos x} = x \tan \frac{x}{2} + 2 \ln \cos \frac{x}{2}$$ PE (331)

6. $$\int \frac{x\, dx}{1 - \cos x} = -x \cot \frac{x}{2} + 2 \ln \cos \frac{x}{2}$$ PE (332)

7. $$\int \frac{x \cos x}{(1 + \sin x)^2}\, dx = -\frac{x}{1 + \sin x} + \tan \left(\frac{x}{2} - \frac{\pi}{4}\right)$$

8. $$\int \frac{x \cos x}{(1 - \sin x)^2}\, dx = \frac{x}{1 - \sin x} + \tan \left(\frac{x}{2} + \frac{\pi}{4}\right)$$

9. $$\int \frac{x \sin x}{(1 + \cos x)^2}\, dx = \frac{x}{1 + \cos x} - \tan \frac{x}{2}$$

10. $$\int \frac{x \sin x}{(1 - \cos x)^2}\, dx = -\frac{x}{1 - \cos x} - \cot \frac{x}{2}$$ MZ 247a

2.648

1. $$\int \frac{x+\sin x}{1+\cos x}\,dx = x\tan\frac{x}{2}$$

2. $$\int \frac{x-\sin x}{1-\cos x}\,dx = -x\cot\frac{x}{2}$$ GU (333)(16)

2.649 $$\int \frac{x^2\,dx}{\left[(ax-b)\sin x+(a+bx)\cos x\right]^2} = \frac{x\sin x+\cos x}{b\left[(ax-b)\sin x+(a+bx)\cos x\right]}$$ GU (333)(17)

2.651 $$\int \frac{dx}{\left[a+(ax+b)\tan x\right]^2} = \frac{\tan x}{a\left[a+(ax+b)\tan x\right]}$$ GU (333)(18)

2.652 $$\int \frac{x\,dx}{\cos(x+t)\cos(x-t)} = \operatorname{cosec} 2t\left\{x\ln\frac{\cos(x-t)}{\cos(x+t)} - L(x+t) + L(x-t)\right\}$$

$$\left[t \neq n\pi; \quad |x| < \left|\frac{\pi}{2} - |t_0|\right|\right],$$

where t_0 is the value of the argument t, which is reduced by multiples of the argument π to lie in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. LO III 288

2.653

1. $$\int \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{2\pi}\,S\left(\sqrt{x}\right)$$ (cf. **8.251** 21)

2. $$\int \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{2\pi}\,C\left(\sqrt{x}\right)$$ (cf. **8.251** 3)

2.654 **Notation**: $\Delta = \sqrt{1-k^2\sin^2 x}, \quad k' = \sqrt{1-k^2}$:

1. $$\int \frac{x\sin x\cos x}{\Delta}\,dx = -\frac{x\Delta}{k^2} + \frac{1}{k^2}E(x,k)$$

2. $$\int \frac{x\sin^3 x\cos x}{\Delta}\,dx = -\frac{k'^2}{9k^4}F(x,k) + \frac{2k^2+5}{9k^4}E(x,k) - \frac{1}{9k^4}\left[3\left(3-\Delta^2\right)x + k^2\sin x\cos x\right]\Delta$$

3. $$\int \frac{x\sin x\cos^3 x}{\Delta}\,dx = -\frac{k'^2}{9k^4}F(x,k) + \frac{7k^2-5}{9k^4}E(x,k) - \frac{1}{9k^4}\left[3\left(\Delta^2-3k'^2\right)x - k^2\sin x\cos x\right]\Delta$$

4. $$\int \frac{x\sin x\,dx}{\Delta^3}\,dx = -\frac{x\cos x}{k'^2\Delta} + \frac{1}{kk'^2}\arcsin\left(k\sin x\right)$$

5. $$\int \frac{x\cos x\,dx}{\Delta^3} = \frac{x\sin x}{\Delta} + \frac{1}{k}\ln\left(k\cos x+\Delta\right)$$

6. $$\int \frac{x\sin x\cos x\,dx}{\Delta^3} = \frac{x}{k^2\Delta} - \frac{1}{k^2}F(x,k)$$

7. $$\int \frac{x\sin^3 x\cos x\,dx}{\Delta^3} = x\frac{2-k^2\sin^2 x}{k^4\Delta} - \frac{1}{k^4}\left[E(x,k)+F(x,k)\right]$$

8. $$\int \frac{x\sin x\cos^3 x\,dx}{\Delta^3} = x\frac{k^2\sin^2 x+k^2-2}{k^4\Delta} + \frac{k'^2}{k^4}F(x,k) + \frac{1}{k^4}E(x,k)$$

2.655 Integrals containing $\sin x^2$ and $\cos x^2$

In integrals containing $\sin x^2$ and $\cos x^2$, it is expedient to make the substitution $x^2 = u$.

1. $$\int x^p \sin x^2\,dx = -\frac{x^{p-1}}{2}\cos x^2 + \frac{p-1}{2}\int x^{p-2}\cos x^2\,dx$$

2. $$\int x^p \cos x^2\,dx = \frac{x^{p-1}}{2}\sin x^2 - \frac{p-1}{2}\int x^{p-2}\sin x^2\,dx$$

3. $$\int x^n \sin x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^k\left[\frac{x^{n-4k+3}\cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1}\sin x^2}{2^{2k}(n-4k+1)!!}\right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\sin x^2\,dx\right\}$$
$$\left[r = \left\lfloor \frac{n}{4}\right\rfloor\right]$$ GU (336)(4a)

4. $$\int x^n \cos x^2\,dx = (n-1)!!\left\{\sum_{k=1}^{r}(-1)^{k-1}\left[\frac{x^{n-4k+3}\sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1}\cos x^2}{2^{2k}(n-4k+1)!!}\right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!}\int x^{n-4r}\cos x^2\,dx\right\}$$
$$\left[r = \left\lfloor \frac{n}{4}\right\rfloor\right]$$ GU (336)(5a)

5. $$\int x \sin x^2\,dx = -\frac{\cos^2 x}{2}$$

6. $$\int x \cos x^2\,dx = -\frac{\sin^2 x}{2}$$

7. $$\int x^2 \sin x^2\,dx = -\frac{x}{2}\cos x^2 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,C(x)$$

8. $$\int x^2 \cos x^2\,dx = \frac{x}{2}\sin x^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,S(x)$$

9. $$\int x^3 \sin x^2\,dx = -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2$$

10. $$\int x^3 \cos x^2\,dx = \frac{x^2}{2}\sin x^2 + \frac{1}{2}\cos x^2$$

2.66 Combinations of trigonometric functions and exponentials

2.661 $$\int e^{ax}\sin^p x\cos^q x\,dx = \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^p x\cos^{q-1}x\,[a\cos x+(p+q)\sin x]\right.$$

$$\left. - pa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\,dx + (q-1)(p+q)\int e^{ax}\sin^p x\cos^{q-2}x\,dx\right\}$$

TI (523)

$$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^{p-1}x\cos^q x\,[a\sin x-(p+q)\cos x]\right.$$

$$\left. + qa\int e^{ax}\sin^{p-1}x\cos^{q-1}x\,dx + (p-1)(p+q)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\right\}$$

TI (524)

$$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^{p-1}x\cos^{q-1}x\left[a\sin x\cos x+q\sin^2 x-p\cos^2 x\right]\right.$$

$$\left. + q(q-1)\int e^{ax}\sin^p x\cos^{q-2}x\,dx + p(p-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\right\}$$

TI (525)

$$= \frac{1}{a^2+(p+q)^2}\left\{e^{ax}\sin^{p-1}x\cos^{q-1}x\left(a\sin x\cos x+q\sin^2 x-p\cos^2 x\right)\right.$$

$$+q(q-1)\int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx$$

$$\left. - (q-p)(p+q-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\right\}$$

TI (526)

$$= \frac{1}{a^2+(p+q)^2}\left[e^{ax}\sin^{p-1}x\cos^{q-1}x\left(a\sin x\cos x+q\sin^2 x-p\cos^2 x\right)\right.$$

$$+p(p-1)\int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx$$

$$\left. + (q-p)(p+q-1)\int e^{ax}\sin^p x\cos^{q-2}x\,dx\right]$$

GU (334)(1a)

For $p=m$ and $q=n$ even integers, the integral $\int e^{ax}\sin^m x\cos^n x\,dx$ can be reduced by means of these formulas to the integral $\int e^{ax}\,dx$. However, when only m or only n is even, they can be reduced to integrals of the form $\int e^{ax}\cos^n x\,dx$ or $\int e^{ax}\sin^m x\,dx$, respectively.

2.662

1. $$\int e^{ax}\sin^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)\,e^{ax}\sin^{n-1}bx + n(n-1)b^2\int e^{ax}\sin^{n-2}bx\,dx\right]$$

2. $$\int e^{ax}\cos^n bx\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1}bx + n(n-1)b^2\int e^{ax}\cos^{n-2}bx\,dx\right]$$

3. $$\int e^{ax}\sin^{2m} bx\,dx$$
$$= \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\sin^{2m-2k-1}bx}{(2m-2k)!\left[a^2+(2m)^2b^2\right]\left[a^2+(2m-2)^2b^2\right]\cdots\left[a^2+(2m-2k)^2b^2\right]}$$
$$\times\left[a\sin bx-(2m-2k)b\cos bx\right] + \frac{(2m)!b^{2m}e^{ax}}{\left[a^2+(2m)^2b^2\right]\left[a^2+(2m-2)^2b^2\right]\cdots\left[a^2+4b^2\right]a}$$
$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}\left(a\cos 2bkx + 2bk\sin 2bkx\right)$$

4. $$\int e^{ax}\sin^{2m+1} bx\,dx$$
$$= \sum_{k=0}^{m}\frac{(2m+1)!b^{2k}e^{ax}\sin^{2m-2k}bx\left[a\sin bx-(2m-2k+1)b\cos bx\right]}{(2m-2k+1)!\left[a^2+(2m+1)^2b^2\right]\left[a^2+(2m-1)^2b^2\right]\cdots\left[a^2+(2m-2k+1)^2b^2\right]}$$
$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\frac{(-1)^k}{a^2+(2k+1)^2b^2}\binom{2m+1}{m-k}\left[a\sin(2k+1)bx-(2k+1)b\cos(2k+1)bx\right]$$

5.[8] $$\int e^{ax}\cos^{2m} bx\,dx = \sum_{k=0}^{m-1}\frac{(2m)!b^{2k}e^{ax}\cos^{2m-2k-1}bx\left[a\cos bx+(2m-2k)b\sin bx\right]}{(2m-2k)!\left[a^2+(2m)^2b^2\right]\left[a^2+(2m-2)^2b^2\right]\cdots\left[a^2+(2m-2k)^2b^2\right]}$$
$$+\frac{(2m)!b^{2m}e^{ax}}{\left[a^2+(2m)^2b^2\right]\left[a^2+(2m-2)^2b^2\right]\cdots\left[a^2+4b^2\right]a}$$
$$= \binom{2m}{m}\frac{e^{ax}}{2^{2m}a} + \frac{e^{ax}}{2^{2m-1}}\sum_{k=1}^{m}\binom{2m}{m-k}\frac{1}{a^2+4b^2k^2}\left[a\cos 2kbx + 2kb\sin 2kbx\right]$$

6. $$\int e^{ax}\cos^{2m+1} bx\,dx$$
$$= \sum_{k=0}^{m}\frac{(2m+1)!b^{2k}e^{ax}\cos^{2m-2k}bx}{(2m-2k+1)!\left[a^2+(2m-1)^2b^2\right]\cdots\left[a^2+(2m-2k+1)^2b^2\right]}$$
$$= \frac{e^{ax}}{2^{2m}}\sum_{k=0}^{m}\binom{2m+1}{m-k}\frac{1}{a^2+(2k+1)^2b^2}\left[a\cos(2k+1)bx+(2k+1)b\sin(2k+1)bx\right]$$

2.663

1. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2+b^2}$$

2. $$\int e^{ax}\sin^2 bx\,dx = \frac{e^{ax}\sin bx\,(a\sin bx - 2b\cos bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)\,a}$$
$$= \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right)$$

3. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\,(a\cos bx + b\sin bx)}{a^2+b^2}$$

4. $$\int e^{ax}\cos^2 bx\,dx = \frac{e^{ax}\cos bx\,(a\cos bx + 2b\sin bx)}{4b^2+a^2} + \frac{2b^2e^{ax}}{(4b^2+a^2)\,a}$$
$$= \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right)$$

2.664

1. $$\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}}{2}\left[\frac{a\sin(b+c)x - (b+c)\cos(b+c)x}{a^2+(b+c)^2} + \frac{a\sin(b-c)x - (b-c)\cos(b-c)x}{a^2+(b-c)^2}\right]$$
GU (334)(6b)

2. $$\int e^{ax}\sin^2 bx\cos cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\cos cx + c\sin cx}{a^2+c^2} - \frac{a\cos(2b+c)x + (2b+c)\sin(2b+c)x}{a^2+(2b+c)^2} - \frac{a\cos(2b-c)x + (2b-c)\sin(2b-c)x}{a^2+(2b-c)^2}\right]$$
GU (334)(6c)

3. $$\int e^{ax}\sin bx\cos^2 cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\sin bx - b\cos bx}{a^2+b^2} + \frac{a\sin(b+2c)x - (b+2c)\cos(b+2c)x}{a^2+(b+2c)^2} + \frac{a\sin(b-2c)x - (b-2c)\cos(b-2c)x}{a^2+(b-2c)^2}\right]$$
GU (334)(6d)

2.665

1. $$\int \frac{e^{ax}\,dx}{\sin^p bx} = -\frac{e^{ax}\,[a\sin bx + (p-2)b\cos bx]}{(p-1)(p-2)b^2\sin^{p-1}bx} + \frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int\frac{e^{ax}\,dx}{\sin^{p-2}bx}$$ TI (530)a

2. $$\int \frac{e^{ax}\,dx}{\cos^p bx} = -\frac{e^{ax}\,[a\cos bx - (p-2)b\sin bx]}{(p-1)(p-2)b^2\cos^{p-1}bx} + \frac{a^2+(p-2)^2b^2}{(p-1)(p-2)b^2}\int\frac{e^{ax}\,dx}{\cos^{p-2}bx}$$ TI (529)a

By successive applications of formulas **2.665** for p a natural number, we obtain integrals of the form $\int\frac{e^{ax}\,dx}{\sin bx}$, $\int\frac{e^{ax}\,dx}{\sin^2 bx}$, $\int\frac{e^{ax}\,dx}{\cos bx}$, $\int\frac{e^{ax}\,dx}{\cos^2 bx}$, which are not expressible in terms of a finite combination of elementary functions.

2.666

1. $$\int e^{ax}\tan^p x\,dx = \frac{e^{ax}}{p-1}\tan^{p-1}x - \frac{a}{p-1}\int e^{ax}\tan^{p-1}x\,dx - \int e^{ax}\tan^{p-2}x\,dx$$ TI (527)

2. $$\int e^{ax}\cot^p x\,dx = -\frac{e^{ax}\cot^{p-1}x}{p-1} + \frac{a}{p-1}\int e^{ax}\cot^{p-1}x\,dx - \int e^{ax}\cot^{p-2}x\,dx$$ TI (528)

3. $$\int e^{ax}\tan x\,dx = \frac{e^{ax}\tan x}{a} - \frac{1}{a}\int\frac{e^{ax}\,dx}{\cos^2 x}$$ (see remark following **2.665**)

4. $$\int e^{ax}\tan^2 x\,dx = \frac{e^{ax}}{a}(a\tan x - 1) - a\int e^{ax}\tan x\,dx$$ (see **2.666** 3) TI 355

5. $$\int e^{ax}\cot x\,dx = \frac{e^{ax}\cot x}{a} + \frac{1}{a}\int \frac{e^{ax}\,dx}{\sin^2 x}$$ (see remark following **2.665**)

6. $$\int e^{ax}\cot^2 x\,dx = -\frac{e^{ax}}{a}(a\cot x + 1) + a\int e^{ax}\cot x\,dx$$

(see **2.666** 5)

Integrals of type $\int R(x, e^{ax}, \sin bx, \cos cx)\,dx$

Notation: $\sin t = -\frac{b}{\sqrt{a^2+b^2}}; \quad \cos t = \frac{a}{\sqrt{a^2+b^2}}.$

2.667

1. $$\begin{aligned}\int x^p e^{ax}\sin bx\,dx &= \frac{x^p e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a\sin bx - b\cos bx)\,dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}}\sin(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax}\sin(bx+t)\,dx\end{aligned}$$

2. $$\begin{aligned}\int x^p e^{ax}\cos bx\,dx &= \frac{x^p e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) - \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a\cos bx + b\sin bx)\,dx \\ &= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}}\cos(bx+t) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax}\cos(bx+t)\,dx\end{aligned}$$

3. $$\int x^n e^{ax}\sin bx\,dx = e^{ax}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}n!x^{n-k+1}}{(n-k+1)!\left(a^2+b^2\right)^{k/2}}\sin(bx+kt)$$

4. $$\int x^n e^{ax}\cos bx\,dx = e^{ax}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}n!x^{n-k+1}}{(n-k+1)!\left(a^2+b^2\right)^{k/2}}\cos(bx+kt)$$

5. $$\int xe^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\sin bx - \left(bx - \frac{2ab}{a^2+b^2}\right)\cos bx\right]$$

6. $$\int xe^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\cos bx + \left(bx - \frac{2ab}{a^2+b^2}\right)\sin bx\right]$$

7. $$\begin{aligned}\int x^2 e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\Bigg\{&\left[ax^2 - \frac{2\left(a^2-b^2\right)}{a^2+b^2}x + \frac{2a\left(a^2-3b^2\right)}{\left(a^2+b^2\right)^2}\right]\sin bx \\ &- \left[bx^2 - \frac{4ab}{a^2+b^2}x + \frac{2b\left(3a^2-b^2\right)}{\left(a^2+b^2\right)^2}\right]\cos bx\Bigg\}\end{aligned}$$

8. $$\int x^2 e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} \left\{ \left[ax^2 - \frac{2\left(a^2-b^2\right)}{a^2+b^2} x + \frac{2a\left(a^2-3b^2\right)}{\left(a^2+b^2\right)^2} \right] \cos bx + \left[bx^2 - \frac{4ab}{a^2+b^2} x + \frac{2b\left(3a^2-b^2\right)}{\left(a^2+b^2\right)^2} \right] \sin bx \right\}$$

GU (335), MZ 274-275

2.67 Combinations of trigonometric and hyperbolic functions

2.671

1. $$\int \sinh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\sinh(ax+b)\cos(cx+d)$$

2. $$\int \sinh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\cosh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\sinh(ax+b)\sin(cx+d)$$

3. $$\int \cosh(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\sin(cx+d) - \frac{c}{a^2+c^2}\cosh(ax+b)\cos(cx+d)$$

4. $$\int \cosh(ax+b)\cos(cx+d)\,dx = \frac{a}{a^2+c^2}\sinh(ax+b)\cos(cx+d) + \frac{c}{a^2+c^2}\cosh(ax+b)\sin(cx+d)$$

GU (354)(1)

2.672

1. $$\int \sinh x \sin x \, dx = \frac{1}{2}\left(\cosh x \sin x - \sinh x \cos x\right)$$

2. $$\int \sinh x \cos x \, dx = \frac{1}{2}\left(\cosh x \cos x + \sinh x \sin x\right)$$

3. $$\int \cosh x \sin x \, dx = \frac{1}{2}\left(\sinh x \sin x - \cosh x \cos x\right)$$

4. $$\int \cosh x \cos x \, dx = \frac{1}{2}\left(\sinh x \cos x + \cosh x \sin x\right)$$

2.673

1. $$\int \sinh^{2m}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{(-1)^{m+n}}{2^{2m+2n-1}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k)c}\binom{2n}{k}\sin[(2n-2k)(cx+d)]$$
$$+\frac{(-1)^n}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(3a)

2. $$\int \sinh^{2m}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{m+n}}{2^{2m+2n-2}}\binom{2m}{m}\sum_{k=0}^{n-1}\frac{(-1)^k}{(2n-2k-1)c}\binom{2n-1}{k}\cos[(2n-2k-1)(cx+d)]$$
$$+\frac{(-1)^{n-1}}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(3b)

3. $$\int \sinh^{2m-1}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{(-1)^j\binom{2m-1}{j}}{(2m-2j-1)a}\cosh[(2m-2j-1)(ax+d)]$$
$$+\frac{(-1)^n}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times\{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(3c)

4. $$\int \sinh^{2m-1}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}}{2^{2m-2n-4}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^{j+k}\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(3d)

5. $$\int \sinh^{2m}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{(-1)^m}{2^{2m+2n}}\binom{2m}{m}\binom{2n}{n}x + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j\binom{2m}{j}}{(2m-2j)a}\sinh[(2m-2j)(ax+b)]$$
$$+\frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)]$$
$$+\frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^j\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(4a)

6. $$\int \sinh^{2m}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^m\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c}\sin[(2n-2k-2)(cx+d)]$$
$$+\frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{(-1)^j\binom{2m}{j}\binom{2-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(4a)

7. $$\int \sinh^{2m-1}(ax+b)\cos^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \cosh[(2m-2j-1)(ax+d)]$$
$$+\frac{1}{2^{2m-2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(4b)

8. $$\int \sinh^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j-1)a\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(4b)

9. $$\int \cosh^{2m}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{(-1)^n\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)]$$
$$+\frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \sinh[(2m-2j)(ax+b)]$$
$$+\frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(5a)

10. $$\int \cosh^{2m-1}(ax+b)\sin^{2n}(cx+d)\,dx$$
$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \sinh[(2m-2j-1)(ax+b)]$$
$$+ \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$
$$\times \{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$
$$+(2n-2k)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(5a)

11. $$\int \cosh^{2m}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1}\binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)]$$
$$+ \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j)a\sinh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(5b)

12. $$\int \cosh^{2m-1}(ax+b)\sin^{2n-1}(cx+d)\,dx$$
$$= \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2a^2+(2n-2k-1)^2c^2}$$
$$\times \{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]\}$$
$$-(2n-2k-1)c\cosh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]$$

GU (354)(5b)

13. $$\int \cosh^{2m}(ax+b)\cos^{2n}(cx+d)\,dx$$

$$= \frac{\binom{2m}{m}\binom{2n}{n}}{2^{2m+2n}}x + \frac{\binom{2m}{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{\binom{2}{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)]$$

$$+\frac{\binom{2n}{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{\binom{2m}{j}}{(2m-2j)a}\sinh[(2m-2j)(ax+b)]$$

$$+\frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m}{j}\binom{2n}{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$

$$+(2n-2k)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(6)

14. $$\int \cosh^{2m-1}(ax+b)\cos^{2n}(cx+d)\,dx$$

$$= \frac{\binom{2n}{n}}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\frac{\binom{2m-1}{j}}{(2m-2j-1)a}\sinh[(2m-2j-1)(ax+b)]$$

$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m-1}{j}\binom{2n}{k}}{(2m-2j-1)^2a^2+(2n-2k)^2c^2}$$

$$\times\{(2m-2j-1)a\sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k)(cx+d)]\}$$

$$+(2n-2k)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k)(cx+d)]$$

GU (354)(6)

15. $$\int \cosh^{2m}(ax+b)\cos^{2n-1}(cx+d)\,dx$$

$$= \frac{\binom{2m}{m}}{2^{2m+2n-2}}\sum_{k=0}^{n-1}\frac{\binom{2n-1}{k}}{(2n-2k-1)c}\sin[(2n-2k-1)(cx+d)]$$

$$+\frac{1}{2^{2m+2n-3}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\binom{2m}{j}\binom{2n-1}{k}}{(2m-2j)^2a^2+(2n-2k-1)^2c^2}$$

$$\times\{(2m-2j)a\sinh[(2m-2j)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$

$$+(2n-2k-1)c\cosh[(2m-2j)(ax+b)]\sin[(2n-2k-1)(cx+d)]$$

GU (354)(6)

16. $$\int \cosh^{2m-1}(ax+b)\cos^{2n-1}(cx+d)\,dx$$
$$= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j}\binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2}$$
$$\times \{(2m-2j-1)a \sinh[(2m-2j-1)(ax+b)]\cos[(2n-2k-1)(cx+d)]\}$$
$$+(2n-2k-1)c\cosh[(2m-2j-1)(ax+b)]\sin[(2n-2k-1)(cx+d)]]$$
GU (354)(6)

2.674

1. $$\int e^{ax}\sinh bx \sin cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\sin cx - c\cos cx\right] - \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\sin cx - c\cos cx\right]$$

2. $$\int e^{ax}\sinh bx \cos cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\cos cx + c\sin cx\right] - \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\cos cx + c\sin cx\right]$$

3. $$\int e^{ax}\cosh bx \sin cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\sin cx - c\cos cx\right] + \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\sin cx - c\cos cx\right]$$

4. $$\int e^{ax}\cosh bx \cos cx\,dx = \frac{e^{(a+b)x}}{2\left[(a+b)^2+c^2\right]}\left[(a+b)\cos cx + c\sin cx\right] + \frac{e^{(a-b)x}}{2\left[(a-b)^2+c^2\right]}\left[(a-b)\cos cx + c\sin cx\right]$$

MZ 379

2.7 Logarithms and Inverse-Hyperbolic Functions

2.71 The logarithm

2.711 $$\int \ln^m x\,dx = x\ln^m x - m\int \ln^{m-1} x\,dx$$
$$= \frac{x}{m+1}\sum_{k=0}^{m}(-1)^k (m+1)m(m-1)\cdots(m-k+1)\ln^{m-k} x$$
$(m>0)$ TI (603)

2.72–2.73 Combinations of logarithms and algebraic functions

2.721

1. $$\int x^n \ln^m x\,dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x\,dx \qquad \text{(see } \mathbf{2.722}\text{)}$$

For $n=-1$

2. $$\int \frac{\ln^m x\,dx}{x} = \frac{\ln^{m+1} x}{m+1}$$

For $n=-1$ and $m=-1$

3. $$\int \frac{dx}{x\ln x} = \ln(\ln x)$$

2.722 $$\int x^n \ln^m x\,dx = \frac{x^{n+1}}{m+1}\sum_{k=0}^{m}(-1)^k (m+1)m(m-1)\cdots(m-k+1)\frac{\ln^{m-k} x}{(n+1)^{k+1}}$$ TI (604)

2.723

1. $$\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right]$$ TI 375

2. $$\int x^n \ln^2 x\,dx = x^{n+1}\left[\frac{\ln^2 x}{n+1} - \frac{2\ln x}{(n+1)^2} + \frac{2}{(n+1)^3}\right]$$ TI 375

3. $$\int x^n \ln^3 x\,dx = x^{n+1}\left[\frac{\ln^3 x}{n+1} - \frac{3\ln^2 x}{(n+1)^2} + \frac{6\ln x}{(n+1)^3} - \frac{6}{(n+1)^4}\right]$$

2.724

1. $$\int \frac{x^n\,dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\ln x)^{m-1}}$$

For $m=1$

2. $$\int \frac{x^n\,dx}{\ln x} = \operatorname{li}\left(x^{n+1}\right)$$

2.725

1. $$\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1}\ln x - \int \frac{(a+bx)^{m+1}\,dx}{x}\right]$$ TI 374

2. $$\int (a+bx)^m \ln x\,dx = \frac{1}{(m+1)b}\left[(a+bx)^{m+1} - a^{m+1}\right]\ln x - \sum_{k=0}^{m}\frac{\binom{m}{k}a^{m-k}b^k x^{k+1}}{(k+1)^2}$$

For $m=-1$, see **2.727** 2.

2.726

1. $$\int (a+bx)\ln x\,dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b}\right]\ln x - \left(ax + \frac{1}{4}bx^2\right)$$

2. $$\int (a+bx)^2 \ln x\,dx = \frac{1}{3b}\left[(a+bx)^3 - a^3\right]\ln x - \left(a^2x + \frac{abx^2}{2} + \frac{b^2x^3}{9}\right)$$

3. $$\int (a+bx)^3 \ln x\, dx = \frac{1}{4b}\left[(a+bx)^4 - a^4\right]\ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4\right)$$

2.727

1.[8] $$\int \frac{\ln x\, dx}{(a+bx)^m} = \frac{1}{b(m-1)}\left[-\frac{\ln x}{(a+bx)^{m-1}} + \int \frac{dx}{x(a+bx)^{m-1}}\right]$$ TI 376

For $m=1$

2.[8] $$\int \frac{\ln x\, dx}{a+bx} = \frac{1}{b}\ln x \ln(a+bx) - \frac{1}{b}\int \frac{\ln(a+bx)\, dx}{x}$$ (see **2.728** 2)

3. $$\int \frac{\ln x\, dx}{(a+bx)^2} = -\frac{\ln x}{b(a+bx)} + \frac{1}{ab}\ln\frac{x}{a+bx}$$

4. $$\int \frac{\ln x\, dx}{(a+bx)^3} = -\frac{\ln x}{2b(a+bx)^2} + \frac{1}{2ab(a+bx)} + \frac{1}{2a^2b}\ln\frac{x}{a+bx}$$

5. $$\int \frac{\ln x\, dx}{\sqrt{a+bx}} = \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} - 2\sqrt{a}\ln\left[\frac{(a+bx)^{1/2} - a^{1/2}}{x^{1/2}}\right]\right\} \qquad [a>0]$$
$$= \frac{2}{b}\left\{(\ln x - 2)\sqrt{a+bx} + 2\sqrt{-a}\arctan\sqrt{\frac{a+bx}{-a}}\right\} \qquad [a<0]$$

2.728

1. $$\int x^m \ln(a+bx)\, dx = \frac{1}{m+1}\left[x^{m+1}\ln(a+bx) - b\int \frac{x^{m+1}\, dx}{a+bx}\right]$$

2.[9] $$\int \frac{\ln(a+bx)}{x} = \ln a \ln x + \frac{bx}{a}\Phi\left(-\frac{bx}{a}, 2, 1\right) \qquad [a>0]$$

2.729

1. $$\int x^m \ln(a+bx)\, dx = \frac{1}{m+1}\left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}}\right]\ln(a+bx) + \frac{1}{m+1}\sum_{k=1}^{m+1}\frac{(-1)^k x^{m-k+2}a^{k-1}}{(m-k+2)b^{k-1}}$$

2. $$\int x\ln(a+bx)\, dx = \frac{1}{2}\left[x^2 - \frac{a^2}{b^2}\right]\ln(a+bx) - \frac{1}{2}\left[\frac{x^2}{2} - \frac{ax}{b}\right]$$

3. $$\int x^2\ln(a+bx)\, dx = \frac{1}{3}\left[x^3 + \frac{a^3}{b^3}\right]\ln(a+bx) - \frac{1}{3}\left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2}\right]$$

4. $$\int x^3\ln(a+bx)\, dx = \frac{1}{4}\left[x^4 - \frac{a^4}{b^4}\right]\ln(a+bx) - \frac{1}{4}\left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3}\right]$$

2.731 $$\int x^{2n}\ln\left(x^2+a^2\right)\, dx = \frac{1}{2n+1}\left\{x^{2n+1}\ln\left(x^2+a^2\right) + (-1)^n 2a^{2n+1}\arctan\frac{x}{a} - 2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2k+1}a^{2n-2k}x^{2k+1}\right\}$$

2.732[7] $$\int x^{2n+1}\ln\left(x^2+a^2\right)\,dx = \frac{1}{2n+2}\left\{\left(x^{2n+2}+(-1)^n a^{2n+2}\right)\ln\left(x^2+a^2\right) + \sum_{k=1}^{n+1}\frac{(-1)^{n-k}}{k}a^{2n-2k+2}x^{2k}\right\}$$

2.733

1. $$\int \ln\left(x^2+a^2\right)\,dx = x\ln\left(x^2+a^2\right) - 2x + 2a\arctan\frac{x}{a}$$ DW

2. $$\int x\ln\left(x^2+a^2\right)\,dx = \frac{1}{2}\left[\left(x^2+a^2\right)\ln\left(x^2+a^2\right) - x^2\right]$$ DW

3. $$\int x^2\ln\left(x^2+a^2\right)\,dx = \frac{1}{3}\left[x^3\ln\left(x^2+a^2\right) - \frac{2}{3}x^3 + 2a^2x - 2a^3\arctan\frac{x}{a}\right]$$ DW

4. $$\int x^3\ln\left(x^2+a^2\right)\,dx = \frac{1}{4}\left[\left(x^4-a^4\right)\ln\left(x^2+a^2\right) - \frac{x^4}{2} + a^2x^2\right]$$ DW

5. $$\int x^4\ln\left(x^2+a^2\right)\,dx = \frac{1}{5}\left[x^5\ln\left(x^2+a^2\right) - \frac{2}{5}x^5 + \frac{2}{3}a^2x^3 - 2a^4x + 2a^5\arctan\frac{x}{a}\right]$$ DW

2.734 $$\int x^{2n}\ln\left|x^2-a^2\right|dx = \frac{1}{2n+1}\left\{x^{2n+1}\ln\left|x^2-a^2\right| + a^{2n+1}\ln\left|\frac{x+a}{x-a}\right| - 2\sum_{k=0}^{n}\frac{1}{2k+1}a^{2n-2k}x^{2k+1}\right\}$$

2.735 $$\int x^{2n+1}\ln\left|x^2-a^2\right|dx = \frac{1}{2n+2}\left\{\left(x^{2n+2}-a^{2n+2}\right)\ln\left|x^2-a^2\right| - \sum_{k=1}^{n+1}\frac{1}{k}a^{2n-2k+2}x^{2k}\right\}$$

2.736

1. $$\int \ln\left|x^2-a^2\right|dx = x\ln\left|x^2-a^2\right| - 2x + a\ln\left|\frac{x+a}{x-a}\right|$$ DW

2. $$\int x\ln\left|x^2-a^2\right|dx = \frac{1}{2}\left\{\left(x^2-a^2\right)\ln\left|x^2-a^2\right| - x^2\right\}$$ DW

3. $$\int x^2\ln\left|x^2-a^2\right|dx = \frac{1}{3}\left\{x^3\ln\left|x^2-a^2\right| - \frac{2}{3}x^3 - 2a^2x + a^3\ln\left|\frac{x+a}{x-a}\right|\right\}$$ DW

4. $$\int x^3\ln\left|x^2-a^2\right|dx = \frac{1}{4}\left\{\left(x^4-a^4\right)\ln\left|x^2-a^2\right| - \frac{x^4}{2} - a^2x^2\right\}$$ DW

5. $$\int x^4\ln\left|x^2-a^2\right|dx = \frac{1}{5}\left\{x^5\ln\left|x^2-a^2\right| - \frac{2}{5}x^5 - \frac{2}{3}a^2x^3 - 2a^4x + a^5\ln\left|\frac{x+a}{x-a}\right|\right\}$$ DW

2.74 Inverse hyperbolic functions

2.741

1. $$\int \operatorname{arcsinh}\frac{x}{a}\,dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2}$$ DW

2. $$\int \operatorname{arccosh}\frac{x}{a}\,dx = x\operatorname{arccosh}\frac{x}{a} - \sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} > 0\right]$$ DW
$$= x\operatorname{arccosh}\frac{x}{a} + \sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} < 0\right]$$ DW

3. $$\int \operatorname{arctanh}\frac{x}{a}\,dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln\left(a^2-x^2\right)$$ DW

4. $$\int \operatorname{arccoth}\frac{x}{a}\,dx = x\operatorname{arccoth}\frac{x}{a} + \frac{a}{2}\ln\left(x^2-a^2\right)$$ DW

2.742

1. $$\int x\operatorname{arcsinh}\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\operatorname{arcsinh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}$$ DW

2. $$\int x\operatorname{arccosh}\frac{x}{a}\,dx = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{arccosh}\frac{x}{a} - \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} > 0\right]$$
$$= \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\operatorname{arccosh}\frac{x}{a} + \frac{x}{4}\sqrt{x^2-a^2} \qquad \left[\operatorname{arccosh}\frac{x}{a} < 0\right]$$
DW

2.8 Inverse Trigonometric Functions

2.81 Arcsines and arccosines

2.811 $$\int \left(\arcsin\frac{x}{a}\right)^n dx = x\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cdot(2k)!\left(\arcsin\frac{x}{a}\right)^{n-2k} + \sqrt{a^2-x^2}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k-1}\binom{n}{2k-1}\cdot(2k-1)!\left(\arcsin\frac{x}{a}\right)^{n-2k+1}$$

2.812 $$\int \left(\arccos\frac{x}{a}\right)^n dx = x\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}\cdot(2k)!\left(\arccos\frac{x}{a}\right)^{n-2k} + \sqrt{a^2-x^2}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-1)^{k}\binom{n}{2k-1}\cdot(2k-1)!\left(\arccos\frac{x}{a}\right)^{n-2k+1}$$

2.813

1.[11] $$\int \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[x\arcsin\frac{x}{|a|} + \sqrt{a^2-x^2}\right]$$

2.[9] $$\int \left(\arcsin\frac{x}{a}\right)^2 dx = x\left(\arcsin\frac{x}{|a|}\right)^2 + 2\sqrt{a^2-x^2}\arcsin\frac{x}{|a|} - 2x$$

3. $$\int \left(\arcsin\frac{x}{a}\right)^3 dx = \operatorname{sign}(a)\left[x\left(\arcsin\frac{x}{|a|}\right)^3 + 3\sqrt{a^2-x^2}\left(\arcsin\frac{x}{|a|}\right)^2 - 6x\arcsin\frac{x}{|a|} - 6\sqrt{a^2-x^2}\right]$$

2.814

1. $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}$$

2. $$\int \left(\arccos\frac{x}{a}\right)^2 dx = x\left(\arccos\frac{x}{a}\right)^2 - 2\sqrt{a^2-x^2}\arccos\frac{x}{a} - 2x$$

3. $$\int \left(\arccos\frac{x}{a}\right)^3 dx = x\left(\arccos\frac{x}{a}\right)^3 - 3\sqrt{a^2-x^2}\left(\arccos\frac{x}{a}\right)^2 - 6x\arccos\frac{x}{a} + 6\sqrt{a^2-x^2}$$

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.821

1. $$\int \operatorname{arccosec}\frac{x}{a}\,dx = \int \arcsin\frac{a}{x}\,dx = x\arcsin\frac{x}{2} + a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[0 < \arcsin\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= x\arcsin\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[-\frac{\pi}{2} < \arcsin\frac{a}{x} < 0\right]$$
DW

2. $$\int \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = x\arccos\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= x\arccos\frac{a}{x} - a\ln\left(x+\sqrt{x^2-a^2}\right) \qquad \left[-\frac{\pi}{2} < \arccos\frac{a}{x} < 0\right]$$
DW

2.822

1.[8] $$\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln\left(a^2+x^2\right)$$ DW

2. $$\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} - \frac{a}{2}\ln\left(a^2+x^2\right)$$ DW

3.[9] $$\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a} - \frac{ax}{2}$$

4.[9] $$\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a}$$

5.[9] $$\int x^2\arctan\frac{x}{a}\,dx = \frac{1}{3}x^3\arctan\frac{x}{a} + \frac{1}{6}a^3\ln\left(x^2+a^2\right) - \frac{ax^2}{6}$$

6.[9] $$\int x^2\operatorname{arccot}\frac{x}{a}\,dx = -\frac{1}{3}x^3\arctan\frac{x}{a} - \frac{1}{6}a^3\ln\left(x^2+a^2\right) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

2.83 Combinations of arcsine or arccosine and algebraic functions

2.831 $$\int x^n\arcsin\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arcsin\frac{x}{a} - \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}$$ (see **2.263** 1, **2.264**, **2.27**)

2.832 $$\int x^n\arccos\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arccos\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{a^2-x^2}}$$ (see **2.263** 1, **2.264**, **2.27**)

1. For $n=-1$, these integrals (that is, $\int\frac{\arcsin x}{x}\,dx$ and $\int\frac{\arccos x}{x}\,dx$) cannot be expressed as a finite combination of elementary functions.

2. $$\int \frac{\arccos x}{x}\,dx = -\frac{\pi}{2}\ln\frac{1}{x} - \int \frac{\arcsin x}{x}\,dx$$

2.833[9]

1. $$\int x \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{|a|} + \frac{x}{4}\sqrt{a^2 - x^2}\right]$$

2. $$\int x \arccos\frac{x}{a}\,dx = \frac{\pi x^2}{4} - \operatorname{sign}(a)\left[\frac{1}{4}\left(2x^2 - a^2\right)\arcsin\frac{x}{|a|} + \frac{x}{4}\sqrt{a^2 - x^2}\right]$$

3. $$\int x^2 \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|} + \frac{1}{9}\left(x^2 + 2a^2\right)\sqrt{a^2 - x^2}\right]$$

4. $$\int x^2 \arccos\frac{x}{a}\,dx = \frac{\pi x^3}{6} - \operatorname{sign}(a)\left[\frac{x^3}{3}\arcsin\frac{x}{|a|} + \frac{1}{9}\left(x^2 + 2a^2\right)\sqrt{a^2 - x^2}\right]$$

5. $$\int x^3 \arcsin\frac{x}{a}\,dx = \operatorname{sign}(a)\left[\left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\arcsin\frac{x}{|a|} + \frac{1}{32}x\left(2x^2 + 3a^2\right)\sqrt{a^2 - x^2}\right]$$

6. $$\int x^3 \arccos\frac{x}{a}\,dx = \frac{\pi x^4}{8} - \operatorname{sign}(a)\left[\frac{\left(8x^4 - 3a^4\right)}{32}\arcsin\frac{x}{|a|} + \frac{1}{32}x\left(2x^2 + 3a^2\right)\sqrt{a^2 - x^2}\right]$$

2.834

1. $$\int \frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\frac{a + \sqrt{a^2 - x^2}}{x}$$

2. $$\int \frac{1}{x^2}\arccos\frac{x}{a}\,dx = -\frac{1}{x}\arccos\frac{x}{a} - \frac{1}{a}\ln\frac{a + \sqrt{a^2 - x^2}}{x}$$

2.835 $$\int \frac{\arcsin x}{(a + bx)^2}\,dx = -\frac{\arcsin x}{b(a + bx)} - \frac{2}{b\sqrt{a^2 - b^2}}\arctan\sqrt{\frac{(a - b)(1 - x)}{(a + b)(1 + x)}} \qquad \left[a^2 > b^2\right]$$

$$= -\frac{\arcsin x}{b(a + bx)} - \frac{1}{b\sqrt{b^2 - a^2}}\ln\frac{\sqrt{(a + b)(1 + x)} + \sqrt{(b - a)(1 - x)}}{\sqrt{(a + b)(1 + x)} - \sqrt{(b - a)(1 - x)}} \qquad \left[a^2 < b^2\right]$$

2.836[8] $$\int \frac{x \arcsin x}{\left(1 + cx^2\right)^2}\,dx = -\frac{\arcsin x}{2c\left(1 + cx^2\right)} + \frac{1}{2c\sqrt{c + 1}}\arctan\frac{\sqrt{c + 1}x}{\sqrt{1 - x^2}} \qquad [c > -1]$$

$$= -\frac{\arcsin x}{2c\left(1 + cx^2\right)} + \frac{1}{4c\sqrt{-(c + 1)}}\ln\frac{\sqrt{1 - x^2} + x\sqrt{-(c + 1)}}{\sqrt{1 - x^2} - x\sqrt{-(c + 1)}} \qquad [c < -1]$$

2.837

1. $$\int \frac{x \arcsin x}{\sqrt{1 - x^2}}\,dx = x - \sqrt{1 - x^2}\arcsin x$$

2. $$\int \frac{x \arcsin x}{\sqrt{1 - x^2}}\,dx = \frac{x^2}{4} - \frac{x}{2}\sqrt{1 - x^2}\arcsin x + \frac{1}{4}(\arcsin x)^2$$

3. $$\int \frac{x^3 \arcsin x}{\sqrt{1 - x^2}}\,dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3}\left(x^2 + 2\right)\sqrt{1 - x^2}\arcsin x$$

2.838

1. $$\int \frac{\arcsin x}{\sqrt{\left(1 - x^2\right)^3}}\,dx = \frac{x \arcsin x}{\sqrt{1 - x^2}} + \frac{1}{2}\ln\left(1 - x^2\right)$$

2. $$\int \frac{x \arcsin x}{\sqrt{\left(1-x^2\right)^3}}\,dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\frac{1-x}{1+x}$$

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.841

1. $$\int x \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arccos\frac{a}{x} - a\sqrt{x^2-a^2}\right\} \qquad \left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{2}\left\{x^2 \arccos\frac{a}{x} + a\sqrt{x^2-a^2}\right\} \qquad \left[\frac{\pi}{2} < \arccos\frac{a}{x} < \pi\right]$$
DW

2. $$\int x^2 \operatorname{arcsec}\frac{x}{a}\,dx = \int \arccos\frac{a}{x}\,dx = \frac{1}{3}\left\{x^3 \arccos\frac{a}{x} - \frac{a}{2}x\sqrt{x^2-a^2} - \frac{a^3}{2}\ln\left(x+\sqrt{x^2-a^2}\right)\right\}$$
$$\left[0 < \arccos\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{3}\left\{x^3 \arccos\frac{a}{x} + \frac{a}{2}x\sqrt{x^2-a^2} + \frac{a^3}{2}\ln\left(x+\sqrt{x^2-z^2}\right)\right\}$$
$$\left[\frac{\pi}{2} < \arccos\frac{a}{x} < \pi\right]$$
DW

3. $$\int x \operatorname{arccosec}\frac{x}{a}\,dx = \int \arcsin\frac{a}{x}\,dx = \frac{1}{2}\left\{x^2 \arcsin\frac{a}{x} + a\sqrt{x^2-a^2}\right\} \qquad \left[0 < \arcsin\frac{a}{x} < \frac{\pi}{2}\right]$$
$$= \frac{1}{2}\left\{x^2 \arcsin\frac{a}{x} - a\sqrt{x^2-a^2}\right\} \qquad \left[-\frac{\pi}{2} < \arcsin\frac{a}{x} < 0\right]$$
DW

2.85 Combinations of the arctangent and arccotangent with algebraic functions

2.851 $$\int x^n \arctan\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arctan\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}$$

2.852

1. $$\int x^n \operatorname{arccot}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arccot}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}$$

For $n=-1$

2. $\displaystyle\int \frac{\arctan x}{x}\,dx$ cannot be expressed as a finite combination of elementary functions.

3. $$\int \frac{\operatorname{arccot} x}{x}\,dx = \frac{\pi}{2}\ln x - \int\frac{\arctan x}{x}\,dx$$

2.853

1. $$\int x \arctan\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\arctan\frac{x}{a} - \frac{ax}{2}$$

2. $$\int x \operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+a^2\right)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}$$

3.[9] $$\int x^2 \arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} + \frac{a^3}{6}\ln\left(x^2+a^2\right) - \frac{ax^2}{6}$$

4.[9] $$\int x^2 \operatorname{arccot}\frac{x}{a}\,dx = -\frac{x^3}{3}\arctan\frac{x}{a} - \frac{a^3}{6}\ln\left(x^2+a^2\right) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

2.854 $$\int \frac{1}{x^2}\arctan\frac{x}{a}\,dx = -\frac{1}{x}\arctan\frac{x}{a} - \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}$$

2.855 $$\int \frac{\arctan x}{(\alpha+\beta x)^2}\,dx = \frac{1}{\alpha^2+\beta^2}\left\{\ln\frac{\alpha+\beta x}{\sqrt{1+x^2}} - \frac{\beta-\alpha x}{\alpha+\beta x}\arctan x\right\}$$

2.856

1. $$\int \frac{x\arctan x}{1+x^2}\,dx = \frac{1}{2}\arctan x\ln\left(1+x^2\right) - \frac{1}{2}\int\frac{\ln\left(1+x^2\right)\,dx}{1+x^2}$$ TI (689)

2. $$\int \frac{x^2\arctan x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) - \frac{1}{2}(\arctan x)^2$$ TI (405)

3. $$\int \frac{x^3\arctan x}{1+x^2}\,dx = -\frac{1}{2}x + \frac{1}{2}\left(1+x^2\right)\arctan x - \int\frac{x\arctan x}{1+x^2}\,dx$$

(see **2.8511**)

4. $$\int \frac{x^4\arctan x}{1+x^2}\,dx = -\frac{1}{6}x^2 + \frac{2}{3}\ln\left(1+x^2\right) + \left(\frac{x^3}{3}-x\right)\arctan x + \frac{1}{2}(\arctan x)^2$$

2.857 $$\int \frac{\arctan x\,dx}{\left(1+x^2\right)^{n+1}} = \left[\sum_{k=1}^{n}\frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!}\frac{x}{\left(1+x^2\right)^{n-k+1}} + \frac{1}{2}\frac{(2n-1)!!}{(2)!!}\arctan x\right]\arctan x$$
$$+\frac{1}{2}\sum_{k=1}^{n}\frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)}\frac{1}{\left(1+x^2\right)^{n-k+1}}$$

2.858 $$\int \frac{x\arctan x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\arctan x + \sqrt{2}\arctan\frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x$$

2.859 $$\int \frac{\arctan x}{\sqrt{\left(a+bx^2\right)^3}}\,dx = \frac{x\arctan x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}}\arctan\sqrt{\frac{a+bx^2}{b-a}} \qquad [a<b]$$
$$= \frac{x\arctan x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}}\ln\frac{\sqrt{a+bx^2}-\sqrt{a-b}}{\sqrt{a+bx^2}+\sqrt{a-b}} \qquad [a>b]$$

3–4 Definite Integrals of Elementary Functions

3.0 Introduction

3.01 Theorems of a general nature

3.011 Suppose that $f(x)$ is integrable[†] over the largest of the intervals $(p,q),(p,r),(r,q)$. Then (depending on the relative positions of the points p, q, and r) it is also integrable over the other two intervals, and we have

$$\int_p^q f(x)\,dx = \int_p^r f(x)\,dx + \int_r^q f(x)\,dx. \qquad \text{FI II 126}$$

3.012 *The first mean-value theorem.* Suppose (1) that $f(x)$ is continuous and that $g(x)$ is integrable over the interval (p,q), (2) that $m \le f(x) \le M$, and (3) that $g(x)$ does not change sign anywhere in the interval (p,q). Then, there exists at least one point ξ (with $p \le \xi \le q$) such that

$$\int_p^q f(x)g(x)\,dx = f(\xi)\int_p^q g(x)\,dx. \qquad \text{FI II 132}$$

3.013 *The second mean-value theorem.* If $f(x)$ is monotonic and non-negative throughout the interval (p,q), where p <q, and if $g(x)$ is integrable over that interval, then there exists at least one point ξ (with $p \le \xi \le q$) such that

1. $$\int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx$$

Under the conditions of Theorem **3.013** 1, if $f(x)$ is nondecreasing, then

2. $$\int_p^q f(x)g(x)\,dx = f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q].$$

If $f(x)$ is monotonic in the interval (p,q), where p <q, and if $g(x)$ is integrable over that interval, then

*We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

†A function $f(x)$ is said to be integrable over the interval (p,q), if the integral $\int_p^q f(x)\,dx$ exists. Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

3. $$\int_p^q f(x)g(x)\,dx = f(p)\int_p^\xi g(x)\,dx + f(q)\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q]\,,$$

or

4. $$\int_p^q f(x)g(x)\,dx = A\int_p^\xi g(x)\,dx + B\int_\xi^q g(x)\,dx \qquad [p \le \xi \le q]\,,$$

where A and B are any two numbers satisfying the conditions

$$A \ge f(p+0) \quad \text{and} \quad B \le f(q-0) \qquad [\text{if } f \text{ decreases}],$$
$$A \le f(p+0) \quad \text{and} \quad B \ge f(q-0) \qquad [\text{if } f \text{ increases}].$$

In particular,

5. $$\int_p^q f(x)g(x)\,dx = f(p+0)\int_p^\xi g(x)\,dx + f(q-0)\int_\xi^q g(x)\,dx$$ FI II 138

3.02 Change of variable in a definite integral

3.020 $$\int_\alpha^\beta f(x)\,dx = \int_\varphi^\psi f[g(t)]g'(t)\,dt; \qquad x = g(t).$$

This formula is valid under the following conditions:

1. $f(x)$ is continuous on some interval $A \le x \le B$ containing the original limits of integration α and β.
2. The equalities $\alpha = g(\varphi)$ and $\beta = g(\psi)$ hold.
3. $g(t)$ and its derivative $g'(t)$ are continuous on the interval $\varphi \le t \le \psi$.
4. As t varies from φ to ψ, the function $g(t)$ always varies in the same direction from $g(\varphi) = \alpha$ to $g(\psi) = \beta$.*

3.021 The integral $\int_\alpha^\beta f(x)\,dx$ can be transformed into another integral with given limits φ and ψ by means of the linear substitution

$$x = \frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}:$$

1. $$\int_\alpha^\beta f(x)\,dx = \frac{\beta-\alpha}{\psi-\varphi}\int_\varphi^\psi f\left(\frac{\beta-\alpha}{\psi-\varphi}t + \frac{\alpha\psi-\beta\varphi}{\psi-\varphi}\right)dt$$

In particular, for $\varphi = 0$ and $\psi = 1$,

*If this last condition is not satisfied, the interval $\varphi \le t \le \psi$ should be partitioned into subintervals throughout each of which the condition is satisfied:

$$\int_\alpha^\beta f(x)\,dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t)\,dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t)\,dt + \cdots + \int_{\varphi_{n-1}}^{\psi} f[g(t)]g'(t)\,dt.$$

2. $$\int_{\alpha}^{\beta} f(x)\,dx = (\beta-\alpha)\int_0^1 f((\beta-\alpha)t+\alpha)\,dt$$

For $\varphi = 0$ and $\psi = \infty$,

3. $$\int_{\alpha}^{\beta} f(x)\,dx = (\beta-\alpha)\int_0^{\infty} f\left(\frac{\alpha+\beta t}{1+t}\right)\frac{dt}{(1+t)^2}$$

3.022 The following formulas also hold:

1. $$\int_{\alpha}^{\beta} f(x)\,dx = \int_{\alpha}^{\beta} f(\alpha+\beta-x)\,dx$$

2. $$\int_0^{\beta} f(x)\,dx = \int_0^{\beta} f(\beta-x)\,dx$$

3. $$\int_{-\alpha}^{\alpha} f(x)\,dx = \int_{-\alpha}^{\alpha} f(-x)\,dx$$

3.03 General formulas

3.031

1. Suppose that a function $f(x)$ is integrable over the interval $(-p,p)$ and satisfies the relation $f(-x) = f(x)$ on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^{p} f(x)\,dx = 2\int_0^p f(x)\,dx.$$ FI II 159

2. Suppose that $f(x)$ is a function that is integrable on the interval $(-p,p)$ and satisfies the relation $f(-x) = -f(x)$ on that interval. (A function satisfying the latter condition is called an *odd* function). Then,

$$\int_{-p}^{p} f(x)\,dx = 0.$$ FI II 159

3.032

1. $$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx,$$

where $f(x)$ is a function that is integrable on the interval $(0,1)$. FI II 159

2. $$\int_0^{2\pi} f(p\cos x + q\sin x)\,dx = 2\int_0^{\pi} f\left(\sqrt{p^2+q^2}\cos x\right)dx,$$

where $f(x)$ is integrable on the interval $\left(-\sqrt{p^2+q^2}, \sqrt{p^2+q^2}\right)$. FI II 160

3. $$\int_0^{\frac{\pi}{2}} f(\sin 2x)\cos x\,dx = \int_0^{\frac{\pi}{2}} f\left(\cos^2 x\right)\cos x\,dx,$$

where $f(x)$ is integrable on the interval $(0,1)$. FI II 161

3.033

1. If $f(x+\pi) = f(x)$ and $f(-x) = f(x)$, then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\,dx$$ LO V 277(3)

2. If $f(x+\pi) = -f(x)$ and $f(-x) = f(x)$, then

$$\int_0^\infty f(x)\frac{\sin x}{x}\,dx = \int_0^{\frac{\pi}{2}} f(x)\cos x\,dx$$ LO V 279(4)

In formulas **3.033**, it is assumed that the integrals in the left members of the formulas exist.

3.034 $\displaystyle\int_0^\infty \frac{f(px) - f(qx)}{x}\,dx = [f(0) - f(+\infty)] \ln\frac{q}{p},$

if $f(x)$ is continuous for $x \geq 0$ and if there exists a finite limit $f(+\infty) = \lim_{x\to+\infty} f(x)$. FI II 633

3.035

1. $$\int_0^\pi \frac{f\left(\alpha + e^{xi}\right) + f\left(\alpha + e^{-xi}\right)}{1 + 2p\cos x + p^2}\,dx = \frac{2\pi}{1-p^2} f(\alpha + p) \qquad [|p| < 1]$$ LA 230(16)

2. $$\int_0^\pi \frac{1 - p\cos x}{1 - 2p\cos x + p^2}\left\{f\left(\alpha + e^{xi}\right) + f\left(\alpha + e^{-xi}\right)\right\} dx = \pi\left\{f(\alpha + p) + f(\alpha)\right\}$$
$[|p| < 1]$ BE 169

3. $$\int_0^\pi \frac{f\left(\alpha + e^{-xi}\right) - f\left(\alpha + e^{xi}\right)}{1 - 2p\cos x + p^2}\sin x\,dx = \frac{\pi}{\pi}\left\{f(\alpha + p) - f(\alpha)\right\}$$
$[|p| < 1]$ BE 169

In formulas **3.035**, it is assumed that the function f is analytic in the closed unit circle with its center at the point α.

3.036

1.[11] $$\int_0^\pi f\left(\frac{\sin^2 x}{1 + 2p\cos x + p^2}\right) dx = \int_0^\pi f\left(\sin^2 x\right) dx \qquad \left[p^2 < 1\right]$$
$$= \int_0^\pi f\left(\frac{\sin^2 x}{p^2}\right) dx \qquad \left[p^2 \geq 1\right]$$
LA 228(6)

2. $$\int_0^\pi F^{(n)}(\cos x)\sin^{2n} x\,dx = (2n-1)!!\int_0^\pi F(\cos x)\cos nx\,dx$$ B 174

3.037 If f is analytic in the circle of radius r and if

$$f\left[r(\cos x + i\sin x)\right] = f_1(r,x) + i f_2(r,x),$$

then

1. $$\int_0^\infty \frac{f_1(r,x)}{p^2 + x^2}\,dx = \frac{\pi}{2p} f\left(re^{-p}\right)$$ LA 230(19)

2. $$\int_0^\infty f_2(r,x)\frac{x\,dx}{p^2 + x^2} = \frac{\pi}{2}\left[f\left(re^{-p}\right) - f(0)\right]$$ LA 230(20)

3. $$\int_0^\infty \frac{f_2(r,x)}{x}\,dx = \frac{\pi}{2}[f(r) - f(0)]$$ LA 230(21)

4. $$\int_0^\infty \frac{f_2(r,x)}{x\left(p^2+x^2\right)}\,dx = \frac{\pi}{2p^2}\left[f(r) - f\left(re^{-p}\right)\right]$$ LA 230(22)

3.038 $$\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{1+x^2}} F\left(qx + p\sqrt{1+x^2}\right) = \int_{-\infty}^{\infty} F\left(p\cosh x + q\sinh x\right)\sinh x\,dx$$
$$= 2q\int_0^\infty F'\left(\operatorname{sign} p\cdot\sqrt{p^2-q^2}\cosh x\right)\sinh^2 x\,dx$$

[If F is a function with a continuous derivative in the interval $(-\infty,\infty)$, all these integrals converge.]

3.04 Improper integrals

3.041 Suppose that a function $f(x)$ is defined on an interval $(p,+\infty)$ and that it is integrable over an arbitrary finite subinterval of the form (p,P). Then, by definition

$$\int_p^{+\infty} f(x)\,dx = \lim_{P\to+\infty}\int_p^P f(x)\,dx,$$

if this limit exists. If it does exist, we say that the integral $\int_p^{+\infty} f(x)\,dx$ exists or that it converges. Otherwise, we say that the integral diverges.

3.042 Suppose that a function $f(x)$ is bounded and integrable in an arbitrary interval $(p,q-\eta)$ (for $0<\eta<q-p$) but is unbounded in every interval $(q-\eta,q)$ to the left of the point q. The point q is then called a *singular point*. Then, by definition,

$$\int_p^q f(x)\,dx = \lim_{\eta\to0}\int_p^{q-\eta} f(x)\,dx,$$

if this limit exists. In this case, we say that the integral $\int_p^q f(x)\,dx$ *exists* or that it *converges*.

3.043 If not only the integral of $f(x)$ but also the integral of $|f(x)|$ exists, we say that the integral of $f(x)$ converges *absolutely*.

3.044 The integral $\int_p^{+\infty} f(x)\,dx$ converges absolutely if there exists a number $\alpha>1$ such that the limit

$$\lim_{x\to+\infty}\left\{x^\alpha|f(x)|\right\}$$

exists. On the other hand, if

$$\lim_{x\to+\infty}\left\{x|f(x)|\right\} = L > 0,$$

the integral $\int_p^{+\infty} |f(x)|\,dx$ diverges.

3.045 Suppose that the upper limit q of the integral $\int_p^q f(x)\,dx$ is a singular point. Then, this integral converges absolutely if there exists a number $\alpha<1$ such that the limit

$$\lim_{x\to q}\left[(q-x)^\alpha|f(x)|\right]$$

exists. On the other hand, if

$$\lim_{x\to q}\left[(q-x)|f(x)|\right] = L > 0,$$

the integral $\int_p^q f(x)\,dx$ diverges.

3.046 Suppose that the functions $f(x)$ and $g(x)$ are defined on the interval $(p, +\infty)$, that $f(x)$ is integrable over every finite interval of the form (p, P), that the integral

$$\int_p^P f(x)\,dx$$

is a bounded function of P, that $g(x)$ is monotonic, and that $g(x) \to 0$ as $x \to +\infty$. Then, the integral

$$\int_p^{+\infty} f(x)g(x)\,dx$$

converges. FI II 577

3.05 The principal values of improper integrals

3.051 Suppose that a function $f(x)$ has a singular point r somewhere inside the interval (p, q), that $f(x)$ is defined at r, and that $f(x)$ is integrable over every portion of this interval that does not contain the point r. Then, by definition

$$\int_p^q f(x)\,dx = \lim_{\substack{\eta\to 0\\ \eta'\to 0}} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta'}^q f(x)\,dx \right\}.$$

Here, the limit must exist for *independent* modes of approach of η and η' to zero. If this limit does not exist but the limit

$$\lim_{\eta\to 0} \left\{ \int_p^{r-\eta} f(x)\,dx + \int_{r+\eta}^q f(x)\,dx \right\}$$

does exist, we say that this latter limit is the *principal value* of the improper integral $\int_p^q f(x)\,dx$, and we say that the integral $\displaystyle\int_p^q f(x)\,dx$ exists in the sense of principal values. FI II 603

3.052 Suppose that the function $f(x)$ is continuous over the interval (p, q) and vanishes at only one point r inside this interval. Suppose that the first derivative $f'(x)$ exists in a neighborhood of the point r. Suppose that $f'(r) \neq 0$ and that the second derivative $f''(r)$ exists at the point r itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$ FI II 605

diverges, but exists in the sense of principal values.

3.053 A divergent integral of a positive function cannot exist in the sense of principal values.

3.054 Suppose that the function $f(x)$ has no singular points in the interval $(-\infty, +\infty)$. Then, by definition

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{P\to-\infty\\ Q\to+\infty}} \int_P^Q f(x)\,dx.$$

Here, the limit must exist for independent approach of P and Q to $\pm\infty$. If this limit does not exist but the limit

$$\lim_{P\to+\infty} \int_{-P}^{+P} f(x)\,dx$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x)\,dx.$$ FI II 607

3.055 The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense). FI II 607

3.1–3.2 Power and Algebraic Functions

3.11 Rational functions

1. $$\int_{-\infty}^{\infty} \frac{p+qx}{r^2+2rx\cos\lambda+x^2}\,dx = \frac{\pi}{r\sin\lambda}\left(p-qr\cos\lambda\right) \qquad \text{(principal value)}$$

(see also **3.194** 8 and **3.252** 1 and 2) BI (22)(14)

3.112[11] Integrals of the form $\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)}$, where

$$g_n(x) = b_0x^{2n-2} + b_1x^{2n-4} + \cdots b_{n-1},$$
$$h_n(x) = a_0x^n + a_1x^{n-1} + \cdots a_n$$

[All roots of $h_n(x)$ lie in the upper half-plane.]

1. $$\int_{-\infty}^{\infty} \frac{g_n(x)\,dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0}\frac{M_n}{\Delta_n},$$ JE

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}, \qquad M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}.$$

2. $$\int_{-\infty}^{\infty} \frac{g_1(x)\,dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0a_1}$$ JE

3.[8] $$\int_{-\infty}^{\infty} \frac{g_2(x)\,dx}{h_2(x)h_2(-x)} = \pi i\frac{-b_0+\frac{a_0b_1}{a_2}}{a_0a_1}$$

4.[11] $$\int_{-\infty}^{\infty} \frac{g_3(x)\,dx}{h_3(x)h_3(-x)} = \pi i\frac{-a_2b_0+a_0b_1-\dfrac{a_0a_1b_2}{a_3}}{a_0\left(a_0a_3-a_1a_2\right)}$$ JE

5. $$\int_{-\infty}^{\infty} \frac{g_4(x)\,dx}{h_4(x)h_4(-x)} = \pi i\frac{b_0\left(-a_1a_4+a_2a_3\right)-a_0a_3b_1+a_0a_1b_2+\dfrac{a_0b_3}{a_4}\left(a_0a_3-a_1a_2\right)}{a_0\left(a_0a_3^2+a_1^2a_4-a_1a_2a_3\right)}$$ JE

6. $$\int_{-\infty}^{\infty} \frac{g_5(x)\,dx}{h_5(x)h_5(-x)} = \pi i\frac{M_5}{a_0\Delta_5},$$

where

$$M_5 = b_0\left(-a_0a_4a_5+a_1a_4^2+a_2^2a_5-a_2a_3a_4\right)+a_0b_1\left(-a_2a_5+a_3a_4\right)$$
$$+a_0b_2\left(a_0a_5-a_1a_4\right)+a_0b_3\left(-a_0a_3+a_1a_2\right)+\frac{a_0b_4}{a_5}\left(-a_0a_1a_5+a_0a_3^2+a_1^2a_4-a_1a_2a_3\right),$$

$$\Delta_5 = a_0^2a_5^2-2a_0a_1a_4a_5-a_0a_2a_3a_5+a_0a_3^2a_4+a_1^2a_4^2+a_1a_2^2a_5-a_1a_2a_3a_4$$ JE

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.121

1. $$\int_0^1 \frac{1}{1-2x\cos\lambda+x^2}\frac{dx}{\sqrt{x}} = 2\operatorname{cosec}\lambda\sum_{k=1}^{\infty}\frac{\sin k\lambda}{2k-1}$$ BI (10)(17)

2. $$\int_0^1 \frac{1}{q-px}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \qquad [0<p<q]$$ BI (10)(9)

3. $$\int_0^1 \frac{dx}{1-2rx+r^2}\sqrt{\frac{1\mp x}{1\pm x}} = \pm\frac{\pi}{4r}\mp\frac{1}{r}\frac{1\mp r}{1\pm r}\arctan\frac{1+r}{1-r}$$ LI (14)(5, 16)

3.13–3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

Notation: In **3.131–3.137** we set: $\alpha = \arcsin\sqrt{\frac{a-c}{a-u}}$, $\beta = \arcsin\sqrt{\frac{c-u}{b-u}}$,

$$\gamma = \arcsin\sqrt{\frac{u-c}{b-c}}, \qquad \delta = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\kappa = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \lambda = \arcsin\sqrt{\frac{a-u}{a-b}},$$

$$\mu = \arcsin\sqrt{\frac{u-a}{u-b}}, \qquad \nu = \arcsin\sqrt{\frac{a-c}{u-c}}, \qquad p = \sqrt{\frac{a-b}{a-c}}, \qquad q = \sqrt{\frac{b-c}{a-c}}.$$

3.131

1. $$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\alpha,p) \qquad [a>b>c\ge u]$$ BY (231.00)

2. $$\int_{u}^{c} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}}F(\beta,p) \qquad [a>b>c>u]$$ BY (232.00)

3. $$\int_{c}^{u} \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\gamma,q) \qquad [a>b\ge u>c]$$ BY (233.00)

4. $$\int_{u}^{b} \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\delta,q) \qquad [a>b>u\ge c]$$ BY (234.00)

5. $$\int_{b}^{u} \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\kappa,p) \qquad [a\ge u>b>c]$$ BY (235.00)

6. $$\int_{u}^{a} \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\lambda,p) \qquad [a>u\ge b>c]$$ BY (236.00)

7. $$\int_{a}^{u} \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}F(\mu,q) \qquad [u>a>b>c]$$ BY (237.00)

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\nu, q) \qquad [u \geq a > b > c]$$ BY (238.00)

3.132

1. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} [c\,F(\beta,p) + (a-c)\,E(\beta,p)] - 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.19)

2. $$\int_c^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\gamma,q) - 2\sqrt{a-c}\,E(\gamma,q)$$
$[a > b \geq u > c]$ BY (233.17)

3. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} \left[(b-a)\,\Pi\left(\delta, q^2, q\right) + a\,F(\delta,q)\right]$$
$[a > b > u \geq c]$ BY (234.16)

4. $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} \left[(b-c)\,\Pi\left(\kappa, p^2, p\right) + c\,F\left(\kappa,p\right)\right]$$
$[a \geq u > b > c]$ BY (235.16)

5. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\lambda,p) + 2\sqrt{a-c}\,E(\lambda,p)$$
$[a > u \geq b > c]$ BY (236.16)

6. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} \left[a(a-b)\,\Pi(\mu,1,q) + b^2\,F(\mu,q)\right]$$
$[u > a > b > c]$ BY (237.16)

3.133

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\alpha,p) - E(\alpha,p)]$$
$[a > b > c \geq u]$ BY (231.08)

2. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\beta,p) - E(\beta,p)] + \frac{2}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.13)

3. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\gamma,q) - \frac{2}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \geq u > c]$ BY (233.09)

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\delta,q) \qquad [a > b > u \geq c]$$ BY (234.05)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F\left(\kappa,p\right) - E\left(\kappa,p\right)] + \frac{2}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.04)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}}\,E(\nu,q) + \frac{2}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\alpha,p) - \frac{2}{(a-b)\sqrt{a-c}}\,F(\alpha,p) - \frac{2}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\beta,p) - \frac{2}{(a-b)\sqrt{a-c}}\,F(\beta,p)$$
$[a > b > c > u]$ BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}}\,F(\gamma,q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\gamma,q) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a > b > u > c]$ BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}}\,F(\lambda,p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\lambda,p) + \frac{2}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a > u > b > c]$ BY (236.09)

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\mu,q) - \frac{2}{(b-c)\sqrt{a-c}}\,F(\mu,q)$$
$[u > a > b > c]$ BY (237.12)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)}\,E(\nu,q) - \frac{2}{(b-c)\sqrt{a-c}}\,F(\nu,q) - \frac{2}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}}\,E(\alpha,p) + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\delta,q) - E(\delta,q)] + \frac{2}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.04)

15. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,E\left(\kappa,p\right)$$
$[a \geq u > b > c]$ BY (235.01)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,E(\lambda,p) - \frac{2}{(b-c)(a-c)}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\mu,q) - E(\mu,q)] + \frac{2}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.13)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}}\,[F(\nu,q) - E(\nu,q)]$$
$[u \geq a > b > c]$ BY (238.03)

3.134

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\alpha,p) - 2(2a-b-c)\,E(\alpha,p)] + \frac{2}{3(a-c)(a-b)}\sqrt{\frac{(c-u)(b-u)}{(a-u)^3}}$$
$[a > b > c \geq u]$ BY (231.08)

2. $$\int_u^c \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\beta,p) - 2(2a-b-c)\,E(\beta,p)] + \frac{2\left[4a^2-3ab-2ac+bc-u(3a-2b-c)\right]}{3(a-b)(a-c)^2}\sqrt{\frac{c-u}{(a-u)^3(b-u)}}$$
$[a > b > c > u]$ BY (232.13)

3. $$\int_c^u \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^3\sqrt{(a-c)^3}}\,[2(2a-b-c)\,E(\gamma,q) - (a-b)\,F(\gamma,q)] - \frac{2\left[5a^2-3ab-3ac+bc-2u(2a-b-c)\right]}{3(a-b)^2(a-c)^2}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$
$[a > b \geq u > c]$ BY (233.09)

4. $$\int_u^b \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[2(2a-b-c)\,E(\delta,q) - (a-b)\,F(\delta,q)] - \frac{2}{3(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)^3}}$$
$[a > b > u \geq c]$ BY (234.05)

5. $$\int_b^u \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\,[(3a-b-2c)\,F(\kappa,p) - 2(2a-b-c)\,E(\kappa,p)] + \frac{2\left[4a^2-2ab-3ac+bc-u(3a-b-2c)\right]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(a-u)^3(u-c)}}$$
$[a > u > b > c]$ BY (235.04)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} = \frac{2}{3(a-b)^2\sqrt{(a-c)^3}}\left[2(2a-b-c)\,E(\nu,q)-(a-b)\,F(\nu,q)\right] + \frac{2\left[4a^2-2ab-3ac+bc+u(b+2c-3a)\right]}{3(a-b)^2(a-c)}\sqrt{\frac{u-b}{(u-a)^3(u-c)}}$$
$[u > a > b > c]$ BY (238.05)

7. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \left[2(a-c)(a+c-2b)\,E(\alpha,p)+(b-c)(3b-a-2c)\,F(\alpha,p)\right] - \frac{2\left[3ab-ac+2bc-4b^2-u(2a-3b+c)\right]}{3(a-b)(b-c)^2}\sqrt{\frac{c-u}{(a-u)(b-u)^3}}$$
$[a > b > c \geq u]$ BY (231.09)

8. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \left[(b-c)(3b-a-2c)\,F(\beta,p)+2(a-c)(a-2b+c)\,E(\beta,p)\right] + \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(a-u)(c-u)}{(b-u)^3}}$$
$[a > b > c > u]$ BY (232.14)

9. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \left[(a-b)(2a-3b+c)\,F(\gamma,q)+2(a-c)(2b-a-c)\,E(\gamma,q)\right] + \frac{2\left[3ab+3bc-ac-5b^2-2u(a-2b+c)\right]}{3(a-b)^2(b-c)^2}\sqrt{\frac{(a-u)(u-c)}{(b-u)^3}}$$
$[a > b > u > c]$ BY (233.10)

10. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \left[(b-c)(3b-2c-a)\,F(\lambda,p)+2(a-c)(a+c-2b)\,E(\lambda,p)\right] + \frac{2\left[3ab+3bc-ac-5b^2+2u(2b-a-c)\right]}{3(a-b)^2(b-c)^2}\sqrt{\frac{(a-u)(u-c)}{(u-b)^3}}$$
$[a > u > b > c]$ BY (236.09)

11. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \left[(a-b)(2a+c-3b)\,F(\mu,q)+2(a-c)(2b-a-c)\,E(\mu,q)\right] + \frac{2}{3(a-b)(b-c)}\sqrt{\frac{(u-a)(u-c)}{(u-b)^3}}$$
$[u > a > b > c]$ BY (237.12)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times [(a-b)(2a+c-3b)\,F(\nu,q) + 2(a-c)(2b-c-a)\,E(\nu,q)] - \frac{2\left[3bc+2ab-ac-4b^2+u(3b-a-2c)\right]}{3(a-b)^2(b-c)}\sqrt{\frac{u-a}{(u-b)^3(u-c)}}$$
$[u \geq a > b > c]$ BY (238.04)

13. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E(\alpha,p) - (b-c)\,F(\alpha,p)] + \frac{2\left[ab-3ac-2bc+4c^2+u(2a+b-3c)\right]}{3(a-c)(b-c)^2}\sqrt{\frac{b-u}{(a-u)(c-u)^3}}$$
$[a > b > c > u]$ By (231.10)

14. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\delta,q) - 2(a+b-2c)\,E(\delta,q)] + \frac{2\left[ab-3ac-2bc+4c^2+u(2a+b-3c)\right]}{3(b-c)^2(a-c)}\sqrt{\frac{b-u}{(a-u)(u-c)^3}}$$
$[a > b > u > c]$ BY (234.04)

15. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E(\kappa,p) - (b-c)\,F(\kappa,p)] + \frac{2}{3(a-c)(b-c)}\sqrt{\frac{(a-u)(u-b)}{(u-c)^3}}$$
$[a \geq u > b > c]$ BY (235.20)

16. $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[2(a+b-2c)\,E(\lambda,p) - (b-c)\,F(\lambda,p)] - \frac{2\left[ab-3ac-3bc+5c^2+2u(a+b-2c)\right]}{3(b-c)^2(a-c)^2}\sqrt{\frac{(a-u)(u-b)}{(u-c)^3}}$$
$[a > u \geq b > c]$ BY (236.10)

17. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\mu,q) - 2(a+b-2c)\,E(\mu,q)] + \frac{2\left[4c^2-ab-2ac-bc+u(3a+2b-5c)\right]}{3(b-c)(a-c)^2}\sqrt{\frac{u-a}{(u-b)(u-c)^3}}$$
$[u > a > b > c]$ BY (237.13)

18. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}}\,[(2a+b-3c)\,F(\nu,q) - 2(a+b-2c)\,E(\nu,q)] + \frac{2}{3(a-c)(b-c)}\sqrt{\frac{(u-a)(u-b)}{(u-c)^3}}$$
$[u \geq a > b > c]$ BY (238.03)

3.135

1.[6]
$$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)\,F(\alpha,p) - (2a-b-c)\,E(\alpha,p)\right] + \frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.13)

2.
$$\int_{u}^{a} \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(b-c)\,F(\lambda,p) - 2(2a-b-c)\,E(\lambda,p)\right] + \frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)}\sqrt{\frac{a-u}{(u-b)(u-c)}}$$
$[a > u > b > c]$ BY (236.15)

3.
$$\int_{a}^{u} \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(2a-b-c)\,E(\mu,q) - 2(a-b)\,F(\mu,q)\right] + \frac{2}{(a-c)(b-c)}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (236.14)

4.
$$\int_{u}^{\infty} \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} = \frac{2}{(a-b)(b-c)^2\sqrt{a-c}}\left[(2a-b-c)\,E(\nu,q) - 2(a-b)\,F(\nu,q)\right] - \frac{2}{(a-b)(b-c)}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.13)

5.
$$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(2b-a-c)\,E(\alpha,p) - (b-c)\,F(\alpha,p)\right] + \frac{2}{(b-c)(a-c)}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY(231.12)

6.
$$\int_{u}^{b} \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}} = \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}}\left[(a-b)\,F(\delta,q) + (2b-a-c)\,E(\delta,q)\right] + \frac{2}{(b-c)(a-c)}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.03)

7.
$$\int_{b}^{u} \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(b-c)\,F(\kappa,p) - (2b-a-c)\,E(\kappa,p)\right] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.15)

8. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}}\left[(a+c-2b)\,E(\nu,q)-(a-b)\,F(\nu,q)\right] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.14)

9. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(b-c)(a-b)^2\sqrt{a-c}}\left[(a+b-2c)\,E(\alpha,p)-2(b-c)\,F(\alpha,p)\right] - \frac{2}{(a-b)(b-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.11)

10. $$\int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a+b-2c)\,E(\beta,p)-2(b-c)\,F(\beta,p)\right] + \frac{2}{(a-b)(a-c)}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.15)

11. $$\int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a-b)\,F(\gamma,q)-(a+b-2c)\,E(\gamma,q)\right] + \frac{2\left[a^2+b^2-ac-bc-u(a+b-2c)\right]}{(a-b)^2(b-c)(a-c)}\sqrt{\frac{u-c}{(a-u)(b-u)}}$$
$[a > b > u > c]$ BY (233.11)

12. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}}\left[(a-b)\,F(\nu,q)-(a+b-2c)\,E(\nu,q)\right] + \frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (238.15)

3.136

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}}$$
$$= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times \left[(b-c)(a+b-2c)\,F(\alpha,p)-2\left(c^2+a^2+b^2-ab-ac-bc\right)E(\alpha,p)\right] + \frac{2[c(a-c)+b(a-b)-u(2a-c-b)]}{(a-b)(a-c)(b-c)^2\sqrt{(a-u)(b-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.14)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times \left[(a-b)(2a-b-c)\,F(\nu,q) - 2\left(a^2+b^2+c^2-ab-ac-bc\right)E(\nu,q)\right] + \frac{2[u(a+b-2c)-a(a-c)-b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (238.16)

3.137

1.[6] $$\int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-r)\sqrt{a-c}}\left[\Pi\left(\alpha, \frac{a-r}{a-c}, p\right) - F(\alpha,p)\right]$$
$[a > b > c \geq u]$ BY (231.15)

2. $$\int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \times \Pi\left(\beta, \frac{r-b}{r-c}, p\right) + \frac{2}{(r-b)\sqrt{a-c}}\,F(\beta,p)$$
$[a > b > c > u, \quad r \neq 0]$ BY (232.17)

3. $$\int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}}\,\Pi\left(\gamma, \frac{b-c}{r-c}, q\right)$$
$[a > b \geq u > c, \quad r \neq c]$ BY (233.02)

4. $$\int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-a)(r-b)\sqrt{a-c}} \times \left[(b-a)\,\Pi\left(\delta, q^2\frac{r-a}{r-b}, q\right) + (r-b)\,F(\delta,q)\right]$$
$[a > b > u \geq c, \quad r \neq b]$ BY (234.18)

5. $$\int_b^u \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(c-r)(b-r)\sqrt{a-c}} \times \left[(c-b)\,\Pi\left(\kappa, p^2\frac{c-r}{b-r}, p\right) + (b-r)\,F(\kappa,p)\right]$$
$[a \geq u > b > c, \quad r \neq b]$ BY (235.17)

6.[8] $$\int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}}\,\Pi\left(\lambda, \frac{a-b}{a-r}, p\right)$$
$[a > u \geq b > c, \quad r \neq a]$ BY (236.02)

7. $$\int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \times \left[(b-a)\,\Pi\left(\mu, \frac{b-r}{a-b}, q\right) + (a-p)\,F(\mu,q)\right]$$
$[u > a > b > c, \quad r \neq a]$ BY (237.17)

8. $$\int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}}\left[\Pi\left(\nu, \frac{r-c}{a-c}, q\right) - F(\nu, q)\right]$$
$[u \geq a > b > c]$ BY (238.06)

3.138

1. $$\int_0^u \frac{dx}{\sqrt{x(1-x)\left(1-k^2x\right)}} = 2\,F\left(\arcsin\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE (532), JA

2. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)\left(k'^2+k^2x\right)}} = 2\,F\left(\arccos\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE(533)

3. $$\int_u^1 \frac{dx}{\sqrt{x(1-x)\left(x-k'^2\right)}} = 2\,F\left(\arcsin\frac{\sqrt{1-u}}{k}, k\right)$$ $[0 < u < 1]$ PE (534)

4. $$\int_0^u \frac{dx}{\sqrt{x(1+x)\left(1+k'^2x\right)}} = 2\,F\left(\arctan\sqrt{u}, k\right)$$ $[0 < u < 1]$ PE (535)

5. $$\int_0^u \frac{dx}{\sqrt{x\left[1+x^2+2\left(k'^2-k^2\right)x\right]}} = F\left(2\arctan\sqrt{u}, k\right)$$
$[0 < u < 1]$ JA

6. $$\int_u^1 \frac{dx}{\sqrt{x\left[k'^2\left(1+x^2\right)+2\left(1+k^2\right)x\right]}} = F\left(\frac{\pi}{2} - 2\arctan\sqrt{u}, k\right)$$
$[0 < u < 1]$ JA

7. $$\int_a^u \frac{dx}{\sqrt{(x-\alpha)\left[(x-m)^2+n^2\right]}} = \frac{1}{\sqrt{p}}\,F\left(2\arctan\sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right)$$
$[\alpha < u]$,

8. $$\int_u^a \frac{dx}{\sqrt{(\alpha-x)\left[(x-m)^2+n^2\right]}} = \frac{1}{\sqrt{p}}\,F\left(2\,\text{arccot}\sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right)$$
$[u < \alpha]$,

where $p = \sqrt{(m-\alpha)^2+n^2}$.

3.139 **Notation** $$\alpha = \arccos\frac{1-\sqrt{3}-u}{1+\sqrt{3}-u}, \qquad \beta = \arccos\frac{\sqrt{3}-1+u}{\sqrt{3}+1-u},$$
$$\gamma = \arccos\frac{\sqrt{3}+1-u}{\sqrt{3}-1+u}, \qquad \delta = \arccos\frac{u-1-\sqrt{3}}{u-1+\sqrt{3}}.$$

1. $$\int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,F\left(\alpha, \sin 75^\circ\right)$$ H 66 (285)

2. $$\int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,F\left(\beta, \sin 75^\circ\right)$$ H 65 (284)

3. $$\int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\gamma, \sin 15^\circ)$$ H 65 (283)

4. $$\int_u^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\delta, \sin 15^\circ)$$ H 65 (282)

5. $$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}} \left\{\Gamma\left(\frac{1}{3}\right)\right\}^3$$ MO 9

6. $$\int_0^1 \frac{x\,dx}{\sqrt{1-x^3}} = \frac{1}{\pi}\frac{\sqrt{3}}{\sqrt[3]{4}} \left\{\Gamma\left(\frac{2}{3}\right)\right\}^3$$ MO 9

7. $$\int_u^1 \sqrt{1-x^3}\,dx = \frac{1}{5}\left\{\sqrt[4]{27}\,F(\beta, \sin 75^\circ) - 2u\sqrt{1-u^3}\right\}$$ BY (244.01)

8. $$\int_u^1 \frac{x\,dx}{\sqrt{1-x^3}} = \left(3^{-\frac{1}{4}} - 3^{\frac{1}{4}}\right) F(\beta, \sin 75^\circ) + 2\sqrt[4]{3}\,E(\beta, \sin 75^\circ) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}$$ BY (244.05)

9. $$\int_u^1 \frac{x^m\,dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1}\int_u^1 \frac{x^{m-3}\,dx}{\sqrt{1-x^3}}$$ BY (244.07)

10. $$\int_1^u \frac{x\,dx}{\sqrt{x^3-1}} = \left(3^{-\frac{1}{4}} + 3^{\frac{1}{4}}\right) F(\gamma, \sin 15^\circ) - 2\sqrt[4]{3}\,E(\gamma, \sin 15^\circ) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}$$ BY (240.05)

11. $$\int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}}\left[F(\alpha, \sin 75^\circ) - 2\,E(\alpha, \sin 75^\circ)\right] + \frac{2}{\sqrt{3}}\frac{\sqrt{1+u+u^2}}{\left(1+\sqrt{3}-u\right)\sqrt{1-u}}$$
$[u \neq 1]$ BY (246.06)

12. $$\int_u^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}}\left[F(\delta, \sin 15^\circ) - 2\,E(\delta, \sin 15^\circ)\right] + \frac{2}{\sqrt{3}}\frac{\sqrt{1+u+u^2}}{\left(u-1+\sqrt{3}\right)\sqrt{u-1}}$$
$[u \neq 1]$ BY (242.03)

13. $$\int_{-\infty}^u \frac{(1-x)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}\left[F(\alpha, \sin 75^\circ) - E(\alpha, \sin 75^\circ)\right]$$ BY (246.07)

14. $$\int_u^1 \frac{(1-x)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}}\left[F(\beta, \sin 75^\circ) - E(\beta, \sin 75^\circ)\right]$$ BY (244.04)

15. $$\int_1^u \frac{(x-1)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2\left(\sqrt{3}-2\right)}{\sqrt{3}}\frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}}\,E(\gamma, \sin 15^\circ)$$ BY (240.08)

16. $$\int_u^\infty \frac{(x-1)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2\left(2-\sqrt{3}\right)}{\sqrt{3}}\frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}}\,E(\delta, \sin 15^\circ)$$ BY (242.07)

17. $$\int_{-\infty}^u \frac{(1-x)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[\frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E(\alpha, \sin 75^\circ)\right]$$ BY (246.08)

18. $$\int_1^u \frac{(x-1)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ)\right]$$ BY (240.04)

19. $$\int_u^\infty \frac{(x-1)\,dx}{\left(1-\sqrt{3}-x\right)^2\sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}}\left[F\left(\delta,\sin 15^\circ\right) - E\left(\delta,\sin 15^\circ\right)\right]$$ BY (242.05)

20. $$\int_{-\infty}^u \frac{\left(x^2+x+1\right)\,dx}{\left(1+\sqrt{3}-x\right)^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,E\left(\alpha,\sin 75^\circ\right)$$ BY (246.01)

21. $$\int_u^1 \frac{\left(x^2+x+1\right)\,dx}{\left(x-1+\sqrt{3}\right)^2\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,E\left(\beta,\sin 75^\circ\right)$$ BY (244.02)

22. $$\int_1^u \frac{\left(x^2+x+1\right)\,dx}{\left(\sqrt{3}+x-1\right)^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,E\left(\gamma,\sin 15^\circ\right)$$ BY (240.01)

23. $$\int_u^\infty \frac{\left(x^2+x+1\right)\,dx}{\left(x-1+\sqrt{3}\right)^2\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,E\left(\delta,\sin 15^\circ\right)$$ BY (242.01)

24. $$\int_1^u \frac{(x-1)\,dx}{\left(x^2+x+1\right)\sqrt{x^3-1}} = \frac{4}{\sqrt[4]{27}}\,E\left(\gamma,\sin 15^\circ\right) - \frac{2+\sqrt{3}}{\sqrt[4]{27}}\,F\left(\gamma,\sin 15^\circ\right) - \frac{2-\sqrt{3}}{\sqrt{3}}\,\frac{2(u-1)\left(\sqrt{3}+1-u\right)}{\left(\sqrt{3}-1+u\right)\sqrt{u^3-1}}$$ BY (240.09)

25. $$\int_{-\infty}^u \frac{\left(1+\sqrt{3}-x\right)^2\,dx}{\left[\left(1+\sqrt{3}-x\right)^2 - 4\sqrt{3}p^2(1-x)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\alpha,p^2,\sin 75^\circ\right)$$ BY (246.02)

26. $$\int_u^1 \frac{\left(1+\sqrt{3}-x\right)^2\,dx}{\left[\left(1+\sqrt{3}-x\right)^2 - 4\sqrt{3}p^2(1-x)\right]\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\beta,p^2,\sin 75^\circ\right)$$ BY (244.03)

27. $$\int_1^u \frac{\left(1-\sqrt{3}-x\right)^2\,dx}{\left[\left(1-\sqrt{3}-x\right)^2 - 4\sqrt{3}p^2(x-1)\right]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\gamma,p^2,\sin 15^\circ\right)$$ BY (240.02)

28. $$\int_u^\infty \frac{\left(1-\sqrt{3}-x\right)^2\,dx}{\left[\left(1-\sqrt{3}-x\right)^2 - 4\sqrt{3}p^2(x-1)\right]\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}}\,\Pi\left(\delta,p^2,\sin 15^\circ\right)$$ BY (242.02)

3.141 **Notation**: In **3.141** and **3.142** we set:

$$\alpha = \arcsin\sqrt{\frac{a-c}{a-u}}, \qquad \beta = \arcsin\sqrt{\frac{c-u}{b-u}}, \qquad \gamma = \arcsin\sqrt{\frac{u-c}{b-c}}$$

$$\delta = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \qquad \kappa = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \lambda = \arcsin\sqrt{\frac{a-u}{a-b}},$$

$$\mu = \arcsin\sqrt{\frac{u-a}{u-b}}, \qquad \nu = \arcsin\sqrt{\frac{a-c}{u-c}}, \qquad p = \sqrt{\frac{a-b}{a-c}}, \qquad q = \sqrt{\frac{b-c}{a-c}}.$$

1. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}}\,dx = 2\sqrt{a-c}\,[F(\beta,p)-E(\beta,p)] + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.06)

2. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\gamma,q)$$
$[a > b \geq u > c]$ BY (233.01)

3. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\delta,q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.06)

4. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}\,[F(\kappa,p)-E(\kappa,p)] + 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.07)

5. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}}\,dx = 2\sqrt{a-c}\,[F(\lambda,p)-E(\lambda,p)]$$
$[a > u \geq b > c]$ BY (236.04)

6. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)}}\,dx = -2\sqrt{a-c}\,E(\mu,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.03)

7. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}}\,dx = \frac{2(b-c)}{\sqrt{a-c}}\,F(\beta,p) - 2\sqrt{a-c}\,E(\beta,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.07)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\gamma,q) - \frac{2(a-b)}{\sqrt{a-c}}\,F(\gamma,q)$$
$[a > b \geq u > c]$ BY (233.04)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\delta,q) - \frac{2(a-b)}{\sqrt{a-c}}\,F(\delta,q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.07)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\kappa,p) - \frac{2(b-c)}{\sqrt{a-c}}\,F(\kappa,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.06)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}}\,dx = 2\sqrt{a-c}\,E(\lambda,p) - \frac{2(b-c)}{\sqrt{a-c}}\,F(\lambda,p)$$
$[a > u \geq b > c]$ BY (236.03)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}}\,dx = \frac{2(a-b)}{\sqrt{a-c}}\,F(\mu,q) - 2\sqrt{a-c}\,E(\mu,q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.04)

13. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}}\,dx = -2\sqrt{a-c}\,E(\beta,p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.08)

14. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}\,[F(\gamma,q) - E(\gamma,q)]$$
$[a > b \geq u > c]$ BY (233.03)

15. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}}\,dx = 2\sqrt{a-c}\,[F(\delta,q) - E(\delta,q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.08)

16. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\kappa,p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.07)

17. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}}\,dx = 2\sqrt{a-c}\,E(\lambda,p)$$
$[a > u \geq b > c]$ BY (236.01)

18. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}}\,dx = 2\sqrt{a-c}\,[F(\mu,q) - E(\mu,q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.05)

19. $$\int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\beta,p) - (b-c)\,F(\beta,p)] + \frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.11)

20. $$\int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\gamma,q) - 2(a-b)\,F(\gamma,q)] - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a > b \geq u > c]$ BY (233.06)

21.[11] $$\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-a)\,F(\delta,q) + (2a-b-c)\,E(\delta,q)] + \frac{2}{3}(b+c-a-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.11)

22. $$\int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\kappa,p) - (b-c)\,F(\kappa,p)] + \frac{2}{3}(b+2c-2a-u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.10)

23.[11] $$\int_u^a \sqrt{\frac{(x-b)(x-c)}{a-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2a-b-c)\,E(\lambda,p)-(b-c)\,F(\lambda,p)] + \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.07)

24. $$\int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(a-b)\,F(\mu,q)+(b+c-2a)\,E(\mu,q)] + \frac{2}{3}(u+2a-2b-c)\sqrt{\frac{(u-a)(u-b)}{u-c}}$$
$[u > a > b > c]$ BY (237.08)

25. $$\int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)\,E(\beta,p)-(b-c)\,F(\beta,p)] + \frac{2}{3}(a+c-b-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.10)

26. $$\int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(2b-a-c)\,E(\gamma,q)+(a-b)\,F(\gamma,q)] - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a > b \geq u > c]$ BY (233.05)

27. $$\int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a-b)\,F(\delta,q)+(2b-a-c)\,E(\delta,q)] + \frac{2}{3}(2a+c-2b-u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.10)

28. $$\int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(b-c)\,F(\kappa,p)+(a+c-2b)\,E(\kappa,p)] + \frac{2}{3}(2b-a-2c+u)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.11)

29. $$\int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+c-2b)\,E(\lambda,p)+(b-c)\,F(\lambda,p)] - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.06)

30.[11] $$\int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+c-2b)\,E(\mu,q)-(a-b)\,F(\mu,q)] + \frac{2}{3}(u+b-a-c)\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.06)

31. $$\int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}}\,dx = \frac{2}{3}\sqrt{a-c}\,[2(b-c)\,F(\beta,p) + (2c-a-b)\,E(\beta,p)] + \frac{2}{3}(a+2b-2c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}}$$
$[a > b > c > u]$ BY (232.09)

32. $$\int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\gamma,q) - (a-b)\,F(\gamma,q)] + \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)}$$
$[a > b \geq u > c]$ BY (233.07)

33. $$\int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\delta,q) - (a-b)\,F(\delta,q)] + \frac{2}{3}(2c-2a-b+u)\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b > u \geq c]$ BY (234.09)

34. $$\int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E\,(\kappa,p) - 2(b-c)\,F\,(\kappa,p)] + \frac{2}{3}(u+c-a-b)\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a \geq u > b > c]$ BY (235.09)

35. $$\int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\lambda,p) - 2(b-c)\,F(\lambda,p)] - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)}$$
$[a > u \geq b > c]$ BY (236.05)

36. $$\int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}}\,dx = \frac{2}{3}\sqrt{a-c}\,[(a+b-2c)\,E(\mu,q) - (a-b)\,F(\mu,q)] + \frac{2}{3}(u+2c-a-2b)\sqrt{\frac{(u-a)(u-c)}{u-b}}$$
$[u > a > b > c]$ BY (237.07)

3.142

1. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,F(\alpha,p) - \frac{2\sqrt{a-c}}{b-c}\,E(\alpha,p) + \frac{2(a-c)}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.05)

2. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}}\,dx = 2\frac{a-b}{(b-c)\sqrt{a-c}}\,F(\delta,q) - \frac{2\sqrt{a-c}}{b-c}\,E(\delta,q) + 2\frac{a-c}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.13)

3. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E\,(\kappa,p) - \frac{2}{\sqrt{a-c}}\,F\,(\kappa,p)$$
$[a \geq u > b > c]$ BY (235.12)

4. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\lambda,p) - \frac{2}{\sqrt{a-c}}\,F(\lambda,p) - \frac{2}{b-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.12)

5. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\mu,q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}}\,F(\mu,q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.10)

6. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\nu,q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}}\,F(\nu,q)$$
$[u \geq a > b > c]$ BY (238.09)

7. $$\int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\alpha,p) - 2\frac{a-b}{b-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c \geq u]$ BY (231.03)

8. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,E(\beta,p)$$ $[a > b > c > u]$ BY (232.01)

9. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\gamma,q) - E(\gamma,q)] + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a > b > u > c]$ BY (233.15)

10. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{c-b}\,E(\lambda,p) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a > u > b > c]$ BY (236.11)

11. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\mu,q) - E(\mu,q)]$$
$[u > a > b > c]$ BY (237.09)

12. $$\int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{a-c}}{b-c}\,[F(\nu,q) - E(\nu,q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u \geq a > b > c]$ BY (238.10)

13. $$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\alpha,p)$$ $[a > b > c \geq u]$ BY (231.01)

14. $$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\beta,p) - \frac{2(a-b)}{a-c}\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.05)

15. $$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\gamma,q) - E(\gamma,q)] + \frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \geq u > c]$ BY (233.13)

16. $$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\delta,q) - E(\delta,q)]$$ $[a > b > u \geq c]$ BY (234.15)

17. $$\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}}\,dx = -\frac{2}{\sqrt{a-c}}\,E(\kappa,p) + 2\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a > u > b > c]$ BY (235.08)

18. $$\int_u^\infty \sqrt{\frac{x-b}{(x-a)^3(x-c)}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\nu,q) - E(\nu,q)] + 2\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u > a > b > c]$ BY (238.07)

19. $$\int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)(c-x)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\alpha,p) - E(\alpha,p)] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}}$$
$[a > b > c > u]$ BY (231.04)

20. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}}\,dx = -\frac{2}{\sqrt{a-c}}\,E(\delta,q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}}$$
$[a > b > u > c]$ BY (234.14)

21. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\kappa,p) - E(\kappa,p)]$$
$[a \geq u > b > c]$ BY (235.03)

22. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,[F(\lambda,p) - E(\lambda,p)] + \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{u-c}}$$
$[a > u \geq b > c]$ BY (236.14)

23. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\mu,q) - 2\frac{b-c}{a-c}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u > a > b > c]$ BY (237.11)

24. $$\int_u^\infty \sqrt{\frac{x-b}{(x-a)(x-c)^3}}\,dx = \frac{2}{\sqrt{a-c}}\,E(\nu,q)$$ $[u \geq a > b > c]$ BY (238.01)

25. $$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\alpha,p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\alpha,p)$$
$[a > b > c \geq u]$ BY (231.07)

26. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\beta,p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\beta,p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a > b > c > u]$ BY (232.03)

27. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\gamma,q) - \frac{2}{\sqrt{a-c}}\,F(\gamma,q) - \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{a-u}}$$
$[a > b \geq u > c]$ BY (233.14)

28. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\delta,q) - \frac{2}{\sqrt{a-c}}\,F(\delta,q)$$
$[a > b > u \geq c]$ BY (234.20)

29. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}}\,F(\kappa,p) - \frac{2\sqrt{a-c}}{a-b}\,E(\kappa,p) + 2\frac{a-c}{a-b}\sqrt{\frac{u-b}{(a-u)(u-c)}}$$
$[a>u>b>c]$ BY (235.13)

30. $$\int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}}\,dx = \frac{2}{\sqrt{a-c}}\,F(\nu,q) - \frac{2\sqrt{a-c}}{a-b}\,E(\nu,q) + \frac{2(a-c)}{a-b}\sqrt{\frac{u-b}{(u-a)(u-c)}}$$
$[u>a>b>c]$ BY (238.08)

31. $$\int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\alpha,p)-E(\alpha,p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}}$$
$[a>b>c\ge u]$ BY (231.06)

32. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\beta,p)-E(\beta,p)]$$
$[a>b>c>u]$ BY (232.04)

33. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}}\,dx = -\frac{2\sqrt{a-c}}{a-b}\,E(\gamma,q) + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{b-u}}$$
$[a>b>u>c]$ BY (233.16)

34. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,[F(\lambda,p)-E(\lambda,p)] + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{u-b}}$$
$[a>u>b>c]$ BY (236.13)

35. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\mu,q)$$
$[u>a>b>c]$ BY (237.01)

36. $$\int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}}\,dx = \frac{2\sqrt{a-c}}{a-b}\,E(\nu,q) - 2\frac{b-c}{a-b}\sqrt{\frac{u-a}{(u-b)(u-c)}}$$
$[u\ge a>b>c]$ BY (238.11)

3.143

1.[6] $$\int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\,F\left(\arctan\frac{\left(1+\sqrt{2}\right)(1-u)}{(1+u)}, 2\sqrt[4]{2}\left(\sqrt{2}-1\right)\right)$$
H 66 (286)

2. $$\int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2}\,F\left(\arccos\frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2}\right)$$
H 66 (287)

3.144 **Notation**: $\alpha = \arcsin\dfrac{1}{\sqrt{u^2-u+1}}$.

1. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\left(x^2-x+1\right)}} = F\left(\alpha, \frac{\sqrt{3}}{2}\right)$$
$[u\ge 1]$ BY (261.50)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3\,(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)\,(u^2-u+1)}} - 4\,E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$

$[u > 1]$ BY (261.54)

3. $$\int_u^\infty \frac{(2x-1)^2\,dx}{\sqrt{x^3(x-1)^3\,(x^2-x+1)}} = 4\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u(u-1)\,(u^2-u-1)}}\right]$$

$[u > 1]$ BY (261.56)

4. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\,(x^2-x+1)^3}} = \frac{4}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right]$$

$[u \geq 1]$ BY (261.52)

5. $$\int_u^\infty \frac{(2x-1)^2\,dx}{\sqrt{x(x-1)\,(x^2-x+1)^3}} = 4\,E\left(\alpha, \frac{\sqrt{3}}{2}\right)$$ $[u > 1]$ BY (261.51)

6. $$\int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}}\,dx = \frac{4}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right)$$

$[u > 1]$ BY (261.53)

7. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3}\left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right)\right] + \frac{1}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$

$[u > 1]$ BY (261.57)

8. $$\int_u^\infty \frac{dx}{(2x-1)^2}\sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$

$[u > 1]$ BY (261.58)

9. $$\int_u^\infty \frac{dx}{(2x-1)^2\sqrt{x(x-1)\,(x^2-x+1)}} = \frac{4}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$

$[u > 1]$ BY (261.55)

10. $$\int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5\,(x^2-x+1)}} = \frac{40}{3}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{4}{3}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)\left(9u^2-9u-1\right)}{3\sqrt{u^3(u-1)^3\,(u^2-u+1)}}$$

$[u > 1]$ BY (261.54)

11. $$\int_u^\infty \frac{dx}{\sqrt{x(x-1)\,(x^2-x+1)^5}} = \frac{44}{27}\,F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{56}{27}\,E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}}$$

$[u > 1]$ BY (261.52)

12. $$\int_u^\infty \frac{dx}{(2x-1)^4\sqrt{x(x-1)\left(x^2-x+1\right)}} = \frac{16}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{8\left(5u^2-5u+2\right)}{9(2u-1)^3}\sqrt{\frac{u(u-1)}{u^2-u+1}}$$
$[u > 1]$ BY (261.55)

3.145

1. $$\int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)\left[(x-m)^2+n^2\right]}} = \frac{1}{\sqrt{pq}} F\left(2\arctan\sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$[\beta < \alpha < u]$

2. $$\int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)\left[(x-m)^2+n^2\right]}} = \frac{1}{\sqrt{pq}} F\left(2\operatorname{arccot}\sqrt{\frac{q(\alpha-u)}{p(u-\beta)}}, \frac{1}{2}\sqrt{\frac{-(p-q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$[\beta < u < \alpha]$

3. $$\int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)\left[(x-m)^2+n^2\right]}} = \frac{1}{\sqrt{pq}} F\left(2\arctan\sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2}\sqrt{\frac{(p+q)^2+(\alpha-\beta)^2}{pq}}\right)$$
$[u < \beta < \alpha]$

where $(m-\alpha)^2+n^2 = p^2$, and $(m-\beta)^2+n^2 = q^2$.*

4. Set

$$(m_1-m)^2+(n_1+n)^2 = p^2, \qquad (m_1-m)^2+(n_1-n)^2 = p_1^2,$$
$$\cot\alpha = \sqrt{\frac{(p+p_1)^2-4n^2}{4n^2-(p-p_1)^2}};$$

then

$$\int_{m-n\tan\alpha}^u \frac{dx}{\sqrt{\left[(x-m)^2+n^2\right]\left[(xm_1)^2+n_1^2\right]}} = \frac{2}{p+p_1} F\left(\alpha+\arctan\frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1}\right)$$
$[m-n\tan\alpha < u < m+n\cot\alpha]$

3.146

1. $$\int_0^1 \frac{1}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4}\sqrt{2}\,\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (13)(6)

2. $$\int_0^1 \frac{x^2}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}$$ BI (13)(7)

*Formulas **3.145** are not valid for $\alpha+\beta = 2m$. In this case, we make the substitution $x-m = z$, which leads to one of the formulas in **3.152**.

3. $$\int_0^1 \frac{x^4}{1+x^4}\frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2}\,\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (13)(8)

3.147 **Notation**: In **3.147–3.151** we set: $\alpha = \arcsin\sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$
$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \kappa = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$
$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$
$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

1. $$\int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\alpha,q)$$
$[a>b>c>d>u]$ BY (251.00)

2. $$\int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\beta,r)$$
$[a>b>c\ge u>d]$ BY (254.00)

3. $$\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\gamma,r)$$
$[a>b>c>u\ge d]$ BY (253.00)

4. $$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\delta,q)$$
$[a>b\ge u>c>d]$ BY (254.00)

5. $$\int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\kappa,q)$$
$[a>b>u\ge c>d]$ BY (255.00)

6. $$\int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\lambda,r)$$
$[a\ge u>b>c>d]$ BY (256.00)

7.[11] $$\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\mu,r)$$
$[a>u\ge b>c>d]$ BY (257.00)

8. $$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\nu, q)$$

$[u > a > b > c > d]$ BY (258.00)

3.148

1.[8] $$\int_u^d \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-c)\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + c\,F(\alpha, q)\right\}$$

$[a > b > c > d > u]$ BY (251.03)

2. $$\int_d^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(d-a)\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right) + a\,F(\beta, r)\right\}$$

$[a > b > c \ge u > d]$ BY (252.11)

3. $$\int_u^c \frac{x\,dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(c-b)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + b\,F(\gamma, r)\right\}$$

$[a > b > c > u \ge d]$ BY (253.11)

4. $$\int_c^u \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(c-d)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + d\,F(\delta, q)\right\}$$

$[a > b \ge u > c > d]$ BY (254.10)

5. $$\int_u^b \frac{x\,dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-a)\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) + a\,F\left(\kappa, q\right)\right\}$$

$[a > b > u \ge c > d]$ BY (255.17)

6.[8] $$\int_b^u \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(b-c)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + c\,F(\lambda, r)\right\}$$

$[a \ge u > b > c > d]$ BY (256.11)

7. $$\int_u^a \frac{x\,dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-d)\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right) + d\,F(\mu, r)\right\}$$

$[a > u \ge b > c > d]$ BY (257.11)

8. $$\int_a^u \frac{x\,dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{(a-b)\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right) + b\,F(\nu, q)\right\}$$

$[u > a > b > c > d]$ BY (258.11)

3.149

1. $$\int_u^d \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(d-x)}}$$
$$= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{(c-d)\,\Pi\left(\alpha, \frac{c(a-d)}{d(a-c)}, q\right) + d\,F(\alpha, q)\right\}$$

$[a > b > c > d > u]$ BY (251.04)

2. $$\int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}} \left\{(a-d)\,\Pi\left(\beta, \frac{a(d-c)}{d(a-c)}, r\right) + d\,F(\beta, r)\right\}$$
$[a > b > c \geq u > d]$ BY (252.12)

3. $$\int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{(b-c)\,\Pi\left(\gamma, \frac{b(c-d)}{c(b-d)}, r\right) + c\,F(\gamma, r)\right\}$$
$[a > b > c > u \geq d]$ BY (253.12)

4. $$\int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{(d-c)\,\Pi\left(\delta, \frac{d(b-c)}{c(b-d)}, q\right) + c\,F(\delta, q)\right\}$$
$[a > b \geq u > c > d]$ BY (254.11)

5. $$\int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{(a-b)\,\Pi\left(\kappa, \frac{a(b-c)}{b(a-c)}, q\right) + b\,F\,(\kappa, q)\right\}$$
$[a > b > u \geq c > d]$ BY (255.18)

6. $$\int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{(c-b)\,\Pi\left(\lambda, \frac{c(a-b)}{b(a-c)}, r\right) + b\,F(\lambda, r)\right\}$$
$[a \geq u > b > c > d]$ BY (256.12)

7. $$\int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{(d-a)\,\Pi\left(\mu, \frac{d(b-a)}{a(b-d)}, r\right) + a\,F(\mu, r)\right\}$$
$[a > u \geq b > c > d]$ BY (257.12)

8. $$\int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{ab\sqrt{(a-c)(b-d)}} \left\{(b-a)\,\Pi\left(\nu, \frac{b(a-d)}{a(b-d)}, q\right) + a\,F(\nu, q)\right\}$$
$[u > a > b > c > d]$ BY (258.12)

3.151

1. $$\int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(d-c)\,\Pi\left(\alpha, \frac{(a-d)(p-c)}{(a-c)(p-d)}, q\right) + (p-d)\,F(\alpha,q)\right]$$
$[a > b > c > d > u, \quad p \neq d]$ BY (251.39)

2. $$\int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(d-a)\,\Pi\left(\beta, \frac{(d-c)(p-a)}{(a-c)(p-d)}, r\right) + (p-d)\,F(\beta,r)\right]$$
$[a > b > c \geq u > d, \quad p \neq d]$ BY (252.39)

3. $$\int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \times \left[(c-b)\,\Pi\left(\gamma, \frac{(c-d)(p-b)}{(b-d)(p-c)}, r\right) + (p-c)\,F(\gamma,r)\right]$$
$[a > b > c > u \geq d, \quad p \neq c]$ BY (253.39)

4. $$\int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(c-d)\,\Pi\left(\delta, \frac{(b-c)(p-d)}{(b-d)(p-c)}, q\right) + (p-c)\,F(\delta,q)\right]$$
$[a > b \geq u > c > d, \quad p \neq c]$ BY (254.39)

5. $$\int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \left[(b-a)\,\Pi\left(\kappa, \frac{(b-c)(p-a)}{(a-c)(p-b)}, q\right) + (p-b)\,F\,(\kappa,q)\right]$$
$[a > b > u \geq c > d, \quad p \neq b]$ BY (255.38)

6. $$\int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \times \left[(b-c)\,\Pi\left(\lambda, \frac{(a-b)(p-c)}{(a-c)(p-b)}, r\right) + (p-b)\,F(\lambda,r)\right]$$
$[a \geq u > b > c > d, \quad p \neq b]$ BY (256.39)

7. $$\int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \left[(a-d)\,\Pi\left(\mu, \frac{(b-a)(p-d)}{(b-d)(p-a)}, r\right) + (p-a)\,F(\mu, r)\right]$$
$[a > u \geq b > c > d, \quad p \neq a]$ BY (257.39)

8. $$\int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \left[(a-b)\,\Pi\left(\nu, \frac{(a-d)(p-b)}{(b-d)(p-a)}, q\right) + (p-a)\,F(\nu, q)\right]$$
$[u > a > b > c > d, \quad p \neq a]$ BY (258.39)

3.152 **Notation**: In **3.152–3.163** we set: $\alpha = \arctan\frac{u}{b}$, $\beta = \operatorname{arccot}\frac{u}{a}$

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \qquad \delta = \arccos\frac{u}{b}, \qquad \varepsilon = \arccos\frac{b}{u}, \qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$
$$\eta = \arcsin\frac{u}{b}, \qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \qquad \kappa = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$
$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \qquad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \qquad \nu = \arcsin\frac{a}{u}, \qquad q = \frac{\sqrt{a^2-b^2}}{a},$$
$$r = \frac{b}{\sqrt{a^2+b^2}}, \qquad s = \frac{a}{\sqrt{a^2+b^2}}, \qquad t = \frac{b}{a}.$$

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}F(\alpha, q)$$ $[a > b > 0]$ H 62(258), BY (221.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a}F(\beta, q)$$ $[a > b > 0]$ H 63 (259), BY (222.00)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\gamma, r)$$ $[b \geq u > 0]$ H 63 (260)

4. $$\int_u^b \frac{dx}{\sqrt{(x^2+a^2)(b^2-x^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\delta, r)$$ $[b > u \geq 0]$ H 63 (261), BY (213.00)

5. $$\int_b^u \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\varepsilon, s)$$ $[u > b > 0]$ H 63 (262), BY (211.00)

6. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{\sqrt{a^2+b^2}}F(\xi, s)$$ $[u > b > 0]$ H 63 (263), BY (212.00)

7. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{1}{a} F(\eta, t)$$ $[a > b \geq u > 0]$ H 63 (264), BY (219.00)

8. $$\int_u^b \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{1}{a} F(\zeta, t)$$ $[a > b > u \geq 0]$ H 63 (265), BY (220.00)

9. $$\int_b^u \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{a} F(\kappa, q)$$ $[a \geq u > b > 0]$ H 63 (266), BY (217.00)

10. $$\int_u^a \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{a} F(\lambda, q)$$ $[a > u \geq b > 0]$ H 63 (257), BY (218.00)

11. $$\int_a^u \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{a} F(\mu, t)$$ $[u > a > b > 0]$ H 63 (268), BY (216.00)

12. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{a} F(\nu, t)$$ $[u \geq a > b > 0]$ H 64(269), BY (215.00)

3.153

1. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}} = u\sqrt{\frac{a^2+u^2}{b^2+u^2}} - a\,E(\alpha, q)$$ $[u > 0, \quad a > b]$ BY (221.09)

2. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}} = \sqrt{a^2+b^2}\,E(\gamma, r) - \frac{a^2}{\sqrt{a^2+b^2}} F(\gamma, r) - u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$ $[b \geq u > 0]$ BY (214.05)

3. $$\int_u^b \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}} = \sqrt{a^2+b^2}\,E(\delta, r) - \frac{a^2}{\sqrt{a^2+b^2}} F(\delta, r)$$ $[b > u \geq 0]$ BY (213.06)

4. $$\int_b^u \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)\left(x^2-b^2\right)}} = \frac{b^2}{\sqrt{a^2+b^2}} F(\varepsilon, s) - \sqrt{a^2+b^2}\,E(\varepsilon, s) + \frac{1}{u}\sqrt{\left(u^2+a^2\right)\left(u^2-b^2\right)}$$ $[u > b > 0]$ BY (211.09)

5. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = a\left\{F(\eta, t) - E(\eta, t)\right\}$$ $[a > b \geq u > 0]$ BY (219.05)

6. $$\int_u^b \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = a\left\{F(\zeta, t) - E(\zeta, t)\right\} + u\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$ $[a > b > u \geq 0]$ BY (220.06)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = a\,E(\kappa, q) - \frac{1}{u}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$ $[a \geq u > b > 0]$ BY (217.05)

8. $$\int_u^a \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = a\,E(\lambda, q)$$ $[a > u \geq b > 0]$ BY (218.06)

9.[6] $$\int_a^u \frac{x^2\,dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = a\left\{F(\mu,t) - E(\mu,t)\right\} + u\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$

$[u > a > b > 0]$ BY (216.06)

10. $$\int_0^1 \frac{x^2\,dx}{\sqrt{\left(1+x^2\right)\left(1+k^2x^2\right)}} = \frac{1}{k^2}\left\{\sqrt{\frac{1+k^2}{2}} - E\left(\frac{\pi}{4}, \sqrt{1-k^2}\right)\right\}$$ BI (14)(9)

3.154

1. $$\int_0^u \frac{x^4\,dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E(\alpha,q) - b^2\,F(\alpha,q)\right\} + \frac{u}{3}\left(u^2-2a^2-b^2\right)\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$

$[a > b, \quad u > 0]$ BY (221.09)

2. $$\int_0^u \frac{x^4\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2a^2-b^2\right)a^2\,F(\gamma,r) - 2\left(a^4-b^4\right)E(\gamma,r)\right\} - \frac{u}{3}\left(2b^2-a^2+u^2\right)\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$

$[a \geq u > 0]$ BY (214.05)

3. $$\int_u^b \frac{x^4\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2a^2-b^2\right)a^2\,F(\delta,r) - 2\left(a^4-b^4\right)E(\delta,r)\right\} + \frac{u}{3}\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}$$

$[b > u \geq 0]$ BY (213.06)

4. $$\int_b^u \frac{x^4\,dx}{\sqrt{\left(a^2+x^2\right)\left(x^2-b^2\right)}} = \frac{1}{3\sqrt{a^2+b^2}}\left\{\left(2b^2-a^2\right)b^2\,F(\varepsilon,s) + 2\left(a^4-b^4\right)E(\varepsilon,s)\right\} + \frac{2b^2-2a^2+u^2}{3u}\sqrt{\left(u^2+a^2\right)\left(u^2-b^2\right)}$$

$[u > b > 0]$ BY (211.09)

5. $$\int_0^u \frac{x^4\,dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\eta,t) - 2\left(a^2+b^2\right)E(\eta,t)\right\} + \frac{u}{3}\sqrt{\left(a^2-u^2\right)\left(b^2-u^2\right)}$$

$[a > b \geq u > 0]$ BY (219.05)

6. $$\int_u^b \frac{x^4\,dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\zeta,t) - 2\left(a^2+b^2\right)E(\zeta,t)\right\} + \frac{u}{3}\left(u^2+a^2+2b^2\right)\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$

$[a > b > u \geq 0]$ BY (220.06)

7. $$\int_b^u \frac{x^4\,dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E\left(\kappa,q\right) - b^2\,F\left(\kappa,q\right)\right\} - \frac{u^2+2a^2+2b^2}{3u}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$

$[a \geq u > b > 0]$ BY (217.05)

8. $$\int_u^a \frac{x^4\,dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{a}{3}\left\{2\left(a^2+b^2\right)E(\lambda,q) - b^2\,F(\lambda,q)\right\} + \frac{u}{3}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$

$[a > u \geq b > 0]$ BY (218.06)

9. $$\int_a^u \frac{x^4\,dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{a}{3}\left\{\left(2a^2+b^2\right)F(\mu,t) - 2\left(a^2+b^2\right)E(\mu,t)\right\} + \frac{u}{3}\left(u^2+2a^2+b^2\right)\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.06)

3.155

1. $$\int_u^a \sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\lambda,q) - 2b^2\,F(\lambda,q)\right\} - \frac{u}{3}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$
$[a > u \ge b > 0]$ BY (218.11)

2. $$\int_a^u \sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\mu,t) - \left(a^2-b^2\right)F(\mu,t)\right\} + \frac{u}{3}\left(u^2-a^2-2b^2\right)\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.10)

3. $$\int_0^u \sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}\,dx = \frac{a}{3}\left\{2b^2\,F(\alpha,q) - \left(a^2+b^2\right)E(\alpha,q)\right\} + \frac{u}{3}\left(u^2+a^2+2b^2\right)\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a > b, \quad u > 0]$ BY (221.08)

4. $$\int_0^u \sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{a^2\,F(\gamma,r) - \left(a^2-b^2\right)E(\gamma,r)\right\} + \frac{u}{3}\left(u^2+2a^2-b^2\right)\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[a \ge u > 0]$ BY (214.12)

5.[9] $$\int_u^b \sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{a^2\,F(\delta,r) + \left(b^2-a^2\right)E(\delta,r)\right\} + \frac{u}{3}\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}$$
$[b > u \ge 0]$ BY (213.13)

6. $$\int_b^u \sqrt{\left(a^2+x^2\right)\left(x^2-b^2\right)}\,dx = \frac{1}{3}\sqrt{a^2+b^2}\left\{\left(b^2-a^2\right)E(\varepsilon,s) - b^2\,F(\varepsilon,s)\right\} + \frac{u^2+a^2-b^2}{3u}\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)}$$
$[u > b > 0]$ BY (211.08)

7. $$\int_0^u \sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\eta,t) - \left(a^2-b^2\right)F(\eta,t)\right\} + \frac{u}{3}\sqrt{\left(a^2-u^2\right)\left(b^2-u^2\right)}$$
$[a > b \ge u > 0]$ BY (219.11)

8. $$\int_u^b \sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E(\zeta,t) - \left(a^2-b^2\right)F(\zeta,t)\right\} + \frac{u}{3}\left(u^2-2a^2-b^2\right)\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a > b > u \ge 0]$ BY (220.05)

9. $$\int_b^u \sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}\,dx = \frac{a}{3}\left\{\left(a^2+b^2\right)E\left(\kappa,q\right) - 2b^2 F\left(\kappa,q\right)\right\} + \frac{u^2-a^2-b^2}{3u}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$
$[a \geq u > b > 0]$ BY (217.09)

3.156

1.[6] $$\int_u^\infty \frac{dx}{x^2\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}} = \frac{1}{ub^2}\sqrt{\frac{b^2+u^2}{a^2+u^2}} - \frac{1}{ab^2}E(\beta,q)$$
$[a \geq b, \quad u > 0]$ BY (222.04)

2. $$\int_u^b \frac{dx}{x^2\sqrt{\left(x^2+a^2\right)\left(b^2-x^2\right)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{a^2 F(\delta,r) - \left(a^2+b^2\right)E(\delta,r)\right\} + \frac{1}{a^2b^2u}\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}$$
$[b > u > 0]$ BY (213.09)

3. $$\int_b^u \frac{dx}{x^2\sqrt{\left(x^2+a^2\right)\left(x^2-b^2\right)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{\left(a^2+b^2\right)E(\varepsilon,s) - b^2 F(\varepsilon,s)\right\}$$
$[u > b > 0]$ BY (211.11)

4. $$\int_u^\infty \frac{dx}{x^2\sqrt{\left(x^2+a^2\right)\left(x^2-b^2\right)}} = \frac{1}{a^2b^2\sqrt{a^2+b^2}}\left\{\left(a^2+b^2\right)E(\xi,s) - b^2 F(\xi,s)\right\} - \frac{1}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$
$[u \geq b > 0]$ BY (212.06)

5. $$\int_u^b \frac{dx}{x^2\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{1}{ab^2}\left\{F(\zeta,t) - E(\zeta,t)\right\} + \frac{1}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a > b > u > 0]$ BY (220.09)

6. $$\int_b^u \frac{dx}{x^2\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{ab^2}E\left(\kappa,q\right)$$
$[a \geq u > b > 0]$ BY (217.01)

7. $$\int_u^a \frac{dx}{x^2\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{ab^2}E(\lambda,q) - \frac{1}{a^2b^2u}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$
$[a > u \geq b > 0]$ BY (218.12)

8. $$\int_a^u \frac{dx}{x^2\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{ab^2}\left\{F(\mu,t) - E(\mu,t)\right\} + \frac{1}{a^2u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.09)

9. $$\int_u^\infty \frac{dx}{x^2\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{ab^2}\left\{F(\nu,t) - E(\nu,t)\right\}$$
$[u \geq a > b > 0]$ BY (215.07)

3.157

1. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{1}{a(p+b^2)}\left\{\frac{b^2}{p}\Pi\left(\alpha, \frac{p+b^2}{p}, q\right) + F(\alpha, q)\right\}$$
$[p \neq 0]$ BY (221.13)

2. $$\int_u^\infty \frac{dx}{(p-x^2)\sqrt{(x^2+a^2)(x^2+b^2)}} = -\frac{1}{a(a^2+p)}\left\{\Pi\left(\beta, \frac{a^2+p}{a^2}, q\right) - F(\beta, q)\right\}$$ BY (222.11)

3. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{p(p+a^2)\sqrt{a^2+b^2}}\left\{a^2\Pi\left(\gamma, \frac{b^2(p+a^2)}{p(a^2+b^2)}, r\right) + p\,F(\gamma, r)\right\}$$
$[b \geq u > 0, \quad p \neq 0]$ BY (214.13)a

4. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{(p-b^2)\sqrt{a^2+b^2}}\Pi\left(\delta, \frac{b^2}{b^2-p}, r\right)$$
$[b > u \geq 0, \quad p \neq b^2]$ BY (213.02)

5. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{p(p-b^2)\sqrt{a^2+b^2}}\left\{b^2\Pi\left(\varepsilon, \frac{p}{p-b^2}, s\right) + \left(p-b^2\right)F(\varepsilon, s)\right\}$$
$[u > b > 0, \quad p \neq b^2]$ BY (211.14)

6. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{(a^2+p)\sqrt{a^2+b^2}}\left\{\Pi\left(\xi, \frac{a^2+p}{a^2+b^2}, s\right) - F(\xi, s)\right\}$$
$[u \geq b > 0]$ BY (212.12)

7. $$\int_0^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{ap}\Pi\left(\eta, \frac{b^2}{p}, t\right)$$ $[a > b \geq u > 0; \quad p \neq b]$ BY (219.02)

8. $$\int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times\left\{\left(b^2-a^2\right)\Pi\left(\zeta, \frac{b^2(p-a^2)}{a^2(p-b^2)}, t\right) + \left(p-b^2\right)F(\zeta, t)\right\}$$
$[a > b > u \geq 0; \quad p \neq b^2]$ BY (220.13)

9. $$\int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)}\left\{b^2\Pi\left(\kappa, \frac{p(a^2-b^2)}{a^2(p-b^2)}, q\right) + \left(p-b^2\right)F(\kappa, q)\right\}$$
$[a \geq u > b > 0; \quad p \neq b^2]$ BY (217.12)

10. $$\int_u^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)}\Pi\left(\lambda, \frac{a^2-b^2}{a^2-p}, q\right)$$
$[a > u \geq b > 0; \quad p \neq a^2]$ BY (218.02)

11. $$\int_a^u \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a(p-a^2)(p-b^2)}\left\{\left(a^2-b^2\right)\Pi\left(\mu, \frac{p-b^2}{p-a^2}, t\right) + \left(p-a^2\right)F(\mu, t)\right\}$$
$[u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2]$ BY (216.12)

12. $$\int_u^\infty \frac{dx}{(x^2-p)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap}\left\{\Pi\left(\nu, \frac{p}{a^2}, t\right) - F(\nu, t)\right\}$$

$[u \geq a > b > 0; \quad p \neq 0]$ BY (215.12)

3.158

1. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left\{a^2 E(\alpha, q) - b^2 F(\alpha, q)\right\}$$

$[a > b; \quad u > 0]$ BY (221.05)

2. $$\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)}\left\{a^2 E(\beta, q) - b^2 F(\beta, q)\right\} - \frac{u}{b^2\sqrt{(a^2+u^2)(b^2+u^2)}}$$

$[a > b, \quad u \geq 0]$ BY (222.05)

3. $$\int_0^u \frac{dx}{\sqrt{(x^2+a^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\left\{F(\alpha, q) - E(\alpha, q)\right\} + \frac{u}{a^2\sqrt{(u^2+a^2)(u^2+b^2)}}$$

$[a > b; \quad u > 0]$ BY (221.06)

4. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2+b^2)}} = \frac{1}{a(a^2-b^2)}\left\{F(\beta, q) - E(\beta, q)\right\}$$

$[a > b, \quad u \geq 0]$ BY (222.03)

5. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}} E(\gamma, r)$$ $[b \geq u > 0]$ BY (214.01)a

6. $$\int_u^b \frac{dx}{\sqrt{(a^2+x^2)^3(b^2-x^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}} E(\delta, r) - \frac{u}{a^2(a^2+b^2)}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$

$[b > u \geq 0]$ BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\left\{F(\varepsilon, s) - E(\varepsilon, s)\right\} + \frac{1}{(a^2+b^2)u}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$

$[u > b > 0]$ BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^3(x^2-b^2)}} = \frac{1}{a^2\sqrt{a^2+b^2}}\left\{F(\xi, s) - E(\xi, s)\right\}$$

$[u \geq b > 0]$ BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)^3}} = \frac{1}{b^2\sqrt{a^2+b^2}}\left\{F(\gamma, r) - E(\gamma, r)\right\} + \frac{u}{b^2\sqrt{(a^2+u^2)(b^2-u^2)}}$$

$[b > u > 0]$ BY (214.10)

10. $$\int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^3}} = \frac{u}{b^2\sqrt{(a^2+u^2)(u^2-b^2)}} - \frac{1}{b^2\sqrt{a^2+b^2}} E(\xi, s)$$

$[u \geq b > 0]$ BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)}} = \frac{1}{a^2\left(a^2-b^2\right)}\left\{a\,E(\eta,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$

$[a > b \geq u > 0]$ BY (219.07)

12. $$\int_u^b \frac{dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)}} = \frac{1}{a\left(a^2-b^2\right)}\,E(\zeta,t)$$ $[a > b > u \geq 0]$ BY (220.10)

13. $$\int_b^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^3\left(x^2-b^2\right)}} = \frac{1}{a\left(a^2-b^2\right)}\left\{F(\kappa,q) - E(\kappa,q) + \frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}}\right\}$$

$[a > u > b > 0]$ BY (217.10)

14. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)^3\left(x^2-b^2\right)}} = \frac{1}{a\left(b^2-a^2\right)}\left\{E(\nu,t) - \frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}}\right\}$$

$[u > a > b > 0]$ BY (215.04)

15. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)^3}} = \frac{1}{ab^2}\,F(\eta,t) - \frac{1}{b^2\left(a^2-b^2\right)}\left\{a\,E(\eta,t) - u\sqrt{\frac{a^2-u^2}{b^2-u^2}}\right\}$$

$[a > b > u > 0]$ BY (219.06)

16. $$\int_u^a \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)}\left\{b^2\,F(\lambda,q) - a^2\,E(\lambda,q) + au\sqrt{\frac{a^2-u^2}{u^2-b^2}}\right\}$$

$[a > u > b > 0]$ BY (218.04)

17. $$\int_a^u \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^3}} = \frac{a}{b^2\left(a^2-b^2\right)}\,E(\mu,t) - \frac{1}{ab^2}\,F(\mu,t)$$

$[u > a > b > 0]$ BY (216.11)

18. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^3}} = \frac{1}{b^2\left(a^2-b^2\right)}\left\{a\,E(\nu,t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\} - \frac{1}{ab^2}\,F(\nu,t)$$

$[u \geq a > b > 0]$ BY (215.06)

3.159

1. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)^3}} = \frac{a}{a^2-b^2}\left\{F(\alpha,q) - E(\alpha,q)\right\}$$

$[a > b, \quad u > 0]$ BY (221.12)

2. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)^3}} = \frac{a}{a^2-b^2}\left\{F(\beta,q) - E(\beta,q)\right\} + \frac{u}{\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$

$[a > b, \quad u \geq 0]$ BY (222.10)

3. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)}} = \frac{1}{a\left(a^2-b^2\right)}\left\{a^2\,E(\alpha,q) - b^2\,F(\alpha,q)\right\} - \frac{u}{\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$

$[a > b, \quad u > 0]$ BY (221.11)

4. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)}} = \frac{1}{a\left(a^2-b^2\right)}\left\{a^2\,E(\beta,q) - b^2\,F(\beta,q)\right\}$$

$[a > b, \quad u \geq 0]$ BY (222.07)

5. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)^3\left(b^2-x^2\right)}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\gamma,r) - E(\gamma,r)\right\}$$

$[b \geq u > 0]$ BY (214.04)

6. $$\int_u^b \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)^3\left(b^2-x^2\right)}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\delta,r) - E(\delta,r)\right\} + \frac{u}{a^2+b^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$

$[b > u \geq 0]$ BY (213.07)

7. $$\int_b^u \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)^3\left(x^2-b^2\right)}} = \frac{1}{\sqrt{a^2+b^2}}\,E(\varepsilon,s) - \frac{a^2}{u\left(a^2+b^2\right)}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$

$[u > b > 0]$ BY (211.13)

8. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)^3\left(x^2-b^2\right)}} = \frac{1}{\sqrt{a^2+b^2}}\,E(\xi,s)$$ $[u \geq b > 0]$ BY (212.01)

9. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)^3}} = \frac{u}{\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}} - \frac{1}{\sqrt{a^2+b^2}}\,E(\gamma,r)$$

$[b > u > 0]$ BY (214.07)

10. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(a^2+x^2\right)\left(x^2-b^2\right)^3}} = \frac{1}{\sqrt{a^2+b^2}}\left\{F(\xi,s) - E(\xi,s)\right\} + \frac{u}{\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)}}$$

$[u > b > 0]$ BY (212.10)

11. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)}} = \frac{1}{a^2-b^2}\left\{a\,E(\eta,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\} - \frac{1}{a}\,F(\eta,t)$$

$[a > b \geq u > 0]$ BY (219.04)

12. $$\int_u^b \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)}} = \frac{a}{a^2-b^2}\,E(\zeta,t) - \frac{1}{a}\,F(\zeta,t)$$

$[a > b > u \geq 0]$ BY (220.08)

13. $$\int_b^u \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)^3\left(x^2-b^2\right)}} = \frac{1}{a\left(a^2-b^2\right)}\left\{b^2\,F\left(\kappa,q\right) - a^2\,E\left(\kappa,q\right) + \frac{a^3}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}}\right\}$$

$[a > u > b > 0]$ BY (217.06)

14. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2-a^2\right)^3\left(x^2-b^2\right)}} = \frac{a}{a^2-b^2}\left\{\frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} - E(\nu,t)\right\} + \frac{1}{a}F(\nu,t)$$

$[u > a > b > 0]$ BY (215.09)

15. $$\int_0^u \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)^3}} = \frac{1}{a^2-b^2}\left\{u\sqrt{\frac{a^2-u^2}{b^2-u^2}} - a\,E(\eta,t)\right\}$$

$[a > b > u > 0]$ BY (219.12)

16. $$\int_u^a \frac{x^2\,dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)^3}} = \frac{1}{a^2-b^2}\left\{a\,F(\lambda,q) - a\,E(\lambda,q) + u\sqrt{\frac{a^2-u^2}{u^2-b^2}}\right\}$$

$[a > u > b > 0]$ BY (218.07)

17. $$\int_a^u \frac{x^2\,dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^3}} = \frac{a}{a^2-b^2}E(\mu,t)$$ $[u > a > b > 0]$ BY (216.01)

18. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^3}} = \frac{1}{a^2-b^2}\left\{a\,E(\nu,t) - \frac{b^2}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\}$$

$[u \geq a > b > 0]$ BY (215.11)

3.161

1. $$\int_u^\infty \frac{dx}{x^4\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E(\beta,q) - b^2\,F(\beta,q)\right\} + \frac{a^2b^2-u^2\left(2a^2+b^2\right)}{3a^2b^4u^3}$$

$[a > b, \quad u > 0]$ BY (222.04)

2. $$\int_u^b \frac{dx}{x^4\sqrt{\left(x^2+a^2\right)\left(b^2-x^2\right)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\left\{a^2\left(2a^2-b^2\right)F(\delta,r) - 2\left(a^4-b^4\right)E(\delta,r)\right\} + \frac{a^2b^2+2u^2\left(a^2-b^2\right)}{3a^4b^4u^3}\sqrt{\left(b^2-u^2\right)\left(a^2+u^2\right)}$$

$[b > u > 0]$ BY (213.09)

3. $$\int_b^u \frac{dx}{x^4\sqrt{\left(x^2+a^2\right)\left(x^2-b^2\right)}} = \frac{2b^2-a^2}{3a^4b^2\sqrt{a^2+b^2}}F(\varepsilon,s) + \frac{2}{3}\frac{\left(a^2-b^2\right)\sqrt{a^2+b^2}}{a^4b^4}E(\varepsilon,s) + \frac{1}{3a^2b^2u^3}\sqrt{\left(u^2+a^2\right)\left(u^2-b^2\right)}$$

$[u > b > 0]$ BY (211.11)

4. $$\int_u^\infty \frac{dx}{x^4\sqrt{\left(x^2+a^2\right)\left(x^2-b^2\right)}} = \frac{1}{3a^4b^4\sqrt{a^2+b^2}}\left\{2\left(a^4-b^4\right)E(\xi,s) + b^2\left(2b^2-a^2\right)F(\xi,s)\right\} - \frac{a^2b^2+u^2\left(2a^2-b^2\right)}{3a^2b^4u^3}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$

$[u \geq b > 0]$ BY (212.06)

5. $$\int_u^b \frac{dx}{x^4\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)}} = \frac{1}{3a^3b^4}\left\{\left\{\left(2a^2+b^2\right)F(\zeta,t) - 2\left(a^2+b^2\right)E(\zeta,t)\right\} + \frac{\left[\left(2a^2+b^2\right)u^2+a^2b^2\right]a}{u^3}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a > b > u > 0]$ BY (220.09)

6. $$\int_b^u \frac{dx}{x^4\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E\left(\kappa,q\right) - b^2\,F\left(\kappa,q\right)\right\} + \frac{1}{3a^2b^2u^3}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$
$[a \geq u > b > 0]$ BY (217.14)

7. $$\int_u^a \frac{dx}{x^4\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}} = \frac{1}{3a^3b^4}\left\{2\left(a^2+b^2\right)E(\lambda,q) - b^2\,F(\lambda,q) - \frac{2\left(a^2+b^2\right)u^2+a^2b^2}{au^3}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}\right\}$$
$[a > u \geq b > 0]$ BY (218.12)

8. $$\int_a^u \frac{dx}{x^4\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{3a^3b^4}\left\{\left\{\left(2a^2+b^2\right)F(\mu,t) - 2\left(a^2+b^2\right)E(\mu,t)\right\} + \frac{\left[\left(a^2+2b^2\right)u^2+a^2b^2\right]b^2}{au^3}\sqrt{\frac{u^2-a^2}{u^2-b^2}}\right\}$$
$[u > a > b > 0]$ BY (216.09)

9. $$\int_u^\infty \frac{dx}{x^4\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)}} = \frac{1}{3a^3b^4}\left\{\left(2a^2+b^2\right)F(\nu,t) - 2\left(a^2+b^2\right)E(\nu,t) + \frac{ab^2}{u^3}\sqrt{\left(u^2-a^2\right)\left(u^2-b^2\right)}\right\}$$
$[u \geq a > b > 0]$ BY (215.07)

3.162

1. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)^5\left(x^2+b^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(3a^2-b^2\right)F(\alpha,q) - 2\left(2a^2-b^2\right)E(\alpha,q)\right\} + \frac{u\left[a^2\left(4a^2-3b^2\right)+u^2\left(3a^2-2b^2\right)\right]}{3a^4\left(a^2-b^2\right)\sqrt{\left(u^2+a^2\right)^3\left(u^2+b^2\right)}}$$
$[a > b, \quad u > 0]$ BY (221.06)

2. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)^5\left(x^2+b^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(3a^2-b^2\right)F(\beta,q)-2\left(2a^2-b^2\right)E(\beta,q)\right\} + \frac{u}{3a^2\left(a^2-b^2\right)}\sqrt{\frac{u^2+b^2}{\left(a^2+u^2\right)^3}}$$
$[a>b, \quad u \geq 0]$ BY (222.03)

3. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)^5}} = \frac{3b^2-a^2}{3ab^2\left(a^2-b^2\right)^2}F(\alpha,q)+\frac{a\left(2a^2-4b^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\alpha,q) + \frac{u}{3b^2\left(a^2-b^2\right)}\sqrt{\frac{u^2+a^2}{\left(u^2+b^2\right)^3}}$$
$[a>b, \quad u>0]$ BY (221.05)

4. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)^5}} = \frac{1}{3ab^4\left(a^2-b^2\right)^2}\left\{2a^2\left(a^2-2b^2\right)E(\beta,q)+b^2\left(3b^2-a^2\right)F(\beta,q)\right\} - \frac{u\left[b^2\left(3a^2-4b^2\right)+u^2\left(2a^2-3b^2\right)\right]}{3b^4\left(a^2-b^2\right)\sqrt{\left(u^2+a^2\right)\left(u^2+b^2\right)^3}}$$
$[a>b, \quad u \geq 0]$ BY (222.05)

5. $$\int_0^u \frac{dx}{\sqrt{\left(a^2+x^2\right)^5\left(b^2-x^2\right)}} = \frac{1}{3a^4\sqrt{\left(a^2+b^2\right)^3}}\left\{2\left(b^2+2a^2\right)E(\gamma,r)-a^2F(\gamma,r)\right\} + \frac{u}{3a^2\left(a^2+b^2\right)}\sqrt{\frac{b^2-u^2}{\left(a^2+u^2\right)^3}}$$
$[b \geq u > 0]$ BY (214.15)

6. $$\int_u^b \frac{dx}{\sqrt{\left(a^2+x^2\right)^5\left(b^2-x^2\right)}} = \frac{1}{3a^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(4a^2+2b^2\right)E(\delta,r)-a^2F(\delta,r)\right\} - \frac{u\left[a^2\left(5a^2+3b^2\right)+u^2\left(4a^2+2b^2\right)\right]}{3a^4\left(a^2+b^2\right)^2}\sqrt{\frac{b^2-u^2}{\left(a^2+u^2\right)^3}}$$
$[b>u>0]$ BY (213.08)

7. $$\int_b^u \frac{dx}{\sqrt{\left(a^2+x^3\right)^5\left(x^2-b^2\right)}} = \frac{1}{3a^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(3a^2+2b^2\right)F(\varepsilon,s)-\left(4a^2+2b^2\right)E(\varepsilon,s)\right\} + \frac{\left(3a^2+b^2\right)u^2+2\left(2a^2+b^2\right)a^2}{3a^2\left(a^2+b^2\right)^2u}\sqrt{\frac{u^2-b^2}{\left(u^2+a^2\right)^3}}$$
$[u>b>0]$ BY (211.05)

8. $$\int_u^\infty \frac{dx}{\sqrt{\left(a^2+x^2\right)^5\left(x^2-b^2\right)}} = \frac{1}{3a^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(3a^2+2b^2\right)F(\xi,s)-\left(4a^2+2b^2\right)E(\xi,s)\right\} + \frac{u}{3a^2\left(a^2+b^2\right)}\sqrt{\frac{u^2-b^2}{\left(a^2+u^2\right)^3}}$$
$[u>b>0]$ BY (212.03)

9. $$\int_0^u \frac{dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)^5}} = \frac{1}{3b^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(2a^2+3b^2\right)F(\gamma,r)-\left(2a^2+4b^2\right)E(\gamma,r)\right\} + \frac{u\left[\left(3a^3+4b^2\right)b^2-\left(2a^2+3b^2\right)u^2\right]}{3b^4\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)^3}}$$
$[b > u > 0]$ BY (214.10)

10. $$\int_u^\infty \frac{dx}{\sqrt{\left(a^2+x^2\right)\left(x^2-b^2\right)^5}} = \frac{1}{3b^4\sqrt{\left(a^2+b^2\right)^3}}\left\{\left(2a^2+4b^2\right)E(\xi,s)-b^2F(\xi,s)\right\} + \frac{u\left[\left(3a^2+4b^2\right)b^2-\left(2a^2+3b^2\right)u^2\right]}{3b^4\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)^3}}$$
$[u > b > 0]$ BY (212.04)

11. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(b^2-x^2\right)^5}} = \frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\eta,t)+\frac{2a\left(2b^2-a^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\eta,t) + \frac{u\left[\left(3a^2-5b^2\right)b^2-2\left(a^2-2b^2\right)u^2\right]}{3b^4\left(a^2-b^2\right)^2\left(b^2-u^2\right)}\sqrt{\frac{a^2-u^2}{b^2-u^2}}$$
$[a > b > a > 0]$ BY (219.06)

12. $$\int_u^a \frac{dx}{\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)^5}} = \frac{3b^2-a^2}{3ab^2\left(a^2-b^2\right)^2}F(\lambda,q)+\frac{2a\left(a^2-2b^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\lambda,q) + \frac{u\left[2\left(2b^2-a^2\right)u^2+\left(3a^2-5b^2\right)b^2\right]}{3b^4\left(a^2-b^2\right)^2\left(u^2-b^2\right)}\sqrt{\frac{a^2-u^2}{u^2-b^2}}$$
$[a > u > b > 0]$ BY (218.04)

13. $$\int_a^u \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^5}} = \frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\mu,t)+\frac{2a\left(2b^2-a^2\right)}{3b^4\left(a^2-b^2\right)^2}E(\mu,t) + \frac{u}{3b^2\left(a^2-b^2\right)\left(u^2-b^2\right)}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.11)

14. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)\left(x^2-b^2\right)^5}} = \frac{\left(4b^2-2a^2\right)a}{3b^4\left(a^2-b^2\right)^2}E(\nu,t)+\frac{2a^2-3b^2}{3ab^4\left(a^2-b^2\right)}F(\nu,t) - \frac{\left(3b^2-a^2\right)u^2-\left(4b^2-2a^2\right)b^2}{3b^2u\left(a^2-b^2\right)^2\left(u^2-b^2\right)}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u \geq a > b > 0]$ BY (215.06)

15. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(b^2-x^2\right)}} = \frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(4a^2-2b^2\right)E(\eta,t)-\left(a^2-b^2\right)F(\eta,t) - \frac{u\left[\left(5a^2-3b^2\right)a^2-\left(4a^2-2b^2\right)u^2\right]}{a\left(a^2-u^2\right)}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a > b \geq u > 0]$ BY (219.07)

16. $$\int_u^b \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(b^2-x^2\right)}}=\frac{2\left(2a^2-b^2\right)}{3a^3\left(a^2-b^2\right)^2}E\left(\zeta,r\right)-\frac{1}{3a^3\left(a^2-b^2\right)}F(\zeta,t)+\frac{u}{3a^2\left(a^2-b^2\right)\left(a^2-u^2\right)}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a>b>u\geq 0]$ BY (220.10)

17. $$\int_b^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^5\left(x^2-b^2\right)}}=\frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(3a^2-b^2\right)F\left(\kappa,q\right)-\left(4a^2-2b^2\right)E\left(\kappa,q\right)\right\}+\frac{2\left(2a^2-b^2\right)a^2+\left(b^2-3a^2\right)u^2}{3a^2u\left(a^2-b^2\right)^2\left(a^2-u^2\right)}\sqrt{\frac{u^2-b^2}{a^2-u^2}},$$
$[a>u>b>0]$ BY (217.10)

18. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)^5\left(x^2-b^2\right)}}=\frac{1}{3a^3\left(a^2-b^2\right)^2}\left\{\left(4a^2-2b^2\right)E(\nu,t)-\left(a^2-b^2\right)F(\nu,t)\right\}+\frac{\left(4a^2-2b^2\right)a^2+\left(b^2-3a^2\right)u^2}{3a^2u\left(a^2-b^2\right)^2\left(u^2-a^2\right)}\sqrt{\frac{u^2-b^2}{u^2-a^2}}$$
$[u>a>b>0]$ BY (215.04)

3.163

1. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)^3}}=\frac{1}{ab^2\left(a^2-b^2\right)^2}\left\{\left(a^2+b^2\right)E(\alpha,q)-2b^2F(\alpha,q)\right\}-\frac{u}{a^2\left(a^2-b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$
$[a>b,\quad u>0]$ BY (221.07)

2. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2+b^2\right)^3}}=\frac{1}{ab^2\left(a^2-b^2\right)^2}\left\{\left(a^2+b^2\right)E(\beta,q)-2b^2F(\beta,q)\right\}-\frac{u}{b^2\left(a^2-b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2+u^2\right)}}$$
$[a>b,\quad u\geq 0]$ BY (222.12)

3. $$\int_0^u \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(b^3-x^2\right)^3}}=\frac{1}{a^2b^2\sqrt{\left(a^2+b^2\right)^3}}\left\{a^2F(\gamma,r)-\left(a^2-b^2\right)E(\gamma,r)\right\}+\frac{u}{b^2\left(a^2+b^2\right)\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}}$$
$[b>u>0]$ BY (214.15)

4. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)^3\left(x^2-b^2\right)^3}}=\frac{b^2-a^2}{a^2b^2\sqrt{\left(a^2+b^2\right)^3}}E(\xi,s)-\frac{1}{a^2\sqrt{\left(a^2+b^2\right)^3}}F(\xi,s)+\frac{u}{b^2\left(a^2+b^2\right)\sqrt{\left(u^2+a^2\right)\left(u^2-b^2\right)}}$$
$[u>b>0]$ BY (212.05)

5. $$\int_0^u \frac{dx}{\sqrt{\left(a^2-x^2\right)^3\left(b^2-x^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)} F(\eta,t) - \frac{a^2+b^2}{ab^2\left(a^2-b^2\right)^2} E(\eta,t) + \frac{\left[a^4+b^4-\left(a^2+b^2\right)u^2\right]u}{a^2b^2\left(a^2-b^2\right)^2\sqrt{\left(a^2-u^2\right)\left(b^2-u^2\right)}}$$ $[a > b > u > 0]$ BY (279.08)

6. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2-a^2\right)^3\left(x^2-b^2\right)^3}} = \frac{1}{ab^2\left(a^2-b^2\right)} F(\nu,t) - \frac{a^2+b^2}{ab^2\left(a^2-b^2\right)^2} E(\nu,t) + \frac{1}{u\left(a^2-b^2\right)\sqrt{\left(u^2-a^2\right)\left(u^2-b^2\right)}}$$ $[u > a > b > 0]$ BY (215.10)

3.164 **Notation**: $\alpha = \arccos\frac{u^2-\rho\overline{\rho}}{u^2+\rho\overline{\rho}}, \qquad r = \frac{1}{2}\sqrt{-\frac{(\rho-\overline{\rho})^2}{\rho\overline{\rho}}}.$

1. $$\int_u^\infty \frac{dx}{\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = \frac{1}{\sqrt{\rho\overline{\rho}}} F(\alpha,r)$$ BY (225.00)

2. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2-\rho\overline{\rho}\right)^2\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = \frac{2u\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}{(\rho+\overline{\rho})^2\left(u^4-\rho^2\overline{\rho}^2\right)} - \frac{1}{(\rho+\overline{\rho})^2\sqrt{\rho\overline{\rho}}} E(\alpha,r)$$ BY (225.03)

3. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2+\rho\overline{\rho}\right)^2\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}} = -\frac{1}{(\rho-\overline{\rho})^2\sqrt{\rho\overline{\rho}}}\left[F(\alpha,r)-E(\alpha,r)\right]$$ BY (225.07)

4. $$\int_u^\infty \frac{x^2\,dx}{\sqrt{\left(x^2+\rho^2\right)^3\left(x^2+\overline{\rho}^2\right)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{\left(\rho^2-\overline{\rho}^2\right)^2} E(\alpha,r) + \frac{1}{(\rho-\overline{\rho})^2\sqrt{\rho\overline{\rho}}} F(\alpha,r) - \frac{2u\left(u^2-\rho\overline{\rho}\right)}{(\rho+\overline{\rho})^2\left(u^2+\rho\overline{\rho}\right)\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}$$ BY (225.05)

5. $$\int_u^\infty \frac{\left(x^2-\rho\overline{\rho}\right)^2\,dx}{\sqrt{\left(x^2+\rho^2\right)^3\left(x^2+\overline{\rho}^2\right)^3}} = -\frac{4\sqrt{\rho\overline{\rho}}}{(\rho-\overline{\rho})^2}\left[F(\alpha,r)-E(\alpha,r)\right] + \frac{2u\left(u^2-\rho\overline{\rho}\right)}{\left(u^2+\rho\overline{\rho}\right)\sqrt{\left(u^2+\rho^2\right)\left(u^2+\overline{\rho}^2\right)}}$$ BY (225.06)

6. $$\int_u^\infty \frac{\sqrt{\left(x^2+\rho^2\right)\left(x^2+\overline{\rho}^2\right)}}{\left(x^2+\rho\overline{\rho}\right)^2}\,dx = \frac{1}{\sqrt{\rho\overline{\rho}}} E(\alpha,r)$$ BY(225.01)

7. $$\int_u^\infty \frac{\left(x^2-\varrho\overline{\varrho}\right)^2\,dx}{\left(x^2+\varrho\overline{\varrho}\right)^2\sqrt{\left(x^2+\varrho^2\right)\left(x^2+\overline{\varrho}^2\right)}} = -\frac{4\sqrt{\varrho\overline{\varrho}}}{(\varrho-\overline{\varrho})^2} E(\alpha,r) + \frac{(\varrho+\overline{\varrho})^2}{(\varrho-\overline{\varrho})^2\sqrt{\varrho\overline{\varrho}}} F(\alpha,r)$$ BY (225.08)

8. $$\int_u^\infty \frac{\left(x^2+\varrho\overline{\varrho}\right)^2\,dx}{\left[\left(x^2+\varrho\overline{\varrho}\right)^2-4p^2\varrho\overline{\varrho}x^2\right]\sqrt{\left(x^2+\varrho^2\right)\left(x^2+\overline{\varrho}^2\right)}} = \frac{1}{\sqrt{\varrho\overline{\varrho}}}\Pi\left(\alpha,p^2,r\right)$$ BY (225.02)

3.165 **Notation**: $\alpha = \arccos \frac{u^2 - a^2}{u^2 + a^2}, \quad r = \frac{\sqrt{a^2 - b^2}}{a\sqrt{2}}.$

1. $$\int_u^a \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} \times F\left[\arctan\left(\frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a - u}{a + u}\right), \frac{2\sqrt{a\sqrt{2}\left(a^2 - b^2\right)}}{a\sqrt{2} + \sqrt{a^2 - b^2}}\right]$$
$[a > b, \quad a > u \geq 0]$ BY (264.00)

2. $$\int_u^\infty \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} F(\alpha, r)$$
$\left[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0\right]$ BY (263.00, 266.00)

3. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a^3}\left[F(\alpha, r) - 2E(\alpha, r)\right] + \frac{\sqrt{u^4 + 2b^2u^2 + a^4}}{a^2u\left(u^2 + a^2\right)}$$
$[a > b > 0, \quad u > 0]$ BY (263.06)

4. $$\int_u^\infty \frac{x^2\, dx}{\left(x^2 + a^2\right)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{4a\left(a^2 - b^2\right)}\left[F(\alpha, r) - E(\alpha, r)\right]$$
$\left[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0\right]$ BY (263.03, 266.05)

5. $$\int_u^\infty \frac{x^2\, dx}{\left(x^2 - a^2\right)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{u\sqrt{u^4 + 2b^2u^2 + a^4}}{2\left(a^2 + b^2\right)\left(u^4 - a^4\right)} - \frac{1}{4a\left(a^2 + b^2\right)} E(\alpha, r)$$
$\left[a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0\right]$ BY (263.05, 266.02)

6. $$\int_u^\infty \frac{x^2\, dx}{\sqrt{\left(x^4 + 2b^2x^2 + a^4\right)^3}} = \frac{a}{2\left(a^4 - b^4\right)} E(\alpha, r) - \frac{1}{4a\left(a^2 - b^2\right)} F(\alpha, r) - \frac{u\left(u^2 - a^2\right)}{2\left(a^2 + b^2\right)\left(u^2 + a^2\right)\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$\left[a^2 > b^2 > -\infty, a^2 > 0, \quad u \geq 0\right]$ BY (263.08, 266.03)

7. $$\int_u^\infty \frac{\left(x^2 - a^2\right)^2\, dx}{\sqrt{\left(x^4 + 2b^2x^2 + a^4\right)^3}} = \frac{a}{a^2 - b^2}\left[F(\alpha, r) - E(\alpha, r)\right] + \frac{u^2 - a^2}{u^2 + a^2} \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$\left[\left|b^2\right| < a^2, \quad u \geq 0\right]$ BY (266.08)

8. $$\int_u^\infty \frac{\left(x^2 + a^2\right)^2\, dx}{\sqrt{\left(x^2 + 2b^2x^2 + a^4\right)^3}} = \frac{a}{a^2 + b^2} E(\alpha, r) - \frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{u^2 - a^2}{u^2 + a^2} \cdot \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}}$$
$\left[\left|b^2\right| < a^2, \quad u \geq 0\right]$ BY (266.06)a

9. $$\int_u^\infty \frac{\left(x^2 - a^2\right)^2\, dx}{\left(x^2 + a^2\right)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{a}{a^2 - b^2} E(\alpha, r) - \frac{a^2 + b^2}{2a\left(a^2 - b^2\right)} F(\alpha, r)$$
$\left[a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0\right]$ BY (263.04, 266.07)

10. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{\left(x^2+a^2\right)^2}\,dx = \frac{1}{2a}\,E(\alpha,r) \qquad \left[a^2>b^2>-\infty,\quad a^2>0,\quad u\ge 0\right]$$
BY (263.01, 266.01)

11. $$\int_u^\infty \frac{\sqrt{x^4+2b^2x^2+a^4}}{\left(x^2-a^2\right)^2}\,dx = \frac{1}{2a}\left[F(\alpha,r)-E(\alpha,r)\right]+\frac{u}{u^4-a^4}\sqrt{u^4+2b^2u^2+a^4}$$
$[a>b>0,\quad u>a]$ BY (263)

12. $$\int_u^\infty \frac{\left(x^2+a^2\right)^2\,dx}{\left[\left(x^2+a^2\right)^2-4a^2p^2x^2\right]\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a}\,\Pi\left(\alpha,p^2,r\right)$$
$[a>b>0,\quad u\ge 0]$ BY (263.02)

3.166 Notation: $$\alpha=\arccos\frac{u^2-1}{u^2+1},\quad \beta=\arctan\left\{\left(1+\sqrt{2}\right)\frac{1-u}{1+u}\right\},$$
$$\gamma=\arccos u,\quad \delta=\arccos\frac{1}{u},\quad \varepsilon=\arccos\frac{1-u^2}{1+u^2},$$
$$r=\frac{\sqrt{2}}{2},\quad q=2\sqrt{3\sqrt{2}-4}=2\sqrt[4]{2}\left(\sqrt{2}-1\right)\approx 0.985171$$

1. $$\int_u^\infty \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}\,F(\alpha,r) \qquad [u\ge 0]$$
H (287), BY (263.50)

2. $$\int_u^\infty \frac{dx}{x^2\sqrt{x^4+1}} = \frac{1}{2}\left[F(\alpha,r)-2\,E(\alpha,r)\right]+\frac{\sqrt{u^4+1}}{u\left(u^2+1\right)}$$
$[u>0]$ BY (263.57)

3. $$\int_u^\infty \frac{x^2\,dx}{\left(x^4+1\right)\sqrt{x^4+1}} = \frac{1}{2}\,E(\alpha,r)-\frac{1}{4}\,F(\alpha,r)-\frac{u\left(u^2-1\right)}{2\left(u^2+1\right)\sqrt{u^4+1}}$$
$[u\ge 0]$ BY (263.59)

4. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2+1\right)^2\sqrt{x^4+1}} = \frac{1}{4}\left[F(\alpha,r)-E(\alpha,r)\right] \qquad [u\ge 0]$$
BY (263.53)

5. $$\int_u^\infty \frac{x^2\,dx}{\left(x^2-1\right)^2\sqrt{x^4+1}} = \frac{u\sqrt{u^4+1}}{2\left(u^4-1\right)}-\frac{1}{4}\,E(\alpha,r) \qquad [u>1]$$
BY (263.55)

6. $$\int_u^\infty \frac{\sqrt{x^4+1}}{\left(x^2-1\right)^2}\,dx = \frac{1}{2}\left[F(\alpha,r)-E(\alpha,r)\right]+\frac{u\sqrt{u^4+1}}{u^4-1}$$
$[u>1]$ BY (263.58)

7. $$\int_u^\infty \frac{\left(x^2-1\right)^2\,dx}{\left(x^2+1\right)^2\sqrt{x^4+1}} = E(\alpha,r)-\frac{1}{2}\,F(\alpha,r) \qquad [u\ge 0]$$
BY (263.54)

8. $$\int_u^\infty \frac{\sqrt{x^4+1}\,dx}{\left(x^2+1\right)^2} = \frac{1}{2}\,E(\alpha,r) \qquad [u\ge 0]$$
BY (263.51)

9. $$\int_u^\infty \frac{\left(x^2+1\right)^2\,dx}{\left[\left(x^2+1\right)^2-4p^2x^2\right]\sqrt{x^4+1}} = \frac{1}{2}\,\Pi\left(\alpha,p^2,r\right) \qquad [u \ge 0]$$ BY (263.52)

10. $$\int_0^u \frac{dx}{\sqrt{x^4+1}} = \frac{1}{2}\,F(\varepsilon,r)$$ H 66(288)

11. $$\int_u^1 \frac{dx}{\sqrt{x^4+1}} = \left(2-\sqrt{2}\right)F(\beta,q) \qquad [0 \le u < 1]$$ BY (264.50)

12. $$\int_u^1 \frac{\left(x^2+x\sqrt{2}+1\right)\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \left(2+\sqrt{2}\right)E(\beta,q) \qquad [0 \le u < 1]$$ BY (264.51)

13. $$\int_u^1 \frac{(1-x)^2\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \frac{1}{\sqrt{2}}\left[F(\beta,q)-E(\beta,q)\right] \qquad [0 \le u < 1]$$ BY (264.55)

14. $$\int_u^1 \frac{(1+x)^2\,dx}{\left(x^2-x\sqrt{2}+1\right)\sqrt{x^4+1}} = \frac{3\sqrt{2}+4}{2}\,E(\beta,q) - \frac{3\sqrt{2}-4}{2}\,F(\beta,q) \qquad [0 \le u < 1]$$ BY (264.56)

15. $$\int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}}\,F(\gamma,r) \qquad [u < 1]$$ H 66 (290), BY (259.75)

16. $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2$$

17. $$\int_1^u \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}\,F(\delta,r) \qquad [u > 1]$$ H 66 (289), BY (260.75)

18.[8] $$\int_u^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \sqrt{2}\,E(\gamma,r) - \frac{1}{\sqrt{2}}\,F(\gamma,r) \qquad [u < 1]$$
$$= \frac{1}{\sqrt{2\pi}}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2 \qquad [u = 0]$$ BY (259.76)

19. $$\int_1^u \frac{x^2\,dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}}\,F(\delta,r) - \sqrt{2}\,E(\delta,r) + \frac{1}{u}\sqrt{u^4-1} \qquad [u > 1]$$ BY (260.77)

20. $$\int_u^1 \frac{x^4\,dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}}\,F(\gamma,r) + \frac{u}{3}\sqrt{1-u^4} \qquad [u < 1]$$ BY (259.76)

21.[3] $$\int_1^u \frac{x^4\,dx}{\sqrt{x^4-1}} = \frac{1}{3\sqrt{2}}\,F(\delta,r) + \frac{1}{3}u\sqrt{u^4-1} \qquad [u > 1]$$ BY (260.77)

22. $$\int_0^u \frac{dx}{\sqrt{x\left(1+x^3\right)}} = \frac{1}{\sqrt[4]{3}}\,F\left(\arccos\frac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) \qquad [u > 0]$$ BY (260.50)

23. $$\int_0^u \frac{dx}{\sqrt{x\left(1-x^3\right)}} = \frac{1}{\sqrt[4]{3}}\,F\left(\arccos\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) \qquad [1 \ge u > 0]$$ BY (259.50)

3.167 **Notation**: In **3.167** and **3.168** we set: $\alpha = \arcsin\sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\beta = \arcsin\sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, \qquad \gamma = \arcsin\sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}},$$

$$\delta = \arcsin\sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, \qquad \kappa = \arcsin\sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\lambda = \arcsin\sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \qquad \mu = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}},$$

$$\nu = \arcsin\sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \qquad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

1. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) - F(\alpha,q)\right\}$$

$[a > b > c > d > u]$ BY (251.05)

2. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left\{\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - F(\beta,r)\right\}$$

$[a > b > c \geq u > d]$ BY (252.14)

3. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(c-b)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (b-d)\,F(\gamma,r)\right\}$$

$[a > b > c > u \geq d]$ BY (253.14)

4. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right)$$

$[a > b \geq u > c > d]$ BY (254.02)

5. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-a)\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) + (a-d)\,F(\kappa,q)\right\}$$

$[a > b > u \geq c > d]$ BY (255.20)

6. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(b-c)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (c-d)\,F(\lambda,r)\right\}$$

$[a \geq u > b > c > d]$ BY (256.13)

7. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right)$$

$[a > u \geq b > c > d]$ BY (257.02)

8. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left\{(a-b)\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right) + (b-d)\,F(\nu, q)\right\}$$
$[u > a > b > c > d]$ BY (258.14)

9. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right)$$
$[a > b > c > d > u]$ BY (251.02)

10. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - (a-c)\,F(\beta, r)\right]$$
$[a > b > c \geq u > d]$ BY (252.13)

11. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) - F(\gamma, r)\right]$$
$[a > b > c > u \geq d]$ BY (253.13)

12. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\delta, \frac{b-c}{b-d}, q\right) - F(\delta, q)\right]$$
$[a > b \geq u > c > d]$ BY (254.12)

13. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-a)\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) + (a-c)\,F(\kappa, q)\right]$$
$[a > b > u \geq c > d]$ BY (259.19)

14. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right)$$
$[a \geq u > b > c > d]$ BY (256.02)

15. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right) + (d-c)\,F(\mu, r)\right]$$
$[a > u \geq b > c > d]$ BY (257.13)

16. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-b)\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right) + (b-c)\,F(\nu, q)\right]$$
$[u > a > b > c > d]$ BY (258.13)

17. $$\int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (b-c)\,F(\alpha, q)\right]$$
$[a > b > c > d > u]$ BY (251.07)

18. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - (a-b)\,F(\beta, r)\right]$$
$[a > b > c \geq u > d]$ BY (252.15)

19. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right)$$
$[a > b > c > u \geq d]$ BY (253.02)

20. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (b-d)\,F(\delta, q)\right]$$
$[a > b \geq u > c > d]$ BY (254.14)

21. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\kappa, \frac{b-c}{a-c}, q\right) - F\left(\kappa, q\right)\right]$$
$[a > b > u \geq c > d]$ BY (255.21)

22. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r)\right]$$
$[a \geq u > b > c > d]$ BY (256.15)

23.[8] $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(a-d)\,\Pi\left(\mu, \frac{b-a}{b-d}, r\right) - (b-d)\,F(\mu, r)\right]$$
$[a > u \geq b > c > d]$ BY (257.15)

24. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\nu, \frac{a-d}{b-d}, q\right)$$
$[u > a > b > c > d]$ BY (258.02)

25. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-d)\,\Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (a-c)\,F(\alpha, q)\right]$$
$[a > b > c > d > u]$ BY (251.06)

26. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\beta, \frac{d-c}{a-c}, r\right)$$
$[a > b > c \geq u > d]$ BY (252.02)

27. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(b-c)\,\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (a-b)\,F(\gamma, r)\right]$$
$[a > b > c > u \geq d]$ BY (253.15)

28. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(d-c)\,\Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (a-d)\,F(\delta, q)\right]$$
$[a > b \geq u > c > d]$ BY (254.13)

29. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\,\Pi\left(\kappa, \frac{b-c}{a-c}, q\right)$$
$[a > b > u \geq c > d]$ BY (255.02)

30. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2}{\sqrt{(a-c)(b-d)}}\left[(c-b)\,\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (a-c)\,F(\lambda, r)\right]$$
$[a \geq u > b > c > d]$ BY (256.14)

31. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\mu, \frac{b-a}{b-d}, r\right) - F(\mu, r)\right]$$
$[a > u \geq b > c > d]$ BY (257.14)

32. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}}\,dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}}\left[\Pi\left(\nu, \frac{a-d}{b-d}, q\right) - F(\nu, q)\right]$$
$[u > a > b > c > d]$ BY (258.15)

3.168

1. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{d-a}\left[\sqrt{\frac{a-c}{b-d}}\,E(\gamma, r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}\right]$$
$[a > b > c > u > d]$ BY (253.06)

2. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\delta, q) - E(\delta, q)]$$
$[a > b \geq u > c > d]$ BY (254.04)

3. $$\int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\kappa, q) - E(\kappa, q)] + \frac{2}{b-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b > u \geq c > d]$ BY (255.09)

4. $$\int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\left[\sqrt{\frac{a-c}{b-d}}\,E(\lambda, r) - \frac{c-d}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}\right]$$
$[a \geq u > b > c > d]$ BY (256.06)

5. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,E(\mu, r)$$
$[a > u \geq b > c > d]$ BY (257.01)

6. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}}\,dx = \frac{2}{a-d}\sqrt{\frac{a-c}{b-d}}\,[F(\nu, q) - E(\nu, q)] + \frac{2}{a-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$
$[u > a > b > c > d]$ BY (258.10)

7. $$\int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(b-c)(a-d)\,F(\gamma, r) - (a-c)(b-d)\,E(\gamma, r)] + \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$
$[a > b > c > u > d]$ BY (253.03)

8. $$\int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d)\,E(\delta,q) - (a-b)(c-d)\,F(\delta,q)]$$

$[a > b \geq u > c > d]$ BY (254.15)

9. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d)\,E(\kappa,q) - (a-b)(c-d)\,F(\kappa,q)] - \frac{2}{c-d}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$

$[a > b > u \geq c > d]$ BY (255.06)

10. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times [(a-c)(b-d)\,E(\lambda,r) - (a-d)(b-c)\,F(\lambda,r)] - \frac{2}{a-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$

$[a \geq u > b > c > d]$ BY (256.03)

11. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}}\,dx = 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}\,E(\mu,r) - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\mu,r)$$

$[a > u \geq b > c > d]$ BY (257.09)

12. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}}\,dx = \frac{2(b-d)}{(a-d)(c-d)}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\nu,q) + 2\frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)}\,E(\nu,q)$$

$[u > a > b > c > d]$ BY (258.09)

13. $$\int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\gamma,r) - E(\gamma,r)] + \frac{2}{c-d}\sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}}$$

$[a > b > c > u > d]$ BY (253.04)

14. $$\int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\delta,q)$$

$[a > b \geq u > c > d]$ BY (254.01)

15. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\kappa,q) - \frac{2(a-d)}{(b-d)(c-d)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b > u \ge c > d]$ BY (255.08)

16. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\lambda,r) - E(\lambda,r)] + \frac{2}{b-d}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a \ge u > b > c > d]$ BY (256.05)

17. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{2}{c-d}\sqrt{\frac{a-c}{b-d}}\,[F(\mu,r) - E(\mu,r)]$$
$[a > u \ge b > c > d]$ BY (257.06)

18. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}}\,dx = \frac{-2}{c-d}\sqrt{\frac{a-c}{b-d}}\,E(\nu,q) + \frac{2}{c-d}\sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}}$$
$[u > a > b > c > d]$ BY (258.05)

19. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\alpha,q) - E(\alpha,q)]$$
$[a > b > c > d > u]$ BY (251.01)

20. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}}\,dx = \frac{-2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\beta,r) + \frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a > b > c \ge u > d]$ BY (252.06)

21. $$\int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\kappa,q) - E(\kappa,q)] + \frac{2}{b-c}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a > b > u > c > d]$ BY (255.05)

22. $$\int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\lambda,r)$$
$[a \ge u > b > c > d]$ BY (256.01)

23. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,E(\mu,r) - \frac{2(c-d)}{(a-c)(b-c)}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a > u \ge b > c > d]$ BY (257.06)

24. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}}\,dx = \frac{2}{b-c}\sqrt{\frac{b-d}{a-c}}\,[F(\nu,q) - E(\nu,q)] + \frac{2}{a-c}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u > a > b > c > d]$ BY (258.06)

25. $$\int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\alpha,q)$$
$[a > b > c > d > u]$ BY (251.01)

26. $$\int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\beta,r)-E(\beta,r)] + \frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a>b>c>u>d]$ BY (252.03)

27. $$\int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{d-c}\sqrt{\frac{b-d}{a-c}}\,E(\kappa,q) + \frac{2}{c-d}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a>b>u>c>d]$ BY (255.03)

28. $$\int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\lambda,r)-E(\lambda,r)]$$
$[a \ge u>b>c>d]$ BY (256.08)

29. $$\int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,[F(\mu,r)-E(\mu,r)] + \frac{2}{a-c}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a>u \ge b>c>d]$ BY (257.03)

30. $$\int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}}\,dx = \frac{2}{c-d}\sqrt{\frac{b-d}{a-c}}\,E(\nu,q) - \frac{2(b-c)}{(a-c)(c-d)}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u>a>b>c>d]$ BY (258.03)

31. $$\int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\alpha,q) - \frac{a-b}{b-c}\frac{2}{\sqrt{(a-c)(b-d)}}\,F(\alpha,q)$$
$[a>b>c>d>u]$ BY (251.08)

32. $$\int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}}\,dx = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\beta,r) - 2\frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\beta,r) + 2\frac{a-c}{(b-c)(c-d)}\sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}}$$
$[a>b>c>u>d]$ BY (252.04)

33. $$\int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}}\,dx = \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}}\,F(\kappa,q) - 2\sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}}\,E(\kappa,q) + \frac{2(a-c)}{(b-c)(c-d)}\sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}}$$
$[a>b>u>c>d]$ BY (255.04)

34. $$\int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\lambda,r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\lambda,r)$$
$[a \ge u>b>c>d]$ BY (256.09)

35. $$\int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\mu,r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}}\,F(\mu,r) - \frac{2}{b-c}\sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}}$$
$[a>u \ge b>c>d]$ BY (257.04)

36. $$\int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)}\,E(\nu,q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\nu,q) - \frac{2}{c-d}\sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}}$$
$[u > a > b > c > d]$ BY (258.04)

37. $$\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\alpha,q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\alpha,q) - \frac{2}{a-b}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.11)

38. $$\int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\beta,r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\beta,r) + \frac{2}{b-c}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c \geq u > d]$ BY (252.07)

39. $$\int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\gamma,r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\gamma,r)$$
$[a > b > c > u \geq d]$ BY (253.07)

40. $$\int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}}\,dx = \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\delta,q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\delta,q) + \frac{2(b-d)}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a > b > u > c > d]$ BY (254.05)

41. $$\int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}}\,dx = \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\mu,r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\mu,r) + \frac{2(b-d)}{(a-b)(b-c)}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a > u > b > c > d]$ BY (257.07)

42. $$\int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)}\,E(\nu,q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}}\,F(\nu,q)$$
$[u > a > b > c > d]$ BY (258.07)

43. $$\int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\alpha,q) - \frac{2(b-c)}{(a-b)(b-d)}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a > b > c > d > u]$

44. $$\int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\beta,r)-E(\beta,r)] + \frac{2}{b-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c \ge u > d]$ BY (252.10)

45. $$\int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\gamma,r)-E(\gamma,r)]$$
$[a > b > c > u \ge d]$ BY (254.08)

46. $$\int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}}\,dx = \frac{2}{b-a}\sqrt{\frac{a-c}{b-d}}\,E(\delta,q) + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a > b \ge u > c > d]$ BY (254.08)

47. $$\int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,[F(\mu,r)-E\,(\mu,r)] + \frac{2}{a-b}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a > u \ge b > c > d]$ BY (257.10)

48. $$\int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}}\,dx = \frac{2}{a-b}\sqrt{\frac{a-c}{b-d}}\,E(\nu,q)$$
$[u > a > b > c > d]$ BY (258.01)

49. $$\int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\alpha,q)-E\,(\alpha,q)] + \frac{2}{b-d}\sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.12)

50. $$\int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\beta,r) - \frac{2(a-b)}{(b-c)(b-d)}\sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}}$$
$[a > b > c \ge u > d]$ BY (252.09)

51. $$\int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,E(\gamma,r)$$
$[a > b > c > u \ge d]$ BY (253.01)

52. $$\int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\delta,q)-E(\delta,q)] + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}}$$
$[a > b > u > c > d]$ BY (254.06)

53. $$\int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{c-b}\sqrt{\frac{a-c}{b-d}}\,E(\mu,r) + \frac{2}{b-c}\sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}}$$
$[a > u > b > c > d]$ BY (257.08)

54. $$\int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}}\,dx = \frac{2}{b-c}\sqrt{\frac{a-c}{b-d}}\,[F(\nu,q)-E(\nu,q)]$$
$[u > a > b > c > d]$ BY (258.08)

55. $$\int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{b-a}\sqrt{\frac{b-d}{a-c}}\,E(\alpha,q) + \frac{2}{a-b}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.09)

56. $$\int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\beta,q) - E(\beta,q)]$$
$[a > b > c \geq u > d]$ BY (252.05)

57. $$\int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\gamma,r) - E(\gamma,r)] + \frac{2}{a-c}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c > u \geq d]$ BY (253.05)

58. $$\int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\delta,q) - \frac{2(a-d)}{(a-b)(a-c)}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b \geq u > c > d]$ BY (254.03)

59. $$\int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,E(\kappa,q)$$
$[a > b > u \geq c > d]$ BY (255.01)

60. $$\int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}}\,dx = \frac{2}{a-b}\sqrt{\frac{b-d}{a-c}}\,[F(\lambda,r) - E(\lambda,r)] + \frac{2}{a-b}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a > u > b > c > d]$ BY (256.10)

61. $$\int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}}\,dx = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\alpha,q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\alpha,q) + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.15)

62. $$\int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\beta,r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\beta,r)$$
$[a > b > c \geq u > d]$ BY (252.08)

63. $$\int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\gamma,r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}}\,F(\gamma,r) - \frac{2}{a-d}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c > u \geq d]$ BY (253.10)

64. $$\int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)}\,E(\delta,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}}\,F(\delta,q) - \frac{2}{a-b}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b \geq u > c > d]$ BY (254.09)

65. $$\int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}}\,dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\kappa,q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\kappa,q)$$
$[a > b > u \geq c > d]$ BY (255.10)

66. $$\int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}}\,dx = \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\lambda,r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\lambda,r) + \frac{2(a-c)}{(a-b)(a-d)}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a > u > b > c > d]$ BY (256.07)

67. $$\int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\alpha,q) - E(\alpha,q)] + \frac{2}{a-d}\sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}}$$
$[a > b > c > d > u]$ BY (251.13)

68. $$\int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\beta,r)$$
$[a > b > c \geq u > d]$ BY (252.01)

69. $$\int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\gamma,r) - \frac{2(a-b)}{(a-c)(a-d)}\sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}}$$
$[a > b > c > u \geq d]$ BY (253.08)

70. $$\int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\delta,q) - E(\delta,q)] + \frac{2}{a-c}\sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}}$$
$[a > b \geq u > c > d]$ BY (254.07)

71. $$\int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{2}{a-d}\sqrt{\frac{b-d}{a-c}}\,[F(\kappa,q) - E(\kappa,q)]$$
$[a > b > u \geq c > d]$ BY (255.07)

72. $$\int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}}\,dx = \frac{-2}{a-d}\sqrt{\frac{b-d}{a-c}}\,E(\lambda,r) + \frac{2}{a-d}\sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}}$$
$[a \geq u > b > c > d]$ BY (256.04)

3.169 **Notation**: In **3.169–3.172**, we set: $\alpha = \arctan\frac{u}{b}$, $\beta = \arctan\frac{a}{u}$,

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \qquad \delta = \arccos\frac{u}{b}, \qquad \varepsilon = \arccos\frac{b}{u}, \qquad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$

$$\eta = \arcsin\frac{u}{b}, \qquad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \qquad \kappa = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$

$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \qquad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \qquad \nu = \arcsin\frac{a}{u}, \qquad q = \frac{\sqrt{a^2-b^2}}{a},$$

$$r = \frac{b}{\sqrt{a^2+b^2}}, \qquad s = \frac{a}{\sqrt{a^2+b^2}} \qquad t = \frac{b}{a}.$$

1. $$\int_0^u \sqrt{\frac{x^2+a^2}{x^2+b^2}}\,dx = a\left\{F(\alpha,q)-E(\alpha,q)\right\} + u\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a > b, \quad u > 0]$ BY (221.03)

2.[6] $$\int_0^u \sqrt{\frac{x^2+b^2}{x^2+a^2}}\,dx = \frac{b^2}{a}\,F(\alpha,q) - a\,E(\alpha,q) + u\sqrt{\frac{a^2+u^2}{b^2+u^2}}$$
$[a > b, \quad u > 0]$ BY (221.04)

3. $$\int_0^u \sqrt{\frac{x^2+a^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\gamma,r) - u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b \ge u > 0]$ BY (214.11)

4. $$\int_u^b \sqrt{\frac{a^2+x^2}{b^2-x^2}}\,dx = \sqrt{a^2+b^2}\,E(\delta,r)$$
$[b > u \ge 0]$ BY (213.01), ZH 64 (273)

5. $$\int_b^u \sqrt{\frac{a^2+x^2}{x^2-b^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\varepsilon,s)-E(\varepsilon,s)\right\} + \frac{1}{u}\sqrt{(u^2+a^2)(u^2-b^2)}$$
$[u > b > 0]$ BY (211.03)

6. $$\int_0^u \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\gamma,r)-E(\gamma,r)\right\} + u\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b \ge u > 0]$ BY (214.03)

7. $$\int_u^b \sqrt{\frac{b^2-x^2}{a^2+x^2}}\,dx = \sqrt{a^2+b^2}\left\{F(\delta,r)-E(\delta,r)\right\}$$
$[b > u \ge 0]$ BY (213.03)

8. $$\int_b^u \sqrt{\frac{x^2-b^2}{a^2+x^2}}\,dx = \frac{1}{u}\sqrt{(a^2+u^2)(u^2-b^2)} - \sqrt{a^2+b^2}\,E(\varepsilon,s)$$
$[u > b > 0]$ BY (211.04)

9. $$\int_0^u \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = a\,E(\eta,t) - \frac{a^2-b^2}{a}\,F(\eta,t)$$
$[a > b \ge u > 0]$ BY (219.03)

10. $$\int_u^b \sqrt{\frac{b^2-x^2}{a^2-x^2}}\,dx = a\,E(\zeta,t) - \frac{a^2-b^2}{a}\,F(\zeta,t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$
$[a > b > u \ge 0]$ BY (220.04)

11. $$\int_b^u \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = a\,E(\kappa,q) - \frac{b^2}{a}\,F(\kappa,q) - \frac{1}{u}\sqrt{(a^2-u^2)(u^2-b^2)}$$
$[a \ge u > b > 0]$ BY (217.04)

12. $$\int_u^a \sqrt{\frac{x^2-b^2}{a^2-x^2}}\,dx = a\,E(\lambda,q) - \frac{b^2}{a}\,F(\lambda,q)$$
$[a > u \ge b > 0]$ BY (218.03)

13. $$\int_a^u \sqrt{\frac{x^2-b^2}{x^2-a^2}}\,dx = \frac{a^2-b^2}{a}\,F(\mu,t) - a\,E(\mu,t) + \mu\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$
$[u > a > b > 0]$ BY (216.03)

14. $$\int_0^u \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\,E(\eta,t)$$
$[a > b \ge u > 0]$ H 64 (276), BY (219.01)

15. $$\int_u^b \sqrt{\frac{a^2-x^2}{b^2-x^2}}\,dx = a\left\{E(\zeta,t) - \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$ $[a > b > u \geq 0]$ BY (220.03)

16. $$\int_b^u \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\kappa,q) - E(\kappa,q)\right\} + \frac{1}{u}\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}$$ $[a \geq u > b > 0]$ BY (217.03)

17. $$\int_u^a \sqrt{\frac{a^2-x^2}{x^2-b^2}}\,dx = a\left\{F(\lambda,q) - E(\lambda,q)\right\}$$ $[a > u \geq b > 0]$ BY (218.09)

18. $$\int_a^u \sqrt{\frac{x^2-a^2}{x^2-b^2}}\,dx = u\sqrt{\frac{u^2-a^2}{u^2-b^2}} - a\,E(\mu,t)$$ $[u > a > b > 0]$ BY (216.04)

3.171

1. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\,E(\varepsilon,s)$$ $[u > b > 0]$ BY (211.01), ZH 64 (274)

2. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{x^2-b^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\,E(\xi,s) - \frac{a^2}{b^2u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$ $[u \geq b > 0]$ BY (212.09)

3. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}} = \frac{a^2-b^2}{ab^2}\,F(\zeta,t) - \frac{a}{b^2}\,E(\zeta,t) + \frac{a^2}{b^2u}\sqrt{\frac{b^2-u^2}{a^2-u^2}}$$ $[a > b > u > 0]$ BY (220.12)

4. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\kappa,q) - \frac{1}{a}\,F(\kappa,q)$$ $[a \geq u > b > 0]$ BY (217.11)

5. $$\int_u^a \frac{dx}{x^2}\sqrt{\frac{a^2-x^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\lambda,q) - \frac{1}{a}f(\lambda,q) - \frac{\sqrt{\left(a^2-u^2\right)\left(u^2-b^2\right)}}{b^2u}$$ $[a > u \geq b > 0]$ BY (218.10)

6. $$\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}\,E(\mu,t) - \frac{a^2-b^2}{ab^2}\,F(\mu,t) - \frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$ $[u > a > b > 0]$ BY (216.08)

7. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+a^2}{x^2+b^2}} = \frac{1}{a}\,F(\beta,q) - \frac{a}{b^2}\,E(\beta,q) + \frac{a^2}{b^2u}\sqrt{\frac{b^2+u^2}{a^2+u^2}}$$ $[a > b, \quad u > 0]$ BY (222.08)

8. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2+b^2}{x^2+a^2}} = \frac{1}{a}\left\{F(\beta,q) - E(\beta,q)\right\} + \frac{1}{u}\sqrt{\frac{b^2+u^2}{a^2+u^2}}$$ $[a > b, \quad u > 0]$ BY (222.09)

9. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2+x^2}} = \frac{\sqrt{\left(b^2-u^2\right)\left(a^2+u^2\right)}}{a^2u} - \frac{\sqrt{a^2+b^2}}{a^2}\,E(\delta,r)$$ $[b > u > 0]$ BY (213.10)

10. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\{F(\varepsilon,s)-E(\varepsilon,s)\}$$ $[a>b>0]$ BY (211.07)

11. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2+x^2}} = \frac{\sqrt{a^2+b^2}}{a^2}\{F(\xi,s)-E(\xi,s)\}+\frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2+u^2}}$$ $[u \ge b>0]$ BY (212.11)

12. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{a^2+x^2}{b^2-x^2}} = \frac{\sqrt{a^2+b^2}}{b^2}\{F(\delta,r)-E(\delta,r)\}+\frac{\sqrt{(b^2-u^2)(a^2+u^2)}}{b^2u}$$ $[b>u>0]$ BY (213.05)

13. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}} = \frac{a}{b^2}E(\nu,t)-\frac{a^2-b^2}{ab^2}F(\nu,t)$$ $[u \ge a>b>0]$ BY (215.08)

14. $$\int_u^b \frac{dx}{x^2}\sqrt{\frac{b^2-x^2}{a^2-x^2}} = \frac{1}{u}\sqrt{\frac{b^2-u^2}{a^2-u^2}}-\frac{1}{a}E(\zeta,t)$$ $[a>b>u>0]$ BY (220.11)

15. $$\int_b^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{a^2-x^2}} = \frac{1}{a}\{F(\kappa,q)-E(\kappa,q)\}$$ $[a \ge u>b>0]$ BY (217.08)

16. $$\int_u^a \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{u^2-x^2}} = \frac{1}{a}\{F(\lambda,q)-E(\lambda,q)\}+\frac{\sqrt{(a^2-u^2)(u^2-b^2)}}{a^2u}$$ $[a>u \ge b>0]$ BY (218.08)

17. $$\int_a^u \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\mu,t)-\frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$ $[u>a>b>0]$ BY (216.07)

18. $$\int_u^\infty \frac{dx}{x^2}\sqrt{\frac{x^2-b^2}{x^2-a^2}} = \frac{1}{a}E(\nu,t)$$ $[u \ge a>b>0]$ BY (215.01), ZH 65 (281)

3.172

1. $$\int_0^u \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\alpha,q)-\frac{a^2-b^2}{a^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$ $[a>b, \quad u>0]$ BY (221.10)

2. $$\int_u^\infty \sqrt{\frac{x^2+b^2}{(x^2+a^2)^3}}\,dx = \frac{1}{a}E(\beta,q)$$ $[a>b, \quad u \ge 0]$ H 64 (271)

3. $$\int_0^u \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\alpha,q)$$ $[a>b, \quad u>0]$ H 64 (270)

4. $$\int_u^\infty \sqrt{\frac{x^2+a^2}{(x^2+b^2)^3}}\,dx = \frac{a}{b^2}E(\beta,q)-\frac{a^2-b^2}{b^2}\frac{u}{\sqrt{(a^2+u^2)(b^2+u^2)}}$$ $[a>b, \quad u \ge 0]$ BY (222.06)

5. $$\int_0^u \sqrt{\frac{b^2-x^2}{(a^2+x^2)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}E(\gamma,r)-\frac{1}{\sqrt{a^2+b^2}}F(\gamma,r)$$ $[b \ge u>0]$ BY (214.08)

6. $$\int_u^b \sqrt{\frac{b^2-x^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\delta,r) - \frac{1}{\sqrt{a^2+b^2}}\,F(\delta,r) - \frac{u}{a^2}\sqrt{\frac{b^2-u^2}{a^2+u^2}}$$
$[b > u \geq 0]$ BY (213.04)

7. $$\int_b^u \sqrt{\frac{x^2-b^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\varepsilon,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}\,F(\varepsilon,s) - \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2+a^2}}$$
$[u > b > 0]$ BY (211.06)

8. $$\int_u^\infty \sqrt{\frac{x^2-b^2}{\left(a^2+x^2\right)^3}}\,dx = \frac{\sqrt{a^2+b^2}}{a^2}\,E(\xi,s) - \frac{b^2}{a^2\sqrt{a^2+b^2}}\,F(\xi,s)$$
$[u \geq b > 0]$ BY (212.08)

9. $$\int_0^u \sqrt{\frac{x^2+a^2}{\left(b^2-x^2\right)^3}}\,dx = \frac{a^2}{b^2\sqrt{a^2+b^2}}\,F(\gamma,r) - \frac{\sqrt{a^2+b^2}}{b^2}\,E(\gamma,r) + \frac{\left(a^2+b^2\right)u}{b^2\sqrt{\left(a^2+u^2\right)\left(b^2-u^2\right)}}$$
$[b > u > 0]$ BY (214.09)

10. $$\int_u^\infty \sqrt{\frac{x^2+a^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{1}{\sqrt{a^2+b^2}}\,F(\xi,s) - \frac{\sqrt{a^2+b^2}}{b^2}\,E(\xi,s) + \frac{\left(a^2+b^2\right)u}{b^2\sqrt{\left(a^2+u^2\right)\left(u^2-b^2\right)}}$$
$[u > b > 0]$ BY (212.07)

11. $$\int_0^u \sqrt{\frac{b^2-x^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{a}\left\{F(\eta,t) - E(\eta,t) + \frac{u}{a}\sqrt{\frac{b^2-u^2}{a^2-u^2}}\right\}$$
$[a > b \geq u > 0]$ BY (219.09)

12. $$\int_u^b \sqrt{\frac{b^2-x^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{a}\left\{F(\zeta,t) - E(\zeta,t)\right\}$$ $[a > b > u \geq 0]$ BY (220.07)

13. $$\int_b^u \sqrt{\frac{x^2-b^2}{\left(a^2-x^2\right)^3}}\,dx = \frac{1}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}} - \frac{1}{a}\,E\left(\kappa,q\right)$$ $[a > u > b > 0]$ BY (217.07)

14. $$\int_u^\infty \sqrt{\frac{x^2-b^2}{\left(x^2-a^2\right)^3}}\,dx = \frac{1}{a}\left[F(\nu,t) - E(\nu,t)\right] + \frac{1}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}}$$
$[u > a > b > 0]$ BY (215.05)

15. $$\int_0^u \sqrt{\frac{a^2-x^2}{\left(b^2-x^2\right)^3}}\,dx = \frac{a}{b^2}\left[F(\eta,t) - E(\eta,t)\right] + \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{b^2-u^2}}$$
$[a > b > u > 0]$ BY (219.10)

16. $$\int_u^a \sqrt{\frac{a^2-x^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{u}{b^2}\sqrt{\frac{a^2-u^2}{u^2-b^2}} - \frac{a}{b^2}\,E(\lambda,q)$$ $[a > u > b > 0]$ BY (218.05)

17. $$\int_a^u \sqrt{\frac{x^2-a^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{a}{b^2}\left[F(\mu,t) - E(\mu,t)\right]$$ $[u > a > b > 0]$ BY (216.05)

18. $$\int_u^\infty \sqrt{\frac{x^2-a^2}{\left(x^2-b^2\right)^3}}\,dx = \frac{a}{b^2}\left[F(\nu,t)-E(\nu,t)\right]+\frac{1}{u}\sqrt{\frac{u^2-a^2}{u^2-b^2}}$$

$[u \geq a > b > 0]$ BY (215.03)

3.173

1. $$\int_u^1 \frac{dx}{x^2}\sqrt{\frac{x^2+1}{1-x^2}} = \sqrt{2}\left[F\left(\arccos u, \frac{\sqrt{2}}{2}\right) - E\left(\arccos u, \frac{\sqrt{2}}{2}\right)\right] + \frac{\sqrt{1-u^4}}{u}$$

$[u < 1]$ BY (259.77)

2. $$\int_1^u \frac{dx}{x^2}\sqrt{\frac{x^2+1}{x^2-1}} = \sqrt{2}\,E\left(\arccos\frac{1}{u}, \frac{\sqrt{2}}{2}\right)$$ $[u > 1]$ BY (260.76)

3.174 **Notation**: In **3.174** and **3.175**, we take: $\alpha = \arccos\dfrac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u}$,

$$\beta = \arccos\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u}, \qquad p = \frac{\sqrt{2+\sqrt{3}}}{2}, \qquad q = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

1. $$\int_0^u \frac{dx}{\left[1+\left(1+\sqrt{3}\right)x\right]^2}\sqrt{\frac{1-x+x^2}{x(1+x)}} = \frac{1}{\sqrt[4]{3}}\,E(\alpha,p)$$ $[u > 0]$ BY (260.51)

2. $$\int_0^u \frac{dx}{\left[1+\left(\sqrt{3}-1\right)x\right]^2}\sqrt{\frac{1+x+x^2}{x(1-x)}} = \frac{1}{\sqrt[4]{3}}\,E(\beta,q)$$ $[1 \geq u > 0]$ BY (259.51)

3. $$\int_0^u \frac{dx}{1-x+x^2}\sqrt{\frac{x(1+x)}{1-x+x^2}} \quad \frac{1}{\sqrt[4]{27}}\,E(\alpha,p) + -\frac{2-\sqrt{3}}{\sqrt[4]{27}}\,F(\alpha,p) - \frac{2\left(2+\sqrt{3}\right)}{\sqrt{3}}\,\frac{1+\left(1-\sqrt{3}\right)u}{1+\left(1+\sqrt{3}\right)u}$$
$$\times\sqrt{\frac{u(1+u)}{1-u+u^2}}$$

$[u > 0]$ BY (260.54)

4. $$\int_0^u \frac{dx}{1+x+x^2}\sqrt{\frac{x(1-x)}{1+x+x^2}} \quad \frac{4}{\sqrt[4]{27}}\,E(\beta,q) - \frac{2+\sqrt{3}}{\sqrt[4]{27}}\,F(\beta,q) - \frac{2\left(2-\sqrt{3}\right)}{\sqrt{3}}\,\frac{1-\left(1+\sqrt{3}\right)u}{1+\left(\sqrt{3}-1\right)u}$$
$$\times\sqrt{\frac{u(1-u)}{1+u+u^2}}$$

$[1 \geq u > 0]$ BY (259.55)

3.175

1. $$\int_0^u \frac{dx}{1+x}\sqrt{\frac{x}{1+x^3}} = \frac{1}{\sqrt[4]{27}}\left[F(\alpha,p) - 2\,E(\alpha,p)\right] + \frac{2}{\sqrt{3}}\,\frac{\sqrt{u\left(1-u+u^2\right)}}{\sqrt{1+u}\left[1+\left(1+\sqrt{3}\right)u\right]}$$

$[u > 0]$ BY (260.55)

2. $$\int_0^u \frac{dx}{1-x}\sqrt{\frac{x}{1-x^3}} = \frac{1}{\sqrt[4]{27}}\left[F(\beta,q) - 2\,E(\beta,q)\right] + \frac{2}{\sqrt{3}}\,\frac{\sqrt{u\left(1+u+u^2\right)}}{\sqrt{1-u}\left[1+\left(\sqrt{3}-1\right)u\right]}$$

$[0 < u < 1]$ BY (259.52)

3.18 Expressions that can be reduced to fourth roots of second-degree polynomials and their products with rational functions

3.181

1. $$\int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b}\left\{2\left[\boldsymbol{E}\left(\frac{1}{\sqrt{2}}\right) + E\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right] - \left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]\right\}$$
$[a \geq u > b]$ BY (271.05)

2. $$\int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} \sqrt{\frac{a-b}{2}} F\left[\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right)\right] + \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}$$
$[u > a > b]$ BY (272.05)

3.182

1. $$\int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}}\left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) + F\left(\arccos\sqrt{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}}\right)\right]$$
$[a \geq u > b]$ BY (271.01)

2. $$\int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}}\, F\left(\arccos\frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}}\right)$$
$[u > a > b]$ BY (272.00)

3.183 **Notation**: In **3.183–3.186** we set:

$$\alpha = \arccos\frac{1}{\sqrt[4]{u^2+1}}, \qquad \beta = \arccos\sqrt[4]{1-u^2}, \qquad \gamma = \arccos\frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

1. $$\int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}}$$
$[u > 0]$ BY (273.55)

2. $$\int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2}\left[2\,E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right)\right]$$ $[0 < u \leq 1]$ BY (271.55)

3. $$\int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\gamma, \frac{1}{\sqrt{2}}\right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}}$$
$[u > 1]$ BY (272.55)

3.184

1. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5}\left[2\,E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right)\right] - \frac{2u}{5}\sqrt[4]{\left(1-u^2\right)^3}$$
[$0 < u \le 1$] BY (271.59)

2. $$\int_1^u \frac{dx}{x^2\sqrt[4]{x^2-1}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\,F\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}} \cdot \frac{\sqrt{u^2-1}}{u}$$
[$u > 1$] BY (272.54)

3.185

1. $$\int_0^u \frac{dx}{\sqrt[4]{\left(x^2+1\right)^3}} = \sqrt{2}\,F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ [$u > 0$] BY (273.50)

2. $$\int_0^u \frac{dx}{\sqrt[4]{\left(1-x^2\right)^3}} = \sqrt{2}\,F\left(\beta, \frac{1}{\sqrt{2}}\right)$$ [$0 < u \le 1$] BY (271.51)

3. $$\int_1^u \frac{dx}{\sqrt[4]{\left(x^2-1\right)^3}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right)$$ [$u > 1$] BY (272.50)

4. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{\left(1-x^2\right)^3}} = \frac{2\sqrt{2}}{3}\,F\left(\beta, \frac{1}{\sqrt{2}}\right) - \frac{2}{3}u\sqrt[4]{1-u^2}$$ [$0 < u \le 1$] BY (271.54)

5. $$\int_0^u \frac{dx}{\sqrt[4]{\left(x^2+1\right)^5}} = 2\sqrt{2}\,E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \sqrt{2}\,F\left(\alpha, \frac{1}{\sqrt{2}}\right)$$
[$u > 0$] BY (273.54)

6. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{\left(x^2+1\right)^5}} = 2\sqrt{2}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2\,E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{2u}{\sqrt[4]{u^2+1}}$$
[$u > 0$] BY (273.56)

7. $$\int_0^u \frac{x^2\,dx}{\sqrt[4]{\left(x^2+1\right)^7}} = \frac{1}{3\sqrt{2}}\,F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{\left(u^2+1\right)^3}}$$ [$u > 0$] BY (273.53)

3.186

1. $$\int_0^u \frac{1+\sqrt{x^2+1}}{\left(x^2+1\right)\sqrt[4]{x^2+1}}\,dx = 2\sqrt{2}\,E\left(\alpha, \frac{1}{\sqrt{2}}\right)$$ [$u > 0$] BY (273.51)

2. $$\int_0^u \frac{dx}{\left(1+\sqrt{1-x^2}\right)\sqrt[4]{1-x^2}} = \sqrt{2}\left[F\left(\beta, \frac{1}{\sqrt{2}}\right) - E\left(\beta, \frac{1}{\sqrt{2}}\right)\right] + \frac{u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}}$$
[$0 < u \le 1$] BY (271.58)

3. $$\int_1^u \frac{dx}{\left(x^2+2\sqrt{x^2-1}\right)\sqrt[4]{x^2-1}} = \frac{1}{2}\left[F\left(\gamma, \frac{1}{\sqrt{2}}\right) - E\left(\gamma, \frac{1}{\sqrt{2}}\right)\right]$$
[$u > 1$] BY (272.53)

4. $$\int_0^u \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{dx}{\sqrt[4]{\left(1-x^2\right)^3}} = \sqrt{2}\left[2\,E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right)\right] - \frac{2u\sqrt[4]{1-u^2}}{1+\sqrt{1-u^2}}$$ $[0 < u \leq 1]$ BY (271.57)

5. $$\int_1^u \frac{x^2\,dx}{\left(x^2+2\sqrt{x^2-1}\right)\sqrt[4]{\left(x^2-1\right)^3}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right)$$ $[u > 1]$ BY (272.51)

3.19–3.23 Combinations of powers of x and powers of binomials of the form $(\alpha+\beta x)$

3.191

1. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\,dx = u^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 185(7)

2. $$\int_u^\infty x^{-\nu}(x-u)^{\mu-1}\,dx = u^{\mu-\nu}\,\mathrm{B}(\nu-\mu,\mu)$$ $[\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ ET II 201(6)

3. $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\,dx = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 774(1)

3.192

1. $$\int_0^1 \frac{x^p\,dx}{(1-x)^p} = p\pi\operatorname{cosec} p\pi$$ $\left[p^2 < 1\right]$ BI (3)(4)

2. $$\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\pi\operatorname{cosec} p\pi$$ $[-1 < p < 0]$ BI (3)(5)

3. $$\int_0^1 \frac{(1-x)^p}{x^{p+1}}\,dx = -\pi\operatorname{cosec} p\pi$$ $[-1 < p < 0]$ BI (4)(6)

4. $$\int_1^\infty (x-1)^{p-\frac{1}{2}}\,\frac{dx}{x} = \pi\sec p\pi$$ $\left[-\frac{1}{2} < p < \frac{1}{2}\right]$ BI (23)(7)

3.193 $$\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n!\,n^{\nu+n}}{\nu(\nu+1)(\nu+2)\ldots(\nu+n)}$$ $[\operatorname{Re}\nu > 0]$ EH I 2

3.194

1. $$\int_0^u \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu}\;{}_2F_1(\nu,\mu;1+\mu;-\beta u)$$ $[|\arg(1+\beta u)| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 310(20)

2.[6] $$\int_u^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu(\nu-\mu)}\;{}_2F_1\left(\nu,\nu-\mu; \quad \nu-\mu+1; \quad -\frac{1}{\beta u}\right)$$ $[\operatorname{Re}\nu > \operatorname{Re}\mu]$ ET I 310(21)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^\nu} = \beta^{-\mu}\,\mathrm{B}(\mu,\nu-\mu)$$ $[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ FI II 775a, ET I 310(19)

4.[11] $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu}\binom{\mu-1}{n}\operatorname{cosec}(\mu\pi)$ $[|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < n+1]$

ET I 308(6)

5. $\displaystyle\int_0^u \frac{x^{\mu-1}\,dx}{1+\beta x} = \frac{u^\mu}{\mu}\; {}_2F_1(1,\mu;1+\mu;-\beta u)$ $[|\arg(1+u\beta)| < \pi, \quad \operatorname{Re}\mu > 0]$

ET I 308(5)

6. $\displaystyle\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+\beta x)^2} = \frac{(1-\mu)\pi}{\beta^\mu}\operatorname{cosec}\mu\pi$ $[0 < \operatorname{Re}\mu < 2]$ BI (16)(4)

7. $\displaystyle\int_0^\infty \frac{x^m\,dx}{(a+bx)^{n+\frac{1}{2}}} = 2^{m+1} m! \frac{(2n-2m-3)!!}{(2n-1)!!}\frac{a^{m-n+\frac{1}{2}}}{b^{m+1}}$

$\left[m < n - \tfrac{1}{2}, \quad a > 0, \quad b > 0\right]$

BI (21)(2)

8. $\displaystyle\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^m} = 2^{-n}\sum_{k=0}^{\infty}\binom{m-n-1}{k}\frac{(-2)^{-k}}{n+k}$ BI (3)(1)

3.195[11] $\displaystyle\int_0^\infty \frac{(1+x)^{p-1}}{(a+x)^{p+1}}\,dx = \frac{1-a^{-p}}{p(a-1)}$ $[p \neq 0, \quad a > 0, \quad a \neq 1]$

$\displaystyle = \frac{\ln a}{a-1}$ $[p = 0, \quad a > 0, \quad a \neq 1]$

$= 1$ $[a = 1]$

LI (19)(6)

3.196

1. $\displaystyle\int_0^u (x+\beta)^\nu (u-x)^{\mu-1}\,dx = \frac{\beta^\nu u^\mu}{\mu}\; {}_2F_1\left(1,-\nu;1+\mu;-\frac{u}{\beta}\right)$

$\left[\left|\arg\frac{u}{\beta}\right| < \pi\right]$ ET II 185(8)

2. $\displaystyle\int_u^\infty (x+\beta)^{-\nu}(x-u)^{\mu-1}\,dx = (u+\beta)^{\mu-\nu}\,\mathrm{B}(\nu-\mu,\mu)$

$\left[\left|\arg\frac{u}{\beta}\right| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0\right]$

ET II 201(7)

3. $\displaystyle\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}\,dx = (b-a)^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)$ $[b > a, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$

EH I 10(13)

4. $\displaystyle\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b}\operatorname{cosec}\nu\pi\left(\frac{b}{b-a}\right)^\nu$ $[a < b, \quad b > 0, \quad 0 < \nu < 1]$ LI (23)(5)

5. $\displaystyle\int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b}\operatorname{cosec}\nu\pi\left(\frac{b}{a-b}\right)^\nu$ $[a > b > 0, \quad 0 < \nu < 1]$ LI (24)(10)

3.197

1. $$\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho}\,dx = \beta^{-\mu}\gamma^{\nu-\varrho}\,\mathrm{B}(\nu,\mu-\nu+\varrho)\;{}_2F_1\left(\mu,\nu;\mu+\varrho;1-\frac{\gamma}{\beta}\right)$$
$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>\operatorname{Re}(\nu-\varrho)]$ ET II 233(9)

2.[11] $$\int_u^\infty x^{-\lambda}(x+\beta)^{\nu}(x-u)^{\mu-1}\,dx = u^{-\lambda}(\beta+u)^{\mu+\nu}\,\mathrm{B}(\lambda-\mu-\nu,\mu)\;{}_2F_1\left(\lambda,\mu;\lambda-\mu;-\frac{\beta}{u}\right)$$
$\left[\left|\arg\frac{u}{\beta}\right|<\pi \text{ or } \left|\frac{\beta}{u}\right|<1, \quad 0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-\nu)\right]$ ET II 201(8)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}\,dx = \mathrm{B}(\lambda,\mu)\;{}_2F_1(\nu,\lambda;\lambda+\mu;\beta)$$
$[\operatorname{Re}\lambda>0, \quad \operatorname{Re}\mu>0, \quad |\beta|<1]$ WH

4. $$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu}\,dx = (1+a)^{-\mu}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad a>-1]$
BI(5)4, EH I 10(11)

5. $$\int_0^\infty x^{\lambda-1}(1+x)^{\nu}(1+\alpha x)^{\mu}\,dx = \mathrm{B}(\lambda,-\mu-\nu-\lambda)\;{}_2F_1(-\mu,\lambda;-\mu-\nu;1-\alpha)$$
$[|\arg\alpha|<\pi, \quad -\operatorname{Re}(\mu+\nu)>\operatorname{Re}\lambda>0]$
EH I 60(12), ET I 310(23)

6. $$\int_1^\infty x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda}\,dx = \alpha^{-\lambda}\,\mathrm{B}(\mu,\nu-\mu)\;{}_2F_1\left(\nu,\mu;\lambda;\alpha^{-1}\right)$$
$[1+\operatorname{Re}\nu>\operatorname{Re}\lambda>\operatorname{Re}\mu, \quad |\arg(\alpha-1)|<\pi]$ EH I 115(6)

7. $$\int_0^\infty x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu}\,dx = \sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2\mu}\frac{\Gamma\left(\mu-\frac{1}{2}\right)}{\Gamma(\mu)}$$
$[\operatorname{Re}\mu>0]$ BI 19(5)

8. $$\int_0^u x^{\nu-1}(x+\alpha)^{\lambda}(u-x)^{\mu-1}\,dx = \alpha^{\lambda}u^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)\;{}_2F_1\left(-\lambda,\nu;\mu+\nu;-\frac{u}{\alpha}\right)$$
$\left[\left|\arg\left(\frac{u}{\alpha}\right)\right|<\pi, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0\right]$
ET II 186(9)

9. $$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu}\,dx = \mathrm{B}(\mu-\lambda,\lambda)\;{}_2F_1(\nu,\mu-\lambda;\mu;1-\beta)$$
$[\operatorname{Re}\mu>\operatorname{Re}\lambda>0]$ EH I 205

10. $$\int_0^1 \frac{x^{q-1}\,dx}{(1-x)^q(1+px)} = \frac{\pi}{(1+p)^q}\operatorname{cosec} q\pi$$
$[0<q<1, \quad p>-1]$ BI (5)(1)

11. $$\int_0^1 \frac{x^{p-\frac{1}{2}}\,dx}{(1-x)^p(1+qx)^p} = \frac{2\,\Gamma\left(p+\frac{1}{2}\right)\Gamma(1-p)}{\sqrt{\pi}}\cos^{2p}\left(\arctan\sqrt{q}\right)\frac{\sin\left[(2p-1)\arctan\left(\sqrt{q}\right)\right]}{(2p-1)\sin\left[\arctan\left(\sqrt{q}\right)\right]}$$
$\left[-\frac{1}{2}<p<1, \quad q>0\right]$ BI (11)(1)

12. $$\int_0^1 \frac{x^{p-\frac{1}{2}}\,dx}{(1-x)^p(1-qx)^p} = \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma(1-p)}{\sqrt{\pi}} \frac{\left(1-\sqrt{q}\right)^{1-2p} - \left(1+\sqrt{q}\right)^{1-2p}}{(2p-1)\sqrt{q}}$$

$\left[-\frac{1}{2} < p < 1, \quad 0 < q < 1\right]$ BI (11)(2)

3.198 $$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}[ax+b(1-x)+c]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}\,\mathrm{B}(\mu,\nu)$$

$[a \ge 0, \quad b \ge 0, \quad c > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 787

3.199 $$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\,dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu}\,\mathrm{B}(\mu,\nu)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad c < a < b]$
EH I 10(14)

3.211 $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\varrho}(1-vx)^{-\sigma}\,dx = \mathrm{B}(\mu,\lambda)\,F_1\left((\lambda,\varrho,\sigma,\lambda+\mu;u,v)\right)$$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ EH I 231(5)

3.212 $$\int_0^\infty \left[(1+ax)^{-p} + (1+bx)^{-p}\right] x^{q-1}\,dx = 2(ab)^{-\frac{q}{2}}\,\mathrm{B}(q,p-q)\cos\left\{q\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$

$[p > q > 0]$ BI (19)(9)

3.213 $$\int_0^\infty \left[(1+ax)^{-p} - (1+bx)^{-p}\right] x^{q-1}\,dx = -2i(ab)^{-\frac{q}{2}}\,\mathrm{B}(q,p-q)\sin\left\{q\arccos\left[\frac{a+b}{2\sqrt{ab}}\right]\right\}$$

$[p > q > 0]$ BI (19)(10)

3.214 $$\int_0^1 \left[(1+x)^{\mu-1}(1-x)^{\nu-1} + (1+x)^{\nu-1}(1-x)^{\mu-1}\right]\,dx = 2^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
LI(1)(15), EH I 10(10)

3.215 $$\int_0^1 \left\{a^\mu x^{\mu-1}(1-ax)^{\nu-1} + (1-a)^\nu x^{\nu-1}[1-(1-a)x]^{\mu-1}\right\}\,dx = \mathrm{B}(\mu,\nu)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad |a| < 1]$
BI (1)(16)

3.216

1. $$\int_0^1 \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 775

2. $$\int_1^\infty \frac{x^{\mu-1}+x^{\nu-1}}{(1+x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 775

3.217 $$\int_0^\infty \left\{\frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1}x^p}\right\}\,dx = \pi\cot p\pi$$ $[0 < p < 1, \quad b > 0]$ BI(18)(13)

3.218 $$\int_0^\infty \frac{x^{2p-1}-(a+x)^{2p-1}}{(a+x)^p x^p}\,dx = \pi\cot p\pi$$ $[p < 1]$ (cf. **3.217**) BI (18)(7)

3.219 $$\int_0^\infty \left\{\frac{x^\nu}{(x+1)^{\nu+1}} - \frac{x^\mu}{(x+1)^{\mu+1}}\right\}\,dx = \psi(\mu+1) - \psi(\nu+1)$$

$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ BI (19)(13)

3.221

1. $$\int_a^\infty \frac{(x-a)^{p-1}}{x-b}\,dx = \pi(a-b)^{p-1}\operatorname{cosec} p\pi$$ $[a > b, \quad 0 < p < 1]$ LI (24)(8)

2. $$\int_{-\infty}^{a} \frac{(a-x)^{p-1}}{x-b}\,dx = -\pi(b-a)^{p-1}\operatorname{cosec} p\pi \qquad [a<b, \quad 0<p<1]$$ LI (24)(8)

3.222

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x} = \beta(\mu) \qquad [\operatorname{Re}\mu > 0]$$ WH

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = \pi\operatorname{cosec}(\mu\pi)a^{\mu-1} \quad \text{for } a>0$$ FI II 718, FI II 737
$$= -\pi\cot(\mu\pi)(-a)^{\mu-1} \quad \text{for } a<0$$ BI(18)(2), ET II 249(28)
$$[0<\operatorname{Re}\mu<1]$$

3.223

1. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta}\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\operatorname{cosec}(\mu\pi)$$
$$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(7)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta}\left[\beta^{\mu-1}\operatorname{cosec}(\mu\pi)+\alpha^{\mu-1}\cot(\mu\pi)\right]$$
$$[|\arg\beta|<\pi, \quad \alpha>0, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(8)

3. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)} = \pi\cot(\mu\pi)\frac{a^{\mu-1}-b^{\mu-1}}{b-a} \qquad [a>b>0, \quad 0<\operatorname{Re}\mu<2]$$ ET I 309(9)

3.224 $$\int_0^\infty \frac{(x+\beta)x^{\mu-1}\,dx}{(x+\gamma)(x+\delta)} = \pi\operatorname{cosec}(\mu\pi)\left\{\frac{\gamma-\beta}{\gamma-\delta}\gamma^{\mu-1}+\frac{\delta-\beta}{\delta-\gamma}\delta^{\mu-1}\right\}$$
$$[|\arg\gamma|<\pi, \quad |\arg\delta|<\pi, \quad 0<\operatorname{Re}\mu<1]$$ ET I 309(10)

3.225

1. $$\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx = (1-p)\pi\operatorname{cosec} p\pi \qquad [-1<p<1]$$ BI (23)(8)

2. $$\int_1^\infty \frac{(x-1)^{1-p}}{x^3}\,dx = \frac{1}{2}p(1-p)\pi\operatorname{cosec} p\pi \qquad [0<p<1]$$ BI (23)(1)

3. $$\int_0^\infty \frac{x^p\,dx}{(1+x)^3} = \frac{\pi}{2}p(1-p)\operatorname{cosec} p\pi \qquad [-1<p<2]$$ BI (16)(5)

3.226

1. $$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x}} = 2\frac{(2n)!!}{(2n+1)!!}$$ BI (8)(1)

2. $$\int_0^1 \frac{x^{n-\frac{1}{2}}\,dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!}\pi.$$ BI (8)(2)

3.227

1. $$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x}\,dx = \beta^{1-\mu}\gamma^{\nu-1}\,\mathrm{B}(\nu,\mu-\nu)\;{}_2F_1\left(\mu-1,\nu;\mu;1-\frac{\gamma}{\beta}\right)$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu]$ ET II 217(9)

2. $$\int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = \pi\gamma^{-\varrho}(\beta-\gamma)^{-\sigma}\operatorname{cosec}(\varrho\pi)\,I_{1-\gamma/\beta}(\sigma,\varrho)$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad -\operatorname{Re}\sigma<\operatorname{Re}\varrho<1]$ ET II 217(10)

3.228

1. $$\int_a^b \frac{(x-a)^{\nu}(b-x)^{-\nu}}{x-c}\,dx = \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{a-c}{b-c}\right)^{\nu}\right] \qquad \text{for } c<a$$
$$= \pi\operatorname{cosec}(\nu\pi)\left[1-\cos(\nu\pi)\left(\frac{c-a}{b-c}\right)^{\nu}\right] \qquad \text{for } a<c<b$$
$$= \pi\operatorname{cosec}(\nu\pi)\left[1-\left(\frac{c-a}{c-b}\right)^{\nu}\right] \qquad \text{for } c>b$$
$[|\operatorname{Re}\nu|<1]$ ET II 250(31)

2. $$\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c}\,dx = \frac{\pi\operatorname{cosec}(\nu\pi)}{b-c}\left|\frac{a-c}{b-c}\right|^{\nu-1} \qquad \text{for } c<a \text{ or } c>b;$$
$$= -\frac{\pi(c-a)^{\nu-1}}{(b-c)^{\nu}}\cot(\nu\pi) \qquad \text{for } a<c<b$$
$[0<\operatorname{Re}\nu<1]$ ET II 250(32)

3. $$\int_a^b \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x-c}\,dx$$
$$= \frac{(b-a)^{\mu+\nu-1}}{b-c}\,\mathrm{B}(\mu,\nu)\;{}_2F_1\left(1,\mu;\mu+\nu;\frac{b-a}{b-c}\right) \qquad \text{for } c<a \text{ or } c>b;$$
$$= \pi(c-a)^{\nu-1}(b-c)^{\mu-1}\cot\mu\pi-(b-a)^{\mu+\nu-2}\,\mathrm{B}(\mu-1,\nu)$$
$$\times\;{}_2F_1\left(2-\mu-\nu,1;2-\mu;\frac{b-c}{b-a}\right) \qquad \text{for } a<c<b$$
$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad \mu+\nu\neq 1,\quad \mu\neq 1,2,\ldots]$ ET II 250(33)

4. $$\int_0^1 \frac{(1-x)^{\nu-1}x^{-\nu}}{a-bx}\,dx = \frac{\pi(a-b)^{\nu-1}}{a^{\nu}}\operatorname{cosec}(\nu\pi)$$
$[0<\operatorname{Re}\nu<1,\quad 0<b<a]$ BI (5)(8)

5. $$\int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c}\,dx = a^{1-\mu}(-c)^{\nu-1}\,\mathrm{B}(\mu-\nu,\nu)\;{}_2F_1\left(\mu-1,\nu;\mu;1+\frac{c}{a}\right) \qquad \text{for } c<0;$$
$$= \pi c^{\nu-1}(a+c)^{1-\mu}\cot[(\mu-\nu)\pi]-\frac{a^{1-\mu-\nu}}{a+c}\,\mathrm{B}(\mu-\nu-1,\nu)$$
$$\times\;{}_2F_1\left(2-\mu,1;2-\mu+\nu;\frac{a}{a+c}\right) \qquad \text{for } c>0$$
$[a>0,\quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu]$ ET II 251(34)

6. $$\int_0^\infty x^{\nu-1}\frac{(\gamma+x)^{-n}}{x+\beta}\,dx = \frac{\pi}{\sin\pi\nu}\frac{\beta^{\nu-1}}{(\gamma-\beta)^n}\left[1-\left(\frac{\gamma}{\beta}\right)^{\nu-1}\sum_{j=0}^{n-1}\frac{(1-\nu)_j}{j!}\left(\frac{\gamma-\beta}{\gamma}\right)^j\right]$$
$[|\arg\beta|<\pi,\quad |\arg\gamma|<\pi,\quad 0<\operatorname{Re}\nu<n]$ AS 256 (6.1.22)

3.229 $$\int_0^1 \frac{x^{\mu-1}\,dx}{(1-x)^\mu(1+ax)(1+bx)} = \frac{\pi\operatorname{cosec}\mu\pi}{a-b}\left[\frac{a}{(1+a)^\mu}-\frac{b}{(1+b)^\mu}\right]$$
$[0<\operatorname{Re}\mu<1]$ BI (5)(7)

3.231

1. $$\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \pi\cot p\pi$$ $[p^2<1]$ BI (4)(4)

2.[11] $$\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\,dx = \pi\operatorname{cosec} p\pi$$ $[p^2<1]$ BI (4)(1)

3. $$\int_0^1 \frac{x^p-x^{-p}}{x-1}\,dx = \frac{1}{p}-\pi\cot p\pi$$ $[p^2<1]$ BI (4)(3)

4. $$\int_0^1 \frac{x^p-x^{-p}}{1+x}\,dx = \frac{1}{p}-\pi\operatorname{cosec} p\pi$$ $[p^2<1]$ BI (4)(2)

5. $$\int_0^1 \frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx = \psi(\nu)-\psi(\mu)$$ $[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0]$
FI II 815, BI(4)(5)

6. $$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x}\,dx = \pi(\cot p\pi-\cot q\pi)$$ $[p>0,\quad q>0]$ FI II 718

3.232 $$\int_0^\infty \frac{(c+ax)^{-\mu}-(c+bx)^{-\mu}}{x}\,dx = c^{-\mu}\ln\frac{b}{a}$$ $[\operatorname{Re}\mu>-1;\quad a>0;\quad b>0;\quad c>0]$
BI (18)(14)

3.233 $$\int_0^\infty\left\{\frac{1}{1+x}-(1+x)^{-\nu}\right\}\frac{dx}{x} = \psi(\nu)+\boldsymbol{C}$$ $[\operatorname{Re}\nu>0]$ EH I 17, WH

3.234

1.[11] $$\int_0^1\left(\frac{x^{q-1}}{1-ax}-\frac{x^{-q}}{a-x}\right)dx = \pi a^{-q}\cot q\pi$$ $[0<q<1,\quad a>0]$ BI (5)(11)

2. $$\int_0^1\left(\frac{x^{q-1}}{1+ax}+\frac{x^{-q}}{a+x}\right)dx = \pi a^{-q}\operatorname{cosec} q\pi$$ $[0<q<1,\quad a>0]$ BI (5)(10)

3.235 $$\int_0^\infty \frac{(1+x)^\mu-1}{(1+x)^\nu}\frac{dx}{x} = \psi(\nu)-\psi(\nu-\mu)$$ $[\operatorname{Re}\nu>\operatorname{Re}\mu>0]$ BI (18)(5)

3.236[10] $$\int_0^1 \frac{x^{\frac{\mu}{2}}\,dx}{\left[(1-x)\left(1-a^2x\right)\right]^{\frac{\mu+1}{2}}} = \frac{(1-a)^{-\mu}-(1+a)^{-\mu}}{2a\mu\sqrt{\pi}}\Gamma\left(1+\frac{\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)$$
$[-2<\mu<1,\quad |a|<1]$ BI (12)(32)

3.237 $$\sum_{n=0}^\infty(-1)^{n+1}\int_n^{n+1}\frac{dx}{x+u} = \ln\frac{u\left[\Gamma\left(\frac{u}{2}\right)\right]^2}{2\left[\Gamma\left(\frac{u+1}{2}\right)\right]^2}$$ $[|\arg u|<\pi]$ ET II 216(1)

3.238

1. $$\int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u}\,dx = -\pi\cot\frac{\nu\pi}{2}|u|^{\nu-1}\operatorname{sign} u \qquad [0 < \operatorname{Re}\nu < 1 \quad u \text{ real}, \quad u \neq 0]$$
ET II 249(29)

2. $$\int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u}\operatorname{sign} x\,dx = \pi\tan\frac{\nu\pi}{2}|u|^{\nu-1} \qquad [0 < \operatorname{Re}\nu < 1 \quad u \text{ real}, \quad u \neq 0]$$
ET II 249(30)

3. $$\int_a^b \frac{(b-x)^{\mu-1}(x-a)^{\nu-1}}{|x-u|^{\mu+\nu}}\,dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^{\mu}|b-u|^{\nu}}\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad 0 < u < a < b \text{ and } 0 < a < b < u]$ MO 7

3.24–3.27 Powers of x, of binomials of the form $\alpha + \beta x^p$ and of polynomials in x

3.241

1. $$\int_0^1 \frac{x^{\mu-1}\,dx}{1+x^p} = \frac{1}{p}\beta\left(\frac{\mu}{p}\right) \qquad [\operatorname{Re}\mu > 0, \quad p > 0]$$
WH, BI (2)(13)

2. $$\int_0^{\infty} \frac{x^{\mu-1}\,dx}{1+x^{\nu}} = \frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu} = \frac{1}{\nu}\,\mathrm{B}\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right) \qquad [\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$$
ET I 309(15)a, BI (17)(10)

3.[11] $$\mathrm{PV}\int_0^{\infty} \frac{x^{p-1}\,dx}{1-x^q} = \frac{\pi}{q}\cot\frac{p\pi}{q} \qquad [p < q]$$
BI (17)(11)

4.[11] $$\int_0^{\infty} \frac{x^{\mu-1}\,dx}{(p+qx^{\nu})^{n+1}} = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu}\frac{\Gamma\left(\frac{\mu}{\nu}\right)\Gamma\left(1+n-\frac{\mu}{\nu}\right)}{\Gamma(1+n)}$$
$$\left[0 < \frac{\mu}{\nu} < n+1, \quad p \neq 0, \quad q \neq 0\right]$$
BI (17)(22)a

5. $$\int_0^{\infty} \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2}\operatorname{cosec}\frac{(p-q)\pi}{q} \qquad [p < 2q]$$
BI (17)(18)

6.[10] $$G(x) = \int_a^b \operatorname{sign}\left[\frac{x}{c} - \left(\frac{b-u}{b-a}\right)^p\right]du = (b-a)F\left[\left(\frac{x}{c}\right)^{1/p}\right]$$
where
$$F(x) = \int_0^1 \operatorname{sign}(x-t)\,dt = \begin{cases} -1 & x \leq 0 \\ 2x-1 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

3.242

1. $$\int_{-\infty}^{\infty} \frac{x^{2m}\,dx}{x^{4n}+2x^{2n}\cos t+1} = \frac{\pi}{n}\sin\left[\frac{(2n-2m-1)}{2n}t\right]\operatorname{cosec} t\operatorname{cosec}\frac{(2m+1)\pi}{2n}$$
$$\left[m < n, \quad t^2 < \pi^2\right]$$
FI II 642

2.[11] $$\int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^c \left(\frac{x^2+1}{x^b+1}\right)\frac{dx}{x^2} = 2^{-1/2-c}(1+a)^{1/2-c}\,\mathrm{B}\left(c-\frac{1}{2},\frac{1}{2}\right)$$

3.243[11] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+x^{2\nu})(1+x^{3\nu})}$$
$$= \frac{\pi}{48\nu}\left[8\operatorname{cosec}(2\rho)+12\operatorname{cosec}(3\rho)-8\operatorname{cosec}\left(2\rho-\frac{4\pi}{3}\right)+8\operatorname{cosec}\left(2\rho-\frac{2\pi}{3}\right)\right.$$
$$\left.-3\operatorname{cosec}\left(\rho-\frac{\pi}{6}\right)\operatorname{cosec}\left(\rho+\frac{\pi}{6}\right)\sec(\rho)\right]$$
where $\rho=\frac{\mu\pi}{6\nu}$, $[0<\operatorname{Re}\mu<5\operatorname{Re}\nu]$ ET I 312(34)

3.244

1. $$\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p>0]$ BI (2)(14)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^q}\,dx = \frac{\pi}{q}\cot\frac{p\pi}{q}$$ $[q>p>0]$ BI (2)(16)

3. $$\int_0^1 \frac{x^{\nu-1}-x^{\mu-1}}{1-x^\nu}\,dx = \frac{1}{\nu}\left[\boldsymbol{C}+\psi\left(\frac{\mu}{\nu}\right)\right]$$ $[\operatorname{Re}\mu>\operatorname{Re}\nu>0]$ BI (2)(17)

4. $$\int_{-\infty}^\infty \frac{x^{2m}-x^{2n}}{1-x^{2l}}\,dx = \frac{\pi}{l}\left[\cot\left(\frac{2m+1}{2l}\pi\right)-\cot\left(\frac{2n+1}{2l}\pi\right)\right]$$ $[m<l,\quad n<l]$ FI II 640

3.245 $$\int_0^\infty \left[x^{\nu-\mu}-x^\nu(1+x)^{-\mu}\right]\,dx = \frac{\nu}{\nu-\mu+1}\,\mathrm{B}(\nu,\mu-\nu)$$ $[\operatorname{Re}\mu>\operatorname{Re}\nu>0]$ BI (16)(13)

3.246 $$\int_0^\infty \frac{1-x^q}{1-x^r}x^{p-1}\,dx = \frac{\pi}{r}\sin\frac{q\pi}{r}\operatorname{cosec}\frac{p\pi}{r}\operatorname{cosec}\frac{(p+q)\pi}{r}$$ $[p+q<r,\quad p>0]$ ET I 331(33), BI (17)(12)

Integrals of the form $\int f\left(x^p\pm x^{-p},x^q\pm x^{-q},\dots\right)\frac{dx}{x}$ can be transformed by the substitution $x=e^t$ or $x=e^{-t}$. For example, instead of $\int_0^1\left(x^{1+p}+x^{1-p}\right)^{-1}\,dx$, we should seek to evaluate $\int_0^\infty \operatorname{sech}px\,dx$ and, instead of $\int_0^1 \frac{x^{n-m-1}+x^{n+m-1}}{1+2x^n\cos a+x^{2n}}\,dx$, we should seek to evaluate $\int_0^\infty \cosh mx\,(\cosh nx-\cos a)^{-1}\,dx$ (see **3.514** 2).

3.247

1.[11] $$\int_0^1 \frac{x^{\alpha-1}(1-x)^{n-1}}{1-\xi x^b}\,dx = (n-1)!\sum_{k=0}^\infty \frac{\xi^k}{(\alpha+kb)(\alpha+kb+1)\dots(\alpha+kb+n-1)}$$ $[b>0,\quad |\xi|<1]$ AD (6704)

2. $$\int_0^\infty \frac{(1-x^p)\,x^{\nu-1}}{1-x^{np}}\,dx = \frac{\pi}{np}\sin\left(\frac{\pi}{n}\right)\operatorname{cosec}\frac{(p+\nu)\pi}{np}\operatorname{cosec}\frac{\pi\nu}{np}$$ $[0<\operatorname{Re}\nu<(n-1)p]$ ET I 311(33)

3.248

1. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu}\,\mathrm{B}\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right)$$ $[\operatorname{Re}\nu > \operatorname{Re}2\mu > 0]$ BI (21)(9)

2. $$\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}$$ BI (8)(14)

3. $$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$ BI (8)(13)

4.[3] $$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}$$

6.* $$\int_{-\infty}^\infty \frac{dx}{\left(1+x^2\right)^2\sqrt{b+ax^2}} = \begin{cases} \dfrac{2}{\sqrt{b-a}}\arctan\left(\sqrt{\dfrac{b}{a}-1}\right) & \text{if } a<b \\ \dfrac{2}{\sqrt{a}} & \text{if } a=b \\ \dfrac{1}{\sqrt{a-b}}\ln\left(\dfrac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right) & \text{if } a>b \end{cases}$$

3.249

1.[0] $$\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^n} = \frac{(2n-3)!!}{2\cdot(2n-2)!!}\frac{\pi}{a^{2n-1}}$$ FI II 743

2.[9] $$\int_0^a \left(a^2-x^2\right)^{n-\frac{1}{2}}\,dx = a^{2n}\frac{(2n-1)!!}{2(2n)!!}\pi.$$ FI II 156

3. $$\int_{-1}^1 \frac{\left(1-x^2\right)^n\,dx}{(a-x)^{n+1}} = 2^{n+1}\,Q_n(a)$$ EH II 181(31)

4. $$\int_0^1 \frac{x^\mu\,dx}{1+x^2} = \frac{1}{2}\,\beta\left(\frac{\mu+1}{2}\right)$$ $[\operatorname{Re}\mu > -1]$ BI (2)(7)

5. $$\int_0^1 \left(1-x^2\right)^{\mu-1}\,dx = 2^{2\mu-2}\,\mathrm{B}(\mu,\mu) = \tfrac{1}{2}\,\mathrm{B}\left(\tfrac{1}{2},\mu\right)$$ $[\operatorname{Re}\mu > 0]$ FI II 784

6. $$\int_0^1 \left(1-\sqrt{x}\right)^{p-1}\,dx = \frac{2}{p(p+1)}$$ $[p > 0]$ BI (7)(7)

7. $$\int_0^1 (1-x^\mu)^{-\frac{1}{\nu}}\,dx = \frac{1}{\mu}\,\mathrm{B}\left(\frac{1}{\mu}, 1-\frac{1}{\nu}\right)$$ $[\operatorname{Re}\mu > 0, \quad |\nu| > 1]$

8.[11] $$\int_{-\infty}^\infty \left(1+\frac{x^2}{n-1}\right)^{-n/2}\,dx = \frac{\sqrt{\pi(n-1)}}{\Gamma\left(\frac{n}{2}\right)}\,\Gamma\left(\frac{n-1}{2}\right)$$ $[n > 1]$

3.251

1. $$\int_0^1 x^{\mu-1}\left(1-x^\lambda\right)^{\nu-1}\,dx = \frac{1}{\lambda}\,\mathrm{B}\left(\frac{\mu}{\lambda},\nu\right)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \lambda > 0]$ FI II 787

2. $$\int_0^\infty x^{\mu-1}\left(1+x^2\right)^{\nu-1}\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right)$$ $\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\left(\nu+\tfrac{1}{2}\mu\right) < 1\right]$

3. $$\int_1^\infty x^{\mu-1}\left(x^p-1\right)^{\nu-1}\,dx = \frac{1}{p}\,\mathrm{B}\left(1-\nu-\frac{\mu}{p},\nu\right)$$ $[p>0, \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu<p-p\operatorname{Re}\nu]$ ET I 311(32)

4. $$\int_0^\infty \frac{x^{2m}\,dx}{\left(ax^2+c\right)^n} = \frac{(2m-1)!!(2n-2m-3)!!\pi}{2\cdot(2n-2)!!a^m c^{n-m-1}\sqrt{ac}}$$ $[a>0, \quad c>0, \quad n>m+1]$ GU (141)(8a)

5. $$\int_0^\infty \frac{x^{2m+1}\,dx}{\left(ax^2+c\right)^n} = \frac{m!(n-m-2)!}{2(n-1)!a^{m+1}c^{n-m-1}}$$ $[ac>0, \quad n>m+1\geq 1]$ GU (141)(8b)

6. $$\int_0^\infty \frac{x^{\mu+1}}{\left(1+x^2\right)^2}\,dx = \frac{\mu\pi}{4\sin\frac{\mu\pi}{2}}$$ $[-2<\operatorname{Re}\mu<2]$ WH

7. $$\int_0^1 \frac{x^\mu\,dx}{\left(1+x^2\right)^2} = -\frac{1}{4}+\frac{\mu-1}{4}\,\beta\left(\frac{\mu-1}{2}\right)$$ $[\operatorname{Re}\mu>1]$ LI (3)(11)

8. $$\int_0^1 x^{q+p-1}\left(1-x^q\right)^{-\frac{p}{q}}\,dx = \frac{p\pi}{q^2}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p]$ BI (9)(22)

9. $$\int_0^1 x^{\frac{q}{p}-1}\left(1-x^q\right)^{-\frac{1}{p}}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{\pi}{p}$$ $[p>1, \quad q>0]$ BI (9)(23)a

10. $$\int_0^1 x^{p-1}\left(1-x^q\right)^{-\frac{p}{q}}\,dx = \frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q}$$ $[q>p>0]$ BI (9)(20)

11. $$\int_0^\infty x^{\mu-1}\left(1+\beta x^p\right)^{-\nu}\,dx = \frac{1}{p}\beta^{-\frac{\mu}{p}}\,\mathrm{B}\left(\frac{\mu}{p},\nu-\frac{\mu}{p}\right)$$ $[|\arg\beta|<\pi, \quad p>0, \quad 0<\operatorname{Re}\mu<p\operatorname{Re}\nu]$ BI (17)(20), EH I 10(16)

3.252

1. $$\int_0^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial c^{n-1}}\left[\frac{1}{\sqrt{ac-b^2}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right]$$ $\left[a>0, \quad ac>b^2\right]$ GW (131)(4)

2. $$\int_{-\infty}^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!\left(ac-b^2\right)^{n-\frac{1}{2}}}$$ $\left[a>0, \quad ac>b^2\right]$ GW (131)(5)

3. $$\int_0^\infty \frac{dx}{\left(ax^2+2bx+c\right)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left\{\frac{1}{\sqrt{c}\left(\sqrt{ac}+b\right)}\right\}$$ $\left[a\geq 0, \quad c>0, \quad b>-\sqrt{ac}\right]$ GW (213)(4)

4. $$\int_0^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2\left(ac-b^2\right)} - \frac{b}{2\left(ac-b^2\right)^{\frac{3}{2}}}\operatorname{arccot}\frac{b}{\sqrt{ac-b^2}}\right\} \quad \text{for } ac > b^2;$$
$$= \frac{(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial c^{n-2}}\left\{\frac{1}{2\left(ac-b^2\right)} + \frac{b}{4\left(b^2-ac\right)^{\frac{3}{2}}}\ln\frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}}\right\} \quad \text{for } b^2 > ac > 0;$$
$$= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad \text{for } ac = b^2$$
$[a > 0, \quad b > 0, \quad n \ge 2]$ GW (141)(5)

5. $$\int_{-\infty}^\infty \frac{x\,dx}{(ax^2+2bx+c)^n} = -\frac{(2n-3)!!\pi b a^{n-2}}{(2n-2)!!\left(ac-b^2\right)^{\frac{(2n-1)}{2}}} \qquad \left[ac > b^2, \quad a > 0, \quad n \ge 2\right]$$
GW (141)(6)

6. $$\int_{-\infty}^\infty \frac{x^m\,dx}{(ax^2+2bx+c)^n} = \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!!\left(ac-b^2\right)^{n-\frac{1}{2}}} \times \sum_{k=0}^{[m/2]}\binom{m}{2k}(2k-1)!!(2n-2k-3)!!\left(\frac{ac-b^2}{b^2}\right)^k$$
$\left[ac > b^2, \quad 0 \le m \le 2n-2\right]$ GW (141)(17)

7. $$\int_0^\infty \frac{x^n\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{c}\left(\sqrt{ac}+b\right)^{n+1}}$$
$\left[a \ge 0, \quad c > 0, \quad b > -\sqrt{ac}\right]$
GW (213)(5a)

8. $$\int_0^\infty \frac{x^{n+1}\,dx}{(ax^2+2bx+c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{a}\left(\sqrt{ac}+b\right)^{n+1}}$$
$\left[a > 0, \quad c \ge 0, \quad b > -\sqrt{ac}\right]$
GW (213)(5b)

9. $$\int_0^\infty \frac{x^{n+\frac{1}{2}}\,dx}{(ax^2+2bx+c)^{n+1}} = \frac{(2n-1)!!\pi}{2^{2n+\frac{1}{2}}\left(b+\sqrt{ac}\right)^{n+\frac{1}{2}} n!\sqrt{a}}$$
$\left[a > 0, \quad c > 0, \quad b + \sqrt{ac} > 0\right]$
LI (21)(19)

10.[6] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(1+2x\cos t+x^2)^\nu} = 2^{\nu-\frac{1}{2}}(\sin t)^{\frac{1}{2}-\nu}\, t\,\Gamma\left(\nu+\frac{1}{2}\right)\mathrm{B}(\mu, 2\nu-\mu)\, P_{\mu-\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t)$$
$[0 < t < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}2\nu]$
ET I 310(22)

11. $$\int_0^\infty \left(1+2\beta x+x^2\right)^{\mu-\frac{1}{2}} x^{-\nu-1}\,dx = 2^{-\mu}\left(\beta^2-1\right)^{\frac{\mu}{2}}\Gamma(1-\mu)\,\mathrm{B}(\nu-2\mu+1,-\nu)\,P^{\mu}_{\nu-\mu}(\beta)$$
$$[\operatorname{Re}\nu<0, \quad \operatorname{Re}(2\mu-\nu)<1, \quad |\arg(\beta\pm 1)|<\pi]$$
EH I 160(33)
$$= -\pi \operatorname{cosec}\nu\pi\, C_\nu^{\frac{1}{2}-\mu}(\beta)$$
$$\left[-2<\operatorname{Re}\left(\tfrac{1}{2}-\mu\right)<\operatorname{Re}\nu<0, \quad |\arg(\beta\pm 1)|<\pi\right]$$
EH I 178(24)

12. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{x^2+2ax\cos t+a^2} = -\pi a^{\mu-2}\operatorname{cosec}t\operatorname{cosec}(\mu\pi)\sin[(\mu-1)t]$$
$$[a>0, \quad 0<|t|<\pi, \quad 0<\operatorname{Re}\mu<2]$$
FI II 738, BI(20)(3)

13. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\left(x^2+2ax\cos t+a^2\right)^2} = \frac{\pi a^{\mu-4}}{2}\operatorname{cosec}\mu\pi\operatorname{cosec}^3 t$$
$$\times\{(\mu-1)\sin t\cos[(\mu-2)t]-\sin[(\mu-1)t]\}$$
$$[a>0, \quad 0<|t|<\pi, \quad 0<\operatorname{Re}\mu<4]$$ LI(20)(8)a, ET I 309(13)

14. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\sqrt{1+2x\cos t+x^2}} = \pi\operatorname{cosec}(\mu\pi)\,P_{\mu-1}(\cos t) \qquad [-\pi<t<\pi, \quad 0<\operatorname{Re}\mu<1]$$
ET I 310(17)

3.253 $$\int_{-1}^1 \frac{(1+x)^{2\mu-1}(1-x)^{2\nu-1}}{\left(1+x^2\right)^{\mu+\nu}}\,dx = 2^{\mu+\nu-2}\,\mathrm{B}(\mu,\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ FI II 787

3.254

1. $$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}\left(x^2+\beta^2\right)^\nu\,dx$$
$$= \beta^{2\nu}u^{\lambda+\mu-1}\,\mathrm{B}(\lambda,\mu)\;{}_3F_2\left(-\nu,\frac{\lambda}{2},\frac{\lambda+1}{2};\frac{\lambda+\mu}{2},\frac{\lambda+\mu+1}{2};\frac{-u^2}{\beta^2}\right)$$
$$\left[\operatorname{Re}\left(\frac{u}{\beta}\right)>0, \quad \lambda>0, \quad \operatorname{Re}\mu>0\right]$$ ET II 186(10)

2.[6] $$\int_u^\infty \left(x^{-\lambda}(x-u)^{\mu-1}\left(x^2+\beta^2\right)\right)^\nu\,dx$$
$$= u^{\mu-\lambda+2\nu}\frac{\Gamma(\mu)\,\Gamma(\lambda-\mu-2\nu)}{\Gamma(\lambda-2\nu)}$$
$$\times\;{}_3F_2\left(-\nu,\frac{\lambda-\mu}{2}-\nu,\frac{1+\lambda-\mu}{2}-\nu;\frac{\lambda}{2}-\nu,\frac{1+\lambda}{2}-\nu;-\frac{\beta^2}{u^2}\right)$$
$$\left[|u|>|\beta| \text{ and } \operatorname{Re}\left(\frac{\beta}{u}\right)>0, \quad 0<\operatorname{Re}\mu<\operatorname{Re}(\lambda-2\nu)\right]$$ ET II 202(9)

3.255 $$\int_0^1 \frac{x^{\mu+\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}}{\left(c+2bx-ax^2\right)^{\mu+1}}\,dx = \frac{\sqrt{\pi}}{\left\{a+\left(\sqrt{c+2b-a}+\sqrt{c}\right)^2\right\}^{\mu+\frac{1}{2}}\sqrt{c+2b-a}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)}$$
$$\left[a+\left(\sqrt{c+2b-a}+\sqrt{c}\right)^2>0, \quad c+2b-a>0, \quad \operatorname{Re}\mu>-\tfrac{1}{2}\right]$$
BI (14)(2)

3.256

1. $$\int_0^1 \frac{x^{p-1}+x^{q-1}}{\left(1-x^2\right)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\cos\left(\frac{q-p}{4}\pi\right)\sec\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\frac{q}{2}\right)$$
$[p>0, \quad q>0, \quad p+q<2]$ BI (8)(25)

2. $$\int_0^1 \frac{x^{p-1}-x^{q-1}}{\left(1-x^2\right)^{\frac{p+q}{2}}}\,dx = \frac{1}{2}\sin\left(\frac{q-p}{4}\pi\right)\operatorname{cosec}\left(\frac{q+p}{4}\pi\right)\mathrm{B}\left(\frac{p}{2},\frac{q}{2}\right)$$
$[p>0, \quad q>0, \quad p+q<2]$ BI (8)(26)

3.257[9] $$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx$$
$$= \frac{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}{2ac^{p+\frac{1}{2}}\,\Gamma(p+1)} \qquad \left[a>0, \quad b<0, \quad c>0, \quad p>-\tfrac{1}{2}\right]$$ BI (20)(4)
$$= \frac{1}{2}\frac{\mathrm{B}\left(p+\frac{1}{2},\frac{1}{2}\right)}{a\,(4ab+x)^{p+\frac{1}{2}}} \qquad \left[a>0, \quad b>0, \quad c>-4ab, \quad p>-\tfrac{1}{2}\right]$$

3.258

1. $$\int_b^\infty \left(x-\sqrt{x^2-a^2}\right)^n dx = \frac{a^2}{2(n-1)}\left(b-\sqrt{b^2-a^2}\right)^{n-1} - \frac{1}{2(n+1)}\left(b-\sqrt{b^2-a^2}\right)^{n+1}$$
$[0<a\le b, \quad n\ge 2]$ GW (215)(5)

2. $$\int_b^\infty \left(\sqrt{x^2+1}-x\right)^n dx = \frac{\left(\sqrt{b^2+1}-b\right)^{n-1}}{2(n-1)} + \frac{\left(\sqrt{b^2+1}-b\right)^{n+1}}{2(n+1)}$$
$[n\ge 2]$ GW (214)(7)

3. $$\int_0^\infty \left(\sqrt{x^2+a^2}-x\right)^n dx = \frac{na^{n+1}}{n^2-1}$$ $[n\ge 2]$ GW (214)(6a)

4. $$\int_0^\infty \frac{dx}{\left(x+\sqrt{x^2+a^2}\right)^n} = \frac{n}{a^{n-1}\left(n^2-1\right)}$$ $[n\ge 2]$ GW (214)(5a)

5. $$\int_0^\infty x^m\left(\sqrt{x^2+a^2}-x\right)^n dx = \frac{n\cdot m!a^{m+n+1}}{(n-m-1)(n-m+1)\ldots(m+n+1)}$$
$[a>0, \quad 0\le m\le n-2]$ GW (214)(6)

6. $$\int_0^\infty \frac{x^m\,dx}{\left(x+\sqrt{x^2+a^2}\right)^n} = \frac{n\cdot m!}{(n-m-1)(n-m+1)\ldots(m+n+1)a^{n-m-1}}$$
$[a>0, \quad 0\le m\le n-2]$ GW (214)(5)

7. $$\int_a^\infty (x-a)^m\left(x-\sqrt{x^2-a^2}\right)^n dx = \frac{n\cdot(n-m-2)!(2m+1)!a^{m+n+1}}{2^m(n+m+1)!}$$
$[a>0, \quad n\ge m+2]$ GH (215)(6)

3.259

1.[6] $$\int_0^1 x^{p-1}(1-x)^{n-1}\left(1+bx^m\right)^l\,dx = (n-1)!\sum_{k=0}^{\infty}\binom{l}{k}\frac{b^k\,\Gamma(p+km)}{\Gamma(p+n+km)}$$
$[|b|<1 \text{ unless } l=0,1,2,\ldots; \quad p,n,p+ml>0]$ BI (1)(14)

2.[11] $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\left(x^m+\beta^m\right)^\lambda\,dx$$
$$= \beta^{m\lambda}u^{\mu+\nu-1}\,\mathrm{B}\left(\mu,\nu\right)$$
$$\times\ {}_{m+1}F_m\left(-\lambda,\frac{\nu}{m},\frac{\nu+1}{m},\ldots,\frac{\nu+m-1}{m};\frac{\mu+\nu}{m},\frac{\mu+\nu+1}{m},\ldots,\frac{\mu+\nu+m-1}{m};\frac{-u^m}{\beta^m}\right)$$
$\left[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad \left|\arg\left(\frac{u}{\beta}\right)\right|<\frac{\pi}{m}\right]$ ET II 186(11)

3.[11] $$\int_0^\infty x^{\lambda-1}\left(1+\alpha x^p\right)^{-\mu}\left(1+\beta x^p\right)^{-\nu}\,dx = \frac{1}{p}\alpha^{-\lambda/p}\,\mathrm{B}\left(\frac{\lambda}{p},\mu+\nu-\frac{\lambda}{p}\right)\,{}_2F_1\left(\nu,\frac{\lambda}{p};\mu+\nu;1-\frac{\beta}{\alpha}\right)$$
$[|\arg\alpha|<\pi, \quad |\arg\beta|<\pi, \quad p>0, \quad 0<\operatorname{Re}\lambda<2\operatorname{Re}(\mu+\nu)]$ ET I 312(35)

3.261

1.[11] $$\mathrm{PV}\int_0^1\frac{(1-x\cos t)\,x^{\mu-1}\,dx}{1-2x\cos t+x^2} = \sum_{k=0}^{\infty}\frac{\cos kt}{\mu+k}$$ $[\operatorname{Re}\mu>0, \quad t\neq 2n\pi]$ BI (6)(9)

2. $$\int_0^1\frac{\left(x^\nu+x^{-\nu}\right)\,dx}{1+2x\cos t+x^2} = \frac{\pi\sin\nu t}{\sin t\sin\nu\pi}$$ $\left[\nu^2<1, \quad t\neq(2n+1)\pi\right]$ BI (6)(8)

3. $$\int_0^1\frac{\left(x^{1+p}+x^{1-p}\right)\,dx}{\left(1+2x\cos t+x^2\right)^2} = \frac{\pi\left(p\sin t\cos pt-\cos t\sin pt\right)}{2\sin^3 t\sin p\pi}$$
$\left[p^2<1, \quad t\neq(2n+1)\pi\right]$ BI (6)(18)

4. $$\int_0^1\frac{x^{\mu-1}}{1+2ax\cos t+a^2x^2}\cdot\frac{dx}{(1-x)^\mu} = \frac{\pi\operatorname{cosec} t\operatorname{cosec}\mu\pi}{\left(1+2a\cos t+a^2\right)^{\frac{\mu}{2}}}\sin\left(t-\mu\arctan\frac{a\sin t}{1+a\cos t}\right)$$
$[a>0, \quad 0<\operatorname{Re}\mu<1]$ BI (6)(21)

3.262 $$\int_0^\infty\frac{x^{-p}\,dx}{1+x^3} = \frac{\pi}{3}\operatorname{cosec}\frac{(1-p)\pi}{3}$$ $[-2<p<1]$ LI (18)(3)

3.263 $$\int_0^\infty\frac{x^\nu\,dx}{(x+\gamma)\left(x^2+\beta^2\right)} = \frac{\pi}{2\left(\beta^2+\gamma^2\right)}\left[\gamma\beta^{\nu-1}\sec\frac{\nu\pi}{2}+\beta^\nu\operatorname{cosec}\frac{\nu\pi}{2}-2\gamma^\nu\operatorname{cosec}(\nu\pi)\right]$$
$[\operatorname{Re}\beta>0, \quad |\arg\gamma|<\pi, \quad -1<\operatorname{Re}\nu<2, \quad \nu\neq 0]$ ET II 216(7)

3.264

1. $$\int_0^\infty\frac{x^{p-1}\,dx}{\left(a^2+x^2\right)\left(b^2-x^2\right)} = \frac{\pi}{2}\frac{a^{p-2}+b^{p-2}\cos\frac{p\pi}{2}}{a^2+b^2}\operatorname{cosec}\frac{p\pi}{2}$$
$[0<p<4, \quad a>0, \quad b>0]$
BI (19)(14)

2. $$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x^2)(\gamma+x^2)} = \frac{\pi}{2}\frac{\gamma^{\frac{\mu}{2}-1}-\beta^{\frac{\mu}{2}-1}}{\beta-\gamma}\operatorname{cosec}\frac{\mu\pi}{2}$$
$$= \frac{\pi}{2(\gamma-\beta)}\left(\frac{1}{\sqrt{\beta}}-\frac{1}{\sqrt{\gamma}}\right) \qquad \left[\mu=\tfrac{1}{2}\right]$$
$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad 0<\operatorname{Re}\mu<4]$ ET I 309(4)

3. $$\int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^2} = \frac{\pi}{2}\left(\frac{1}{a^2b^{1/2}}-\frac{1}{2a(a+b)^{3/2}}-\frac{1}{a^2(a+b)^{1/2}}\right)$$ MC

4. $$\int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^3} = \frac{\pi}{4}\left(\frac{2}{a^3b^{1/2}}-\frac{3}{4a(a+b)^{5/2}}-\frac{1}{a^2(a+b)^{3/2}}-\frac{2}{a^3(a+b)^{1/2}}\right)$$

5. $$\int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^4} = \frac{\pi}{4}\left(\frac{2}{a^4b^{1/2}}-\frac{5}{8a(a+b)^{7/2}}-\frac{3}{4a^2(a+b)^{5/2}}\right.$$
$$\left.-\frac{1}{a^3(a+b)^{3/2}}-\frac{2}{a^4(a+b)^{1/2}}\right)$$

6. $$\int_0^\infty \frac{dx}{(b+x^2)(a+b+x^2)^n}$$
$$= \frac{\pi}{2}\frac{1}{a^nb^{1/2}}-\frac{1}{2a(a+b)^{n-1/2}}\,\mathrm{B}\left(n-\frac{1}{2},\frac{1}{2}\right)\ {}_2F_1\left(1-n,1;\ \frac{3}{2}-n;\ \frac{a+b}{a}\right)$$
AS 263 (6.6.3.2)
$$= \frac{\pi}{2}\frac{1}{a^nb^{1/2}}-\frac{\pi}{2a^n(a+b)^{n-1/2}}\sum_{j=0}^{n-1}\frac{\left(\frac{1}{2}\right)_j}{j!}\left(\frac{a}{a+b}\right)^j$$
$[n>0, \quad a+b>0]$

7. $$\int_0^\infty \frac{x^2\,dx}{(x^2+\alpha^2)(x^2+\beta^2)(x^2+\gamma^2)} = \frac{\pi}{2\alpha(\beta^2-\gamma^2)}\left[\frac{\beta}{\beta+\alpha}-\frac{\gamma}{\gamma+\alpha}\right] = \frac{\pi}{2(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)}$$

3.265 $$\int_0^1 \frac{1-x^{\mu-1}}{1-x}\,dx = \psi(\mu)+\boldsymbol{C} \qquad [\operatorname{Re}\mu>0]$$ FI II 796, WH, ET I 16(13)
$$= \psi(1-\mu)+\boldsymbol{C}-\pi\cot(\mu\pi) \qquad [\operatorname{Re}\mu>0]$$ EH I 16(15)a

3.266 $$\int_0^\infty \frac{(x^\nu-a^\nu)\,dx}{(x-a)(\beta+x)} = \frac{\pi}{a+\beta}\left\{\beta^\nu\operatorname{cosec}(\nu\pi)-a^\nu\cot(\nu\pi)-\frac{a^\nu}{\pi}\ln\frac{\beta}{a}\right\}$$
$[|\arg\beta|<\pi, \quad |\operatorname{Re}\nu|<1, \quad \nu\neq 0]$
ET II 216(8)

3.267

1. $$\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n+1)}$$ BI (9)(6)

2. $$\int_0^1 \frac{x^{3n-1}\,dx}{\sqrt[3]{1-x^3}} = \frac{(n-1)!\,\Gamma\left(\frac{2}{3}\right)}{3\,\Gamma\left(n+\frac{2}{3}\right)}$$ BI (9)(7)

3.* $\int_0^1 \frac{x^{3n-2}\,dx}{\sqrt[3]{1-x^3}} = \frac{\Gamma\left(n-\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{3\,\Gamma\left(n+\frac{1}{3}\right)}$

3.268

1. $\int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p}\right) dx = \ln p$ BI (5)(14)

2. $\int_0^1 \frac{1-x^\mu}{1-x} x^{\nu-1}\,dx = \psi(\mu+\nu) - \psi(\nu)$ $[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0]$ BI (2)(3)

3. $\int_0^1 \left[\frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}}\right] dx = n\boldsymbol{C} + \sum_{k=1}^{n} \psi\left(\mu + \frac{n-k}{n}\right)$

$[\operatorname{Re}\mu > 0]$ BI (13)(10)

3.269

1. $\int_0^1 \frac{x^p - x^{-p}}{1-x^2} x\,dx = \frac{\pi}{2}\cot\frac{p\pi}{2} - \frac{1}{p}$ $\left[p^2 < 1\right]$ BI (4)(12)

2. $\int_0^1 \frac{x^p - x^{-p}}{1+x^2} x\,dx = \frac{1}{p} - \frac{\pi}{2}\operatorname{cosec}\frac{p\pi}{2}$ $\left[p^2 < 1\right]$ BI (4)(8)

3. $\int_0^1 \frac{x^\mu - x^\nu}{1-x^2}\,dx = \frac{1}{2}\psi\left(\frac{\nu+1}{2}\right) - \frac{1}{2}\psi\left(\frac{\mu+1}{2}\right)$ $[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ BI (2)(9)

3.271

1. $\int_0^\infty \frac{x^p - x^q}{x-1}\frac{dx}{x+a} = \frac{\pi}{1+a}\left(\frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi}\right)$

$\left[p^2 < 1, \quad q^2 < 1, \quad a > 0\right]$ BI (19)(2)

2. $\int_0^\infty \frac{x^p - a^p}{x-a}\frac{x^p - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{\frac{a^{2p}-1}{\sin(2p\pi)} - \frac{1}{\pi}a^p\ln a\right\}$

$\left[p^2 < \frac{1}{4}\right]$ BI (19)(3)

3. $\int_0^\infty \frac{x^p - a^p}{x-a}\frac{x^{-p} - 1}{x-1}\,dx = \frac{\pi}{a-1}\left\{2\left(a^p - 1\right)\cot p\pi - \frac{1}{\pi}\left(a^p + 1\right)\ln a\right\}$

$\left[p^2 < 1\right]$ BI (18)(9)

4. $\int_0^\infty \frac{x^p - a^p}{x-a}\frac{1 - x^{-p}}{1-x} x^q\,dx = \frac{\pi}{a-1}\left\{\frac{a^{p+q}-1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]}\right\}\frac{\sin p\pi}{\sin q\pi}$

$\left[(p+q)^2 < 1, \quad (p-q)^2 < 1\right]$ BI (19)(4)

5. $\int_0^\infty \left(\frac{x^p - x^{-p}}{1-x}\right)^2 dx = 2\left(1 - 2p\pi\cot 2p\pi\right)$ $\left[0 < p^2 < \frac{1}{4}\right]$ BI (16)(3)

3.272

1. $\int_0^1 \frac{x^{n-1} + x^{n-\frac{1}{2}} - 2x^{2n-1}}{1-x}\,dx = 2\ln 2$ BI (8)(8)

2. $$\int_0^1 \frac{x^{n-1}+x^{n-\frac{2}{3}}+x^{n-\frac{1}{3}}-3x^{3n-1}}{1-x}\,dx = 3\ln 3$$ BI (8)(9)

3.273

1. $$\int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1}x^{n+1}\sin nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\sin kt}{\Gamma(p+k)}$$
$[p>0]$ BI (6)(13)

2. $$\int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1}x^{n+1}\cos nt}{1-2ax\cos t + a^2x^2}(1-x)^{p-1}\,dx = \Gamma(p)\sum_{k=1}^{n}\frac{(k-1)!a^{k-1}\cos kt}{\Gamma(p+k)}$$
$[p>0]$ BI (6)(14)

3. $$\int_0^1 x\frac{\sin t - x^n\sin[(n+1)t] + x^{n+1}\sin nt}{1-2x\cos t + x^2}\,dx = \sum_{k=1}^{n}\frac{\sin kt}{k+1}$$ BI (6)(12)

4. $$\int_0^1 \frac{1 - x\cos t - x^{n+1}\cos[(n+1)t] + x^{n+2}\cos nt}{1-2x\cos t + x^2}\,dx = \sum_{k=0}^{n}\frac{\cos kt}{k+1}$$ BI (6)(11)

3.274

1. $$\int_0^\infty \frac{x^{\mu-1}(1-x)}{1-x^n}\,dx = \frac{\pi}{n}\sin\frac{\pi}{n}\operatorname{cosec}\frac{\mu\pi}{n}\operatorname{cosec}\frac{(\mu+1)\pi}{n}$$
$[0<\operatorname{Re}\mu<n-1]$ BI (20)(13)

2. $$\int_0^1 \frac{1-x^n}{(1+x)^{n+1}}\frac{dx}{1-x} = \frac{1}{2^{n+1}}\sum_{k=1}^{n}\frac{2^k}{k}$$ BI (5)(3)

3. $$\int_0^\infty \frac{x^q-1}{x^p-x^{-p}}\frac{dx}{x} = \frac{\pi}{2p}\tan\frac{q\pi}{2p}$$ $[p>q]$ BI (18)(6)

3.275

1. $$\int_0^1 \left(\frac{x^{n-1}}{1-x^{1/p}} - \frac{px^{np-1}}{1-x}\right)dx = p\ln p$$ $[p>0]$ BI (13)(9)

2. $$\int_0^1 \left(\frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x}\right)dx = \boldsymbol{C} + \frac{1}{n}\sum_{k=1}^{n}\psi\left(m+\frac{n-k}{n}\right)$$ BI (5)(13)

3. $$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q}\right)dx = \ln q$$ $[q>0]$ BI (5)(12)

4. $$\int_0^\infty \left(\frac{1}{1+x^{2^n}} - \frac{1}{1+x^{2^m}}\right)\frac{dx}{x} = 0.$$ BI (18)(17)

3.276

1.[10] $$\int_0^\infty \frac{\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx}{x^2} = \frac{1}{2|b|}\frac{\mathrm{B}\left(p+\frac{1}{2},\frac{1}{2}\right)}{\left(2a\left(b+|b|\right)+c\right)^{p+\frac{1}{2}}}$$
$\left[a>0,\quad c>-4ac,\quad p>-\frac{1}{2}\right]$

2.[10] $$\int_0^\infty \left(a + \frac{b}{x^2}\right)\left[\left(ax + \frac{b}{x}\right)^2 + c\right]^{-p-1} dx = \frac{\mathrm{B}\left(p + \frac{1}{2}, \frac{1}{2}\right)}{(4ab + c)^{p+\frac{1}{2}}}$$

$\left[a > 0, \quad b > 0, \quad c > -4ac, \quad p > -\frac{1}{2}\right]$

3.277

1.[11] $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{1+x^2}+\beta\right]^\nu}{\sqrt{1+x^2}}\, dx = 2^{\frac{\mu}{2}-1}\left(\beta^2 - 1\right)^{\frac{\nu}{2}+\frac{\mu}{4}} \Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)\, P^{\nu+\frac{\mu}{2}}_{\frac{\mu}{2}-1}(\beta)$$

$[\operatorname{Re}\beta > -1, \quad 0 < \operatorname{Re}\mu < 1 - \operatorname{Re}\nu]$

ET I 310(25)

2. $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{\beta^2+x^2}+x\right]^\nu}{\sqrt{\beta^2+x^2}}\, dx = \frac{\beta^{\mu+\nu-1}}{2^\mu}\,\mathrm{B}\left(\mu, \frac{1-\mu-\nu}{2}\right)$$

$[\operatorname{Re}\beta > 0, \quad 0 < \operatorname{Re}\mu < 1 - \operatorname{Re}\nu]$

ET I 311(28)

3. $$\int_0^\infty \frac{x^{\mu-1}\left[\cos t \pm i \sin t\sqrt{1+x^2}\right]^\nu}{\sqrt{1+x^2}}\, dx = 2^{\frac{\mu-1}{2}} \sin^{\frac{1-\mu}{2}} t \frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}$$
$$\times\left[\pi^{-\frac{1}{2}}\, Q^{\frac{\mu+1}{2}}_{-\frac{\mu+1}{2}-\nu}(\cos t) \mp \frac{i}{2}\pi^{\frac{1}{2}}\, P^{-\frac{\mu+1}{2}-\nu}_{\frac{\mu-1}{2}}(\cos t)\right]$$

$[\operatorname{Re}\mu > 0]$ ET I 311 (27)

4. $$\int_0^\infty \frac{x^{\mu-1}\left[\sqrt{\left(\beta^2-1\right)\left(x^2+1\right)}+\beta\right]^\nu}{\sqrt{x^2+1}}\, dx$$
$$= \frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}} e^{-\frac{1}{2}i\pi(\mu-1)} \frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma(1-\mu-\nu)}{\Gamma(-\nu)}\left(\beta^2-1\right)\frac{1-\mu}{4}\, Q^{\frac{\mu-1}{2}}_{-\frac{\mu+1}{2}-\nu}(\beta)$$

$[\operatorname{Re}\beta > 1, \quad \operatorname{Re}\nu < 0, \quad \operatorname{Re}\mu < 1 - \operatorname{Re}\nu]$ ET I 311(26)

5. $$\int_u^\infty \frac{(x-u)^{\mu-1}\left(\sqrt{x+1}-\sqrt{x-1}\right)^{2\nu}}{\sqrt{x^2-1}}\, dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} e^{\left(\mu-\frac{1}{2}\right)\pi i}\left(u^2-1\right)^{\frac{2\mu-1}{4}}\, Q^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(u)$$

$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1 + \operatorname{Re}\nu]$ ET II 202(10)

6. $$\int_1^\infty \frac{x^{\mu-1}\left[\left(x-\sqrt{x^2-1}\right)^\nu + \left(x-\sqrt{x^2-1}\right)^{-\nu}\right]}{\sqrt{x^2-1}}\, dx = 2^{-\mu}\,\mathrm{B}\left(\frac{1-\mu+\nu}{2}, \frac{1-\mu-\nu}{2}\right)$$

$[\operatorname{Re}\mu < 1 + \operatorname{Re}\nu]$ ET I 311(29)

7. $$\int_0^u \frac{(u-x)^{\mu-1}\left[\left(\sqrt{x+2}+\sqrt{x}\right)^{2\nu} + \left(\sqrt{x+2}-\sqrt{x}\right)^{2\nu}\right]}{\sqrt{x(x+2)}}\, dx = 2\frac{2\mu+1}{2}\sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}}\, P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(u+1)$$

$[|\arg u| < \pi, \quad \operatorname{Re}\mu > 0]$ ET II 186(12)

3.278[8]

1. $$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^q \frac{dx}{1-x^2} = 0$$ $[pq > 1]$

3.3–3.4 Exponential Functions

3.31 Exponential functions

3.310[11] $\int_0^\infty e^{-px}\,dx = \frac{1}{p}$ [$\operatorname{Re} p > 0$]

3.311

1. $\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}$ LO III 284a

2. $\int_0^\infty \frac{e^{-\mu x}}{1+e^{-x}}\,dx = \beta(\mu)$ [$\operatorname{Re}\mu > 0$] EH I 20(3), ET I 144(7)

3.[11] $\int_{-\infty}^\infty \frac{e^{-px}}{1+e^{-qx}}\,dx = \frac{\pi}{|q|}\operatorname{cosec}\frac{p\pi}{q}$
[$q > p > 0$ or $0 > p > q$] (cf. **3.241** 2) BI (28)(7)

4. $\int_0^\infty \frac{e^{-qx}\,dx}{1-ae^{-px}} = \sum_{k=0}^\infty \frac{a^k}{q+kp}$ [$0 < a < 1$] BI (27)(7)

5. $\int_0^\infty \frac{1-e^{\nu x}}{e^x-1}\,dx = \psi(\nu) + \boldsymbol{C} + \pi\cot(\pi\nu)$ [$\operatorname{Re}\nu < 1$] (cf. **3.265**) EH I 16(16)

6. $\int_0^\infty \frac{e^{-x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) + \boldsymbol{C}$ [$\operatorname{Re}\nu > 0$] WH, EH I 16(14)

7. $\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{1-e^{-x}}\,dx = \psi(\nu) - \psi(\mu)$ [$\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0$] (cf. **3.231** 5)
BI (27)(8)

8. $\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = \pi b^{\mu-1}\cot(\mu\pi)$ [$b > 0, \quad 0 < \operatorname{Re}\mu < 1$] ET I 120(14)a

9. $\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b+e^{-x}} = \pi b^{\mu-1}\operatorname{cosec}(\mu\pi)$ [$|\arg b| < \pi, \quad 0 < \operatorname{Re}\mu < 1$]
ET I 120(15)a

10.[11] $\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\frac{p\pi}{p+q}$ [$p > 0, \quad q > 0$] GW (311)(16c)

11. $\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\,dx = \frac{1}{r-s}\left[\psi\left(\frac{r-q}{r-s}\right) - \psi\left(\frac{r-p}{r-s}\right)\right]$
[$r > s, r > p, r > q$] GW (311)(16)

12. $\int_0^\infty \frac{a^x-b^x}{c^x-d^x}\,dx = \frac{1}{\ln\frac{c}{d}}\left[\psi\left(\frac{\ln\frac{c}{b}}{\ln\frac{c}{d}}\right) - \psi\left(\frac{\ln\frac{c}{a}}{\ln\frac{c}{d}}\right)\right]$ [$c > a > 0, \quad b > 0, \quad d > 0$]
GW (311)(16a)

13.* $\int_0^\infty \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\operatorname{cosec}\left(\frac{\pi p}{p+q}\right)$

3.312

1. $$\int_0^\infty \left(1 - e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x}\,dx = \beta\,\mathrm{B}(\beta\mu, \nu) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0]$$
LI(25)(13), EH I 11(24)

2. $$\int_0^\infty \left(1 - e^{-x}\right)^{-1} \left(1 - e^{-\alpha x}\right) \left(1 - e^{-\beta x}\right) e^{-px}\,dx = \psi(p+\alpha) + \psi(p+\beta) - \psi(p+\alpha+\beta) - \psi(p)$$
$[\operatorname{Re} p > 0, \quad \operatorname{Re} p > -\operatorname{Re}\alpha, \quad \operatorname{Re} p > -\operatorname{Re}\beta, \quad \operatorname{Re} p > -\operatorname{Re}(\alpha+\beta)]$ ET I 145(15)

3.[11] $$\int_0^\infty \left(1 - e^{-x}\right)^{\nu-1} \left(1 - \beta e^{-x}\right)^{-\varrho} e^{-\mu x}\,dx = \mathrm{B}(\mu, \nu)\ {}_2F_1(\varrho, \mu; \mu+\nu; \beta)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad |\arg(1-\beta)| < \pi]$ EH I 116(15)

3.313

1.[7] $$\mathrm{PV}\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{1 - e^{-x}} = \pi\cot\pi\mu \qquad [0 < \operatorname{Re}\mu < 1]$$

2.[7] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(1 + e^{-x}\right)^\nu} = \mathrm{B}(\mu, \nu-\mu) \qquad [0 < \operatorname{Re}\mu < \operatorname{Re}\nu]$$

3.314 $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(e^{\beta/\gamma} + e^{-x/\gamma}\right)^\nu} = \gamma\exp\left[\beta\left(\mu - \frac{\nu}{\gamma}\right)\right] \mathrm{B}(\gamma\mu, \nu - \gamma\mu)$$
$\left[\operatorname{Re}\left(\frac{\nu}{\gamma}\right) > \operatorname{Re}\mu > 0, \quad |\operatorname{Im}\beta| < \pi\operatorname{Re}\gamma\right]$ ET I 120(21)

3.315

1. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(e^\beta + e^{-x}\right)^\nu \left(e^\gamma + e^{-x}\right)^\varrho} = \exp[\gamma(\mu-\varrho) - \beta\nu]\,\mathrm{B}(\mu, \nu+\varrho-\mu)\ {}_2F_1\left(\nu, \mu; \nu+\varrho; 1 - e^{\nu-\beta}\right)$$
$[|\operatorname{Im}\beta| < \pi, \quad |\operatorname{Im}\gamma| < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}(\nu+\varrho)]$ ET I 121(22)

2. $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\left(\beta + e^{-x}\right)\left(\gamma + e^{-x}\right)} = \frac{\pi\left(\beta^{\mu-1} - \gamma^{\mu-1}\right)}{\gamma - \beta}\operatorname{cosec}(\mu\pi)$$
$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad \beta \neq \gamma, \quad 0 < \operatorname{Re}\mu < 2]$ ET I 120(18)

3.316 $$\int_{-\infty}^\infty \frac{\left(1 + e^{-x}\right)^\nu - 1}{\left(1 + e^{-x}\right)^\mu}\,dx = \psi(\mu) - \psi(\mu-\nu) \qquad [\operatorname{Re}\mu > \operatorname{Re}\nu > 0]$$
(cf. **3.235**)
BI (28)(8)

3.317

1. $$\int_{-\infty}^\infty \left(\frac{1}{1 + e^{-x}} - \frac{1}{\left(1 + e^{-x}\right)^\mu}\right) dx = \boldsymbol{C} + \psi(\mu) \qquad [\operatorname{Re}\mu > 0]$$
(cf. **3.233**) BI (28)(10)

2. $$\int_{-\infty}^\infty \left(\frac{1}{\left(1 + e^{-x}\right)^\nu} - \frac{1}{\left(1 + e^{-x}\right)^\mu}\right) dx = \psi(\mu) - \psi(\nu) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$
(cf. **3.219**)
BI (28)(11)

3.318

1. $$\int_0^\infty \frac{\left[\beta+\sqrt{1-e^{-x}}\right]^{-\nu}+\left[\beta-\sqrt{1-e^{-x}}\right]^{-\nu}}{\sqrt{1-e^{-x}}} e^{-\mu x}\,dx = \frac{2^{\mu+1} e^{(\mu-\nu)\pi i}\left(\beta^2-1\right)^{(\mu-\nu)/2}\Gamma(\mu)\,Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)}$$
$[\operatorname{Re}\mu>0]$ ET I 145(18)

2.[7] $$\int_u^\infty \frac{1}{\sqrt{1-e^{-2x}}}\left(e^{-u}\sqrt{1-e^{-2x}}-e^{-x}\sqrt{1-e^{-2u}}\right)^{\nu} e^{-\mu x}\,dx = \frac{2^{-\frac{1}{2}(\mu+\nu)}\sqrt{\pi}e^{-\frac{u}{2}(\mu+\nu)}\,\Gamma(\mu)\,\Gamma(\nu+1)\,P_{-\frac{1}{2}(\mu-\nu)}^{-\frac{1}{2}(\mu+\nu)}\left(\sqrt{1-e^{-2u}}\right)}{\Gamma[(\mu+\nu+1)/2]}$$
$[u>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1]$ ET I 145(19)

3.32–3.34 Exponentials of more complicated arguments

3.321

1.[11] $$\frac{\sqrt{\pi}}{2}\Phi(u)=\frac{\sqrt{\pi}}{2}\operatorname{erf}(u)=\int_0^u e^{-x^2}\,dx=\sum_{k=0}^\infty \frac{(-1)^k u^{2k+1}}{k!(2k+1)} = e^{-u^2}\sum_{k=0}^\infty \frac{2^k u^{2k+1}}{(2k+1)!!}$$
(cf. **8.25**) AD 6.700

2. $$\int_0^u e^{-q^2x^2}\,dx = \frac{\sqrt{\pi}}{2q}\Phi(qu)$$ $[q>0]$

3. $$\int_0^\infty e^{-q^2x^2}\,dx=\frac{\sqrt{\pi}}{2q}$$ $[q>0]$ FI II 624

4.* $$\int_0^u xe^{-q^2x^2}\,dx = \frac{1}{2q^2}\left[1-e^{-q^2u^2}\right]$$

5.* $$\int_0^u x^2e^{-q^2x^2}\,dx = \frac{1}{2q^3}\left[\frac{\sqrt{\pi}}{2}\Phi(qu)-que^{-q^2u^2}\right]$$

6.* $$\int_0^u x^3e^{-q^2x^2}\,dx = \frac{1}{2q^4}\left[1-\left(1+q^2u^2\right)e^{-q^2u^2}\right]$$

7.* $$\int_0^u x^4e^{-q^2x^2}\,dx = \frac{1}{2q^5}\left[\frac{3\sqrt{\pi}}{4}\Phi(qu)-\left(\frac{3}{2}+q^2u^2\right)que^{-q^2u^2}\right]$$

3.322

1.[11] $$\int_u^\infty \exp\left(-\frac{x^2}{4\beta}-\gamma x\right)dx = \sqrt{\pi\beta}e^{\beta\gamma^2}\left[1-\Phi\left(\gamma\sqrt{\beta}+\frac{u}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0]$ ET I 146(21)

2. $$\int_0^\infty \exp\left(-\frac{x^2}{4\beta}-\gamma x\right)dx = \sqrt{\pi\beta}\exp\left(\beta\gamma^2\right)\left[1-\Phi\left(\gamma\sqrt{\beta}\right)\right]$$
$[\operatorname{Re}\beta>0]$ NT 27(1)a

3.[11] $\mathrm{PV}\int_0^\infty e^{\pm i\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}e^{\pm\pi i/4}$ $[\lambda > 0]$ PBM 343 (2.3.15(2))

3.323

1.[11] $\int_1^\infty \exp\left(-qx - x^2\right)\,dx = \frac{\sqrt{\pi}}{2}e^{q^2/4}\left[1 - \Phi\left(1 + \frac{1}{2}q\right)\right]$ BI (29)(4)

2.[10] $\int_{-\infty}^\infty \exp\left(-p^2x^2 \pm qx\right)\,dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p}$ $[\operatorname{Re} p^2 > 0]$ BI (28)(1)

3.[11] $\int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)\,dx = 2^{-\frac{3}{2}}\frac{\gamma}{\beta}e^{\frac{\gamma^4}{2\beta^2}}K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right)$

$\left[|\arg\beta| < \frac{\pi}{4}, \quad |\arg\gamma| < \frac{\pi}{4}\right]$ ET I 147(34)a

3.324

1. $\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right)\,dx = \sqrt{\frac{\beta}{\gamma}}K_1\left(\sqrt{\beta\gamma}\right)$ $[\operatorname{Re}\beta \ge 0, \quad \operatorname{Re}\gamma > 0]$ ET I 146(25)

2.[11] $\int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right]\,dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right)$ $[b \ge 0]$

3.325 $\int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{ab}\right)$ $[a > 0, \quad b > 0]$ FI II 644

3.326

1.[8] $\int_0^\infty \exp\left(-x^\mu\right)\,dx = \frac{1}{\mu}\Gamma\left(\frac{1}{\mu}\right)$ $[\operatorname{Re}\mu > 0]$ BI (26)(4)

2.[10] $\int_0^\infty x^m \exp\left(-\beta x^n\right)\,dx = \frac{\Gamma(\gamma)}{n\beta^\gamma} \qquad \gamma = \frac{m+1}{n}$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re} m > 0, \quad \operatorname{Re} n > 0]$

3.* $\int_0^\infty (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right)}{n\beta^{1/n}}$

$[\operatorname{Re} n > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg b| < \pi]$

4.* $$\int_0^u (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{2}{n}, \beta(u-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}}$$

$[\operatorname{Re} n > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg b| < \pi, \quad |\arg(u-b)| < \pi]$

5.* $\int_u^\infty (x-a)\exp\left(-\beta(x-b)^n\right)\,dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}}$

$[\operatorname{Re} n > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg(u-b)| < \pi]$

Exponentials of exponentials

3.327 $$\int_0^\infty \exp\left(-ae^{nx}\right)\,dx = -\frac{1}{n}\,\mathrm{Ei}(-a)$$ $[n \geq 1, \quad \operatorname{Re} a \geq 0, \quad a \neq 0]$ LI (26)(5)

3.328 $$\int_{-\infty}^\infty \exp\left(-e^{x}\right)e^{\mu x}\,dx = \Gamma(\mu)$$ $[\operatorname{Re}\mu > 0]$ NH 145(14)

3.329 $$\int_0^\infty \left[\frac{a\exp\left(-ce^{ax}\right)}{1-e^{-ax}} - \frac{b\exp\left(-ce^{bx}\right)}{1-e^{-bx}}\right]dx = e^{-c}\ln\frac{b}{a}$$ $[a > 0, \quad b > 0, \quad c > 0]$ BI (27)(12)

3.331

1. $$\int_0^\infty \exp\left(-\beta e^{-x} - \mu x\right)\,dx = \beta^{-\mu}\,\gamma(\mu,\beta)$$ $[\operatorname{Re}\mu > 0]$ ET I 147(36)

2. $$\int_0^\infty \exp\left(-\beta e^{x} - \mu x\right)\,dx = \beta^{\mu}\,\Gamma(-\mu,\beta)$$ $[\operatorname{Re}\beta > 0]$ ET I 147(37)

3.[11] $$\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\exp\left(\beta e^{-x} - \mu x\right)\,dx = \mathrm{B}(\mu,\nu)\beta^{-\frac{\mu+\nu}{2}}e^{\frac{\beta}{2}}\,M_{\frac{\nu-\mu}{2},\frac{\nu+\mu-1}{2}}(\beta)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 147(38)

4. $$\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\exp\left(-\beta e^{x} - \mu x\right)\,dx = \Gamma(\nu)\beta^{\frac{\mu-1}{2}}e^{-\frac{\beta}{2}}\,W_{\frac{1-\mu-2\nu}{2},\frac{-\mu}{2}}(\beta)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$ ET I 147(39)

3.332 $$\int_0^\infty \left(1-e^{-x}\right)^{\nu-1}\left(1-\lambda e^{-x}\right)^{-\varrho}\exp\left(\beta e^{-x} - \mu x\right)\,dx = \mathrm{B}(\mu,\nu)\,\Phi_1(\mu,\varrho,\nu,\lambda,\beta)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad |\arg(1-\lambda)| < \pi]$ ET I 147(40)

3.333

1.[3] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)-1} = \Gamma(\mu)\,\zeta(\mu)$$ $[\operatorname{Re}\mu > 1]$ ET I 121(24)

2.[3] $$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{\exp\left(e^{-x}\right)+1} = \left(1-2^{1-\mu}\right)\Gamma(\mu)\,\zeta(\mu)$$ $[\operatorname{Re}\mu > 0, \quad \mu \neq 1]$
$$= \ln 2$$ $[\mu = 1]$
ET I 121(25)

3.* $$\int_0^\infty \left(\frac{\tanh(x)}{x^3} - \frac{1}{x^2\cosh^2(x)}\right)dx = \frac{7\,\zeta(3)}{\pi^2}$$

3.334[11] $$\int_0^\infty \left(e^x-1\right)^{\nu-1}\exp\left[-\frac{\beta}{e^x-1} - \mu x\right]dx = \Gamma(\mu-\nu+1)e^{\frac{\beta}{2}}\beta^{\frac{\nu-1}{2}}\,W_{\frac{\nu-2\mu-1}{2},-\frac{\nu}{2}}(\beta)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > \operatorname{Re}\nu - 1]$
ET I 137(41)

Exponentials of hyperbolic functions

3.335 $$\int_0^\infty \left(e^{\nu x} + e^{-\nu x}\cos\nu\pi\right)\exp\left(-\beta\sinh x\right)\,dx = -\pi\left[\mathbf{E}_\nu(\beta) + Y_\nu(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ EH II 35(34)

3.336

1. $$\int_0^\infty \exp\left(-\nu x - \beta \sinh x\right)\, dx = \pi \operatorname{cosec} \nu\pi \left[\mathbf{J}_\nu(\beta) - J_\nu(\beta)\right]$$
$$\left[|\arg\beta| < \frac{\pi}{2} \text{ and } |\arg\beta| = \frac{\pi}{2} \text{ for } \operatorname{Re}\nu > 0; \quad \nu \text{ is not an integer}\right]$$ WA 341(2)

2. $$\int_0^\infty \exp\left(nx - \beta \sinh x\right)\, dx = \frac{1}{2}\left[S_n(\beta) - \pi\,\mathbf{E}_n(\beta) - \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0; \quad n = 0, 1, 2, \ldots]$$ WA 342(6)

3. $$\int_0^\infty \exp\left(-nx - \beta \sinh x\right)\, dx = \frac{1}{2}(-1)^{n+1}\left[S_n(\beta) + \pi\,\mathbf{E}_n(\beta) + \pi\, Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0; \quad n = 0, 1, 2, \ldots]$$ EH II 84(47)

3.337

1. $$\int_{-\infty}^\infty \exp\left(-\alpha x - \beta \cosh x\right)\, dx = 2\, K_\alpha(\beta) \qquad \left[|\arg\beta| < \frac{\pi}{2}\right]$$ WA 201(7)

2. $$\int_{-\infty}^\infty \exp\left(-\nu x + i\beta \cosh x\right)\, dx = i\pi e^{\frac{i\nu\pi}{2}}\, H_\nu^{(1)}(\beta) \qquad [0 < \arg z < \pi]$$ EH II 21(27)

3. $$\int_{-\infty}^\infty \exp\left(-\nu x - i\beta \cosh x\right)\, dx = -i\pi e^{-\frac{i\nu\pi}{2}}\, H_\nu^{(2)}(\beta) \qquad [-\pi < \arg z < 0]$$ EH II 21(30)

Exponentials of trigonometric functions and logarithms

3.338

1. $$\int_0^\pi \left\{\exp i\left[(\nu-1)x - \beta\sin x\right] - \exp i\left[(\nu+1)x - \beta \sin x\right]\right\}\, dx = 2\pi\left[\mathbf{J}_\nu'(\beta) + i\,\mathbf{E}_\nu'(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ EH II 36

2. $$\int_0^\pi \exp\left[\pm i\left(\nu x - \beta \sin x\right)\right]\, dx = \pi\left[\mathbf{J}_\nu(\beta) \pm i\,\mathbf{E}_\nu(\beta)\right] \qquad [\operatorname{Re}\beta > 0]$$ EH II 35(32)

3.[10] $$\int_0^\infty \exp\left[-\gamma\left(x - \beta\sin x\right)\right]\, dx = \frac{1}{\gamma} + 2\sum_{k=1}^\infty \frac{\gamma\, J_k(k\beta)}{\gamma^2 + k^2} \qquad [\operatorname{Re}\gamma > 0]$$ WA 619(4)

4.[6] $$\int_{-\pi}^\pi \frac{\exp\left[\dfrac{a + b\sin x + c\cos x}{1 + p\sin x + q\cos x}\right]}{1 + p\sin x + q\cos x}\, dx = \frac{2\pi}{\sqrt{1 - p^2 - q^2}} e^{-\alpha}\, I_0(\beta),$$
$$\text{with } \alpha = \frac{bp + cq - a}{1 - p^2 - q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2 - b^2 - c^2}{1 - p^2 - q^2}}; \qquad \left[p^2 + q^2 < 1\right]$$

5.* $$\int_0^{\pi/4} \exp\left[-\sum_{n=1}^\infty \frac{\tan^{2n} x}{n + \frac{1}{2}}\right]\, dx = \ln\sqrt{2}$$

3.339[6] $$\int_0^{\pi} \exp(z\cos x)\,dx = \pi\, I_0(z)$$ BI (277)(2)a

3.341 $$\int_0^{\frac{\pi}{2}} \exp(-p\tan x)\,dx = \operatorname{ci}(p)\sin p - \operatorname{si}(p)\cos(p) \qquad [p>0]$$ BI (271)(2)a

3.342[11] $$\int_0^1 \exp(-px\ln x)\,dx = \int_0^1 x^{-px}\,dx = \sum_{k=1}^{\infty} \frac{p^{k-1}}{k^k}$$ BI (29)(1)

3.35 Combinations of exponentials and rational functions

3.351

1.[8] $$\int_0^u x^n e^{-\mu x}\,dx = \frac{n!}{\mu^{n+1}} - e^{-u\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1}\,\gamma(n+1, \mu u)$$
$[u>0, \quad \operatorname{Re}\mu > 0, n = 0, 1, 2, \ldots]$ ET I 134(5)

2.[11] $$\int_u^{\infty} x^n e^{-\mu x}\,dx = e^{-u\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} = \mu^{-n-1}\,\Gamma(n+1, \mu u)$$
$[u>0, \quad \operatorname{Re}\mu > 0, n = 0, 1, 2, \ldots]$ ET I 33(4)

3. $$\int_0^{\infty} x^n e^{-\mu x}\,dx = n!\mu^{-n-1} \qquad [\operatorname{Re}\mu > 0]$$ ET I 133(3)

4. $$\int_u^{\infty} \frac{e^{-px}\,dx}{x^{n+1}} = (-1)^{n+1} \frac{p^n \operatorname{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n} \sum_{k=0}^{n-1} \frac{(-1)^k p^k u^k}{n(n-1)\ldots(n-k)}$$
$[p>0]$ NT 21(3)

5. $$\int_1^{\infty} \frac{e^{-\mu x}\,dx}{x} = -\operatorname{Ei}(-\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (104)(10)

6. $$\int_{-\infty}^{u} \frac{e^x}{x}\,dx = \operatorname{li}(e^u) = \operatorname{Ei}(u) \qquad [u<0]$$

7.[9] $$\int_0^u x e^{-\mu x}\,dx = \frac{1}{\mu^2} - \frac{1}{\mu^2} e^{-\mu u}(1+\mu u) \qquad [u>0]$$

8.[11] $$\int_0^u x^2 e^{-\mu x}\,dx = \frac{2}{\mu^3} - \frac{1}{\mu^3} e^{-\mu u}\left(2 + 2\mu u + \mu^2 u^2\right) \qquad [u>0]$$

9.[7] $$\int_0^u x^3 e^{-\mu x}\,dx = \frac{6}{\mu^4} - \frac{1}{\mu^4} e^{-\mu u}\left(6 + 6\mu u + 3\mu^2 u^2 + \mu^3 u^3\right)$$
$[u>0]$

3.352

1. $$\int_0^u \frac{e^{-\mu x}\,dx}{x+\beta} = e^{\mu\beta}\left[\operatorname{Ei}(-\mu u - \mu\beta) - \operatorname{Ei}(-\mu\beta)\right] \qquad [u \ge 0, \quad |\arg\beta| < \pi]$$ ET II 217(12)

2. $$\int_u^{\infty} \frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu} \operatorname{Ei}(-\mu u - \mu\beta) \qquad [u \ge 0, \quad |\arg(u+\beta)| < \pi, \quad \operatorname{Re}\mu > 0]$$
ET I 134(6), JA

3. $\displaystyle\int_u^v \frac{e^{-\mu x}\,dx}{x+\alpha} = e^{\alpha\mu}\left\{\operatorname{Ei}[-(\alpha+v)\mu] - \operatorname{Ei}[-(\alpha+u)\mu]\right\}$ $[-\alpha < n, \text{ and } -\alpha > v, \quad \operatorname{Re}\mu > 0]$
ET I 134 (7)

4. $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{x+\beta} = -e^{\beta\mu}\operatorname{Ei}(-\mu\beta)$ $[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$ ET II 217(11)

5.[7] $\displaystyle\int_u^\infty \frac{e^{-px}\,dx}{a-x} = e^{-pa}\operatorname{Ei}(pa-pu)$
$\left[p > 0, \quad a < u; \text{ for } a > u, \text{ one should replace } \operatorname{Ei}(pa-pu) \text{ in this formula with } \overline{\operatorname{Ei}}(pa-pu)\right]$
ET II 251(37)

6.[8] $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{a-x} = e^{-\mu a}\operatorname{Ei}(a\mu)$
$[a < 0, \quad \operatorname{Re}\mu > 0]$ BI (91)(4)

7. $\displaystyle\int_{-\infty}^\infty \frac{e^{ipx}\,dx}{x-a} = i\pi e^{iap}$ $[p > 0]$ ET II 251(38)

3.353

1. $\displaystyle\int_u^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = e^{-u\mu}\sum_{k=1}^{n-1}\frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\operatorname{Ei}[-(u+\beta)\mu]$
$[n \ge 2, \quad |\arg(u+\beta)| < \pi, \quad \operatorname{Re}\mu > 0]$
ET I 134(10)

2.[7] $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{(x+\beta)^n} = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(k-1)!(-\mu)^{n-k-1}\beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!}e^{\beta\mu}\operatorname{Ei}(-\beta\mu)$
$[n \ge 2, \quad |\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$
ET I 134(9), BI (92)(2)

3. $\displaystyle\int_0^\infty \frac{e^{-px}\,dx}{(a+x)^2} = pe^{\alpha p}\operatorname{Ei}(-ap) + \frac{1}{a}$ $[p > 0, \quad a > 0]$
LI (281)(28), LI (281)(29)

4. $\displaystyle\int_0^1 \frac{xe^x}{(1+x)^2}\,dx = \frac{e}{2} - 1.$ BI (80)(6)

5.[7] $\displaystyle\int_0^\infty \frac{x^n e^{-\mu x}}{x+\beta}\,dx = (-1)^{n-1}\beta^n e^{\beta\mu}\operatorname{Ei}(-\beta\mu) + \sum_{k=1}^{n}(k-1)!(-\beta)^{n-k}\mu^{-k}$
$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$
BI (91)(3)a, LET I 135(11)

3.354

1. $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2+x^2} = \frac{1}{\beta}\left[\operatorname{ci}(\beta\mu)\sin\beta\mu - \operatorname{si}(\beta\mu)\cos\beta\mu\right]$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ BI (91)(7)

2. $\displaystyle\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2+x^2} = -\operatorname{ci}(\beta\mu)\cos\beta\mu - \operatorname{si}(\beta\mu)\sin\beta\mu$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ BI (91)(8)

3.[7] $\displaystyle\int_0^\infty \frac{e^{-\mu x}\,dx}{\beta^2-x^2} = \frac{1}{2\beta}\left[e^{-\beta\mu}\operatorname{Ei}(\beta\mu) - e^{\beta\mu}\operatorname{Ei}(-\beta\mu)\right]$ $[|\arg(\pm\beta)| < \pi, \quad \operatorname{Re}\mu > 0]$ BI (91)(14)

4. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\beta^2 - x^2} = \frac{1}{2}\left[e^{-\beta\mu}\operatorname{Ei}(\beta\mu) + e^{\beta\mu}\operatorname{Ei}(-\beta\mu)\right]$$
$\left[|\arg(\pm\beta)| < \pi, \quad \operatorname{Re}\mu > 0;\ \text{for } \beta > 0 \text{ one should replace } \operatorname{Ei}(\beta\mu) \text{ in this formula with } \overline{\operatorname{Ei}}(\beta\mu)\right]$ BI (91)(15)

5.[8] $$\int_{-\infty}^\infty \frac{e^{-ipx}\,dx}{a^2 + x^2} = \frac{\pi}{a}e^{-|ap|}$$ $[a \neq 0, \quad p \text{ real}]$ ET I 118(1)a

3.355

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\left(\beta^2 + x^2\right)^2} = \frac{1}{2\beta^3}\left\{\operatorname{ci}(\beta\mu)\sin\beta\mu - \operatorname{si}(\beta\mu)\cos\beta\mu\right\} - \beta\mu\left[\operatorname{ci}(\beta\mu)\cos\beta\mu + \operatorname{si}(\beta\mu)\sin\beta\mu\right]$$ LI (92)(6)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\left(\beta^2 + x^2\right)^2} = \frac{1}{2\beta^2}\left\{-\beta\mu\left[\operatorname{ci}(\beta\mu)\sin\beta\mu - \operatorname{si}(\beta\mu)\cos\beta\mu\right]\right\}$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ BI (92)(7)

3.[3] $$\int_0^\infty \frac{e^{-px}\,dx}{\left(a^2 - x^2\right)^2} = \frac{1}{4a^3}\left[(ap-1)e^{ap}\operatorname{Ei}(-ap) + (1+ap)e^{-ap}\operatorname{Ei}(ap)\right]$$
$\left[\operatorname{Im}\left(a^2\right) > 0, \quad p > 0\right]$ BI (92)(8)

4.[3] $$\int_0^\infty \frac{xe^{-px}\,dx}{\left(a^2 - x^2\right)^2} = \frac{1}{4a^2}\left\{-2 + ap\left[e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap)\right]\right\}$$
$\left[\operatorname{Im}\left(a^2\right) > 0, \quad p > 0\right]$ LI (92)(9)

3.356

1. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2 + x^2}\,dx = (-1)^{n-1}a^{2n}\left[\operatorname{ci}(ap)\cos ap + \operatorname{si}(ap)\sin ap\right] + \frac{1}{p^{2n}}\sum_{k=1}^{n}(2n-2k+1)!\left(-a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(12)

2. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2 + x^2}\,dx = (-1)^n a^{2n-1}\left[\operatorname{ci}(ap)\sin ap - \operatorname{si}(ap)\cos ap\right] + \frac{1}{p^{2n-1}}\sum_{k=1}^{n}(2n-2k)!\left(-a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(11)

3. $$\int_0^\infty \frac{x^{2n+1}e^{-px}}{a^2 - x^2}\,dx = \frac{1}{2}a^{2n}\left[e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap)\right] - \frac{1}{p^{2n}}\sum_{k=1}^{n}(2n-2k+1)!\left(a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(17)

4. $$\int_0^\infty \frac{x^{2n}e^{-px}}{a^2 - x^2}\,dx = \frac{1}{2}a^{2n-1}\left[e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap)\right] - \frac{1}{p^{2n-1}}\sum_{k=1}^{n}(2n-2k)!\left(a^2p^2\right)^{k-1}$$
$[p > 0]$ BI (91)(16)

3.357

1. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a^2}\left\{\operatorname{ci}(a\mu)\left(\sin a\mu+\cos a\mu\right) + \operatorname{si}(a\mu)\left(\sin a\mu-\cos a\mu\right) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(18)

2. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2a}\left\{\operatorname{ci}(a\mu)\left(\sin a\mu-\cos a\mu\right) - \operatorname{si}(a\mu)\left(\sin a\mu+\cos a\mu\right) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(19)

3. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3+a^2x+ax^2+x^3} = \frac{1}{2}\left\{-\operatorname{ci}(a\mu)\left(\sin a\mu+\cos a\mu\right) - \operatorname{si}(a\mu)\left(\sin a\mu-\cos a\mu\right) - e^{a\mu}\operatorname{Ei}(-a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(20)

4. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a^2}\left\{\operatorname{ci}(a\mu)\left(\sin a\mu-\cos a\mu\right) - \operatorname{si}(a\mu)\left(\sin a\mu+\cos a\mu\right) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(21)

5. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2a}\left\{-\operatorname{ci}(a\mu)\left(\sin a\mu+\cos a\mu\right) - \operatorname{si}(a\mu)\left(\sin a\mu-\cos a\mu\right) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(22)

6. $$\int_0^\infty \frac{x^2e^{-\mu x}\,dx}{a^3-a^2x+ax^2-x^3} = \frac{1}{2}\left\{\operatorname{ci}(a\mu)\left(\cos a\mu-\sin a\mu\right) + \operatorname{si}(a\mu)\left(\cos a\mu+\sin a\mu\right) + e^{-a\mu}\operatorname{Ei}(a\mu)\right\}$$
$[\operatorname{Re}\mu>0, \quad a>0]$ BI (92)(23)

3.358

1. $$\int_0^\infty \frac{e^{-px}}{a^4-x^4}\,dx = \frac{1}{4a^3}\left\{e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) + 2\operatorname{ci}(ap)\sin ap - 2\operatorname{si}(ap)\cos ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(18)

2. $$\int_0^\infty \frac{xe^{-px}\,dx}{a^4-x^4} = \frac{1}{4a^2}\left\{e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) - 2\operatorname{ci}(ap)\cos ap - 2\operatorname{si}(ap)\sin ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(19)

3. $$\int_0^\infty \frac{x^2e^{-px}\,dx}{a^4-x^4} = \frac{1}{4a}\left\{e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) - 2\operatorname{ci}(ap)\sin ap + 2\operatorname{si}(ap)\cos ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(20)

4. $$\int_0^\infty \frac{x^3e^{-px}\,dx}{a^4-x^4} = \frac{1}{4}\left\{e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) + 2\operatorname{ci}(ap)\cos ap + 2\operatorname{si}(ap)\sin ap\right\}$$
$[p>0, \quad a>0]$ BI (91)(21)

5. $$\int_0^\infty \frac{x^{4n} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4} a^{4n-3} \left[e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) + 2\operatorname{ci}(ap)\sin ap - 2\operatorname{si}(ap)\cos ap\right] - \frac{1}{p^{4n-3}} \sum_{k=1}^{n} (4n-4k)!\left(a^4 p^4\right)^{k-1}$$
$[p > 0, \quad a > 0]$ BI (91)(22)

6. $$\int_0^\infty \frac{x^{4n+1} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4} a^{4n-2} \left[e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) - 2\operatorname{ci}(ap)\cos ap - 2\operatorname{si}(ap)\sin ap\right] - \frac{1}{p^{4n-2}} \sum_{k=1}^{n} (4n-4k+1)!\left(a^4 p^4\right)^{k-1}$$
$[p > 0, \quad a > 0]$ BI (91)(23)

7. $$\int_0^\infty \frac{x^{4n+2} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4} a^{4n-1} \left[e^{-ap}\operatorname{Ei}(ap) - e^{ap}\operatorname{Ei}(-ap) - 2\operatorname{ci}(ap)\sin ap + 2\operatorname{si}(ap)\cos ap\right] - \frac{1}{p^{4n-1}} \sum_{k=1}^{n} (4n-4k+2)!\left(a^4 p^4\right)^{k-1}$$
$[p > 0, \quad a > 0]$ BI (91)(24)

8. $$\int_0^\infty \frac{x^{4n+3} e^{-px}}{a^4 - x^4}\,dx = \frac{1}{4} a^{4n} \left[e^{ap}\operatorname{Ei}(-ap) + e^{-ap}\operatorname{Ei}(ap) + 2\operatorname{ci}(ap)\cos ap + 2\operatorname{si}(ap)\sin ap\right] - \frac{1}{p^{4n}} \sum_{k=1}^{n} (4n-4k+3)!\left(a^4 p^4\right)^{k-1}$$
$[p > 0, \quad a > 0]$ BI (91)(25)

3.359 $$\int_{-\infty}^\infty \frac{(i-x)^n}{(i+x)^n} \frac{e^{-ipx}}{i+x^2}\,dx = (-1)^{n-1} 2\pi p e^{-p} L_{n-1}(2p) \qquad \text{for } p > 0;$$
$$= 0 \qquad \text{for } p < 0.$$
ET I 118(2)

3.36–3.37 Combinations of exponentials and algebraic functions

3.361

1.[8] $$\int_0^u \frac{e^{-qx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{q}}\,\Phi\left(\sqrt{qu}\right)$$ $[q > 0]$

2.[8] $$\int_0^\infty \frac{e^{-qx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{q}}$$ $[q > 0]$ BI(98)(10)

3.[8] $$\int_{-1}^\infty \frac{e^{-qx}}{\sqrt{1+x}}\,dx = e^q\sqrt{\frac{\pi}{q}}$$ $[q > 0]$ BI (104)(16)

3.362

1. $$\int_1^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x-1}} = \sqrt{\frac{\pi}{\mu}} e^{-\mu}$$ $[\operatorname{Re}\mu > 0]$ BI (104)(11)a

2. $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{\mu}} e^{\beta\mu}\left[1 - \Phi\left(\sqrt{\beta\mu}\right)\right]$$ $[\operatorname{Re}\mu > 0, \quad |\arg\beta| < \pi]$ ET I 135(18)

3.363

1. $$\int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x}\,dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u}\left[1-\Phi\left(\sqrt{u\mu}\right)\right]$$ $[\operatorname{Re}\mu > 0]$ ET I 136(23)

2. $$\int_u^\infty \frac{e^{-\mu x}\,dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}}\left[1-\Phi\left(\sqrt{u\mu}\right)\right]$$ $[u>0, \quad \operatorname{Re}\mu \ge 0]$ ET I 136(26)

3.364

1. $$\int_0^2 \frac{e^{-px}\,dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p)$$ $[p>0]$ GW (312)(7a)

2. $$\int_{-1}^1 \frac{e^{2x}\,dx}{\sqrt{1-x^2}} = \pi I_0(2)$$ BI (277)(2)a

3. $$\int_0^\infty \frac{e^{-px}\,dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}} K_0\left(\frac{ap}{2}\right)$$ $[a>0, \quad p>0]$ GW (312)(8a)

3.365

1. $$\int_0^u \frac{xe^{-\mu x}\,dx}{\sqrt{u^2-x^2}} = \frac{\pi u}{2}\left[\mathbf{L}_1(\mu u) - I_1(\mu u)\right] + u$$ $[u>0, \quad \operatorname{Re}\mu > 0]$ ET I 136(28)

2. $$\int_u^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2-u^2}} = u K_1(u\mu)$$ $[u>0, \quad \operatorname{Re}\mu > 0]$ ET I 136(29)

3.366

1. $$\int_0^{2u} \frac{(u-x)e^{-\mu x}\,dx}{\sqrt{2ux-x^2}} = \pi u e^{-u\mu} I_1(u\mu)$$ $[\operatorname{Re}\mu > 0]$ ET I 136(31)

2. $$\int_0^\infty \frac{(x+\beta)e^{-\mu x}\,dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu)$$ $[\operatorname{Re}\mu > 0, \quad |\arg\beta| < \pi]$ ET I 136(30)

3. $$\int_0^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2}\left[\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)\right] - \beta$$ $\left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 136(27)

3.367 $$\int_0^\infty \frac{e^{-\mu x}\,dx}{(1+\cos t + x)\sqrt{x^2+2x}} = \frac{\exp\left(2\mu\cos^2\frac{t}{2}\right)}{\sin t}\left(t - \sin t\int_0^u K_0(v) e^{-v\cos t}\,dv\right)$$ $[\operatorname{Re}\mu > 0]$ ET I 136(33)

3.368 $$\int_0^\infty \frac{e^{-\mu x}\,dx}{x+\sqrt{x^2+\beta^2}} = \frac{\pi}{2\beta\mu}\left[\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)\right] - \frac{1}{\beta^2\mu^2}$$ $\left[|\arg\beta| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 136(32)

3.369[11] $$\int_0^\infty \frac{e^{-\mu x}\,dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu} e^{a\mu}\left(1-\Phi\left(\sqrt{a\mu}\right)\right)$$ $[|\arg a| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 135(20)

3.371[11] $$\int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x}\,dx = \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\mu^{-n-\frac{1}{2}}$$
$$= \sqrt{\pi}2^{-n}\mu^{-n-1/2}(2n-1)!!$$ $[n \ge 0]$

$[\operatorname{Re}\mu > 0]$ ET I 135(17)

3.372 $$\int_0^\infty x^{n-\frac{1}{2}}(2+x)^{n-\frac{1}{2}}e^{-px}\,dx = \frac{(2n-1)!!}{p^n}e^p\,K_n(p) \qquad [p>0, \quad n=0,1,2,\ldots]$$ GW (312)(8)

3.373 $$\int_0^\infty \left[\left(x+\sqrt{x^2+\beta^2}\right)^n + \left(x-\sqrt{x^2+\beta^2}\right)^n\right]e^{-\mu x}\,dx = 2\beta^{n+1}\,O_n(\beta\mu)$$ $[\operatorname{Re}\mu>0]$ WA 05(1)

3.374

1. $$\int_0^\infty \frac{\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \frac{1}{2}\left[S_n(\mu)-\pi\,\mathbf{E}_n(\mu)-\pi\,Y_n(\mu)\right]$$ $[\operatorname{Re}\mu>0]$ ET I 37(35)

2. $$\int_0^\infty \frac{\left(x-\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}e^{-\mu x}\,dx = -\frac{1}{2}\left[S_n(\mu)+\pi\,\mathbf{E}_n(\mu)+\pi\,Y_n(\mu)\right]$$ $[\operatorname{Re}\mu>0]$ ET I 137(36)

3.38–3.39 Combinations of exponentials and arbitrary powers

3.381

1. $$\int_0^u x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\,\gamma(\nu,\mu u) \qquad [\operatorname{Re}\nu>0]$$ EH I 266(22), EH II 133(1)

2. $$\int_0^u x^{p-1}e^{-x}\,dx = \sum_{k=0}^\infty (-1)^k\frac{u^{p+k}}{k!(p+k)} = e^{-u}\sum_{k=0}^\infty \frac{u^{p+k}}{p(p+1)\ldots(p+k)}$$ AD 6.705

3.[8] $$\int_u^\infty x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\,\Gamma(\nu,\mu u) \qquad [u>0, \quad \operatorname{Re}\mu>0]$$ EH I 256(21), EH II 133(2)

4. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\mu^\nu}\,\Gamma(\nu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ FI II 779

5. $$\int_0^\infty x^{\nu-1}e^{-(p+iq)x}\,dx = \Gamma(\nu)\left(p^2+q^2\right)^{-\frac{\nu}{2}}\exp\left(-i\nu\arctan\frac{q}{p}\right)$$ $[p>0, \quad \operatorname{Re}\nu>0 \text{ and } p=0, \quad 0<\operatorname{Re}\nu<1]$ EH I 12(32)

6. $$\int_u^\infty \frac{e^{-x}}{x^\nu}\,dx = u^{-\frac{\nu}{2}}e^{-\frac{u}{2}}\,W_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(u) \qquad [u>0]$$ WH

7. $$\int_0^\infty x^{k-1}e^{i\mu x}\,dx = \frac{\Gamma(k)}{(-i\mu)^k} \qquad [0<\operatorname{Re}(k)<1, \quad \mu\neq 0]$$ GH2 62 (313.14)

8.* $$\int_0^u x^m e^{-\beta x^n}\,dx = \frac{\gamma\left(v,\beta u^n\right)}{n\beta^v} \qquad v=\frac{m+1}{n} \qquad [u>0, \quad \operatorname{Re}v>0, \quad \operatorname{Re}n>0, \quad \operatorname{Re}\beta>0]$$

9.* $$\int_u^\infty x^m e^{-\beta x^n}\,dx = \frac{\Gamma\left(v,\beta u^n\right)}{n\beta^v} \qquad v=\frac{m+1}{n} \qquad [u>0, \quad \operatorname{Re}v>0, \quad \operatorname{Re}n>0, \quad \operatorname{Re}\beta>0]$$

10.* $$\int_0^\infty x^m e^{-\beta x^n}\,dx = \frac{\gamma\left(v, \beta u^n\right) + \Gamma\left(v, \beta u^n\right)}{n\beta^v}$$

$$v = \frac{m+1}{n} \qquad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$ See also 3.326 1

11.* $$\int_{-\infty}^\infty x^{2m} e^{-\beta x^{2n}}\,dx = 2\int_0^\infty x^{2m} e^{-\beta x^{2n}}\,dx = \frac{2\left(\gamma\left(v, \beta u^n\right) + \Gamma\left(v, \beta u^n\right)\right)}{n\beta^v} = \frac{\Gamma(v)}{n\beta^v}$$

$$v = \frac{2m+1}{2n} \qquad [u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} n > 0, \quad \operatorname{Re} \beta > 0]$$

3.382

1.[6] $$\int_0^u (u-x)^\nu e^{-\mu x}\,dx = (-\mu)^{-\nu-1} e^{-u\mu}\,\gamma(\nu+1, -u\mu) \qquad [\operatorname{Re}\nu > -1, \quad u > 0]$$ ET I 137(6)

2. $$\int_u^\infty (x-u)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{-u\mu}\,\Gamma(\nu+1) \qquad [u > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0]$$ ET I 137(5), ET II 202(11)

3. $$\int_0^\infty (1+x)^{-\nu} e^{-\mu x}\,dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}}\; W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(\mu) \qquad [\operatorname{Re}\mu > 0]$$ WH

4. $$\int_0^\infty (x+\beta)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{\beta\mu}\,\Gamma(\nu+1, \beta\mu) \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$$ ET I 137(4), ET II 233(10)

5. $$\int_0^u (a+x)^{\mu-1} e^{-x}\,dx = e^a[\gamma(\mu, a+u) - \gamma(\mu, a)] \qquad [\operatorname{Re}\mu > 0]$$ EH II 139

6. $$\int_{-\infty}^\infty (\beta+ix)^{-\nu} e^{-ipx}\,dx = 0 \qquad [\text{for } p > 0]$$

$$= \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)} \qquad [\text{for } p < 0]$$

$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0]$$ ET I 118(4)

7. $$\int_{-\infty}^\infty (\beta-ix)^{-\nu} e^{-ipx}\,dx = \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)} \qquad [\text{for } p > 0]$$

$$= 0 \qquad [\text{for } p < 0]$$

$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0]$$ ET I 118(3)

3.383

1.[11] $$\int_0^u x^{\nu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \mathrm{B}(\mu, \nu) u^{\mu+\nu-1}\;{}_1F_1(\nu; \mu+\nu; \beta u)$$

$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 187(14)

2.[11] $$\int_0^u x^{\mu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \sqrt{\pi}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right)\Gamma(\mu)\, I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

$$[\operatorname{Re}\mu > 0]$$ ET II 187(13)

3. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1} e^{-\beta x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta u > 0]$$ ET II 202(12)

4.[11] $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{-\beta x}\,dx = \beta^{-\frac{\mu+\nu}{2}} u^{\frac{\mu+\nu-2}{2}}\,\Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right) W_{\frac{\nu-\mu}{2},\frac{1-\mu-\nu}{2}}(\beta u)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta u > 0]$ ET II 202(13)

5.[11] $$\int_0^\infty e^{-px}x^{q-1}(1+ax)^{-\nu}\,dx$$
$$= \frac{\pi^2}{p^q\,\Gamma(\nu)\sin[\pi(q-\nu)]}\left[\left(\frac{p}{a}\right)^\nu \frac{L_{-\nu}^{\nu-q}\left(\frac{p}{a}\right)}{\sin(\pi\nu)\,\Gamma(1-q)} - \left(\frac{p}{a}\right)^q \frac{L_{-q}^{q-\nu}\left(\frac{p}{a}\right)}{\sin(\pi q)\,\Gamma(1-\nu)}\right] \qquad [\nu \neq \pm 1, \pm 2, \ldots]$$
$$= \frac{\Gamma(q)}{p^q} \qquad [\nu = 0]$$

$[\operatorname{Re} q > 0, \quad \operatorname{Re} p > 0, \quad \operatorname{Re} a > 0]$

6. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu+\frac{1}{2}}e^{-\mu x}\,dx = 2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\mu^{-\frac{1}{2}}e^{\frac{\beta\mu}{2}}\,D_{1-2\nu}\left(\sqrt{2\beta\mu}\right)$$

$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu \geq 0, \quad \mu \neq 0]$ ET I 39(20), EH II 119(2)a

7. $$\int_0^\infty x^{\nu-1}(x+\beta)^{-\nu-\frac{1}{2}}e^{-\mu x}\,dx = 2^\nu\,\Gamma(\nu)\beta^{-\frac{1}{2}}e^{\frac{\beta\mu}{2}}\,D_{-2\nu}\left(\sqrt{2\beta\mu}\right)$$

$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu \geq 0]$
ET I 139(21), EH II 119(1)a

8. $$\int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}}\,\Gamma(\nu)\,K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right)$$

$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
ET II 233(11), EH II 19(16)a, EH II 82(22)a

9. $$\int_u^\infty \frac{(x-u)^\nu e^{-\mu x}}{x}\,dx = u^\nu\,\Gamma(\nu+1)\,\Gamma(-\nu, u\mu) \qquad [u > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0]$$

ET I 138(8)

10. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{x+\beta}\,dx = \beta^{\nu-1}e^{\beta\mu}\,\Gamma(\nu)\,\Gamma(1-\nu, \beta\mu) \qquad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$

EH II 137(3)

3.384

1. $$\int_{-1}^1 (1-x)^{\nu-1}(1+x)^{\mu-1}e^{-ipx}\,dx = 2^{\mu+\nu-1}\,\mathrm{B}(\mu,\nu)e^{ip}\;{}_1F_1(\mu;\nu+\mu;-2ip)$$

$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0]$ ET I 119(13)

2. $$\int_u^v (x-u)^{2\mu-1}(v-x)^{2\nu-1}e^{-px}\,dx$$
$$= \mathrm{B}(2\mu, 2\nu)(v-u)^{\mu+\nu-1}p^{-\mu-\nu}\exp\left(-p\frac{u+v}{2}\right) M_{\mu-\nu,\mu+\nu-\frac{1}{2}}(vp-up)$$

$[v > u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 139(23)

3. $$\int_u^\infty (x+\beta)^{2\nu-1}(x-u)^{2\varrho-1}e^{-\mu x}\,dx = \frac{(u+\beta)^{\nu+\varrho-1}}{\mu^{\nu+\varrho}}\exp\left[\frac{(\beta-u)\mu}{2}\right]\Gamma(2\varrho)\,W_{\nu-\varrho,\nu+\varrho-\frac{1}{2}}(u\mu+\beta\mu)$$

$[u > 0, \quad |\arg(\beta+u)| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\varrho > 0]$ ET I 139(22)

4. $$\int_u^\infty (x+\beta)^{\nu}(x-u)^{-\nu}e^{-\mu x}\,dx = \frac{1}{\mu}\nu\pi\operatorname{cosec}(\nu\pi)e^{-\frac{(\beta+u)\mu}{2}}\,\mathrm{k}_{2\nu}\left[\frac{(\beta+u)\mu}{2}\right]$$

$[\nu \neq 0, \quad u > 0, \quad |\arg(u+\beta)| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu < 1]$ ET I 139(17)

5. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu+\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{\mu}}2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\,D_{1-2\nu}\left(2\sqrt{u\mu}\right)$$

$[u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$

ET I 139(18)

6. $$\int_u^\infty (x-u)^{\nu-1}(x+u)^{-\nu-\frac{1}{2}}e^{-\mu x}\,dx = \frac{1}{\sqrt{u}}2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\,D_{-2\nu}\left(2\sqrt{u\mu}\right)$$

$[u > 0, \quad \operatorname{Re}\mu \geq 0, \quad \operatorname{Re}\nu > 0]$

ET I 139(19)

7.[6] $$\int_{-\infty}^\infty (\beta-ix)^{-\mu}\,(\gamma-ix)^{-\nu}e^{-ipx}\,dx = \frac{2\pi e^{-\beta p}p^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\;{}_1F_1(\nu;\mu+\nu;(\beta-\gamma)p) \qquad [\text{for } p > 0]$$

$$= 0 \qquad [\text{for } p < 0]$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}(\mu+\nu) > 1]$ ET I 119(10)

8.[6] $$\int_{-\infty}^\infty (\beta+ix)^{-\mu}(\gamma+ix)^{-\nu}e^{-ipx}\,dx = 0 \qquad [\text{for } p > 0]$$

$$= \frac{2\pi e^{\gamma p}(-p)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\;{}_1F_1[\mu;\mu+\nu;(\beta-\gamma)p] \qquad [\text{for } p < 0]$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}(\mu+\nu) > 1]$ ET I 19(11)

9.[6] $$\int_{-\infty}^\infty (\beta+ix)^{-2\mu}(\gamma-ix)^{-2\nu}e^{-ipx}\,dx$$

$$= 2\pi(\beta+\gamma)^{-\mu-\nu}\frac{p^{\mu+\nu-1}}{\Gamma(2\nu)}\exp\left(\frac{\beta-\gamma}{2}p\right)W_{\nu-\mu,\frac{1}{2}-\nu-\mu}(\beta p+\gamma p) \qquad [\text{for } p > 0]$$

$$= 2\pi(\beta+\gamma)^{-\mu-\nu}\frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)}\exp\left(\frac{\beta-\gamma}{2}p\right)W_{\mu-\nu,\frac{1}{2}-\nu-\mu}(-\beta p-\gamma p) \qquad [\text{for } p < 0]$$

$\left[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}(\mu+\nu) > \tfrac{1}{2}\right]$ ET I 19(12)

3.385[11] $$\int_0^1 x^{\nu-1}(1-x)^{\lambda-1}(1-\beta x)^{-\varrho}e^{-\mu x}\,dx = \mathrm{B}(\nu,\lambda)\Phi_1(\nu,\varrho,\lambda+\nu,-\mu,\beta)$$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > 0, \quad |\arg(1-\beta)| < \pi]$ ET I 39(24)

3.386

1. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^{n} (\beta_k + ix)^{\nu_k} e^{-ipx}\, dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p} \beta_0^{\nu_0} \prod_{k=1}^{n} (\beta_0 + \beta_k)^{\nu_k}$$

$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n} \operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x, \quad p > 0\right]$$ ET I 118(8)

2. $$\int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^{n} (\beta_k + ix)^{\nu_k} e^{-ipx}\, dx}{\beta_0 + ix} = 0$$

$$\left[\operatorname{Re}\nu_0 > -1, \quad \operatorname{Re}\beta_k > 0, \quad \sum_{k=0}^{n} \operatorname{Re}\nu_k < 1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x, \quad p > 0\right]$$ ET I 119(9)

3.387

1.[6] $$\int_{-1}^{1} \left(1 - x^2\right)^{\nu-1} e^{-\mu x}\, dx = \sqrt{\pi} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, I_{\nu-\frac{1}{2}}(\mu)$$

$$\left[\operatorname{Re}\nu > 0, \quad |\arg\mu| < \frac{\pi}{2}\right]$$ WA 172(2)a

2.[6] $$\int_{-1}^{1} \left(1 - x^2\right)^{\nu-1} e^{i\mu x}\, dx = \sqrt{\pi} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, J_{\nu-\frac{1}{2}}(\mu)$$

$$[\operatorname{Re}\nu > 0]$$ WA 25(3), WA 48(4)a

3. $$\int_{1}^{\infty} \left(x^2 - 1\right)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, K_{\nu-\frac{1}{2}}(\mu)$$

$$\left[|\arg\mu| < \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right]$$ WA 190(4)a

4. $$\int_{1}^{\infty} \left(x^2 - 1\right)^{\nu-1} e^{i\mu x}\, dx$$

$$= i\frac{\sqrt{\pi}}{2} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, H^{(1)}_{\frac{1}{2}-\nu}(\mu) \qquad [\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(28)a

$$= -i\frac{\sqrt{\pi}}{2} \left(-\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, H^{(2)}_{\frac{1}{2}-\nu}(-\mu) \qquad [\operatorname{Im}\mu < 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(29)a

5. $$\int_{0}^{u} \left(u^2 - x^2\right)^{\nu-1} e^{\mu x}\, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) \left[I_{\nu-\frac{1}{2}}(u\mu) + \mathbf{L}_{\nu-\frac{1}{2}}(u\mu)\right]$$

$$[u > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 188(20)a

6. $$\int_{u}^{\infty} \left(x^2 - u^2\right)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\, K_{\nu-\frac{1}{2}}(u\mu)$$

$$[u > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 203(17)a

7.[11] $$\int_0^\infty \left(x^2+u^2\right)^{\nu-1} e^{-\mu x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu)\left[\mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - Y_{\nu-\frac{1}{2}}(u\mu)\right]$$

$[|\arg u| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 138(10)

3.388

1. $$\int_0^{2u} \left(2ux - x^2\right)^{\nu-1} e^{-\mu x}\,dx = \sqrt{\pi}\left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} e^{-u\mu}\,\Gamma(\nu)\,I_{\nu-\frac{1}{2}}(u\mu)$$

$[u > 0, \quad \operatorname{Re}\nu > 0]$ ET I 138(14)

2. $$\int_0^\infty \left(2\beta x + x^2\right)^{\nu-1} e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\beta\mu}\,\Gamma(\nu)\,K_{\nu-\frac{1}{2}}(\beta\mu)$$

$[|\arg\beta| < \pi, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0]$ ET I 138(13)

3. $$\int_0^\infty \left(x^2 + ix\right)^{\nu-1} e^{-\mu x}\,dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\,\Gamma(\nu)\,H^{(2)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 138(15)

4. $$\int_0^\infty \left(x^2 - ix\right)^{\nu-1} e^{-\mu x}\,dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\,\Gamma(\nu)\,H^{(1)}_{\nu-\frac{1}{2}}\left(\frac{\mu}{2}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 138(16)

3.389

1. $$\int_0^u x^{2\nu-1}\left(u^2 - x^2\right)^{\varrho-1} e^{\mu x}\,dx = \frac{1}{2}\,\mathrm{B}(\nu,\varrho)u^{2\nu+2\varrho-2}\;{}_1F_2\left(\nu;\frac{1}{2},\nu+\varrho;\frac{\mu^2u^2}{4}\right) + \frac{\mu}{2}\,\mathrm{B}\left(\nu+\frac{1}{2},\varrho\right)u^{2\nu+2\varrho-1}\;{}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\nu+\varrho+\frac{1}{2};\frac{\mu^2u^2}{4}\right)$$

$[\operatorname{Re}\varrho > 0, \quad \operatorname{Re}\nu > 0]$ ET II 188(21)

2.[7] $$\int_0^\infty x^{2\nu-1}\left(u^2 + x^2\right)^{\varrho-1} e^{-\mu x}\,dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi}\,\Gamma(1-\varrho)}\,G^{31}_{13}\left(\frac{\mu^2u^2}{4}\,\middle|\,\begin{matrix}1-\nu\\ 1-\varrho-\nu,0,\frac{1}{2}\end{matrix}\right)$$

$\left[|\arg u| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0\right]$ ET II 234(15)a

3.[7] $$\int_0^u x\left(u^2 - x^2\right)^{\nu-1} e^{\mu x}\,dx = \frac{u^{2\nu}}{2\nu} + \frac{\sqrt{\pi}}{2}\left(\frac{\mu}{2}\right)^{\frac{1}{2}-\nu} u^{\nu+\frac{1}{2}}\,\Gamma(\nu)\left[I_{\nu+\frac{1}{2}}(\mu u) + \mathbf{L}_{\nu+\frac{1}{2}}(\mu u)\right]$$

$[\operatorname{Re}\nu > 0]$ ET II 188(19)a

4. $$\int_u^\infty x\left(x^2 - u^2\right)^{\nu-1} e^{-\mu x}\,dx = 2^{\nu-\frac{1}{2}}\left(\sqrt{\pi}\right)^{-1}\mu^{\frac{1}{2}-\nu}u^{\nu+\frac{1}{2}}\,\Gamma(\nu)\,K_{\nu+\frac{1}{2}}(u\mu)$$

$[\operatorname{Re}(u\mu) > 0]$ ET II 203(16)a

5. $$\int_{-\infty}^\infty \frac{(ix)^{-\nu}e^{-ipx}\,dx}{\beta^2 + x^2} = \pi\beta^{-\nu-1}e^{-|p|\beta}$$

$\left[|\nu| < 1, \quad \operatorname{Re}\beta > 0, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x\right]$ ET I 118(5)

6. $$\int_0^\infty \frac{x^\nu e^{-\mu x}}{\beta^2+x^2}\,dx = \frac{1}{2}\,\Gamma(\nu)\beta^{\nu-1}\left[\exp\left(i\mu\beta + i\frac{(\nu-1)\pi}{2}\right)\right.$$
$$\left.\times\,\Gamma(1-\nu, i\beta\mu) + \exp\left(-i\beta\mu - i\frac{(\nu-1)\pi}{2}\right)\Gamma\left(1-\nu, -i\beta\mu\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -1]$ ET II 218(22)

7. $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}\,dx}{1+x^2} = \pi\operatorname{cosec}(\nu\pi)V_\nu(2\mu,0)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 138(9)

8. $$\int_{-\infty}^\infty \frac{(\beta+ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{-p\gamma}$$
$[\operatorname{Re}\nu > -1, \quad p > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ ET I 118(6)

9.[6] $$\int_{-\infty}^\infty \frac{(\beta-ix)^{-\nu}e^{-ipx}}{\gamma^2+x^2}\,dx = \frac{\pi}{\gamma}(\beta+\gamma)^{-\nu}e^{\gamma p}$$
$[p < 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > -1]$ ET I 118(7)

3.391 $$\int_0^\infty \left[\left(\sqrt{x+2\beta}+\sqrt{x}\right)^{2\nu} - \left(\sqrt{x+2\beta}-\sqrt{x}\right)^{2\nu}\right]e^{-\mu x}\,dx = 2^{\nu+1}\frac{\nu}{\mu}\beta^\nu e^{\beta\mu}\,K_\nu(\beta\mu)$$
$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 140(30)

3.392

1. $$\int_0^\infty \left(x+\sqrt{1+x^2}\right)^\nu e^{-\mu x}\,dx = \frac{1}{\mu}\,S_{1,\nu}(\mu) + \frac{\nu}{\mu}\,S_{0,\nu}(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(25)

2. $$\int_0^\infty \left(\sqrt{1+x^2}-x\right)^\nu e^{-\mu x}\,dx = \frac{1}{\mu}\,S_{1,\nu}(\mu) - \frac{\nu}{\mu}\,S_{0,\nu}(\mu)$$
$[\operatorname{Re}\mu > 0]$ ET I 140(26)

3. $$\int_0^\infty \frac{\left(x+\sqrt{1+x^2}\right)^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = \pi\operatorname{cosec}\nu\pi\left[\mathbf{J}_{-\nu}(\mu) - J_{-\nu}(\mu)\right]$$
$[\operatorname{Re}\mu > 0]$ ET I 140(27), EH II 35(33)

4. $$\int_0^\infty \frac{\left(\sqrt{1+x^2}-x\right)^\nu}{\sqrt{1+x^2}}e^{-\mu x}\,dx = S_{0,\nu}(\mu) - \nu\,S_{-1,\nu}(\mu)$$ $[\operatorname{Re}\mu > 0]$ ET I 140(28)

3.393 $$\int_0^\infty \frac{\left(x+\sqrt{x^2+4\beta^2}\right)^{2\nu}}{\sqrt{x^3+4\beta^2x}}e^{-\mu x}\,dx$$
$$= \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2}\beta^{2\nu}}\left[J_{\nu+1/4}(\beta\mu)\,Y_{\nu-1/4}(\beta\mu) - J_{\nu-1/4}(\beta\mu)\,Y_{\nu+1/4}(\beta\mu)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$ ET I 140(33)

3.394 $$\int_0^\infty \frac{\left(1+\sqrt{1+x^2}\right)^{\nu+1/2}}{x^{\nu+1}\sqrt{1+x^2}}e^{-\mu x}\,dx = \sqrt{2}\,\Gamma(-\nu)\,D_\nu\left(\sqrt{2i\mu}\right)D_\nu\left(\sqrt{-2i\mu}\right)$$
$[\operatorname{Re}\mu \geq 0, \quad \operatorname{Re}\nu < 0]$ ET I 140(32)

3.395

1. $$\int_1^\infty \frac{\left(\sqrt{x^2-1}+x\right)^\nu + \left(\sqrt{x^2-1}+x\right)^{-\nu}}{\sqrt{x^2-1}} e^{-\mu x}\,dx = 2\,K_\nu(\mu)$$

$[\operatorname{Re}\mu > 0]$ ET I 140(29)

2. $$\int_1^\infty \frac{\left(x+\sqrt{x^2-1}\right)^{2\nu} + \left(x-\sqrt{x^2-1}\right)^{2\nu}}{\sqrt{x\left(x^2-1\right)}} e^{-\mu x}\,dx = \sqrt{\frac{2\mu}{\pi}}\,K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right)$$

$[\operatorname{Re}\mu > 0]$ ET I 140(34)

3. $$\int_0^\infty \frac{\left(x+\sqrt{x^2+1}\right)^\nu + \cos\nu\pi\left(x+\sqrt{x^2+1}\right)^{-\nu}}{\sqrt{x^2+1}} e^{-\mu x}\,dx = -\pi\left[\mathbf{E}_\nu(\mu) + Y_\nu(\mu)\right]$$

$[\operatorname{Re}\mu > 0]$ EH II 35(34)

3.41–3.44 Combinations of rational functions of powers and exponentials

3.411

1. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}-1} = \frac{1}{\mu^\nu}\,\Gamma(\nu)\,\zeta(\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 1]$ FI II 792a

2. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}-1} = (-1)^{n-1}\left(\frac{2\pi}{p}\right)^{2n}\frac{B_{2n}}{4n}$$ $[n = 1, 2, \ldots]$ FI II 721a

3. $$\int_0^\infty \frac{x^{\nu-1}\,dx}{e^{\mu x}+1} = \frac{1}{\mu^\nu}\left(1-2^{1-\nu}\right)\Gamma(\nu)\,\zeta(\nu)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 792a, WH

4. $$\int_0^\infty \frac{x^{2n-1}\,dx}{e^{px}+1} = \left(1-2^{1-2n}\right)\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n}$$ $[n = 1, 2, \ldots]$ BI(83)(2), EH I 39(25)

5. $$\int_0^{\ln 2} \frac{x\,dx}{1-e^{-x}} = \frac{\pi^2}{12}$$ BI (104)(5)

6.[8] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-\beta e^{-x}}\,dx = \Gamma(\nu)\sum_{n=0}^\infty (\mu+n)^{-\nu}\beta^n = \Gamma(\nu)\,\Phi(\beta,\nu,\mu)$$

$[\operatorname{Re}\mu > 0 \text{ and either } |\beta| \le 1, \quad \beta \ne 1, \quad \operatorname{Re}\nu > 0; \text{ or } \beta = 1, \quad \operatorname{Re}\nu > 1]$ EH I 27(3)

7.[11] $$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-e^{-\beta x}}\,dx = \frac{1}{\beta^\nu}\,\Gamma(\nu)\,\zeta\left(\nu, \frac{\mu}{\beta}\right)$$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 1]$

ET I 144(10)

8. $$\int_0^\infty \frac{x^{n-1}e^{-px}}{1+e^x}\,dx = (n-1)!\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(p+k)^n}$$ $[p > -1; \quad n = 1, 2, \ldots]$ BI (83)(9)

9. $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x-1} = \frac{\pi^2}{6} - 1$$ (cf. **4.231** 3) BI (82)(1)

10. $$\int_0^\infty \frac{xe^{-2x}\,dx}{e^{-x}+1} = 1 - \frac{\pi^2}{12}$$ (cf. **4.251** 6) BI (82)(2)

11. $$\int_0^\infty \frac{xe^{-3x}}{e^{-x}+1}\,dx = \frac{\pi^2}{12} - \frac{3}{4}$$ (cf. **4.251** 5) BI (82)(3)

12.[11] $$\int_0^\infty \frac{xe^{-(2n-1)x}}{1+e^x}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2}$$ (cf. **4.251** 6) BI (82)(5)

13.[11] $$\int_0^\infty \frac{xe^{-2nx}}{1+e^x}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2}$$ (cf. **4.251** 5) BI (82)(4)

14.[7] $$\int_0^\infty \frac{x^2e^{-nx}}{1-e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{1}{k^3} = 2\left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3}\right)$$ $[n = 1, 2, \ldots]$ (cf. **4.261** 12) BI (82)(9)

15.[7] $$\int_0^\infty \frac{x^2e^{-nx}}{1+e^{-x}}\,dx = 2\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1}\left(\frac{3}{2}\,\zeta(3) + 2\sum_{k=1}^{n-1} \frac{(-1)^k}{k^3}\right)$$ $[n = 1, 2, \ldots]$ (cf. **4.261** 11) LI (82)(10)

16. $$\int_{-\infty}^\infty \frac{x^2e^{-\mu x}}{1+e^{-x}}\,dx = \pi^3 \csc^3 \mu\pi \left(2 - \sin^2 \mu\pi\right)$$ $[0 < \operatorname{Re}\mu < 1]$ ET I 120(17)a

17. $$\int_0^\infty \frac{x^3e^{-nx}}{1-e^{-x}}\,dx = \frac{\pi^4}{15} - 6\sum_{k=1}^{n-1} \frac{1}{k^4}$$ (cf. **4.262** 5) BI (82)(12)

18.[11] $$\int_0^\infty \frac{x^3e^{-nx}}{1+e^{-x}}\,dx = 6\sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^4} = (-1)^{n+1}\left(\frac{7}{120}\pi^4 + 6\sum_{k=1}^{n-1} \frac{(-1)^k}{k^4}\right)$$ (cf. **4.262** 4) LI (82)(13)

19.[9] $$\int_0^\infty e^{-px}\left(e^{-x}-1\right)^n \frac{dx}{x} = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p+n-k)$$ LI (89)(10)

20.[9] $$\int_0^\infty e^{-px}\left(e^{-x}-1\right)^n \frac{dx}{x^2} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p+n-k)\ln(p+n-k)$$ LI (89)(15)

21. $$\int_0^\infty x^{n-1}\frac{1-e^{-mx}}{1-e^x}\,dx = (n-1)!\sum_{k=1}^{m} \frac{1}{k^n}$$ (cf. **4.272** 11) LI (83)(8)

22.[7] $$\int_0^\infty \frac{x^{p-1}}{e^{rx}-q}\,dx = \frac{1}{qr^p}\,\Gamma(p)\sum_{k=1}^{\infty} \frac{q^k}{k^p} = \Gamma(p)r^{-p}\,\Phi(q,p,1)$$ $[p > 0, \quad r > 0, \quad -1 < q < 1]$ BI (83)(5)

23. $$\int_{-\infty}^\infty \frac{xe^{\mu x}\,dx}{\beta+e^x} = \pi\beta^{\mu-1}\operatorname{cosec}(\mu\pi)\left[\ln\beta - \pi\cot(\mu\pi)\right]$$ $[|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 1]$ BI (101)(5), ET I 120(16)a

24. $$\int_{-\infty}^\infty \frac{xe^{\mu x}}{e^{\nu x}-1}\,dx = \left(\frac{\pi}{\nu}\operatorname{cosec}\frac{\mu\pi}{\nu}\right)^2$$ $[\operatorname{Re}\nu > \operatorname{Re}\mu > 0]$ (cf. **4.254** 2) LI (101)(3)

25. $$\int_0^\infty x\frac{1+e^{-x}}{e^x-1}\,dx = \frac{\pi^2}{3}-1$$ (cf. **4.231** 4) BI (82)(6)

26. $$\int_0^\infty x\frac{1-e^{-x}}{1+e^{-3x}}e^{-x}\,dx = \frac{2\pi^2}{27}$$ LI (82)(7)a

27. $$\int_0^\infty \frac{1-e^{-\mu x}}{1+e^x}\frac{dx}{x} = \ln\left[\frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\sqrt{\pi}\right]$$ $[\operatorname{Re}\mu > -1]$ BI (93)(4)

28. $$\int_0^\infty \frac{e^{-\nu x}-e^{-\mu x}}{e^{-x}+1}\frac{dx}{x} = \ln\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (93)(6)

29. $$\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1+e^{rx}}\frac{dx}{x} = \ln\left[\tan\frac{p\pi}{2r}\cot\frac{q\pi}{2r}\right]$$ $[|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0]$ BI (103)(3)

30. $$\int_{-\infty}^\infty \frac{e^{px}-e^{qx}}{1-e^{rx}}\frac{dx}{x} = \ln\left[\sin\frac{p\pi}{r}\operatorname{cosec}\frac{q\pi}{r}\right]$$ $[|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0]$ BI (103)(4)

31. $$\int_0^\infty \frac{e^{-qx}+e^{(q-p)x}}{1-e^{-px}}x\,dx = \left(\frac{\pi}{p}\operatorname{cosec}\frac{q\pi}{p}\right)^2$$ $[0 < q < p]$ BI (82)(8)

32. $$\int_0^\infty \frac{e^{-px}-e^{(p-q)x}}{e^{-qx}+1}\frac{dx}{x} = \ln\cot\frac{p\pi}{2q}$$ $[0 < p < q]$ BI (93)(7)

3.412 $$\int_0^\infty \left\{\frac{a+be^{-px}}{ce^{px}+g+he^{-px}} - \frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}}\right\}\frac{dx}{x} = \frac{a+b}{c+g+h}\ln\frac{p}{q}$$ $[p > 0, \quad q > 0]$ BI (96)(7)

3.413

1. $$\int_0^\infty \frac{\left(1-e^{-\beta x}\right)\left(1-e^{-\gamma x}\right)e^{-\mu x}}{1-e^{-x}}\frac{dx}{x} = \ln\frac{\Gamma(\mu)\,\Gamma(\beta+\gamma+\mu)}{\Gamma(\mu+\beta)\,\Gamma(\mu+\gamma)}$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\mu > -\operatorname{Re}\beta, \quad \operatorname{Re}\mu > -\operatorname{Re}\gamma, \quad \operatorname{Re}\mu > -\operatorname{Re}(\beta+\gamma)]$ (cf. **4.267** 25) BI (93)(13)

2. $$\int_0^\infty \frac{\left\{1-e^{(q-p)x}\right\}^2}{e^{qx}-e^{(q-2p)x}}\frac{dx}{x} = \ln\operatorname{cosec}\frac{q\pi}{2p}$$ $[0 < q < p]$ BI (95)(6)

3. $$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-x}}\frac{1+e^{-(2n+1)x}}{x}\,dx = \ln\left\{\frac{q(q+2)(q+4)\cdots(q+2n)(p+1)(p+3)\cdots(p+2n-1)}{p(p+2)(p+4)\cdots(p+2n)(q+1)(q+3)\cdots(q+2n-1)}\right\}$$ $[\operatorname{Re}p > -2n, \quad \operatorname{Re}q > -2n]$ (cf. **4.267** 14) BI (93)(11)

3.414 $$\int_0^\infty \frac{\left(1-e^{-\beta x}\right)\left(1-e^{-\gamma x}\right)\left(1-e^{-\delta x}\right)e^{-\mu x}}{1-e^{-x}}\frac{dx}{x} = \ln\frac{\Gamma(\mu)\,\Gamma(\mu+\beta+\gamma)\,\Gamma(\mu+\beta+\delta)\,\Gamma(\mu+\gamma+\delta)}{\Gamma(\mu+\beta)\,\Gamma(\mu+\gamma)\,\Gamma(\mu+\delta)\,\Gamma(\mu+\beta+\gamma+\delta)}$$ $[2\operatorname{Re}\mu > |\operatorname{Re}\beta| + |\operatorname{Re}\gamma| + |\operatorname{Re}\delta|]$ (cf. **4.267** 31) BI (93)(14), ET I 145(17)

3.415

1. $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}-1)} = \frac{1}{2}\left[\ln\left(\frac{\beta\mu}{2\pi}\right) - \frac{\pi}{\beta\mu} - \psi\left(\frac{\beta\mu}{2\pi}\right)\right]$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$

BI (97)(20), EH I 18(27)

2.[11] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2(e^{2\pi x}-1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta}\psi'(\beta) \sim \frac{1}{4\beta^4}\sum_{k=0}^\infty \frac{|B_{2k+2}|}{\beta^{2k}}$$

[asymptotic expansion for $\operatorname{Re}\beta > 0$] BI(97)(22), EH I 22(12)

3.[11] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)(e^{\mu x}+1)} = \frac{1}{2}\left[\psi\left(\frac{\beta\mu}{2\pi}+\frac{1}{2}\right) - \ln\left(\frac{\beta\mu}{2\pi}\right)\right]$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$

4.[8] $$\int_0^\infty \frac{x\,dx}{(x^2+\beta^2)^2(e^{2\pi x}+1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta}\psi'\left(\beta+\frac{1}{2}\right)$$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$

3.416

1. $$\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i}\frac{dx}{e^{2\pi x}-1} = \frac{1}{2}\frac{2n-1}{2n+1}$$ $[n = 1, 2, \ldots]$ BI (88)(4)

2. $$\int_0^\infty \frac{(1+ix)^{2n}-(1-ix)^{2n}}{i}\frac{dx}{e^{\pi x}+1} = \frac{1}{2n+1}$$ $[n = 1, 2, \ldots]$ BI (87)(1)

3.[8] $$\int_0^\infty \frac{(1+ix)^{2n-1}-(1-ix)^{2n-1}}{i}\frac{dx}{e^{\pi x}+1} = \frac{1}{2n}\left[1-2^{2^n}B_{2n}\right]$$

$[n = 1, 2, \ldots]$ BI (87)(2)

3.417

1. $$\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x+b^2e^{-x}} = \frac{\pi}{2ab}\ln\frac{b}{a}$$ $[ab > 0]$ (cf. **4.231** 8) BI (101)(1)

2. $$\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x-b^2e^{-x}} = \frac{\pi^2}{4ab}$$ (cf. **4.231** 10) LI (101)(2)

3.418

1.[6] $$\int_0^\infty \frac{x\,dx}{e^x+e^{-x}-1} = \frac{1}{3}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right] = 1.1719536193\ldots$$ LI (88)(1)

2.[6] $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x+e^{-x}-1} = \frac{1}{6}\left[\psi'\left(\frac{1}{3}\right) - \frac{5}{6}\pi^2\right] = 0.3118211319\ldots$$ LI (88)(2)

3. $$\int_0^{\ln 2} \frac{x\,dx}{e^x+2e^{-x}-2} = \frac{\pi}{8}\ln 2$$ BI (104)(7)

3.419

1. $$\int_{-\infty}^{\infty} \frac{x\,dx}{(\beta+e^x)(1+e^{-x})} = \frac{(\ln\beta)^2}{2(\beta-1)} \qquad [|\arg\beta| < \pi] \qquad (\text{cf. } \mathbf{4.232}\ 2)$$
BI (101)(16)

2. $$\int_{-\infty}^{\infty} \frac{x\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\pi^2+(\ln\beta)^2}{2(\beta+1)} \qquad [|\arg\beta| < \pi] \qquad (\text{cf. } \mathbf{4.232}\ 3)$$
BI (101)(17)

3. $$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\left[\pi^2+(\ln\beta)^2\right]\ln\beta}{3(\beta+1)} \qquad [|\arg\beta| < \pi] \qquad (\text{cf. } \mathbf{4.261}\ 4)$$
BI (102)(6)

4. $$\int_{-\infty}^{\infty} \frac{x^3\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\left[\pi^2+(\ln\beta)^2\right]^2}{4(\beta+1)} \qquad [|\arg\beta| < \pi] \qquad (\text{cf. } \mathbf{4.262}\ 3)$$
BI (102)(9)

5. $$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\left[\pi^2+(\ln\beta)^2\right]^2}{15(\beta+1)}\left[7\pi^2+3(\ln\beta)^2\right]\ln\beta$$
(cf. **4.263** 1) BI (102)(10)

6.[11] $$\int_{-\infty}^{\infty} \frac{x^5\,dx}{(\beta+e^x)(1-e^{-x})} = \frac{\left[\pi^2+(\ln\beta)^2\right]^2}{6(\beta+1)}\left[3\pi^2+(\ln\beta)^2\right]$$
(cf. **4.264** 3) BI (102)(11)

7. $$\int_{-\infty}^{\infty} \frac{(x-\ln\beta)\,x\,dx}{(\beta-e^x)(1-e^{-x})} = \frac{-\left[4\pi^2+(\ln\beta)^2\right]\ln\beta}{6(\beta-1)} \qquad [|\arg\beta| < \pi] \qquad (\text{cf. } \mathbf{4.257}\ 4)$$
BI (102)(7)

3.421

1. $$\int_0^{\infty} \left(e^{-\nu x}-1\right)^n \left(e^{-\rho x}-1\right)^m e^{-\mu x}\frac{dx}{x^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{l=0}^{m}(-1)^l\binom{m}{l} \times \{(m-l)\rho+(n-k)\nu+\mu\}\ln\left[(m-l)\rho+(n-k)\nu+\mu\right]$$
$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\rho > 0]$ BI (89)(17)

2. $$\int_0^{\infty} \left(1-e^{-\nu x}\right)^n \left(1-e^{-\rho x}\right) e^{-x}\frac{dx}{x^3} = \frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\rho+k\nu+1)^2 \times \ln(\rho+k\nu+1) + \frac{1}{2}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(k\nu+1)^2\ln(k\nu+1)$$
$[n \geq 2, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\rho > 0]$ BI (89)(31)

3. $$\int_{-\infty}^{\infty} \frac{xe^{-\mu x}\,dx}{(\beta+e^{-x})(\gamma+e^{-x})} = \frac{\pi\left(\beta^{\mu-1}\ln\beta-\gamma^{\mu-1}\ln\gamma\right)}{(\beta-\gamma)\sin\mu\pi} + \frac{\pi^2\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\cos\mu\pi}{(\gamma-\beta)\sin^2\mu\pi}$$
$[|\arg\beta|<\pi, \quad |\arg\gamma|<\pi, \quad \beta\neq\gamma. \quad 0<\operatorname{Re}\mu<2]$ ET I 120(19)

4. $$\int_0^{\infty} \left(e^{-px}-e^{-qx}\right)\left(e^{-rx}-e^{-sx}\right)e^{-x}\frac{dx}{x} = \ln\frac{(p+s+1)(q+r+1)}{(p+r+1)(q+s+1)}$$
$[p+s>-1, \quad p+r>-1, \quad q>p]$ (cf. **4.267** 24) BI (89)(11)

5. $$\int_0^{\infty} \left(1-e^{-px}\right)\left(1-e^{-qx}\right)\left(1-e^{-rx}\right)e^{-x}\frac{dx}{x}$$
$$= (p+q+1)\ln(p+q+1)$$
$$+(p+r+1)\ln(p+r+1)+(q+r+1)\ln(q+r+1)$$
$$-(p+1)\ln(p+1)-(q+1)\ln(q+1)-(r+1)\ln(r+1)$$
$$-(p+q+r)\ln(p+q+r)$$
$[p>0, \quad q>0, \quad r>0]$ (cf. **4.268** 3) BI (89)(14)

3.422 $$\int_{-\infty}^{\infty} \frac{x(x-a)e^{\mu x}\,dx}{(\beta-e^{x})(1-e^{-x})} = \frac{-\pi^2}{e^a-1}\operatorname{cosec}^2\mu\pi\left[(e^{\alpha\mu}+1)\ln\mu-2\pi\cot\mu\pi\,(e^{\alpha\mu}-1)\right]$$
$[a>0, \quad |\arg\beta|<\pi, \quad |\operatorname{Re}\mu|<1]$ (cf. **4.257** 5) BI (102)(8)a

3.423

1. $$\int_0^{\infty} \frac{x^{\nu-1}}{(e^x-1)^2}\,dx = \Gamma(\nu)\left[\zeta(\nu-1)-\zeta(\nu)\right]$$ $[\operatorname{Re}\nu>2]$ ET I 313(10)

2.[6] $$\int_0^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(e^x-1)^2}\,dx = \Gamma(\nu)\left[\zeta(\nu-1,\mu+2)-(\mu+1)\,\zeta(\nu,\mu+2)\right]$$
$[\operatorname{Re}\mu>-2, \quad \operatorname{Re}\nu>2]$ ET I 313(11)

3.[8] $$\int_0^{\infty} \frac{x^q e^{-px}\,dx}{(1-ae^{-px})^2} = \frac{\Gamma(q+1)}{ap^{q+1}}\sum_{k=1}^{\infty}\frac{a^k}{k^q}$$ $[a<1, \quad q>-1, \quad p>0]$ BI (85)(13)

4.[7] $$\int_0^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(1-\beta e^{-x})^2}\,dx = \Gamma(\nu)\left[\Phi(\beta;\nu-1;\mu)-(\mu-1)\,\Phi(\beta;\nu;\mu)\right]$$
$[\operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0, \quad |\arg(1-\beta)|<\pi]$ (cf. **9.550**) ET I 313(12)

5. $$\int_{-\infty}^{\infty} \frac{xe^x\,dx}{(\beta+e^x)^2} = \frac{1}{\beta}\ln\beta$$ $[|\arg\beta|<\pi]$ (cf. **4.231** 5)
BI (101)(10)

6.* $$\int_0^{t} x^5\frac{e^{-x}}{(1-e^{-x})^2}\,dx = 120\,\zeta(5)-\sum_{k=1}^{\infty}\frac{e^{-kt}}{k^5}\left(y^5+5y^4+20y^3+60y^2+120y+120\right)$$
$$= 120\,\zeta(5)-\frac{t^5e^{-t/2}}{2\sinh(t/2)}-5\sum_{k=1}^{\infty}\frac{e^{-kt}}{k^5}\left(y^4+4y^3+12y^2+24y+24\right)$$
$y=kt$

3.424

1.[7] $$\int_0^\infty \frac{(1+a)e^x - a}{(1-e^x)^2} e^{-ax} x^n\,dx = n!\,\zeta(n,a)$$ $[a > -1, \quad n = 1, 2, \ldots]$ BI (85)(15)

2. $$\int_0^\infty \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n\,dx = n! \sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n}$$ $[a > -1, \quad n = 1, 2, \ldots]$ BI (85)(14)

3. $$\int_{-\infty}^\infty \frac{a^2 e^x + b^2 e^{-x}}{(a^2 e^x - b^2 e^{-x})^2} x^2\,dx = \frac{\pi^2}{2ab}$$ $[ab > 0]$ BI (102)(3)a

4. $$\int_{-\infty}^\infty \frac{a^2 e^x - b^2 e^{-x}}{(a^2 e^x + b^2 e^{-x})^2} x^2\,dx = \frac{\pi}{ab} \ln \frac{b}{a}$$ $[ab > 0]$ BI (102)(1)

5. $$\int_0^\infty \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} x^2\,dx = \frac{2}{3}\pi^2 - 2$$ BI (85)(7)

3.425

1.[7] $$\int_{-\infty}^\infty \frac{x e^x\,dx}{(a^2 + b^2 e^{2x})^n} = \frac{\sqrt{\pi}\,\Gamma\left(n - \frac{1}{2}\right)}{4a^{2n-1} b\,\Gamma(n)} \left[2 \ln \frac{a}{2b} - \boldsymbol{C} - \psi\left(n - \frac{1}{2}\right)\right]$$ $[ab > 0, \quad n > 0]$ BI(101)(13), LI(101)(13)

2.[7] $$\int_{-\infty}^\infty \frac{\left(a^2 e^x - e^{-x}\right) x^2\,dx}{(a^2 e^x + e^{-x})^{p+1}} = -\frac{1}{a^{p+1}} \mathrm{B}\left(\frac{p}{2}, \frac{p}{2}\right) \ln a$$ $[a > 0, \quad p > 0]$ BI (102)(5)

3.426

1. $$\int_{-\infty}^\infty \frac{(e^x - ae^{-x})\, x^2\,dx}{(a+e^x)^2 (1+e^{-x})^2} = \frac{(\ln a)^2}{a-1}$$ BI (102)(12)

2. $$\int_{-\infty}^\infty \frac{(e^x - ae^{-x})\, x^2\,dx}{(a+e^x)^2 (1-e^{-x})^2} = \frac{\pi^2 + (\ln a)^2}{a+1}$$ BI (102)(13)

3.427

1. $$\int_0^\infty \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x} - 1}\right) dx = \psi(\mu)$$ $[\operatorname{Re}\mu > 0]$ (cf. **4.281** 4) WH

2.[7] $$\int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x}\,dx = \boldsymbol{C}$$ (cf. **4.281** 1) BI (94)(1)

3. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{1+e^{-x}}\right) \frac{e^{-2x}}{x}\,dx = \frac{1}{2} \ln \frac{\pi}{4}$$ BI (94)(5)

4. $$\int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right) \frac{e^{-\mu x}}{x}\,dx = \ln \Gamma(\mu) - \left(\mu - \frac{1}{2}\right) \ln \mu + \mu - \frac{1}{2} \ln(2\pi)$$ $[\operatorname{Re}\mu > 0]$ WH

5. $$\int_0^\infty \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1}\right) \frac{dx}{x} = -\frac{1}{2} \ln \pi$$ BI (94)(6)

6. $$\int_0^\infty \left(\frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu\right) \frac{e^{-x}}{x}\,dx = -\ln \Gamma(\mu) - \ln \sin(\pi\mu) + \ln \pi$$ $[\operatorname{Re}\mu < 1]$ EH I 21(6)

7. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu x}}{x}\right) dx = \ln\mu - \psi(\nu)$$ (cf. **4.281** 5) BI (94)(3)

8. $$\int_0^\infty \left(\frac{n}{x} - \frac{e^{-\mu x}}{1-e^{-x/n}}\right) e^{-x}\, dx = n\,\psi(n\mu+n) - n\ln n$$
$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots]$ BI (94)(4)

9. $$\int_0^\infty \left(\mu - \frac{1-e^{-\mu x}}{1-e^{-x}}\right) \frac{e^{-x}}{x}\, dx = \ln\Gamma(\mu+1)$$ $[\operatorname{Re}\mu > -1]$ WH

10. $$\int_0^\infty \left(\nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1}\right) \frac{dx}{x} = \ln\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)}$$
$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > 0]$ BI (94)(8)

11. $$\int_0^\infty \left[(1-e^x)^{-1} + x^{-1} - 1\right] e^{-xz}\, dx = \psi(z) - \ln z$$ $[\operatorname{Re} z > 0]$ EH I 18(24)

3.428

1. $$\int_0^\infty \left(\nu e^{-\mu x} - \frac{1}{\mu}e^{-x} - \frac{1}{\mu}\frac{e^{-1} - e^{-\mu\nu x}}{1-e^{-x}}\right) \frac{dx}{x} = \frac{1}{\mu}\ln\Gamma(\mu\nu) - \nu\ln\mu$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (94)(18)

2. $$\int_0^\infty \left(\frac{n-1}{2} + \frac{n-1}{1-e^{-x}} + \frac{e^{(1-\mu)x}}{1-e^{x/n}} + \frac{e^{-n\mu x}}{1-e^{-x}}\right) e^{-x}\frac{dx}{x} = \frac{n-1}{2}\ln 2\pi - \left(n\mu + \frac{1}{2}\right)\ln n$$
$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots]$ BI (94)(14)

3. $$\int_0^\infty \left(n\mu - \frac{n-1}{2} - \frac{n}{1-e^{-x}} - \frac{e^{(1-\mu)x}}{1-e^{x/n}}\right) \frac{e^{-x}}{x}\, dx = \sum_{k=0}^{n-1} \ln\Gamma\left(\mu - \frac{k}{n} + 1\right)$$
$[\operatorname{Re}\mu > 0, \quad n = 1, 2, \ldots]$ BI (94)(13)

4. $$\int_0^\infty \left(\frac{e^{-\nu x}}{1-e^x} - \frac{e^{-\mu\nu x}}{1-e^{\mu x}} - \frac{e^x}{1-e^x} + \frac{e^{\mu x}}{1-e^{\mu x}}\right) \frac{dx}{x} = \nu\ln\mu$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ LI (94)(15)

5. $$\int_0^\infty \left[\frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1-e^{-\mu x}} + \left(a\mu - \frac{\mu+1}{2}\right) e^{-\mu x} + (1-a\mu)e^{-x}\right] \frac{dx}{x}$$
$$= \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - a\mu\right)\ln\mu$$
$[\operatorname{Re}\mu > 0]$ BI (94)(16)

6. $$\int_0^\infty \left[\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu\nu x}}{1-e^{-\mu x}} - \frac{(\mu-1)e^{-\mu x}}{1-e^{-\mu x}} - \frac{\mu-1}{2}e^{-\mu}x\right] \frac{dx}{x} = \frac{\mu-1}{2}\ln(2\pi) + \left(\frac{1}{2} - \mu\nu\right)\ln\mu$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ (cf. **4.267** 37) BI (94)(17)

7. $$\int_0^\infty \left[1 - e^{-x} - \frac{(1-e^{-\nu x})(1-e^{-\mu x})}{1-e^{-x}}\right] \frac{dx}{x} = \ln \mathrm{B}(\mu, \nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (94)(12)

3.429 $$\int_0^\infty \left[e^{-x} - (1+x)^{-\mu}\right] \frac{dx}{x} = \psi(\mu)$$ $[\operatorname{Re}\mu > 0]$ NH 184(7)

3.431

1. $$\int_0^\infty \left(e^{-\mu x} - 1 + \mu x - \frac{1}{2}\mu^2 x^2\right) x^{\nu-1}\,dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu}\,\Gamma(\nu+3)$$
$[\operatorname{Re}\mu > 0, \quad -2 > \operatorname{Re}\nu > -3]$ LI (90)(5)

2. $$\int_0^\infty \left[x^{-1} - \frac{1}{2}x^{-2}(x+2)\left(1-e^{-x}\right)\right] e^{-px}\,dx = -1 + \left(p + \frac{1}{2}\right)\ln\left(1 + \frac{1}{p}\right)$$
$[\operatorname{Re}p > 0]$ ET I 144(6)

3.432

1. $$\int_0^\infty x^{\nu-1} e^{-mx}\left(e^{-x}-1\right)^n\,dx = \Gamma(\nu)\sum_{k=0}^n (-1)^k \binom{n}{k}\frac{1}{(n+m-k)^\nu}$$
$[n = 0, 1, \ldots, \operatorname{Re}\nu > 0]$ LI (90)(10)

2. $$\int_0^\infty \left[x^{\nu-1}e^{-x} - e^{-\mu x}\left(1-e^{-x}\right)^{\nu-1}\right]\,dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ LI (81)(14)

3.433 $$\int_0^\infty x^{p-1}\left[e^{-x} + \sum_{k=1}^n (-1)^k \frac{x^{k-1}}{(k-1)!}\right]\,dx = \Gamma(p) \qquad [-n < p < -n+1, \quad n = 0, 1, \ldots]$$
FI II 805

3.434

1. $$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}}\,dx = \frac{\mu^\rho - \nu^\rho}{\rho}\,\Gamma(1-\rho)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\rho < 1]$ BI (90)(6)

2. $$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{x}\,dx = \ln\frac{\nu}{\mu}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 634

3.435

1. $$\int_0^\infty \left\{(x+1)e^{-x} - e^{-\frac{x}{2}}\right\}\frac{dx}{x} = 1 - \ln 2$$
LI (89)(19)

2.[11] $$\int_0^\infty \frac{1-e^{-\mu x}}{x(x+\beta)}\,dx = \frac{1}{\beta}\left[\ln(\beta\mu) + \boldsymbol{C} - e^{\beta\mu}\operatorname{Ei}(-\beta\mu)\right]$$
$[|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$ ET II 217 (18)

3. $$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} = \boldsymbol{C}$$
FI II 7 95, 802

4. $$\int_0^\infty \left(e^{-\mu x} - \frac{1}{1+ax}\right)\frac{dx}{x} = \ln\frac{a}{\mu} - \boldsymbol{C}$$
$[a > 0, \quad \operatorname{Re}\mu > 0]$ BI (92)(10)

3.436 $$\int_0^\infty \left\{\frac{e^{-npx} - e^{-nqx}}{n} - \frac{e^{-mpx} - e^{-mqx}}{m}\right\}\frac{dx}{x^2} = (q-p)\ln\frac{m}{n} \qquad [p > 0, \quad q > 0]$$
BI (89)(28)

3.437 $$\int_0^\infty \left\{pe^{-x} - \frac{1-e^{-px}}{x}\right\}\frac{dx}{x} = p\ln p - p \qquad [p > 0]$$
BI (89)(24)

3.438

1. $$\int_0^\infty \left\{\left(\frac{1}{2}+\frac{1}{x}\right)e^{-x} - \frac{1}{x}e^{-\frac{x}{2}}\right\}\frac{dx}{x} = \frac{\ln 2 - 1}{2}$$ BI (89)(19)

2.[7] $$\int_0^\infty \left\{\frac{p^2}{6}e^{-x} - \frac{p^2}{2x} - \frac{p}{x^2} - \frac{1-e^{-px}}{x^3}\right\}\frac{dx}{x} = \frac{p^2}{6}\ln p - \frac{11}{36}p^3$$ $[p > 0]$ BI (89)(33)

3. $$\int_0^\infty \left(e^{-x} - e^{-2x} - \frac{1}{x}e^{-2x}\right)\frac{dx}{x} = 1 - \ln 2$$ BI (89)(25)

4. $$\int_0^\infty \left\{\left(p - \frac{1}{2}\right)e^{-x} + \frac{x+2}{2x}\left(e^{-px} - e^{-\frac{x}{2}}\right)\right\}\frac{dx}{x} = \left(p-\frac{1}{2}\right)(\ln p - 1)$$ $[p > 0]$ BI (89)(22)

3.439 $$\int_0^\infty \left\{(p-q)e^{-rx} + \frac{1}{mx}\left(e^{-mpx} - e^{-mqx}\right)\right\}\frac{dx}{x} = p\ln p - q\ln q - (p-q)\left(1 + \ln\frac{r}{m}\right)$$ $[p > 0, \quad q > 0, \quad r > 0]$ LI(89)(26), LI(89)(27)

3.441 $$\int_0^\infty \left\{(p-r)e^{-qx} + (r-q)e^{-px} + (q-p)e^{-rx}\right\}\frac{dx}{x^2} = (r-q)p\ln p + (p-r)q\ln q + (q-p)r\ln r$$ $[p > 0, \quad q > 0, \quad r > 0]$ (cf. **4.268** 6) BI (89)(18)

3.442

1. $$\int_0^\infty \left\{1 - \frac{x+2}{2x}\left(1 - e^{-x}\right)\right\}e^{-qx}\frac{dx}{x} = -1 + \left(q + \frac{1}{2}\right)\ln\frac{q+1}{q}$$ $[q > 0]$ BI (89)(23)

2. $$\int_0^\infty \left(\frac{e^{-x}-1}{x} + \frac{1}{1+x}\right)\frac{dx}{x} = \boldsymbol{C} - 1$$ BI (92)(16)

3. $$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2}\right)\frac{dx}{x} = -\boldsymbol{C} + \ln\frac{a}{p}$$ $[p > 0]$ BI (92)(11)

3.443

1. $$\int_0^\infty \left\{\frac{e^{-x}p^2}{2} - \frac{p}{x} + \frac{1-e^{-px}}{x^2}\right\}\frac{dx}{x} = \frac{p^2}{2}\ln p - \frac{3}{4}p^2$$ $[p > 0]$ BI (89)(32)

2. $$\int_0^\infty \frac{\left(1-e^{-px}\right)^n e^{-qx}}{x^3}\,dx = \frac{1}{2}\sum_{k=2}^n (-1)^{k-1}\binom{n}{k}(q+kp)^2\ln(q+kp)$$ $[n > 2, \quad q > 0, \quad pn + q > 0]$ (cf. **4.268** 4) BI (89)(30)

3. $$\int_0^\infty \left(1-e^{-px}\right)^2 e^{-qx}\frac{dx}{x^2} = (2p+q)\ln(2p+q) - 2(p+q)\ln(p+q) + q\ln q$$ $[q > 0, \quad 2p > -q]$ (cf. **4.268** 2) BI (89)(13)

3.45 Combinations of powers and algebraic functions of exponentials

3.451

1. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}}\,dx = \frac{4}{3}\left(\frac{4}{3}-\ln 2\right)$$ BI (99)(1)

2. $$\int_0^\infty xe^{-x}\sqrt{1-e^{-2x}}\,dx = \frac{\pi}{4}\left(\frac{1}{2}+\ln 2\right)$$ (cf. **4.241** 9) BI (99)(2)

3.452

1. $$\int_0^\infty \frac{x\,dx}{\sqrt{e^x-1}} = 2\pi\ln 2$$ FI II 643a,BI(99)(4)

2. $$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = 4\pi\left\{(\ln 2)^2+\frac{\pi^2}{12}\right\}$$ BI (99)(5)

3. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^x-1}} = \frac{\pi}{2}\left[2\ln 2-1\right]$$ BI (99)(6)

4. $$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1-\ln 2$$ BI (99)(8)

5. $$\int_0^\infty \frac{xe^{-2x}\,dx}{\sqrt{e^x-1}} = \frac{3}{4}\pi\left(\ln 2-\frac{7}{12}\right)$$ BI (99)(7)

3.453

1. $$\int_0^\infty \frac{xe^x}{a^2e^x-(a^2-b^2)}\frac{dx}{\sqrt{e^x-1}} = \frac{2\pi}{ab}\ln\left(1+\frac{b}{a}\right)$$ $[ab>0]$ (cf. **4.298** 17) BI (99)(16)

2. $$\int_0^\infty \frac{xe^x\,dx}{\left[a^2e^x-(a^2+b^2)\right]\sqrt{e^x-1}} = \frac{2\pi}{ab}\arctan\frac{b}{a}$$ $[ab>0]$ (cf. **4.298** 18) BI (99)(17)

3.454

1.[11] $$\int_0^\infty \frac{xe^{-2nx}\,dx}{\sqrt{e^{2x}-1}} = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}\left\{\ln 2+\sum_{k=1}^{2n}\frac{(-1)^k}{k}\right\}$$ LI (99)(10)

2. $$\int_0^\infty \frac{xe^{-(2n-1)x}\,dx}{\sqrt{e^{2x}-1}} = -\frac{(2n-2)!!}{(2n-1)!!}\left\{\ln 2+\sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right\}$$ LI (99)(9)

3.455

1. $$\int_0^\infty \frac{x^2e^x\,dx}{\sqrt{(e^x-1)^3}} = 8\pi\ln 2$$ BI (99)(11)

2. $$\int_0^\infty \frac{x^3e^x\,dx}{\sqrt{(e^x-1)^3}} = 24\pi\left[(\ln 2)^2+\frac{\pi^2}{12}\right]$$ BI (99)(12)

3.456

1. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{e^{3x}-1}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3+\frac{\pi}{3\sqrt{3}}\right]$$ BI (99)(13)

2. $$\int_0^\infty \frac{x\,dx}{\sqrt[3]{\left(e^{3x}-1\right)^2}} = \frac{\pi}{3\sqrt{3}}\left[\ln 3 - \frac{\pi}{3\sqrt{3}}\right]$$ (cf. **4.244** 3) BI (99)(14)

3.457

1. $$\int_0^\infty xe^{-x}\left(1-e^{-2x}\right)^{n-1/2}\,dx = \frac{(2n-1)!!}{4\cdot(2n)!!}\pi\left[\boldsymbol{C}+\psi(n+1)+2\ln 2\right]$$ (cf. **4.241** 5) BI (99)(3)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{\left(a+e^x\right)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}}\left[\ln(4a)-3\boldsymbol{C}-2\,\psi(2n)-\psi(n)\right]$$ BI (101)(12)

3. $$\int_{-\infty}^\infty \frac{x\,dx}{\left(a^2e^x+e^{-x}\right)^\mu} = -\frac{1}{2a^\mu}\,\mathrm{B}\left(\frac{\mu}{2},\frac{\mu}{2}\right)\ln a$$ $[a>0, \quad \operatorname{Re}\mu>0]$ BI (101)(14)

3.458

1.[7] $$\int_0^{\ln 2} xe^x\left(e^x-1\right)^{p-1}\,dx = \frac{1}{p}\left[\ln 2+\sum_{k=0}^\infty \frac{(-1)^{k-1}}{p+k+1}\right]$$ BI (104)(4)

2. $$\int_{-\infty}^\infty \frac{xe^x\,dx}{\left(a+e^x\right)^{\nu+1}} = \frac{1}{\nu a^\nu}\left[\ln a-\boldsymbol{C}-\psi(\nu)\right]$$ $[a>0]$

$$= \frac{1}{\nu a^\nu}\left[\ln a-\sum_{k=1}^{\nu-1}\frac{1}{k}\right]$$ $[a>0, \quad \nu=1,2,\ldots]$

BI (101)(11)

3.46–3.48 Combinations of exponentials of more complicated arguments and powers

3.461

1. $$\int_u^\infty \frac{e^{-p^2x^2}}{x^{2n}}\,dx = \frac{(-1)^n 2^{n-1}p^{2n-1}\sqrt{\pi}}{(2n-1)!!}\left[1-\Phi(pu)\right] + \frac{e^{-p^2u^2}}{2u^{2n-1}}\sum_{k=0}^{n-1}\frac{(-1)^k 2^{k+1}(pu)^{2k}}{(2n-1)(2n-3)\cdots(2n-2k-1)}$$ $[p>0]$ NT 21(4)

2. $$\int_0^\infty x^{2n}e^{-px^2}\,dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}$$ $[p>0, \quad n=0,1,\ldots]$ FI II 743

3. $$\int_0^\infty x^{2n+1}e^{-px^2}\,dx = \frac{n!}{2p^{n+1}}$$ $[p>0]$ BI (81)(7)

4. $$\int_{-\infty}^\infty (x+ai)^{2n}e^{-x^2}\,dx = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\sum_{k=0}^n(-1)^k\frac{(2a)^{2k}n!}{(2k)!(n-k)!}$$ BI (100)(12)

5.[11] $$\int_u^\infty e^{-\mu x^2}\frac{dx}{x^2} = \frac{1}{u}e^{-\mu u^2}-\sqrt{\mu\pi}\left[1-\Phi\left(u\sqrt{\mu}\right)\right]$$ $\left[|\arg\mu|<\frac{\pi}{2}, \quad u>0\right]$ ET I 135(19)a

6.* $$\int_0^\infty \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = b\,K_1(ab)$$ $[\operatorname{Re}a>0, \quad \operatorname{Re}b>0]$

7.* $$\int_0^\infty x^2 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{2b}{a^2}\,K_1(ab) + \frac{b^2}{a}\,K_0(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

8.* $$\int_0^\infty x^4 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{12b^2}{a^3}\,K_2(ab) + \frac{3b^3}{a^2}\,K_1(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

9.* $$\int_0^\infty x^6 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{90b^3}{a^4}\,K_3(ab) + \frac{15b^4}{a^3}\,K_2(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

3.462

1. $$\int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x}\,dx = (2\beta)^{-\nu/2}\,\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$

EH II 119(3)a, ET I 313(13)

2.[8] $$\int_{-\infty}^\infty x^n e^{-px^2+2qx}\,dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\frac{d^{n-1}}{dq^{n-1}}\left(qe^{q^2/p}\right) \qquad [p > 0] \qquad \text{BI (100)(8)}$$

$$= n!e^{q^2/p}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{(n-2k)!(k)!}\left(\frac{p}{4q^2}\right)^k \qquad [p > 0] \qquad \text{LI (100)(8)}$$

3.[11] $$\int_{-\infty}^\infty (ix)^\nu e^{-\beta^2 x^2 - iqx}\,dx = 2^{-\frac{\nu}{2}}\sqrt{\pi}\beta^{-\nu-1}\exp\left(-\frac{q^2}{8\beta^2}\right) D_\nu\left(\frac{q}{\beta\sqrt{2}}\right)$$

$\left[\operatorname{Re}\beta^2 > 0, \quad \operatorname{Re}\nu > -1, \quad \arg ix = \frac{\pi}{2}\operatorname{sign} x\right]$ ET I 121(23)

4. $$\int_{-\infty}^\infty x^n \exp\left[-(x-\beta)^2\right]\,dx = (2i)^{-n}\sqrt{\pi}\,H_n(i\beta)$$ EH II 195(31)

5.[11] $$\int_0^\infty x e^{-\mu x^2 - 2\nu x}\,dx = \frac{1}{2\mu} - \frac{\nu}{2\mu}\sqrt{\frac{\pi}{\mu}}e^{\frac{\nu^2}{\mu}}\left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

$\left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 146(31)a

6. $$\int_{-\infty}^\infty x e^{-px^2+2qx}\,dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{p}\right) \qquad [\operatorname{Re} p > 0] \qquad \text{BI (100)(7)}$$

7.[11] $$\int_0^\infty x^2 e^{-\mu x^2 - 2\nu x}\,dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{2\nu^2+\mu}{4}e^{\frac{\nu^2}{\mu}}\left[1 - \operatorname{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

$\left[|\arg\nu| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET I 146(32)

8. $$\int_{-\infty}^\infty x^2 e^{-\mu x^2+2\nu x}\,dx = \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}}\left(1 + 2\frac{\nu^2}{\mu}\right)e^{\frac{\nu^2}{\mu}} \qquad [|\arg\nu| < \pi, \quad \operatorname{Re}\mu > 0] \qquad \text{BI (100)(8)a}$$

9.* $$\int_0^\infty e^{-\beta x^n \pm a}\,dx = \frac{e^{\pm a}}{n\beta^{1/n}}\,\Gamma\left(\frac{1}{n}\right) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re} n > 0]$$

10.* $\int_0^\infty (x-a)e^{-\beta(x-a)}\,dx = e^{a\beta}\frac{(1-a\beta)}{\beta^2}$ $[\operatorname{Re}\beta>0]$

11.* $\int_0^\infty (x-a)e^{-\beta(x+a)}\,dx = e^{-a\beta}\frac{(1-a\beta)}{\beta^2}$ $[\operatorname{Re}\beta>0]$

12.* $\int_0^\infty (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\,\Gamma\left(m+1,\pm\frac{pb}{a}\right)$ $\left[p>0,\quad \left|\arg\left(\frac{b}{a}\right)\right|<\pi\right]$

13.* $\int_u^\infty (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\,\Gamma\left(m+1,pu\pm\frac{pb}{a}\right)$

$\left[p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

14.* $\int_0^u (ax\pm b)^m e^{-px}\,dx = \frac{a^m e^{\pm pb/a}}{p^{m+1}}\left[\Gamma\left(m+1,\pm\frac{pb}{a}\right)-\Gamma\left(m+1,pu\pm\frac{pb}{a}\right)\right]$

$\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

15.* $\int_0^\infty \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\,\Gamma\left(-n+1,\pm\frac{pb}{a}\right)$ $\left[p>0,\quad \left|\arg\left(\frac{b}{a}\right)\right|<\pi\right]$

16.* $\int_u^\infty \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\,\Gamma\left(-n+1,pu\pm\frac{pb}{a}\right)$

$\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

17.* $\int_0^u \frac{e^{-px}}{(ax\pm b)^n}\,dx = \frac{p^{n-1}e^{\pm pb/a}}{a^n}\left[\Gamma\left(-n+1,\pm\frac{pb}{a}\right)-\Gamma\left(-n+1,pu\pm\frac{pb}{a}\right)\right]$

$\left[u>0,\quad p>0,\quad \left|\arg\left(\frac{b}{a}\pm u\right)\right|<\pi\right]$

18.* $\int_0^\infty \left(\frac{x-a}{b}\right)^j \exp\left(-\beta\left(\frac{x-a}{b}\right)^k\right)dx = \frac{b\,\Gamma\left(\frac{j+1}{k},\beta\left(-\frac{a}{b}\right)^k\right)}{k\beta^{(j+1)/k}}$

$\left[\arg\left(-\frac{a}{b}\right)>0,\quad \operatorname{Re}b>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}k>0\right]$

19.* $\int_u^\infty \frac{e^{-\beta x^n}}{x^m}\,dx = \frac{\Gamma(z,\beta u^n)}{n\beta^z}\qquad z=\frac{1-m}{n}$ $[u>0,\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}n>0,\quad \operatorname{Re}z>0]$

20.* $\int_0^\infty \frac{\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = K_0(ab)$ $[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

21.* $\int_0^\infty \frac{x^2\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{b}{a}\,K_1(ab)$ $[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

22.* $\int_0^\infty \frac{x^4\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{3b^2}{a^2}\,K_1(ab)$ $[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

23.* $\int_0^\infty \frac{x^6\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = \frac{15b^3}{a^3}\,K_3(ab)$ $[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$

24.* $$\int_0^\infty \frac{x^{2n}\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = (2n-1)!!\left(\frac{b}{a}\right)^n K_n(ab)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

25.* $$\int_0^\infty \frac{\exp\left(-px^2\right)}{\sqrt{a^2+x^2}}\,dx = \frac{1}{2}\exp\left(\frac{a^2p}{2}\right)K_0\left(\frac{a^2p}{2}\right)$$ $[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$

3.463 $$\int_0^\infty \left(e^{-x^2} - e^{-x}\right)\frac{dx}{x} = \frac{1}{2}\boldsymbol{C}$$ BI (89)(5)

3.464 $$\int_0^\infty \left(e^{-\mu x^2} - e^{-\nu x^2}\right)\frac{dx}{x^2} = \sqrt{\pi}\left(\sqrt{\nu}-\sqrt{\mu}\right)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ FI II 645

3.465 $$\int_0^\infty \left(1+2\beta x^2\right)e^{-\mu x^2}\,dx = \frac{\mu+\beta}{2}\sqrt{\frac{\pi}{\mu^3}}$$ $[\operatorname{Re}\mu > 0]$ ET I 136(24)a

3.466

1. $$\int_0^\infty \frac{e^{-\mu^2x^2}}{x^2+\beta^2}\,dx = [1-\Phi(\beta\mu)]\frac{\pi}{2\beta}e^{\beta^2\mu^2}$$ $\left[\operatorname{Re}\beta > 0, \quad |\arg\mu| < \frac{\pi}{4}\right]$ NT 19(13)

2. $$\int_0^\infty \frac{x^2e^{-\mu^2x^2}}{x^2+\beta^2}\,dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi\beta}{2}e^{\mu^2\beta^2}[1-\Phi(\beta\mu)]$$ $\left[\operatorname{Re}\beta > 0, \quad |\arg\mu| < \frac{\pi}{4}\right]$ ET II 217(16)

3. $$\int_0^1 \frac{e^{x^2}-1}{x^2}\,dx = \sum_{k=1}^\infty \frac{1}{k!(2k-1)}$$ FI II 683

3.467 $$\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}$$ BI (92)(12)

3.468

1. $$\int_{u\sqrt{2}}^\infty \frac{e^{-x^2}}{\sqrt{x^2-u^2}}\frac{dx}{x} = \frac{\pi}{4u}[1-\Phi(u)]^2$$ $[u > 0]$ NT 33(17)

2. $$\int_0^\infty \frac{xe^{-\mu x^2}\,dx}{\sqrt{a^2+x^2}} = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{a^2\mu}\left[1-\Phi\left(a\sqrt{\mu}\right)\right]$$ $[\operatorname{Re}\mu > 0, \quad a > 0]$ NT 19(11)

3.469

1. $$\int_0^\infty e^{-\mu x^4-2\nu x^2}\,dx = \frac{1}{4}\sqrt{\frac{2\nu}{\mu}}\exp\left(\frac{\nu^2}{2\mu}\right)K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right)$$ $[\operatorname{Re}\mu \ge 0]$ ET I 146(23)

2. $$\int_0^\infty \left(e^{-x^4} - e^{-x}\right)\frac{dx}{x} = \frac{3}{4}\boldsymbol{C}$$ BI (89)(7)

3. $$\int_0^\infty \left(e^{-x^4} - e^{-x^2}\right)\frac{dx}{x} = \frac{1}{4}\boldsymbol{C}$$ BI (89)(6)

3.471

1. $$\int_0^u \exp\left(-\frac{\beta}{x}\right)\frac{dx}{x^2} = \frac{1}{\beta}\exp\left(-\frac{\beta}{u}\right)$$ ET II 188(22)

2. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{\nu-1}{2}}u^{\frac{2\mu+\nu-1}{2}}\exp\left(-\frac{\beta}{2u}\right)\Gamma(\mu)\,W_{\frac{1-2\mu-\nu}{2},\frac{\nu}{2}}\left(\frac{\beta}{u}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0, \quad u > 0]$

ET II 187(18)

3. $$\int_0^u x^{-\mu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{-\mu}u^{\mu-1}\,\Gamma(\mu)\exp\left(-\frac{\beta}{u}\right)$$
$[\operatorname{Re}\mu > 0, \quad u > 0]$ ET II 187(16)

4. $$\int_0^u x^{-2\mu}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \frac{1}{\sqrt{\pi u}}\beta^{\frac{1}{2}-\mu}e^{-\frac{\beta}{2u}}\,\Gamma(\mu)\,K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$$
$[u > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$
ET II 187(17)

5. $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \mathrm{B}(1-\mu-\nu,\mu)u^{\mu+\nu-1}\;{}_1F_1\left(1-\mu-\nu;1-\nu;\frac{\beta}{u}\right)$$
$[0 < \operatorname{Re}\mu < \operatorname{Re}(1-\nu), \quad u > 0]$
ET II 203(15)

6. $$\int_u^\infty x^{-2\mu}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \sqrt{\frac{\pi}{u}}\beta^{\frac{1}{2}-\mu}\,\Gamma(\mu)\exp\left(\frac{\beta}{2u}\right)I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$$
$[\operatorname{Re}\mu > 0, \quad u > 0]$ ET II 202(14)

7. $$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\,\Gamma(1-\mu-\nu)e^{\frac{\beta}{2\gamma}}\,W_{\frac{\nu-1}{2}+\mu,-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right)$$
$[|\arg\gamma| < \pi, \quad \operatorname{Re}(1-\mu) > \operatorname{Re}\nu > 0]$
ET II 234(13)a

8. $$\int_0^u x^{-2\mu}\left(u^2-x^2\right)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}}u^{\mu-\frac{3}{2}}\,\Gamma(\mu)\,K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right)$$
$[\operatorname{Re}\beta > 0, \quad u > 0, \quad \operatorname{Re}\mu > 0]$
ET II 188(23)a

9. $$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu\left(2\sqrt{\beta\gamma}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$
ET II 82(23)a, LET I 146(29)

10. $$\int_0^\infty x^{\nu-1}\exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right]dx = 2\beta^\nu e^{\frac{i\nu\pi}{2}}\,K_{-\nu}(\beta\mu)$$
$\left[\operatorname{Im}\mu > 0, \quad \operatorname{Im}\left(\beta^2\mu\right) < 0;\ \text{note that}\ K_{-\nu} \equiv K_\nu\right]$ EH II 82(24)

11. $$\int_0^\infty x^{\nu-1}\exp\left[\frac{i\mu}{2}\left(x+\frac{\beta^2}{x}\right)\right]dx = i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}}\,H_{-\nu}^{(1)}(\beta\mu)$$
$\left[\operatorname{Im}\mu > 0, \quad \operatorname{Im}\left(\beta^2\mu\right) > 0\right]$
EH II 21(33)

12. $$\int_0^\infty x^{\nu-1}\exp\left(-x-\frac{\mu^2}{4x}\right)dx = 2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu)$$
$\left[|\arg\mu| < \frac{\pi}{2}, \operatorname{Re}\mu^2 > 0;\ \text{note that}\ K_{-\nu} \equiv K_\nu\right]$ WA 203(15)

13. $$\int_0^\infty \frac{x^{\nu-1}e^{-\frac{\beta}{x}}}{x+\gamma}\,dx = \gamma^{\nu-1}e^{\frac{\beta}{\gamma}}\,\Gamma(1-\nu)\,\Gamma\left(\nu,\frac{\beta}{\gamma}\right)$$
$[|\arg\gamma| < \pi, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu < 1]$
ET II 218(19)

14. $$\int_0^1 \frac{\exp\left(1-\frac{1}{x}\right)-x^{\nu}}{x(1-x)}\,dx = \psi(\nu)$$ $[\operatorname{Re}\nu > 0]$ BI (80)(7)

15. $$\int_0^\infty x^{-\frac{1}{2}} e^{-\gamma x-\beta/x}\,dx = \sqrt{\frac{\pi}{\gamma}}e^{-2\sqrt{\beta\gamma}}$$ $[\operatorname{Re}\beta \ge 0, \quad \operatorname{Re}\gamma > 0]$ ET 245 (5.6.1)

16. $$\int_0^\infty x^{n-\frac{1}{2}} e^{-px-q/x}\,dx = (-1)^n\sqrt{\pi}\frac{\partial^n}{\partial p^n}\left(p^{-1/2}e^{-2\sqrt{pq}}\right)$$ $[\operatorname{Re}p > 0, \quad \operatorname{Re}q > 0]$ PBM 344 (2.3.16(2))

3.472

1. $$\int_0^\infty \left(\exp\left(-\frac{a}{x^2}\right)-1\right)e^{-\mu x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\left[\exp\left(-2\sqrt{a\mu}\right)-1\right]$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}a > 0]$ ET I 146(30)

2. $$\int_0^\infty x^2\exp\left(-\frac{a}{x^2}-\mu x^2\right)dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu^3}}\left(1+2\sqrt{a\mu}\right)\exp\left(-2\sqrt{a\mu}\right)$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}a > 0]$ ET I 146(26)

3. $$\int_0^\infty \exp\left(-\frac{a}{x^2}-\mu x^2\right)\frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(-2\sqrt{a\mu}\right)$$ $[\operatorname{Re}\mu > 0, \quad a > 0]$ ET I 146(28)a

4. $$\int_0^\infty \exp\left[-\frac{1}{2a}\left(x^2+\frac{1}{x^2}\right)\right]\frac{dx}{x^4} = \sqrt{\frac{a\pi}{2}}(1+a)e^{-1/a}$$ $[a > 0]$ BI (98)(14)

5. $$\int_0^\infty x^{-n-1/2}e^{-px-q/x}\,dx = (-1)^n\sqrt{\frac{\pi}{p}}\frac{\partial^n}{\partial q^n}e^{-2\sqrt{pq}}$$ $[\operatorname{Re}p > 0, \quad \operatorname{Re}q > 0]$ PBM 344 (2.3.16(3))

3.473 $$\int_0^\infty \exp\left(-x^n\right)x^{(m+1/2)n-1}\,dx = \frac{(2m-1)!!}{2^m n}\sqrt{\pi}$$ BI (98)(6)

3.474

1. $$\int_0^1 \left\{\frac{n\exp\left(1-x^{-n}\right)}{1-x^n}-\frac{x^{np}}{1-x}\right\}\frac{dx}{x} = \frac{1}{n}\sum_{k=1}^{n}\psi\left(p+\frac{k-1}{n}\right)$$ $[p > 0]$ BI (80)(8)

2. $$\int_0^1 \left\{\frac{n\exp\left(1-x^{-n}\right)}{1-x^n}-\frac{\exp\left(1-\frac{1}{x}\right)}{1-x}\right\}\frac{dx}{x} = -\ln n$$ BI (80)(9)

3.475

1.[7] $$\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-\frac{1}{1+x^{2^{n+1}}}\right\}\frac{dx}{x} = -\frac{1}{2^n}\boldsymbol{C}$$ $[n \in \mathbb{Z}]$ BI (92)(14)

2. $$\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-\frac{1}{1+x^2}\right\}\frac{dx}{x} = -2^{-n}\boldsymbol{C}$$ BI (92)(13)

3. $$\int_0^\infty \left\{\exp\left(-x^{2^n}\right)-e^{-x}\right\}\frac{dx}{x} = \left(1-2^{-n}\right)\boldsymbol{C}$$ BI (89)(8)

3.476

1. $$\int_0^\infty [\exp(-\nu x^p) - \exp(-\mu x^p)] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu} \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ BI (89)(3)

2. $$\int_0^\infty [\exp(-x^p) - \exp(-x^q)] \frac{dx}{x} = \frac{p-q}{pq} \boldsymbol{C} \qquad [p > 0, \quad q > 0]$$ BI (89)(9)

3.477

1.[10] $$\int_{-\infty}^\infty \frac{e^{-a|x|}}{x-u} \, dx = e^{-au} \gamma(0, -au) - e^{au} \gamma(0, au) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Im} u \neq 0, \quad \arg u \neq 0]$$ MC

2.[8] $$\int_{-\infty}^\infty \frac{\operatorname{sign} x \exp(-a|x|)}{x-u} \, dx = -[\exp(a|u|) \operatorname{Ei}(-a|u|) - \exp(-a|u|) \operatorname{Ei}(a|u|)]$$
$$[a > 0]$$ ET II 251(36)

3.478

1. $$\int_0^\infty x^{\nu-1} \exp(-\mu x^p) \, dx = \frac{1}{p} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad p > 0]$$ BI(81)(8)a, ET I 313(15, 16)

2. $$\int_0^\infty x^{\nu-1} [1 - \exp(-\mu x^p)] \, dx = -\frac{1}{|p|} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right)$$
$$[\operatorname{Re}\mu > 0 \text{ and } -p < \operatorname{Re}\nu < 0 \text{ for } p > 0, \quad 0 < \operatorname{Re}\nu < -p \text{ for } p < 0]$$ ET I 313(18, 19)

3.[11] $$\int_0^u x^{\nu-1} (u-x)^{\mu-1} \exp(\beta x^n) \, dx = \mathrm{B}(\mu, \nu) u^{\mu+\nu-1} {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \ldots, \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \ldots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right)$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad n = 2, 3, \ldots]$$ ET II 187(15)

4. $$\int_0^\infty x^{\nu-1} \exp\left(-\beta x^p - \gamma x^{-p}\right) dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\beta\gamma}\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$$ ET I 313(17)

3.479

1. $$\int_0^\infty \frac{x^{\nu-1} \exp\left(-\beta\sqrt{1+x}\right)}{\sqrt{1+x}} \, dx = \frac{2}{\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu} \Gamma(\nu) \, K_{\frac{1}{2}-\nu}(\beta)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$$ ET I 313(14)

2.[11] $$\int_0^\infty \frac{x^{\nu-1} \exp\left(i\mu\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \, dx = i \frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2}\right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) H^{(1)}_{\frac{1-\nu}{2}}(\mu)$$
$$[\operatorname{Im}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ EH II 83(30)

3.481

1. $$\int_{-\infty}^\infty x e^x \exp(-\mu e^x) \, dx = -\frac{1}{\mu} (\boldsymbol{C} + \ln\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (100)(13)

2. $$\int_{-\infty}^{\infty} x e^{x} \exp\left(-\mu e^{2x}\right) dx = -\frac{1}{4}\left[\boldsymbol{C} + \ln(4\mu)\right]\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0]$$ BI (100)(14)

3.482

1.[3] $$\int_0^{\infty} \exp\left(nx - \beta\sinh x\right) dx = \frac{1}{2}\left[S_n(\beta) - \pi\,\mathbf{E}_n(\beta) - \pi\,Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(11)

2. $$\int_0^{\infty} \exp\left(-nx - \beta\sinh x\right) dx = (-1)^{n+1}\frac{1}{2}\left[S_n(\beta) + \pi\,\mathbf{E}_n(\beta) + \pi\,Y_n(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(12)

3. $$\int_0^{\infty} \exp\left(-\nu x - \beta\sinh x\right) dx = \frac{\pi}{\sin\nu\pi}\left[\mathbf{J}_\nu(\beta) - J_\nu(\beta)\right]$$
$$[\operatorname{Re}\beta > 0]$$ ET I 168(13)

3.483 $$\int_{-\infty}^{\infty} \frac{\exp\left(\nu\operatorname{arcsinh} x - iax\right)}{\sqrt{1+x^2}}\,dx = \begin{cases} 2\exp\left(-\dfrac{i\nu\pi}{2}\right) K_\nu(a) & \text{for } a > 0, \\ 2\exp\left(\dfrac{i\nu\pi}{2}\right) K_\nu(-a) & \text{for } a < 0 \end{cases} \qquad [|\operatorname{Re}\nu| < 1]$$
ET I 122(32)

3.484 $$\int_0^{\infty} \left[\left(1+\frac{a}{qx}\right)^{qx} - \left(1+\frac{a}{px}\right)^{px}\right]\frac{dx}{x} = \left(e^a - 1\right)\ln\frac{q}{p} \qquad [p > 0, \quad q > 0]$$ BI (89)(34)

3.485 $$\int_0^{\pi/2} \exp\left(-\tan^2 x\right) dx = \frac{\pi e}{2}\left[1 - \Phi(1)\right]$$

3.486[6] $$\int_0^1 x^{-x}\,dx = \int_0^1 e^{-x\ln x}\,dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627\ldots$$ FI II 483

3.487

1.* $$\int_0^{\pi/4} \exp\left[-\sum_{k=0}^{\infty}\left(\frac{\tan^{2k+1} x}{k+\frac{1}{2}}\right)\right] dx = \ln 2$$

3.5 Hyperbolic Functions

3.51 Hyperbolic functions

3.511

1. $$\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a} \qquad [a > 0]$$

2. $$\int_0^{\infty} \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\frac{a\pi}{2b} \qquad [b > |a|]$$ BI (27)(10)a

3. $$\int_0^{\infty} \frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\sec\frac{a\pi}{2b} - \frac{1}{b}\beta\left(\frac{a+b}{2b}\right) \qquad [b > |a|]$$ GW (351)(3b)

4. $$\int_0^{\infty} \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\sec\frac{a\pi}{2b} \qquad [b > |a|]$$ BI (4)(14)a

5. $$\int_0^\infty \frac{\sinh ax \cosh bx}{\sinh cx}\,dx = \frac{\pi}{2c}\frac{\sin\frac{a\pi}{c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}}$$ $[c > |a| + |b|]$ BI (27)(11)

6. $$\int_0^\infty \frac{\cosh ax \cosh bx}{\cosh cx}\,dx = \frac{\pi}{c}\frac{\cos\frac{a\pi}{2c}\cos\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}}$$ $[c > |a| + |b|]$ BI (27)(5)a

7. $$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh cx}\,dx = \frac{\pi}{c}\frac{\sin\frac{a\pi}{2c}\sin\frac{b\pi}{2c}}{\cos\frac{a\pi}{c}+\cos\frac{b\pi}{c}}$$ $[c > |a| + |b|]$ BI (27)(6)a

8.[11] $$\int_0^\infty \frac{dx}{\cosh^2 x} = 1$$ BI (98)(25)

9. $$\int_{-\infty}^\infty \frac{\sinh^2 ax}{\sinh^2 x}\,dx = 1 - a\pi\cot a\pi$$ $\left[a^2 < 1\right]$ BI (16)(3)a

10. $$\int_0^\infty \frac{\sinh ax \sinh bx}{\cosh^2 bx}\,dx = \frac{a\pi}{2b^2}\sec\frac{a\pi}{2b}$$ $[b > |a|]$ BI (27)(16)a

3.512

1. $$\int_0^\infty \frac{\cosh 2\beta x}{\cosh^{2\nu} ax}\,dx = \frac{4^{\nu-1}}{a}\,\mathrm{B}\left(\nu+\frac{\beta}{a}, \nu-\frac{\beta}{a}\right)$$ $[\operatorname{Re}(\nu\pm\beta) > 0, \quad a > 0, \quad \beta > 0]$

LI(27)(17)a, EH I 11(26)

2. $$\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x}\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2}\right)$$ $[\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) < 0]$

EH I 11(23)

3.513

1. $$\int_0^\infty \frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\ln\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}$$ $[ab \neq 0]$ GW (351)(8)

2. $$\int_0^\infty \frac{dx}{a+b\cosh x} = \frac{2}{\sqrt{b^2-a^2}}\arctan\frac{\sqrt{b^2-a^2}}{a+b}$$ $\left[b^2 > a^2\right]$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}}$$ $\left[b^2 < a^2\right]$

GW (351)(7)

3. $$\int_0^\infty \frac{dx}{a\sinh x+b\cosh x} = \frac{2}{\sqrt{b^2-a^2}}\arctan\frac{\sqrt{b^2-a^2}}{a+b}$$ $\left[b^2 > a^2\right]$
$$= \frac{1}{\sqrt{a^2-b^2}}\ln\frac{a+b+\sqrt{a^2-b^2}}{a+b-\sqrt{a^2-b^2}}$$ $\left[a^2 > b^2\right]$

GW (351)(9)

4. $$\int_0^\infty \frac{dx}{a+b\cosh x+c\sinh x} = \frac{2}{\sqrt{b^2-a^2-c^2}}\left[\arctan\frac{\sqrt{b^2-a^2-c^2}}{a+b+c}+\epsilon\pi\right]$$
$$\left[\text{when } b^2>a^2+c^2;\ \text{and} \begin{cases}\epsilon=0 & \text{for } (b-a)(a+b+c)>0\\ |\epsilon|=1 & \text{for } (b-a)(a+b+c)<0\\ \epsilon=1 & \text{for } a<b+c\\ \epsilon=-1 & \text{for } a>b+c\end{cases}\right]$$
$$= \frac{1}{\sqrt{a^2-b^2+c^2}}\ln\frac{a+b+c+\sqrt{a^2-b^2+c^2}}{a+b+c-\sqrt{a^2-b^2+c^2}} \qquad \left[b^2<a^2+c^2, \quad a^2\neq b^2\right]$$
$$= \frac{1}{c}\ln\frac{a+c}{a} \qquad [a=b\neq 0, \quad c\neq 0]$$
$$= \frac{2(a-b)}{c(a-b-c)} \qquad \left[b^2=a^2+c^2, \quad c(a-b-c)<0\right]$$
GW (351)(6)

3.514

1. $$\int_0^\infty \frac{dx}{\cosh ax+\cos t} = \frac{t}{a}\operatorname{cosec} t \qquad [0<t<\pi, \quad a>0]$$ BI (27)(22)a

2. $$\int_0^\infty \frac{\cosh ax-\cos t_1}{\cosh bx-\cos t_2}\,dx = \frac{\pi}{b}\frac{\sin\frac{a(\pi t_2)}{b}}{\sin t_2\sin\frac{a}{b}\pi} - \frac{\pi t_2}{b\sin t_2}\cos t_1$$
$$[0<|a|<b, \quad 0<t_2<\pi]$$ BI (6)(20)a

3. $$\int_0^\infty \frac{\cosh ax\,dx}{(\cosh x+\cos t)^2} = \frac{\pi\left(-\cos t\sin at+a\sin t\cos at\right)}{\sin^3 t\sin a\pi}$$
$$\left[0<a^2<1, \quad 0<t<\pi\right]$$ BI (6)(18)a

4. $$\int_0^\infty \frac{\sinh ax\sinh bx}{(\cosh ax+\cos t)^2}\,dx = \frac{b\pi}{a^2}\operatorname{cosec} t\operatorname{cosec}\frac{b\pi}{a}\sin\frac{bt}{a} \qquad [0<|b|<a, \quad 0<t<\pi]$$ BI (27)(27)a

3.515 $$\int_{-\infty}^\infty \left(1-\frac{\sqrt{2}\cosh x}{\sqrt{\cosh 2x}}\right)dx = -\ln 2$$ BI (21)(12)a

3.516

1. $$\int_0^\infty \frac{dx}{\left(z+\sqrt{z^2-1}\cosh x\right)^\mu} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\left(z+\sqrt{z^2-1}\cosh x\right)^\mu} = Q_{\mu-1}(z)$$
$$[\operatorname{Re}\mu>-1]$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of z in the z-plane cut from -1 to $+1$ provided $\mu<0$. If $\mu>0$, this formula ceases to be valid for points at which the denominator vanishes. CO, WH

1. $$\int_0^\infty \frac{dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{n+1}} = Q_n(\beta)$$ EH II 181(32)

2. $$\int_0^\infty \frac{\cosh\gamma x\,dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{\nu+1}} = \frac{e^{-i\gamma\pi}\,\Gamma(\nu-\gamma+1)\,Q_\nu^\gamma(\beta)}{\Gamma(\nu+1)}$$

$[\operatorname{Re}(\nu\pm\gamma)>-1, \quad \nu\neq-1,-2,-3,\ldots]$ EH I 157(12)

3. $$\int_0^\infty \frac{\sinh^{2\mu}x\,dx}{\left(\beta+\sqrt{\beta^2-1}\cosh x\right)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}\,\Gamma(\nu-2\mu+1)\,\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{\pi}\,(\beta^2-1)^{\frac{\mu}{2}}\,\Gamma(\nu+1)}\,Q_{\nu-\mu}^\mu(\beta)$$

$[\operatorname{Re}(\nu-2\mu+1)>0, \quad \operatorname{Re}(\nu+1)>0]$ EH I 155(2)

3.517

1. $$\int_0^\infty \frac{\cosh\left(\gamma+\frac{1}{2}\right)x\,dx}{(\beta+\cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\left(\beta^2-1\right)^{-\frac{\nu}{2}}\frac{\Gamma(\nu+\gamma+1)\,\Gamma(\nu-\gamma)\,P_\gamma^{-\nu}(\beta)}{\Gamma\left(\nu+\frac{1}{2}\right)}$$

$[\operatorname{Re}(\nu-\gamma)>0, \quad \operatorname{Re}(\nu+\gamma+1)>0]$ EH I 156(11)

2. $$\int_0^a \frac{\cosh\left(\gamma+\frac{1}{2}\right)x\,dx}{(\cosh a-\cosh x)^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\sinh^\nu a}\,P_\gamma^\nu(\cosh a)$$

$\left[\operatorname{Re}\nu<\frac{1}{2}, \quad a>0\right]$ EH I 156(8)

3.518

1. $$\int_0^\infty \frac{\sinh^{2\mu}x\,dx}{(\cosh a+\sinh a\cosh x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi}\sinh^\mu a}\frac{\Gamma(\nu-2\mu+1)\,\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\nu+1)}\,Q_{\nu-\mu}^\mu(\cosh a)$$

$[\operatorname{Re}(\nu+1)>0, \quad \operatorname{Re}(\nu-2\mu+1)>0, \quad a>0]$ EH I 155(3)a

2.[10] $$\int_0^\infty \frac{\sinh^{2\mu+1}x\,dx}{(\beta+\cosh x)^{\nu+1}} = 2^\mu\left(\beta^2-1\right)^{\frac{\mu-\nu}{2}}\Gamma(\nu-2\mu)\,\Gamma(\mu+1)\,P_\mu^{\mu-\nu}(\beta)$$

$[\operatorname{Re}(\nu-\mu)>\operatorname{Re}\mu>-1$, β does not lie on the ray $(-\infty,+1)$ of the real axis$]$ EH I 155(1)

3. $$\int_0^\infty \frac{\sinh^{2\mu-1}x\cosh x\,dx}{\left(1+a\sinh^2 x\right)^\nu} = \frac{1}{2}a^{-\mu}\,\mathrm{B}(\mu,\nu-\mu)$$

$[\operatorname{Re}\nu>\operatorname{Re}\mu>0, \quad a>0]$ EH I 11(22)

4.[7] $$\int_0^\infty \frac{\sinh^{\mu-1}x\,(\cosh x+1)^{\nu-1}\,dx}{(\beta+\cosh x)^\varrho} = 2^{\mu+\nu-\rho}\,\mathrm{B}\left(\frac{1}{2}\mu,\varrho+2-\mu-\nu\right) \times {}_2F_1\left(\varrho,\varrho+2-\mu-\nu;2-\frac{1}{2}\mu-\nu;\frac{1}{2}-\frac{1}{2}\beta\right)$$

$[\operatorname{Re}\mu>0, \quad \operatorname{Re}(\varrho-\mu-\nu)>-2, \quad |\arg(1+\beta)|<\pi]$ EH I 115(11)

5.[6] $$\int_0^\infty \frac{\sinh^{\mu-1}x\,(\cosh x-1)^{\nu-1}\,dx}{(\beta+\cosh x)^\varrho} = 2^{-(2-\mu-\nu+\varrho)}\,{}_2F_1\left(\varrho,2-\mu-\nu+\varrho;1+\varrho-\frac{\mu}{2};\frac{1-\beta}{2}\right) \times \mathrm{B}\left(2-\mu-\nu+\varrho,-1+\nu+\frac{\mu}{2}\right)$$

$[\beta\notin(-\infty,-1), \quad \operatorname{Re}(2+\varrho)\operatorname{Re}(\mu+\nu), \quad \operatorname{Re}(2\nu+\mu)>2]$ EH I 115(10)

6.[7] $$\int_0^\infty \frac{\sinh^{\mu-1} x \cosh^{\nu-1} x}{\left(\cosh^2 x - \beta\right)^\varrho}\,dx = {}_2F_1\left(\varrho, 1+\varrho-\frac{\mu+\nu}{2}; 1+\varrho-\frac{\nu}{2}; \beta\right) 2\,\mathrm{B}\left(\frac{\mu}{2}, 1+\varrho-\frac{\mu+\nu}{2}\right)$$
$[\beta \notin (1,\infty), \quad \operatorname{Re}\mu > 0, \quad 2\operatorname{Re}(1+\varrho) > \operatorname{Re}(\mu+\nu)]$ EH I 115(9)

3.519 $$\int_0^{\pi/2} \frac{\sinh\left[(r-p)\right]\tan x}{\sinh\left(r\tan x\right)}\,dx = \pi \sum_{k=1}^\infty \frac{1}{k\pi+r} \sin\frac{pk\pi}{r}$$ $\left[p^2 < r^2\right]$ BI (274)(13)

3.52–3.53 Combinations of hyperbolic functions and algebraic functions

3.521

1. $$\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$ $[a > 0]$ GW (352)(2b)

2. $$\int_0^\infty \frac{x\,dx}{\cosh x} = 2\boldsymbol{G} = \pi\ln 2 - 4\,L\left(\frac{\pi}{4}\right) = 1.831931188\ldots$$ LI III 225(103a), BI(84)(1)a

3. $$\int_1^\infty \frac{dx}{x\sinh ax} = -2\sum_{k=0}^\infty \operatorname{Ei}[-(2k+1)a]$$ $[a > 0]$ LI (104)(14)

4. $$\int_1^\infty \frac{dx}{x\cosh ax} = 2\sum_{k=0}^\infty (-1)^{k+1} \operatorname{Ei}[-(2k+1)a]$$ $[a > 0]$ LI (104)(13)

3.522

1. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\sinh ax} = \frac{\pi}{2ab} + \pi\sum_{k=1}^\infty \frac{(-1)^k}{ab+k\pi}$$ $[a > 0, \quad b > 0]$

2. $$\int_0^\infty \frac{x\,dx}{(b^2+x^2)\sinh \pi x} = \frac{1}{2b} - \beta(b+1)$$ $[b > 0]$ BI(97)(16), GW(352)(8)

3. $$\int_0^\infty \frac{dx}{(b^2+x^2)\cosh ax} = \frac{2\pi}{b}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2ab+(2k-1)\pi}$$ $[a > 0, \quad b > 0]$ BI (97)(5)

4. $$\int_0^\infty \frac{dx}{(b^2+x^2)\cosh \pi x} = \frac{1}{b}\,\beta\left(b+\frac{1}{2}\right)$$ $[b > 0]$ BI (97)(4)

5. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \pi x} = \ln 2 - \frac{1}{2}$$ BI (97)(7)

6. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh \pi x} = 2 - \frac{\pi}{2}$$ BI (97)(1)

7. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \frac{\pi x}{2}} = \frac{\pi}{2} - 1$$ BI (97)(8)

8. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh \frac{\pi x}{2}} = \ln 2$$ BI (97)(2)

9. $$\int_0^\infty \frac{x\,dx}{(1+x^2)\sinh \frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi + 2\ln\left(\sqrt{2}+1\right)\right] - 2$$ BI (97)(9)

10. $$\int_0^\infty \frac{dx}{(1+x^2)\cosh\frac{\pi x}{4}} = \frac{1}{\sqrt{2}}\left[\pi - 2\ln\left(\sqrt{2}+1\right)\right]$$ BI (97)(3)

3.523

1. $$\int_0^\infty \frac{x^{\beta-1}}{\sinh ax}\,dx = \frac{2^\beta - 1}{2^{\beta-1}a^\beta}\,\Gamma(\beta)\,\zeta(\beta)$$ $[\operatorname{Re}\beta > 1, \quad a > 0]$ WH

2. $$\int_0^\infty \frac{x^{2n-1}}{\sinh ax}\,dx = \frac{2^{2n}-1}{2n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$ $[a > 0, \quad n = 1, 2, \ldots]$ WH, GW(352)(2a)

3. $$\int_0^\infty \frac{x^{\beta-1}}{\cosh ax}\,dx = \frac{2}{(2a)^\beta}\,\Gamma(\beta)\,\Phi\left(-1,\beta,\frac{1}{2}\right) = \frac{2}{(2a)^\beta}\,\Gamma(\beta)\sum_{k=0}^\infty (-1)^k\left(\frac{2}{2k+1}\right)^\beta$$ $[\operatorname{Re}\beta > 0, \quad a > 0]$ EH I 35, ET I 322(1)

4. $$\int_0^\infty \frac{x^{2n}}{\cosh ax}\,dx = \left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|$$ $[a > 0]$ BI(84)(12)a, GW(352)(1a)

5. $$\int_0^\infty \frac{x^2\,dx}{\cosh x} = \frac{\pi^3}{8}$$ (cf. **4.261** 6) BI (84)(3)

6. $$\int_0^\infty \frac{x^3\,dx}{\sinh x} = \frac{\pi^4}{8}$$ (cf. **4.262** 1 and 2) BI (84)(5)

7. $$\int_0^\infty \frac{x^4\,dx}{\cosh x} = \frac{5}{32}\pi^5$$ BI (84)(7)

8. $$\int_0^\infty \frac{x^5}{\sinh x}\,dx = \frac{\pi^6}{4}$$ BI (84)(8)

9. $$\int_0^\infty \frac{x^6}{\cosh x}\,dx = \frac{61}{128}\pi^7$$ BI (84)(9)

10. $$\int_0^\infty \frac{x^7}{\sinh x}\,dx = \frac{17}{16}\pi^8$$ BI (84)(10)

11. $$\int_0^\infty \frac{x^{1/2}\,dx}{\cosh x} = \sqrt{\pi}\sum_{k=0}^\infty (-1)^k \frac{1}{(2k+1)^{3/2}}$$ BI (98)(7)a

12. $$\int_0^\infty \frac{dx}{x^{1/2}\cosh x} = 2\sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{1/2}}$$ BI (98)(25)a

3.524

1. $$\int_0^\infty x^{\mu-1}\frac{\sinh\beta x}{\sinh\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\zeta\left[\mu,\frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right] - \zeta\left[\mu,\frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]\right\}$$ $[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > -1]$ ET I 323(10)

2.[11] $$\int_0^\infty x^{2m}\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m}}{da^{2m}}\left(\tan\frac{a\pi}{2b}\right)$$ $[b > |a|]$ BI (112)(20)a

3. $$\int_0^\infty \frac{\sinh ax}{\sinh bx}\frac{dx}{x^p} = \Gamma(1-p)\sum_{k=0}^\infty \left\{\frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}}\right\}$$
$[b > |a|, \quad p < 1]$ BI (131)(2)a

4.[11] $$\int_0^\infty x^{2m+1}\frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m+1}}{da^{2m+1}}\left(\sec\frac{a\pi}{2b}\right)$$ $[b > |a|]$ BI (112)(18)a

5. $$\int_0^\infty x^{\mu-1}\frac{\cosh\beta x}{\sinh\gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu}\left\{\zeta\left[\mu, \frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right] + \zeta\left[\mu, \frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right]\right\}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > 1]$
ET I 323(12)

6. $$\int_0^\infty x^{2m}\frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m}}{da^{2m}}\left(\sec\frac{a\pi}{2b}\right)$$ $[b > |a|]$ BI(112)(17)

7. $$\int_0^\infty \frac{\cosh ax}{\cosh bx}\cdot\frac{dx}{x^p} = \Gamma(1-p)\sum_{k=0}^\infty (-1)^k\left\{\frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}}\right\}$$
$[b > |a|, \quad p < 1]$ BI(131)(1)a

8. $$\int_0^\infty x^{2m+1}\frac{\cosh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\frac{d^{2m+1}}{da^{2m+1}}\left(\tan\frac{a\pi}{2b}\right)$$ $[b > |a|]$ BI (112)(19)a

9.[8] $$\int_0^\infty x^2\frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi^3}{4b^3}\sin\frac{a\pi}{2b}\sec^3\frac{a\pi}{2b}$$ $[b > |a|]$ BI (84)(18)

10. $$\int_0^\infty x^4\frac{\sinh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^5\cdot\sin\frac{a\pi}{2b}\cdot\left(2+\sin^2\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(17)a

11. $$\int_0^\infty x^6\frac{\sinh ax}{\sinh bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^7\sin\frac{a\pi}{2b}\left(45-30\cos^2\frac{a\pi}{2b}+2\cos^4\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(21)a

12. $$\int_0^\infty x\frac{\sinh ax}{\cosh bx}\,dx = \frac{\pi^2}{4b^2}\sin\frac{a\pi}{2b}\sec^2\frac{a\pi}{2b}$$ $[b > |a|]$ BI (84)(15)a

13. $$\int_0^\infty x^3\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4\sin\frac{a\pi}{2b}\cdot\left(6-\cos^2\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(14)a

14. $$\int_0^\infty x^5\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^6\sin\frac{a\pi}{2b}\left(120-60\cos^2\frac{a\pi}{2b}+\cos^4\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(18)a

15. $$\int_0^\infty x^7\frac{\sinh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8\sin\frac{a\pi}{2b}\left(5040-4200\cos^2\frac{a\pi}{2b}+546\cos^4\frac{a\pi}{2b}-\cos^6\frac{a\pi}{2b}\right)$$
$[b > |a|]$ BI (82)(22)a

16. $$\int_0^\infty x\frac{\cosh ax}{\sinh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^2$$ $[b > |a|]$ BI (84)(16)a

17. $$\int_0^\infty x^3 \frac{\cosh ax}{\sinh bx}\,dx = 2\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^4\left(1+2\sin^2\frac{a\pi}{2b}\right) \qquad [b>|a|]$$ BI (82)(15)a

18. $$\int_0^\infty x^5 \frac{\cosh ax}{\sinh bx}\,dx = 8\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^6\left(15-15\cos^2\frac{a\pi}{2b}+2\cos^4\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(19)a

19. $$\int_0^\infty x^7 \frac{\cosh ax}{\sinh bx}\,dx = 16\left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^8\left(315-420\cos^2\frac{a\pi}{2b}+126\cos^4\frac{a\pi}{2b}-4\cos^6\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI(82)(23)a

20. $$\int_0^\infty x^2 \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi^3}{8b^3}\left(2\sec^3\frac{a\pi}{2b}-\sec\frac{a\pi}{2b}\right) \qquad [b>|a|]$$ BI (84)(17)a

21. $$\int_0^\infty x^4 \frac{\cosh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^5\left(24-20\cos^2\frac{a\pi}{2b}+\cos^4\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(16)a

22. $$\int_0^\infty x^6 \frac{\cosh ax}{\cosh bx}\,dx = \left(\frac{\pi}{2b}\sec\frac{a\pi}{2b}\right)^7\left(720-840\cos^2\frac{a\pi}{2b}+182\cos^4\frac{a\pi}{2b}-\cos^6\frac{a\pi}{2b}\right)$$
$[b>|a|]$ BI (82)(20)a

23. $$\int_0^\infty \frac{\sinh ax}{\cosh bx}\cdot\frac{dx}{x} = \ln\tan\left(\frac{a\pi}{4b}+\frac{\pi}{4}\right) \qquad [b>|a|]$$ BI (95)(3)a

3.525

1. $$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cdot\frac{dx}{1+x^2} = -\frac{a}{2}\cos a+\frac{1}{2}\sin a\ln\left[2\left(1+\cos a\right)\right]$$
$[\pi\ge|a|]$ BI (97)(10)a

2. $$\int_0^\infty \frac{\sinh ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\sin a+\frac{1}{2}\cos a\ln\frac{1-\sin a}{1+\sin a} \qquad [\pi\ge2|a|]$$ BI (97)(11)a

3. $$\int_0^\infty \frac{\cosh ax}{\sinh \pi x}\cdot\frac{x\,dx}{1+x^2} = \frac{1}{2}\left(a\sin a-1\right)+\frac{1}{2}\cos a\ln\left[2\left(1+\cos a\right)\right]$$
$[\pi>|a|]$ BI (97)(12)a

4. $$\int_0^\infty \frac{\cosh ax}{\sinh \frac{\pi}{2}x}\cdot\frac{x\,dx}{1+x^2} = \frac{\pi}{2}\cos a-1+\frac{1}{2}\sin a\ln\frac{1+\sin a}{1-\sin a}$$
$\left[\frac{\pi}{2}>|a|\right]$ BI (97)(13)a

5. $$\int_0^\infty \frac{\sinh ax}{\cosh \pi x}\cdot\frac{x\,dx}{1+x^2} = -2\sin\frac{a}{2}+\frac{\pi}{2}\sin a-\cos a\ln\tan\frac{a+\pi}{4}$$
$[\pi>|a|]$ GW (352)(12)

6. $$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\cdot\frac{dx}{1+x^2} = 2\cos\frac{a}{2}-\frac{\pi}{2}\cos a-\sin a\ln\tan\frac{a+\pi}{4}$$
$[\pi>|a|]$ GW (352)(11)

7. $$\int_0^\infty \frac{\sinh ax}{\sinh bx} \cdot \frac{dx}{c^2+x^2} = \frac{\pi}{c} \sum_{k=1}^\infty \frac{\sin \frac{k(b-a)}{b}\pi}{bc+k\pi} \qquad [b \geq |a|]$$ BI (97)(18)

8. $$\int_0^\infty \frac{\cosh ax}{\sinh bx} \cdot \frac{x\,dx}{c^2+x^2} = \frac{\pi}{2bc} + \pi \sum_{k=1}^\infty \frac{\cos \frac{k(b-a)}{b}\pi}{bc+k\pi} \qquad [b > |a|]$$ BI (97)(19)

3.526

1. $$\int_0^\infty \frac{\sinh ax \cosh bx}{\cosh cx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \left\{ \tan \frac{(a+b+c)\pi}{4c} \cot \frac{(b+c-a)\pi}{4c} \right\}$$ $$[c > |a| + |b|]$$ BI (93)(10)a

2. $$\int_0^\infty \frac{\sinh^2 ax}{\sinh bx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \sec \frac{a}{b}\pi \qquad [b > |2a|]$$ BI (95)(5)a

3. $$\int_0^\infty \frac{x^{\mu-1}}{\sinh \beta x \cosh \gamma x}\,dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \Phi\left[-1, \mu, \frac{1}{2}\left(1+\frac{\beta}{\gamma}\right)\right] + \Phi\left[-1, \mu, \frac{1}{2}\left(1-\frac{\beta}{\gamma}\right)\right]\right\}$$ $$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, \quad \operatorname{Re}\mu > 0]$$ ET I 323(11)

3.527

1. $$\int_0^\infty \frac{x^{\mu-1}}{\sinh^2 ax}\,dx = \frac{4}{(2a)^\mu} \Gamma(\mu)\,\zeta(\mu-1) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\mu > 2]$$ BI (86)(7)a

2. $$\int_0^\infty \frac{x^{2m}}{\sinh^2 ax}\,dx = \frac{\pi^{2m}}{a^{2m+1}} |B_{2m}| \qquad [a > 0, \quad m = 1, 2, \ldots]$$ BI(86)(5)a

3.[6] $$\int_0^\infty \frac{x^{\mu-1}}{\cosh^2 ax}\,dx = \frac{4}{(2a)^\mu}\left(1 - 2^{2-\mu}\right)\Gamma(\mu)\,\zeta(\mu-1) \qquad [\operatorname{Re} a > 0, \quad \operatorname{Re}\mu > 0, \quad \mu \neq 2]$$ $$= \frac{1}{a^2}\ln 2 \qquad [\operatorname{Re} a > 0, \quad \mu = 2]$$ BI (86)(6)a

4. $$\int_0^\infty \frac{x\,dx}{\cosh^2 ax} = \frac{\ln 2}{a^2} \qquad [a \neq 0]$$ LO III 396

5. $$\int_0^\infty \frac{x^{2m}}{\cosh^2 ax}\,dx = \frac{\left(2^{2m}-2\right)\pi^{2m}}{(2a)^{2m}a} |B_{2m}| \qquad [a > 0, \quad m = 1, 2, \ldots]$$ BI(86)(2)a

6. $$\int_0^\infty x^{\mu-1} \frac{\sinh ax}{\cosh^2 ax}\,dx = \frac{2\,\Gamma(\mu)}{a^\mu} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\mu-1}} \qquad [\operatorname{Re}\mu > 1, \quad a > 0]$$ BI (86)(15)a

7. $$\int_0^\infty \frac{x \sinh ax}{\cosh^2 ax}\,dx = \frac{\pi}{2a^2} \qquad [a > 0]$$ BI (86)(8)a

8. $$\int_0^\infty x^{2m+1} \frac{\sinh ax}{\cosh^2 ax}\,dx = \frac{2m+1}{a}\left(\frac{\pi}{2a}\right)^{2m+1} |E_{2m}| \qquad [a > 0, \quad m = 0, 1, \ldots]$$ BI (86)(12)a

9. $$\int_0^\infty x^{2m+1} \frac{\cosh ax}{\sinh^2 ax}\,dx = \frac{2^{2m+1}-1}{a^2(2a)^{2m}} (2m+1)!\,\zeta(2m+1)$$ $$[a \neq 0, \quad m = 1, 2, \ldots]$$ BI (86)(13)a

10.[11] $$\int_0^\infty x^{2m} \frac{\cosh ax}{\sinh^2 ax}\, dx = \frac{2^{2m}-1}{a} \left(\frac{\pi}{a}\right)^{2m} |B_{2m}|$$ $[a > 0, \quad m = 1, 2, \ldots]$ BI (86)(14)a

11.[8] $$\int_0^\infty \frac{x \sinh ax}{\cosh^{2\mu+1} ax}\, dx = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\Gamma\left(\mu + \frac{1}{2}\right)}$$ $[\mu > 0, \quad a > 0]$ LI (86)(9)

12. $$\int_{-\infty}^\infty \frac{x^2\, dx}{\sinh^2 x} = \frac{\pi^2}{3}$$ BI (102)(2)a

13. $$\int_0^\infty x^2 \frac{\cosh ax}{\sinh^2 ax}\, dx = \frac{\pi^2}{2a^3}$$ $[a > 0]$ BI (86)(11)a

14.[11] $$\int_0^\infty x^2 \frac{\sinh x}{\cosh^2 x}\, dx = 4\boldsymbol{G}$$ $[a \neq 0]$ BI (86)(10)a

15.[10] $$\int_0^\infty \frac{\tanh \frac{x}{2}\, dx}{\cosh x} = \ln 2$$ BI (93)(17)a

16.* $$\int_0^\infty x^{\mu-1} \frac{\cosh ax}{\sinh^2 ax} = \frac{2\,\Gamma(\mu)\,\zeta(\mu-1)}{a^\mu} \left(1 - 2^{1-\mu}\right)$$

3.528

1. $$\int_0^\infty \frac{(1+xi)^{2n-1} - (1-xi)^{2n-1}}{i \sinh \frac{\pi x}{2}}\, dx = 2$$ BI (87)(8)

2. $$\int_0^\infty \frac{(1+xi)^{2n} - (1-xi)^{2n}}{i \sinh \frac{\pi x}{2}}\, dx = (-1)^{n+1} 2|E_{2n}| + 2$$ $[n = 0, 1, \ldots]$ BI (87)(7)

3.529

1. $$\int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x}\right) \frac{dx}{x} = -\ln 2$$ BI (94)(10)a

2. $$\int_0^\infty \frac{\cosh ax - 1}{\sinh bx} \cdot \frac{dx}{x} = -\ln\cos\frac{a\pi}{2b}$$ $[b > |a|]$ GW (352)(66)

3. $$\int_0^\infty \left(\frac{a}{\sinh ax} - \frac{b}{\sinh bx}\right) \frac{dx}{x} = (b-a)\ln 2$$ BI (94)(11)a

3.531

1.[7] $$\int_0^\infty \frac{x\, dx}{2\cosh x - 1} = \frac{4}{\sqrt{3}} \left[\frac{\pi}{3} \ln 2 - L\left(\frac{\pi}{3}\right)\right] = 1.1719536193\ldots$$ [see **8.26** for $L(x)$] LI (88)(1)

2.[10] $$\int_0^\infty \frac{x\, dx}{\cosh 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t}$$ LO III 402

3. $$\int_0^\infty \frac{x^2\, dx}{\cosh x + \cos t} = \frac{t}{3} \cdot \frac{\pi^2 - t^2}{\sin t}$$ $[0 < t < \pi]$ BI (88)(3)a

4. $$\int_0^\infty \frac{x^4\, dx}{\cosh x + \cos t} = \frac{t}{15} \frac{\left(\pi^2 - t^2\right)\left(7\pi^2 - 3t^2\right)}{\sin t}$$ $[0 < t < \pi]$ BI (88)(4)a

5.[3] $$\int_0^\infty \frac{x^{2m}\,dx}{\cosh x - \cos 2a\pi} = 2(2m)!\operatorname{cosec} 2a\pi \sum_{k=1}^{\infty} \frac{\sin 2ka\pi}{k^{2m+1}} \qquad \left[0 < a < 1, \quad a \neq \tfrac{1}{2}\right]$$
$$= 2\left(2^{2m-1} - 1\right)\pi^{2m}|B_{2m}| \qquad \left[a = \tfrac{1}{2}\right]$$
BI (88)(5)a

6.[3] $$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh x - \cos t}$$
$$= \frac{i\,\Gamma(\mu)}{\sin t}\left[e^{-it}\,\Phi\left(e^{-it},\mu,1\right) - e^{it}\,\Phi\left(e^{it},\mu,1\right)\right] \qquad [\operatorname{Re}\mu > 0, \quad 0 < t < 2\pi, \quad t \neq \pi]$$ ET I 323(5)
$$= \left(2 - 2^{3-\mu}\right)\Gamma(\mu)\,\zeta(\mu-1) \qquad [\mu \neq 2, \quad t = \pi]$$
$$= 2\ln 2 \qquad [\mu = 2, \quad t = \pi]$$

7. $$\int_0^\infty \frac{x^\mu\,dx}{\cosh x + \cos t} = \frac{2\,\Gamma(\mu+1)}{\sin t}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k^{\mu+1}} \qquad [\mu > -1, \quad 0 < t < \pi]$$ BII (96)(14)a

8. $$\int_0^u \frac{x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2}\operatorname{cosec} 2t\,[L(\theta+t) - L(\theta-t) - 2\,L(t)]$$
$$[\theta = \arctan(\tanh u \cot t)\,, \quad t \neq n\pi]$$
LO III 402

3.532

1.[11] $$\int_0^\infty \frac{x^n\,dx}{a\cosh x + b\sinh x} = \frac{2n!}{a+b}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{n+1}}\left(\frac{b-a}{b+a}\right)^k$$
$$[a > 0, \quad b > 0, \quad n > -1]$$ GW (352)(5)

2. $$\int_0^u \frac{x\cosh x\,dx}{\cosh 2x - \cos 2t} = \frac{1}{2}\operatorname{cosec} t\left\{L\left(\frac{\theta+t}{2}\right) - L\left(\frac{\theta-t}{2}\right) + L\left(\pi - \frac{\psi+t}{2}\right)\right.$$
$$\left. + L\left(\frac{\psi-t}{2}\right) - 2\,L\left(\frac{t}{2}\right) - 2\,L\left(\frac{\pi-t}{2}\right)\right\}$$
$$\left[\tan\frac{\theta}{2} = \tanh\frac{u}{2}\cot\frac{t}{2}, \quad \tan\frac{\psi}{2} = \coth\frac{u}{2}\cot\frac{t}{2}; \quad t \neq n\pi\right]$$ LO III 288a

3.533

1. $$\int_0^\infty \frac{x\cosh x\,dx}{\cosh 2x - \cos 2t} = \operatorname{cosec} t\left[\frac{\pi}{2}\ln 2 - L\left(\frac{t}{2}\right) - L\left(\frac{(\pi-t)}{2}\right)\right]$$
$$[t \neq m\pi]$$ LO III 403

2.[6] $$\int_0^\infty x\frac{\sinh ax\,dx}{(\cosh ax - \cos t)^2} = \frac{\pi - t}{a^2}\operatorname{cosec} t \qquad [a > 0, \quad 0 < t < \pi]$$ (cf. **3.514** 1)
BI (88)(11)a

3. $$\int_0^\infty x^3\frac{\sinh x\,dx}{(\cosh x + \cos t)^2} = \frac{t\left(\pi^2 - t^2\right)}{\sin t} \qquad [0 < t < \pi]$$ (cf. **3.531** 3)
BI (88)(13)

4.[11] $$\int_0^\infty x^{2m+1} \frac{\sinh x\,dx}{(\cosh x - \cos 2a\pi)^2} = 2(2m+1)!\operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \qquad \left[0 < a < 1, \quad a \neq \tfrac{1}{2}\right]$$
$$= 2(2m+1)\left(2^{2m-1}-1\right)\pi^{2m}|B_{2m}| \qquad \left[a = \tfrac{1}{2}\right]$$
BI (88)(14)

3.534

1. $$\int_0^1 \sqrt{1-x^2}\cosh ax\,dx = \frac{\pi}{2a} I_1(a)$$ WA 94(9)

2. $$\int_0^1 \frac{\cosh ax}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} I_0(a)$$ WA 94(9)

3.535 $$\int_0^1 \frac{x}{\sqrt{\cosh 2a - \cosh 2ax}} \cdot \frac{dx}{\sinh ax} = \frac{\pi}{2\sqrt{2}a^2} \cdot \frac{\arcsin(\tanh a)}{\sinh a} \qquad [a > 0]$$ BI (80)(11)

3.536

1.[11] $$\int_0^\infty \frac{x^2}{\cosh^2 x}\,dx = \frac{\pi^2}{12}$$ BI (98)(7)

2. $$\int_0^\infty \frac{x^2 \tanh x^2\,dx}{\cosh^2 x} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}$$ BI (98)(8)

3. $$\int_0^\infty \sinh(\nu \operatorname{arcsinh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx = \frac{\sin\frac{\mu\pi}{2}\sin\frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu)\,\Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$$[-1 < \operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|]$$ ET I 324(14)

4. $$\int_0^\infty \cosh(\nu \operatorname{arccosh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}}\,dx = \frac{\cos\frac{\mu\pi}{2}\cos\frac{\nu\pi}{2}}{2^\mu \pi} \Gamma(\mu)\,\Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right)$$
$$[0 < \operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|]$$ ET I 324(15)

3.54 Combinations of hyperbolic functions and exponentials

3.541

1. $$\int_0^\infty e^{-\mu x}\sinh^\nu \beta x\,dx = \frac{1}{2^{\nu+1}\beta} \mathrm{B}\left(\frac{\mu}{2\beta} - \frac{\nu}{2}, \nu+1\right) \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -1, \operatorname{Re}\mu > \operatorname{Re}\beta\nu]$$
EH I 11(25), ET I 163(5)

2. $$\int_0^\infty e^{-\mu x} \frac{\sinh \beta x}{\sinh bx}\,dx = \frac{1}{2b}\left[\psi\left(\frac{1}{2} + \frac{\mu+\beta}{2b}\right) - \psi\left(\frac{1}{2} + \frac{\mu-\beta}{2b}\right)\right]$$
$$[\operatorname{Re}(\mu + b \pm \beta) > 0]$$ EH I 16(14)a

3. $$\int_{-\infty}^\infty e^{-\mu x} \frac{\sinh \mu x}{\sinh \beta x}\,dx = \frac{\pi}{2\beta} \tan\frac{\mu\pi}{\beta} \qquad [\operatorname{Re}\beta > 2|\operatorname{Re}\mu|]$$ BI (18)(6)

4. $$\int_0^\infty e^{-x} \frac{\sinh ax}{\sinh x}\,dx = \frac{1}{a} - \frac{\pi}{2}\cot\frac{a\pi}{2} \qquad [0 < a < 2]$$ BI (4)(3)

5. $$\int_0^\infty \frac{e^{-px}\,dx}{(\cosh px)^{2q+1}} = \frac{2^{2q-2}}{p}\mathrm{B}(q,q) - \frac{1}{2qp} \qquad [p > 0, \quad q > 0]$$ LI (27)(19)

6. $$\int_0^\infty e^{-\mu x} \frac{dx}{\cosh x} = \beta\left(\frac{\mu+1}{2}\right)$$ $[\operatorname{Re}\mu > -1]$ ET I 163(7)

7. $$\int_0^\infty e^{-\mu x} \tanh x \, dx = \beta\left(\frac{\mu}{2}\right) - \frac{1}{\mu}$$ $[\operatorname{Re}\mu > 0]$ ET I 163(9)

8. $$\int_0^\infty \frac{e^{-\mu x}}{\cosh^2 x} \, dx = \mu\,\beta\left(\frac{\mu}{2}\right) - 1$$ $[\operatorname{Re}\mu > 0]$ ET I 163(8)

9. $$\int_0^\infty e^{-\mu x} \frac{\sinh \mu x}{\cosh^2 \mu x} \, dx = \frac{1}{\mu}(1 - \ln 2)$$ $[\operatorname{Re}\mu > 0]$ LI (27)(15)

10. $$\int_0^\infty e^{-qx} \frac{\sinh px}{\sinh qx} \, dx = \frac{1}{p} - \frac{\pi}{2q} \cot \frac{p\pi}{2q}$$ $[0 < p < 2q]$ BI (27)(9)a

3.542

1. $$\int_0^\infty e^{-\mu x} (\cosh \beta x - 1)^\nu \, dx = \frac{1}{2^\nu \beta} \mathrm{B}\left(\frac{\mu}{\beta} - \nu, 2\nu + 1\right)$$
$\left[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\mu > \operatorname{Re}\beta\nu\right]$ ET I 163(6)

2. $$\int_0^\infty e^{-\mu x} (\cosh x - \cosh u)^{\nu-1} \, dx = -i\sqrt{\frac{2}{\pi}} e^{i\pi\nu}\, \Gamma(\nu) \sinh^{\nu-\frac{1}{2}} u \; Q_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh u)$$
$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > \operatorname{Re}\nu - 1]$
EH I 155(4), ET I 164(23)

3.543

1. $$\int_{-\infty}^\infty \frac{e^{-ibx} \, dx}{\sinh x + \sinh t} = -\frac{i\pi e^{itb}}{\sinh \pi b \cosh t}\left(\cosh \pi b - e^{-2itb}\right)$$
$[t > 0]$ ET I 121(30)

2. $$\int_0^\infty \frac{e^{-\mu x}}{\cosh x - \cos t} \, dx = 2 \operatorname{cosec} t \sum_{k=1}^\infty \frac{\sin kt}{\mu + k}$$ $[\operatorname{Re}\mu > -1, \quad t \neq 2n\pi]$ BI (6)(10)a

3. $$\int_0^\infty \frac{1 - e^{-x} \cos t}{\cosh x - \cos t} e^{-(\mu-1)x} \, dx = 2 \sum_{k=0}^\infty \frac{\cos kt}{\mu + k}$$ $[\operatorname{Re}\mu > 0, \quad t \neq 2n\pi]$ BI (6)(9)a

4. $$\int_0^\infty \frac{e^{px} - \cos t}{(\cosh px + \cos t)^2} \, dx = \frac{1}{p}\left(t \operatorname{cosec} t + \frac{1}{1 + \cos t}\right)$$ $[p > 0]$ BI (27)(26)a

3.544 $$\int_u^\infty \frac{\exp\left[-\left(n + \frac{1}{2}\right) x\right]}{\sqrt{2(\cosh x - \cosh u)}} \, dx = Q_n(\cosh u), \qquad [u > 0]$$ EH II 181(33)

3.545

1. $$\int_0^\infty \frac{\sinh ax}{e^{px} + 1} \, dx = \frac{\pi}{2p} \operatorname{cosec} \frac{a\pi}{p} - \frac{1}{2a}$$ $[p > a, \quad p > 0]$ BI (27)(3)

2. $$\int_0^\infty \frac{\sinh ax}{e^{px} - 1} \, dx = \frac{1}{2a} - \frac{\pi}{2p} \cot \frac{a\pi}{p}$$ $[p > a, \quad p > 0]$ BI (27)(9)

3.546

1. $$\int_0^\infty e^{-\beta x^2} \sinh ax\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\beta}} \exp\frac{a^2}{4\beta}\,\Phi\left(\frac{a}{2\sqrt{\beta}}\right)$$ $[\operatorname{Re}\beta > 0]$ ET I166(38)a

2. $$\int_0^\infty e^{-\beta x^2} \cosh ax\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\frac{a^2}{4\beta}$$ $[\operatorname{Re}\beta > 0]$ FI II 720a

3. $$\int_0^\infty e^{-\beta x^2} \sinh^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left(\exp\frac{a^2}{\beta} - 1\right)$$ $[\operatorname{Re}\beta > 0]$ ET I 166(40)

4. $$\int_0^\infty e^{-\beta x^2} \cosh^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left(\exp\frac{a^2}{\beta} + 1\right)$$ $[\operatorname{Re}\beta > 0]$ ET I 166(41)

3.547

1. $$\int_0^\infty \exp(-\beta\sinh x)\sinh\gamma x\,dx = \frac{\pi}{2}\cot\frac{\gamma\pi}{2}\left[J_\gamma(\beta) - \mathbf{J}_\gamma(\beta)\right] - \frac{\pi}{2}\left[\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)\right] = \gamma\,S_{-1,\gamma}(\beta)$$ $[\operatorname{Re}\beta > 0]$ WA 341(5), ET I 168(14)a

2. $$\int_0^\infty \exp(-\beta\cosh x)\sinh\gamma x\sinh x\,dx = \frac{\gamma}{\beta}K_\gamma(\beta)$$

3. $$\int_0^\infty \exp(-\beta\sinh x)\cosh\gamma x\,dx = \frac{\pi}{2}\tan\frac{\pi\gamma}{2}\left[\mathbf{J}_\gamma(\beta) - J_\gamma(\beta)\right] - \frac{\pi}{2}\left[\mathbf{E}_\gamma(\beta) + Y_\gamma(\beta)\right] = S_{0,\gamma}(\beta)$$ $[\operatorname{Re}\beta > 0, \quad \gamma \text{ not an integer}]$ ET I 168(16)a, WA 341(4), EH II 84(50)

4. $$\int_0^\infty \exp(-\beta\cosh x)\cosh\gamma x\,dx = K_\gamma(\beta)$$ $[\operatorname{Re}\beta > 0]$ ET I 168(16)a, WA 201(5)

5. $$\int_0^\infty \exp(-\beta\sinh x)\sinh\gamma x\cosh x\,dx = \frac{\gamma}{\beta}S_{0,\gamma}(\beta)$$ $[\operatorname{Re}\beta > 0]$ ET I 168(7), EH II 85(51)

6. $$\int_0^\infty \exp(-\beta\sinh x)\sinh[(2n+1)x]\cosh x\,dx = O_{2n+1}(\beta)$$ $[\operatorname{Re}\beta > 0]$ ET I 167(5)

7. $$\int_0^\infty \exp(-\beta\sinh x)\cosh\gamma x\cosh x\,dx = \frac{1}{\beta}S_{1,\gamma}(\beta)$$ $[\operatorname{Re}\beta > 0]$

8. $$\int_0^\infty \exp(-\beta\sinh x)\cosh 2nx\cosh x\,dx = O_{2n}(\beta)$$ $[\operatorname{Re}\beta > 0]$ ET I 168(6)

9. $$\int_0^\infty \exp(-\beta\cosh x)\sinh^{2\nu} x\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right)K_\nu(\beta)$$ $\left[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ EH II 82(20)

10.[11] $$\int_0^\infty \exp\left[-2\left(\beta\coth x + \mu x\right)\right]\sinh^{2\nu} x\,dx = \frac{1}{2}\beta^\nu\,\Gamma\left(\mu - \nu\right)\,W_{-\mu,\nu-\frac{1}{2}}(4\beta)$$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > \operatorname{Re}\nu]$

11. $$\int_0^\infty \exp\left(-\frac{\beta^2}{2}\sinh x\right)\sinh^{\nu-1} x\cosh^\nu x\,dx = -\pi\,D_\nu\left(\beta e^{i\pi/4}\right)D_\nu\left(\beta e^{-i\pi/4}\right)$$ $\left[\operatorname{Re}\nu > 0, \quad |\arg\beta| \le \frac{\pi}{4}\right]$ EH II 120(10)

12. $$\int_0^\infty \frac{\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\nu+\frac{1}{4}}(\beta)\,J_{\nu-\frac{1}{4}}(\beta) + Y_{\nu+\frac{1}{4}}(\beta)\,Y_{\nu-\frac{1}{4}}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ EH I 169(20)

13. $$\int_0^\infty \frac{\exp(-2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx = \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\nu+\frac{1}{4}}(\beta)\,Y_{\nu-\frac{1}{4}}(\beta) - J_{\nu-\frac{1}{4}}(\beta)\,Y_{\nu+\frac{1}{4}}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(21)

14. $$\int_0^\infty \frac{\exp(-2\beta\sinh x)\sinh 2\nu x}{\sqrt{\sinh x}}\,dx = \frac{1}{4i}\sqrt{\frac{\pi^3\beta}{2}}\left\{e^{\nu\pi i}\,H^{(1)}_{\frac{1}{2}+\nu}(\beta)\,H^{(2)}_{\frac{1}{2}-\nu}(\beta) - e^{-\nu\pi i}\,H^{(1)}_{\frac{1}{2}-\nu}(\beta)\,H^{(2)}_{\frac{1}{2}+\nu}(\beta)\right\}$$
$[\operatorname{Re}\beta > 0]$ ET I 170(24)

15. $$\int_0^\infty \frac{\exp(-2\beta\sinh x)\cosh 2\nu x}{\sqrt{\sinh x}}\,dx = \frac{1}{4}\sqrt{\frac{\pi^3\beta}{2}}\left\{e^{\nu\pi i}\,H^{(1)}_{\frac{1}{2}+\nu}(\beta)\,H^{(2)}_{\frac{1}{2}-\nu}(\beta) + e^{-\nu\pi i}\,H^{(1)}_{\frac{1}{2}-\nu}(\beta)\,H^{(2)}_{\frac{1}{2}+\nu}(\beta)\right\}$$
$[\operatorname{Re}\beta > 0]$ ET I 170(25)

16. $$\int_0^\infty \frac{\exp(-2\beta\cosh x)\cosh 2\nu x}{\sqrt{\cosh x}}\,dx = \sqrt{\frac{\beta}{\pi}}\,K_{\nu+\frac{1}{4}}(\beta)\,K_{\nu-\frac{1}{4}}(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 170(26)

17.[8] $$\int_0^\infty \frac{\exp[-2\beta(\cosh x - 1)]\cosh 2\nu x}{\sqrt{\cosh x}}\,dx = \sqrt{\frac{\beta}{\pi}}\cdot e^{2\beta}\,K_{\nu+\frac{1}{4}}(\beta)\,K_{\nu-\frac{1}{4}}(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 170(27)

18. $$\int_0^\infty \frac{\cos\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(-2\nu x - 2\beta\sinh x) + \sin\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx$$
$$= \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\frac{1}{4}+\nu}(\beta)\,J_{\frac{1}{4}-\nu}(\beta) + Y_{\frac{1}{4}+\nu}(\beta)\,Y_{\frac{1}{4}-\nu}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(22)

19. $$\int_0^\infty \frac{\sin\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(-2\nu x - 2\beta\sinh x) - \cos\left[\left(\nu+\frac{1}{4}\right)\pi\right]\exp(2\nu x - 2\beta\sinh x)}{\sqrt{\sinh x}}\,dx$$
$$= \frac{1}{2}\sqrt{\pi^3\beta}\left[J_{\frac{1}{4}+\nu}(\beta)\,Y_{\frac{1}{4}-\nu}(\beta) - J_{\frac{1}{4}-\nu}(\beta)\,Y_{\frac{1}{4}+\nu}(\beta)\right]$$
$[\operatorname{Re}\beta > 0]$ ET I 169(23)

20. $$\int_0^\infty \frac{\exp[-\beta(\cosh x - 1)]\cosh\nu x\sinh x}{\sqrt{\cosh x(\cosh x - 1)}}\,dx = e^\beta\,K_\nu(\beta)$$
$[\operatorname{Re}\beta > 0]$ ET I 169(19)

3.548

1. $$\int_0^\infty e^{-\mu x^4}\sinh ax^2\,dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\exp\left(\frac{a^2}{8\mu}\right)I_{\frac{1}{4}}\left(\frac{a^2}{8\mu}\right)$$
$[\operatorname{Re}\mu > 0, \quad a \geq 0]$ ET I 166(42)

2. $$\int_0^\infty e^{-\mu x^4}\cosh ax^2\,dx = \frac{\pi}{4}\sqrt{\frac{a}{2\mu}}\exp\left(\frac{a^2}{8\mu}\right)I_{-\frac{1}{4}}\left(\frac{a^2}{8\mu}\right)$$
$[\operatorname{Re}\mu > 0, \quad a > 0]$ ET I 166(43)

3.549

1. $$\int_0^\infty e^{-\beta x} \sinh\left[(2n+1)\operatorname{arcsinh} x\right] dx = O_{2n+1}(\beta) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } \mathbf{3.547}\ 6)$$ ET I 167(5)

2. $$\int_0^\infty e^{-\beta x} \cosh\left(2n\operatorname{arcsinh} x\right) dx = O_{2n}(\beta) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } \mathbf{3.547}\ 8)$$ ET I 168(6)

3. $$\int_0^\infty e^{-\beta x} \sinh\left(\nu\operatorname{arcsinh} x\right) dx = \frac{\nu}{\beta} S_{0,\nu}(\beta) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } \mathbf{3.547}5)$$ ET I 168(7)

4. $$\int_0^\infty e^{-\beta x} \cosh\left(\nu\operatorname{arcsinh} x\right) dx = \frac{1}{\beta} S_{1,\nu}(\beta) \qquad [\operatorname{Re}\beta > 0] \qquad (\text{cf. } \mathbf{3.547}\ 7)$$

A number of other integrals containing hyperbolic functions and exponentials, depending on $\operatorname{arcsinh} x$ or $\operatorname{arccosh} x$, can be found by first making the substitution $x = \sinh t$ or $x = \cosh t$.

3.55–3.56 Combinations of hyperbolic functions, exponentials, and powers

3.551

1. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \sinh\gamma x\, dx = \frac{1}{2}\Gamma(\mu)\left[(\beta-\gamma)^{-\mu} - (\beta+\gamma)^{-\mu}\right]$$ $$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\beta > |\operatorname{Re}\gamma|]$$ ET I 164(18)

2. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \cosh\gamma x\, dx = \frac{1}{2}\Gamma(\mu)\left[(\beta-\gamma)^{-\mu} + (\beta+\gamma)^{-\mu}\right]$$ $$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > |\operatorname{Re}\gamma|]$$ ET I 164(19)

3. $$\int_0^\infty x^{\mu-1} e^{-\beta x} \coth x\, dx = \Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu, \frac{\beta}{2}\right) - \beta^{-\mu}\right]$$ $$[\operatorname{Re}\mu > 1, \quad \operatorname{Re}\beta > 0]$$ ET I 164(21)

4. $$\int_0^\infty x^n e^{-(p+mq)x} \sinh^m qx\, dx = 2^{-m} n! \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(p+2kq)^{n+1}}$$ $$[p > 0, \quad q > 0, \quad m < p + qm]$$ LI (81)(4)

5.[11] $$\int_0^1 \frac{e^{-\beta x}}{x} \sinh\gamma x\, dx = \frac{1}{2}\left[\ln\frac{\beta+\gamma}{\beta-\gamma} + \operatorname{Ei}(\gamma-\beta) - \operatorname{Ei}(-\gamma-\beta)\right]$$ $$[\beta > \gamma]$$ BI (80)(4)

6. $$\int_0^\infty \frac{e^{-\beta x}}{x} \sinh\gamma x\, dx = \frac{1}{2}\ln\frac{\beta+\gamma}{\beta-\gamma} \qquad [\operatorname{Re}\beta > |\operatorname{Re}\gamma|]$$ ET I 163(12)

7. $$\int_1^\infty \frac{e^{-\beta x}}{x} \cosh\gamma x\, dx = \frac{1}{2}\left[-\operatorname{Ei}(\gamma-\beta) - \operatorname{Ei}(-\gamma-\beta)\right] \qquad [\operatorname{Re}\beta > |\operatorname{Re}\gamma|]$$ ET I 164(15)

8.[6] $$\int_0^\infty x e^{-x} \coth x\, dx = \frac{\pi^2}{4} - 1$$ BI (82)(6)

9. $$\int_0^\infty e^{-\beta x} \tanh x \frac{dx}{x} = \ln\frac{\beta}{4} + 2\ln\frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4}+\frac{1}{2}\right)}$$ $[\operatorname{Re}\beta > 0]$ ET I 164(16)

10.[6] $$\int_0^\infty x e^{-x} \coth(x/2)\, dx = \frac{\pi^2}{3} - 1$$

3.552

1. $$\int_0^\infty \frac{x^{\mu-1}e^{-\beta x}}{\sinh x}\, dx = 2^{1-\mu}\,\Gamma(\mu)\,\zeta\left[\mu, \frac{1}{2}(\beta+1)\right]$$ $[\operatorname{Re}\mu > 1, \quad \operatorname{Re}\beta > -1]$ ET I 164(20)

2. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\sinh ax}\, dx = \frac{1}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m}$$ $[a > 0, \quad m = 1, 2, \dots]$ EH I 38(24)a

3. $$\int_0^\infty \frac{x^{\mu-1}e^{-x}}{\cosh x}\, dx = 2^{1-\mu}\left(1 - 2^{1-\mu}\right)\Gamma(\mu)\,\zeta(\mu)$$ $[\operatorname{Re}\mu > 0, \quad \mu \neq 1]$

$$= \ln 2$$ $[\text{if } \mu = 1]$

EH I 32(5)

4. $$\int_0^\infty \frac{x^{2m-1}e^{-ax}}{\cosh ax}\, dx = \frac{1-2^{1-2m}}{2m}|B_{2m}|\left(\frac{\pi}{a}\right)^{2m}$$ $[a > 0, \quad m = 1, 2, \dots]$ EH I 39(25)a

5. $$\int_0^\infty \frac{x^2 e^{-2nx}}{\sinh x}\, dx = 4\sum_{k=n}^\infty \frac{1}{(2k+1)^3}$$ $[n = 0, 1, 2, \dots]$ (cf. **4.261** 13)

BI(84)(4)

6.[11] $$\int_0^\infty \frac{x^3 e^{-2nx}}{\sinh x}\, dx = \frac{\pi^4}{8} - 12\sum_{k=1}^n \frac{1}{(2k-1)^4}$$ $[n = 0, 1, \dots]$ (cf. **4.262** 6)

BI (84)(6)

3.553

1. $$\int_0^\infty \frac{\sinh^2 ax}{\sinh x}\frac{e^{-x}\, dx}{x} = \frac{1}{2}\ln\left(a\pi \operatorname{cosec} a\pi\right)$$ $[a < 1]$ BI (95)(7)

2.[11] $$\int_0^\infty \frac{\sinh^2 \frac{x}{2}}{\cosh x}\cdot\frac{e^{-x}\, dx}{x} = \frac{1}{2}\ln\frac{4}{\pi}$$ (cf. **4.267** 2) BI (95)(4)

3.554

1.[11] $$\int_0^\infty e^{-\beta x}\left(1 - \operatorname{sech} x\right)\frac{dx}{x} = 2\ln\frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)} - \ln\frac{\beta}{4}$$ $[\operatorname{Re}\beta > 0]$ ET I 164(17)

2. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x} - \operatorname{cosech} x\right) dx = \psi\left(\frac{\beta+1}{2}\right) - \ln\frac{\beta}{2}$$ $[\operatorname{Re}\beta > 0]$ ET I 163(10)

3. $$\int_0^\infty \left[\frac{\sinh\left(\frac{1}{2}-\beta\right)x}{\sinh\frac{x}{2}} - (1-2\beta)e^{-x}\right]\frac{dx}{x} = 2\ln\Gamma(\beta) - \ln\pi + \ln\left(\sin\pi\beta\right)$$

$[0 < \operatorname{Re}\beta < 1]$ EH I 21(7)

4. $$\int_0^\infty e^{-\beta x}\left(\frac{1}{x}-\coth x\right)dx=\psi\left(\frac{\beta}{2}\right)-\ln\frac{\beta}{2}+\frac{1}{\beta}\qquad [\operatorname{Re}\beta>0]$$ ET I 163(11)

5. $$\int_0^\infty\left\{-\frac{\sinh qx}{\sinh\frac{x}{2}}+2qe^{-x}\right\}\frac{dx}{x}=2\ln\Gamma\left(q+\frac{1}{2}\right)+\ln\cos\pi q-\ln\pi$$
$$\left[q^2<\tfrac{1}{2}\right]$$ WH

6. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\left(\coth x-1\right)dx=2^{1-\mu}\,\Gamma(\mu)\,\zeta\left(\mu,\frac{\beta}{2}+1\right)$$
$$[\operatorname{Re}\beta>0;\quad \operatorname{Re}\mu>1]$$ ET I 164(22)

3.555

1. $$\int_0^\infty\frac{\sinh^2 ax}{1-e^{px}}\cdot\frac{dx}{x}=\frac{1}{4}\ln\left(\frac{p}{2a\pi}\sin\frac{2a\pi}{p}\right)\qquad [0<2|a|<p]$$ (cf. **3.545** 2) BI (93)(15)

2. $$\int_0^\infty\frac{\sinh^2 ax}{e^x+1}\cdot\frac{dx}{x}=-\frac{1}{4}\ln\left(a\pi\cot a\pi\right)\qquad \left[a<\tfrac{1}{2}\right]$$ (cf. **3.545** 1) BI (93)(9)

3.556

1. $$\int_{-\infty}^\infty x\frac{1-e^{px}}{\sinh x}\,dx=-\frac{\pi^2}{2}\tan^2\frac{p\pi}{2}\qquad [p<1]$$ (cf. **4.255** 3) BI (101)(4)

2. $$\int_0^\infty\frac{1-e^{-px}}{\sinh x}\cdot\frac{1-e^{-(p+1)x}}{x}\,dx=2p\ln 2\qquad [p>-1]$$ BI (95)(8)

3.557

1. $$\int_0^\infty\frac{e^{-px}-e^{-qx}}{\cosh x-\cos\frac{m}{n}\pi}\cdot\frac{dx}{x}$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q+k}{2n}\right)\Gamma\left(\frac{p+k}{2n}\right)}{\Gamma\left(\frac{n+p+k}{2n}\right)\Gamma\left(\frac{q+k}{2n}\right)}\qquad [m+n\ \text{odd}]$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{n+q-k}{n}\right)\Gamma\left(\frac{p+k}{n}\right)}{\Gamma\left(\frac{n+p-k}{n}\right)\Gamma\left(\frac{q+k}{n}\right)}\qquad [m+n\ \text{even}]$$
$$[p>-1,\quad q>-1]$$ BI (96)(1)

2. $$\int_0^\infty\frac{\left(1-e^{-x}\right)^2}{\cosh x+\cos\frac{m}{n}\pi}\cdot\frac{dx}{x}$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n+k+1}{2n}\right)\right]^2\Gamma\left(\frac{k+2}{2n}\right)\Gamma\left(\frac{k}{2n}\right)}{\left[\Gamma\left(\frac{k+1}{2n}\right)\right]^2\Gamma\left(\frac{n+k}{2n}\right)\Gamma\left(\frac{n+k+2}{2n}\right)}\qquad [m+n\ \text{odd}]$$
$$=2\operatorname{cosec}\left(\frac{m}{n}\pi\right)\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\times\ln\frac{\left[\Gamma\left(\frac{n-k+1}{n}\right)\right]^2\Gamma\left(\frac{k+2}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\left[\Gamma\left(\frac{k+1}{n}\right)\right]^2\Gamma\left(\frac{n-k}{n}\right)\Gamma\left(\frac{n-k+2}{n}\right)}\qquad [m+n\ \text{even}]$$
BI (96)(2)

3. $$\int_0^\infty \left[e^{-x}\tan\frac{m}{2n}\pi - \frac{e^{-px}\sin\frac{m}{n}\pi}{\cosh x + \cos\frac{m}{n}\pi}\right]\cdot\frac{dx}{x}$$
$$= \tan\left(\frac{m}{2n}\pi\right)\ln(2n) + 2\sum_{k=1}^{n-1}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n+k}{2n}\right)}{\Gamma\left(\frac{p+k}{2n}\right)} \qquad [m+n \text{ odd}]$$
$$= \tan\left(\frac{m}{2n}\pi\right)\ln n + 2\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\left(\frac{km}{n}\pi\right)\ln\frac{\Gamma\left(\frac{p+n-k}{n}\right)}{\Gamma\left(\frac{p+k}{n}\right)} \qquad [m+n \text{ even}]$$
BI (96)(3)

4. $$\int_0^\infty \frac{1+e^{-x}}{\cosh x+\cos a}\cdot\frac{dx}{x^{1-p}} = 2\sec\frac{a}{2}\,\Gamma(p)\sum_{k=1}^\infty(-1)^{k-1}\frac{\cos\left(k-\frac{1}{2}\right)a}{k^p}$$
$[p>0]$ LI (96)(5)

5. $$\int_0^\infty \frac{x^q e^{-\frac{x}{2}}\cosh\frac{x}{2}}{\cosh x+\cos\lambda}\,dx = \frac{\Gamma(q+1)}{\cos\frac{\lambda}{2}}\sum_{k=1}^\infty(-1)^{k-1}\frac{\cos\left(k-\frac{1}{2}\right)\lambda}{k^{q+1}}$$
$[q>-1]$ LI (96)(5)a

6. $$\int_0^\infty x\frac{e^{-x}-\cos a}{\cosh x-\cos a}\,dx = |a|\pi - \frac{a^2}{2} - \frac{\pi^2}{3}$$ BI (88)(8)

7. $$\int_0^\infty x^{2m+1}\frac{e^{-x}-\cos a\pi}{\cosh x-\cos a\pi}\,dx = 2\cdot(2m+1)!\sum_{k=1}^\infty\frac{\cos ka\pi}{k^{2m+2}}$$ BI (88)(6)

3.558

1. $$\int_0^\infty x\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = \frac{2n\pi^2}{3} - 4\sum_{k=1}^{n-1}\frac{n-k}{k^2}$$ BI (85)(3)

2. $$\int_0^\infty x\frac{1-(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = \frac{n\pi^2}{3} + 4\sum_{k=1}^{n-1}(-1)^k\frac{n-k}{k^2}$$ LI (85)(1)

3. $$\int_0^\infty x^2\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = 8n\,\zeta(3) - 8\sum_{k=1}^{n-1}\frac{n-k}{k^3}$$ BI (85)(5)

4. $$\int_0^\infty x^2 e^x\frac{1-e^{-2nx}}{\sinh^2 x}\,dx = 8n\sum_{k=1}^\infty\frac{1}{(2k-1)^3} - 8\sum_{k=1}^{n-1}\frac{n-k}{(2k-1)^3}$$ LI (85)(6)

5. $$\int_0^\infty x^2\frac{1+(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = 6n\,\zeta(3) - 8\sum_{k=1}^{n-1}\frac{n-k}{k^3}$$ LI (85)(4)

6. $$\int_0^\infty x^3\frac{1-e^{-nx}}{\sinh^2\frac{x}{2}}\,dx = \frac{4}{15}n\pi^4 - 24\sum_{k=1}^{n-1}\frac{n-k}{k^4}$$ BI (85)(9)

7. $$\int_0^\infty x^3\frac{1+(-1)^n e^{-nx}}{\cosh^2\frac{x}{2}}\,dx = \frac{7}{30}n\pi^4 + 24\sum_{k=1}^{n-1}(-1)^k\frac{n-k}{k^4}$$ BI (85)(8)

3.559 $$\int_0^\infty e^{-x}\left[a-\frac{1}{2}+\frac{(1-e^{-x})(1-ax)-xe^{-x}}{4\sinh^2\frac{x}{2}}e^{(2-a)x}\right]\frac{dx}{x}=a-\frac{1}{2}+\ln\Gamma(a)-\frac{1}{2}\ln(2\pi)\qquad [a>0]$$ BI (96)(6)

3.561 $$\int_0^\infty \frac{e^{-2x}\tanh\frac{x}{2}}{x\cosh x}\,dx = 2\ln\frac{\pi}{2\sqrt{2}}$$ BI (93)(18)

3.562

1. $$\int_0^\infty x^{2\mu-1}e^{-\beta x^2}\sinh\gamma x\,dx=\frac{1}{2}\Gamma(2\mu)(2\beta)^{-\mu}\exp\left(\frac{\gamma^2}{8\beta}\right)\left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right)-D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right]$$ $\left[\operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}\beta>0\right]$ ET I 166(44)

2. $$\int_0^\infty x^{2\mu-1}e^{-\beta x^2}\cosh\gamma x\,dx=\frac{1}{2}\Gamma(2\mu)(2\beta)^{-\mu}\exp\left(\frac{\gamma^2}{8\beta}\right)\left[D_{-2\mu}\left(-\frac{\gamma}{\sqrt{2\beta}}\right)+D_{-2\mu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right]$$ $[\operatorname{Re}\mu>0,\quad \operatorname{Re}\beta>0]$ ET I 166(45)

3. $$\int_0^\infty xe^{-\beta x^2}\sinh\gamma x\,dx=\frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\qquad [\operatorname{Re}\beta>0]$$ BI(81)(12)a,ET I 165(34)

4. $$\int_0^\infty xe^{-\beta x^2}\cosh\gamma x\,dx=\frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right)+\frac{1}{2\beta}$$ $[\operatorname{Re}\beta>0]$ ET I 166(35)

5. $$\int_0^\infty x^2e^{-\beta x^2}\sinh\gamma x\,dx=\frac{\sqrt{\pi}\left(2\beta+\gamma^2\right)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\Phi\left(\frac{\gamma}{2\sqrt{\beta}}\right)+\frac{\gamma}{4\beta^2}$$ $[\operatorname{Re}\beta>0]$ ET I 166(36)

6. $$\int_0^\infty x^2e^{-\beta x^2}\cosh\gamma x\,dx=\frac{\sqrt{\pi}\left(2\beta+\gamma^2\right)}{8\beta^2\sqrt{\beta}}\exp\left(\frac{\gamma^2}{4\beta}\right)\qquad [\operatorname{Re}\beta>0]$$ ET I 166(37)

3.6–4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

1. $$\int_0^{2\pi}(1-\cos x)^n\sin nx\,dx=0$$ BI (68)(10)

2. $$\int_0^{2\pi}(1-\cos x)^n\cos nx\,dx=(-1)^n\frac{\pi}{2^{n-1}}$$ BI (68)(11)

3. $$\int_0^{\pi}(\cos t+i\sin t\cos x)^n\,dx=\int_0^{\pi}(\cos t+i\sin t\cos x)^{-n-1}\,dx=\pi P_n(\cos t)$$ EH I 158(23)a

3.612

1.[6] $$\int_0^\pi \frac{\sin nx \cos mx}{\sin x}\,dx = 0 \qquad \text{for } n \le m;$$
$$= \pi \qquad \text{for } n > m, \quad \text{if } m+n \text{ is odd and positive}$$
$$= 0 \qquad \text{for } n > m, \quad \text{if } m+n \text{ is even}$$
LI (64)(3)

2. $$\int_0^\pi \frac{\sin nx}{\sin x}\,dx = 0 \qquad \text{for } n \text{ even}$$
$$= \pi \qquad \text{for } n \text{ odd}$$
BI (64)(1, 2)

3. $$\int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx = \frac{\pi}{2}$$ FI II 145

4. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin x}\,dx = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{k-1}}{2n-1}\right)$$ GW (332)(21b)

5. $$\int_0^\pi \frac{\sin 2nx}{\cos x}\,dx = 2\int_0^{\pi/2} \frac{\sin 2nx}{\cos x}\,dx = (-1)^{n-1}4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}\right)$$ GW (332)(22a)

6. $$\int_0^\pi \frac{\cos(2n+1)x}{\cos x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x}\,dx = (-1)^n\pi$$ GW (332)(22b)

7. $$\int_0^{\pi/2} \frac{\sin 2nx \cos x}{\sin x}\,dx = \frac{\pi}{2}$$ LI (45)(17)

3.613

1.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1 + a\cos x} = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n \qquad \left[a^2 < 1, \quad n \ge 0\right]$$ BI (64)(12)

2.[6] $$\int_0^\pi \frac{\cos nx\,dx}{1 - 2a\cos x + a^2} = \frac{\pi a^n}{1-a^2} \qquad \left[a^2 < 1, \quad n \ge 0\right]$$
$$= \frac{\pi}{\left(a^2-1\right)a^n} \qquad \left[a^2 > 1, \quad n \ge 0\right]$$
BI (65)(3)

3. $$\int_0^\pi \frac{\sin nx \sin x\,dx}{1 - 2a\cos x + a^2} = \frac{\pi}{2}a^{n-1} \qquad \left[a^2 < 1, \quad n \ge 1\right]$$
$$= \frac{\pi}{2a^{n+1}} \qquad \left[a^2 > 1, \quad n \ge 1\right]$$
BI(65)(4), GW(332)(34a)

4.[10] $$\int_0^\pi \frac{\cos nx \cos x\, dx}{1 - 2a\cos x + a^2} = \frac{\pi}{2}\cdot\frac{1+a^2}{1-a^2}a^{n-1} \qquad [a^2 < 1, \quad n \geq 1]$$
$$= \frac{\pi}{2a^{n+1}}\cdot\frac{a^2+1}{a^2-1} \qquad [a^2 > 1, \quad n \geq 1]$$
$$= \frac{\pi a}{1-a^2} \qquad [n = 0, \quad a^2 < 1]$$
$$= \frac{\pi}{a\left(a^2-1\right)} \qquad [n = 0, \quad a^2 > 1]$$

BI(65)(5), GW(332)(34b)

5. $$\int_0^\pi \frac{\cos(2n-1)x\, dx}{1 - 2a\cos 2x + a^2} = \int_0^\pi \frac{\cos 2nx \cos x\, dx}{1 - 2a\cos 2x + a^2} = 0 \qquad [a^2 \neq 1]$$ BI (65)(9, 10)

6. $$\int_0^\pi \frac{\cos(2n-1)x \cos 2x\, dx}{1 - 2a\cos 2x + a^2} = 0 \qquad [a^2 \neq 1]$$ BI (65)(12)

7. $$\int_0^\pi \frac{\sin 2nx \sin x\, dx}{1 - 2a\cos 2x + a^2} = \int_0^\pi \frac{\sin(2n-1)x \sin 2x\, dx}{1 - 2a\cos 2x + a^2} = 0$$
$$[a^2 \neq 1]$$ BI (65)(6, 7)

8. $$\int_0^\pi \frac{\sin(2n-1)x \sin x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2}\cdot\frac{a^{n-1}}{1+a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2}\cdot\frac{1}{(1+a)a^n} \qquad [a^2 > 1]$$

BI (65)(8)

9. $$\int_0^\pi \frac{\cos(2n-1)x \cos x\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2}\cdot\frac{a^{n-1}}{1-a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2}\cdot\frac{1}{(a-1)a^n} \qquad [a^2 > 1]$$

BI (65)(11)

10. $$\int_0^\pi \frac{\sin nx - a\sin(n-1)x}{1 - 2a\cos x + a^2} \sin mx\, dx = 0 \qquad \text{for } m < n$$
$$= \frac{\pi}{2}a^{m-n} \qquad \text{for } m \geq n$$
$$[a^2 < 1]$$ LI (65)(13)

11.[6] $$\int_0^\pi \frac{\cos nx - a\cos(n-1)x}{1 - 2a\cos x + a^2} \cos mx\, dx = \frac{\pi}{2}\left(a^{|m|-n} - 1\right)$$
$$[a^2 < 1]$$ BI (65)(14)

12. $$\int_0^\pi \frac{\sin nx - a\sin[(n+1)x]}{1 - 2a\cos x + a^2}\, dx = 0 \qquad [a^2 < 1]$$ BI (68)(13)

13. $$\int_0^\pi \frac{\cos nx - a\cos[(n+1)x]}{1 - 2a\cos x + a^2}\, dx = \pi a^n \qquad [a^2 < 1]$$ BI (68)(14)

3.614[7] $$\int_0^\pi \frac{\sin x}{a^2 - 2ab\cos x + b^2} \cdot \frac{\sin px \cdot dx}{1 - 2a^p \cos px + a^{2p}}$$
$$= \frac{\pi b^{p-1}}{2a^{p+1}\left(1 - b^p\right)} \qquad [0 < b \le a \le 1, \quad p = 1, 2, 3, \ldots]$$
$$= \frac{\pi a^{p-1}}{2b\left(b^p - a^{2p}\right)} \qquad \left[0 < a \le 1, \quad a^2 < b, \quad p = 1, 2, 3, \ldots\right]$$
BI (66)(9)

3.615

1. $$\int_0^{\pi/2} \frac{\cos 2nx\, dx}{1 - a^2\sin^2 x} = \frac{(-1)^n \pi}{2\sqrt{1-a^2}} \left(\frac{1 - \sqrt{1-a^2}}{a}\right)^{2n} \qquad \left[a^2 < 1\right]$$ BI (47)(27)

2. $$\int_0^\pi \frac{\cos x \sin 2nx\, dx}{1 + (a + b\sin x)^2} = -\frac{\pi}{b} \sin\left\{2n \arctan\sqrt{\frac{s}{2}}\right\} \tan^{2n}\left(\frac{1}{2}\arccos\sqrt{\frac{s}{2a^2}}\right)$$

3. $$\int_0^\pi \frac{\cos x \cos(2n+1)x\, dx}{1 + (a + b\sin x)^2} = \frac{\pi}{b} \cos\left\{(2n+1) \arctan\sqrt{\frac{s}{2}}\right\} \tan^{2n+1}\left(\frac{1}{2}\arccos\sqrt{\frac{s}{2a^2}}\right)$$
where $s = -\left(1 + b^2 - a^2\right) + \sqrt{\left(1 + b^2 - a^2\right)^2 + 4a^2}$ BI (65)(21, 22)

3.616

1. $$\int_0^\pi \left(1 - 2a\cos x + a^2\right)^n\, dx = \pi \sum_{k=0}^{n} \binom{n}{k}^2 a^{2k}$$ BI (63)(1)

2.[10] $$\int_0^\pi \frac{dx}{\left(1 - 2a\cos x + a^2\right)^n} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{\left(1 - 2a\cos x + a^2\right)^n}$$
$$= \frac{\pi}{\left(1-a^2\right)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2\,(n-k-1)!} \left(\frac{a^2}{1-a^2}\right)^k \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi}{\left(a^2-1\right)^n} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{(k!)^2\,(n-k-1)!} \frac{1}{\left(a^2-1\right)^k} \qquad \left[a^2 > 1\right]$$
BI (331)(63)

3. $$\int_0^\pi \left(1 - 2a\cos x + a^2\right)^n \cos nx\, dx = (-1)^n \pi a^n$$ BI (63)(2)

4. $$\int_0^\pi \left(1 - 2a\cos x + a^2\right)^n \cos mx\, dx$$
$$= \frac{1}{2}\int_0^{2\pi} \left(1 - 2a\cos x + a^2\right)^n \cos mx\, dx$$
$$= 0 \qquad [n < m]$$
$$= \pi(-a)^m \left(1 + a^2\right)^{n-m} \sum_{k=0}^{[(n-m)/2]} \binom{n}{k}\binom{n-k}{m+k} \left(\frac{a}{1+a^2}\right)^{2k} \qquad [n \ge m]$$
GW (332)(35a)

5. $$\int_0^{2\pi} \frac{\sin nx\, dx}{\left(1 - 2a\cos 2x + a^2\right)^m} = 0$$ GW (332)(32a)

6. $$\int_0^{\pi} \frac{\sin x\,dx}{(1-2a\cos 2x+a^2)^m} = \frac{1}{2(m-1)a}\left[\frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}}\right] \qquad [a \neq 0, \quad \pm 1]$$

GW (332)(32c)

7. $$\int_0^{\pi} \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m} = \frac{1}{2}\int_0^{2\pi} \frac{\cos nx\,dx}{(1-2a\cos x+a^2)^m}$$
$$= \frac{a^{2m+n-2}\pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(\frac{1-a^2}{a^2}\right)^k \qquad [a^2<1]$$
$$= \frac{\pi}{a^n\left(a^2-1\right)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k}\binom{2m-k-2}{m-1}\left(a^2-1\right)^k \qquad [a^2>1]$$

GW (332)(31)

8. $$\int_0^{\pi/2} \frac{\cos 2nx\,dx}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^{n+1}} = \binom{2n}{n}\frac{\left(b^2-a^2\right)^n}{(2ab)^{2n+1}}\pi$$

$[a>0, \quad b>0]$ GW (332)(30b)

3.617[10] $$\int_0^{\pi} \frac{dx}{\left(1-2a\cos x+a^2\right)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n\left(\frac{2\sqrt{|a|}}{|1+a|}\right), \qquad |a| \neq 1$$

with
$$F_n(k) = \int_0^{\pi/2} \frac{dx}{\left(1-k^2\sin^2 x\right)^{n+1/2}}$$

where the $F_n(k)$ satisfies the recurrence relation
$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1}\frac{dF_n(k)}{dk}, \qquad n = 0, 1, 2, \ldots$$

and
$$F_0(k) = \boldsymbol{K}(k) \equiv \int_0^{\pi/2} \frac{dx}{\left(1-k^2\sin^2 x\right)^{1/2}}$$

is the complete elliptic integral of the first kind.
Introducing the complete elliptic integral of the second kind
$$\boldsymbol{E}(k) = \int_0^{\pi/2} \left(1-k^2\sin^2 x\right)^{1/2} dx$$

the derivatives
$$\frac{d\,\boldsymbol{K}(k)}{dk} = \frac{\boldsymbol{E}(k)}{k\left(1-k^2\right)} - \frac{\boldsymbol{K}(k)}{k}, \qquad \frac{d\,\boldsymbol{E}(k)}{dk} = \frac{\boldsymbol{E}(k)-\boldsymbol{K}(k)}{k}$$

combined with the recurrence relation lead to

$$F_1(k) = F_0(k) + k\frac{d\,F_0(k)}{dk}$$
$$= \boldsymbol{K}(k) + \frac{\boldsymbol{E}(k)}{1-k^2} - \boldsymbol{K}(k) = \frac{\boldsymbol{E}(k)}{1-k^2},$$
$$F_2(k) = \frac{\boldsymbol{E}(k)}{1-k^2} + \frac{k}{3}\frac{d}{dk}\left(\frac{\boldsymbol{E}(k)}{1-k^2}\right)$$
$$= \frac{1}{3\left(1-k^2\right)}\left[\left(\frac{4-2k^2}{1-k^2}\right)\boldsymbol{E}(k) - \boldsymbol{K}(k)\right]$$

3.62 Powers of trigonometric functions

3.621

1. $$\int_0^{\pi/2} \sin^{\mu-1} x\,dx = \int_0^{\pi/2} \cos^{\mu-1} x\,dx = 2^{\mu-2}\,\mathrm{B}\left(\frac{\mu}{2}, \frac{\mu}{2}\right)$$ FI II 789

2. $$\int_0^{\pi/2} \sin^{3/2} x\,dx = \int_0^{\pi/2} \cos^{3/2} x\,dx = \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

3. $$\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2} \cos^{2m} x\,dx = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$$ FI II 151

4. $$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \int_0^{\pi/2} \cos^{2m+1} x\,dx = \frac{(2m)!!}{(2m+1)!!}$$ FI II 151

5. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$$ LO V 113(50), LO V 122, FI II 788

6.* $$\int_0^{\pi/2} \sqrt{\sin x}\,dx = \sqrt{\frac{2}{\pi}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2$$

7.* $$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{2\sqrt{2\pi}}$$

3.622

1. $$\int_0^{\pi/2} \tan^{\pm\mu} x\,dx = \frac{\pi}{2}\sec\frac{\mu\pi}{2} \qquad [|\operatorname{Re}\mu| < 1]$$ BI (42)(1)

2. $$\int_0^{\pi/4} \tan^{\mu} x\,dx = \frac{1}{2}\beta\left(\frac{\mu+1}{2}\right) \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(1)

3. $$\int_0^{\pi/4} \tan^{2n} x\,dx = (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}$$ BI (34)(2)

4.[11] $$\int_0^{\pi/4} \tan^{2n+1} x\,dx = (-1)^n\frac{\ln 2}{2} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}$$ BI (34)(3)

3.623

1. $$\int_0^{\pi/2} \tan^{\mu-1} x \cos^{2\nu-2} x\,dx = \int_0^{\pi/2} \cot^{\mu-1} x \sin^{2\nu-2} x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{\mu}{2}, \nu - \frac{\mu}{2}\right)$$ $[0 < \operatorname{Re}\mu < 2\operatorname{Re}\nu]$ BI(42)(6), BI(45)(22)

2.[6] $$\int_0^{\pi/4} \tan^{\mu} x \sin^2 x\,dx = \frac{1+\mu}{4}\beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(4)

3.[6] $$\int_0^{\pi/4} \tan^{\mu} x \cos^2 x\,dx = \frac{1-\mu}{4}\beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \qquad [\operatorname{Re}\mu > -1]$$ BI (34)(5)

3.624

1. $$\int_0^{\pi/4} \frac{\sin^p x}{\cos^{p+2} x}\,dx = \frac{1}{p+1}$$ $[p > -1]$ GW (331)(34b)

2.[3] $$\int_0^{\pi/2} \frac{\sin^{\mu-\frac{1}{2}} x}{\cos^{2\mu-1} x}\,dx = \int_0^{\pi/2} \frac{\cos^{\mu-\frac{1}{2}} x}{\sin^{2\mu-1} x}\,dx = \frac{1}{2}\left\{\frac{\Gamma\left(\frac{\mu}{2}+\frac{1}{4}\right)\Gamma(1-\mu)}{\Gamma\left(\frac{5}{4}-\frac{\mu}{2}\right)}\right\}$$ $\left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right]$ LI (55)(12)

3.[11] $$\int_0^{\pi/4} \frac{\cos^{n-\frac{1}{2}}(2x)}{\cos^{2n+1}(x)}\,dx = \pi\frac{(2n)!!}{2^{2n+1}\,(n!)^2}$$ BI (38)(3)

4.[8] $$\int_0^{\pi/4} \frac{\cos^{\mu} 2x}{\cos^{2(\mu+1)} x}\,dx = 2^{2\mu}\,\mathrm{B}(\mu+1,\mu+1)$$ $[\operatorname{Re}\mu > -1]$ BI (35)(1)

5. $$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{\cos^{\mu} 2x}\,dx = 2^{1-2\mu}\,\mathrm{B}(2\mu-1,1-\mu) = \frac{\Gamma\left(\mu-\frac{1}{2}\right)\Gamma(1-\mu)}{2\sqrt{\pi}}$$ $\left[\frac{1}{2} < \operatorname{Re}\mu < 1\right]$ BI (35)(4)

6.[6] $$\int_0^{\pi/2} \left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{a\pi}{2} - \frac{1}{2}\sin\pi a\,[2a\,\beta(a) - 1]\,,$$ $[a > 0]$

3.625

1. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x\cos^p 2x}{\cos^{2p+2n+1} x}\,dx = \frac{(n-1)!}{2}\cdot\frac{\Gamma(p+1)}{\Gamma(p+n+1)} = \frac{(n-1)!}{2(p+n)(p+n-1)\cdots(p+1)} = \frac{1}{2}\,\mathrm{B}(n,p+1)$$ $[p > -1]$ (cf. **3.251** 1) BI (35)(2)

2. $$\int_0^{\pi/4} \frac{\sin^{2n} x\cos^p 2x}{\cos^{2p+2n+2} x}\,dx = \tfrac{1}{2}\,\mathrm{B}\left(n+\tfrac{1}{2},p+1\right)$$ $[p > -1]$ (cf. **3.251** 1) BI (35)(3)

3. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x\cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x}\,dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!}$$ BI (38)(6)

4.[8] $$\int_0^{\pi/4} \frac{\sin^{2n} x\cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x}\,dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!}\cdot\frac{\pi}{2}$$ BI (38)(7)

3.626

1. $$\int_0^{\pi/4} \frac{\sin^{2n-1} x}{\cos^{2n+2} x}\sqrt{\cos 2x}\,dx = \frac{(2n-2)!!}{(2n+1)!!}$$ (cf. **3.251** 1) BI (38)(4)

2. $$\int_0^{\pi/4} \frac{\sin^{2n} x}{\cos^{2n+3} x}\sqrt{\cos 2x}\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}$$ (cf. **3.251** 1) BI (38)(5)

3.627 $$\int_0^{\pi/2} \frac{\tan^{\mu} x}{\cos^{\mu} x}\,dx = \int_0^{\pi/2} \frac{\cot^{\mu} x}{\sin^{\mu} x}\,dx = \frac{\Gamma(\mu)\,\Gamma\left(\frac{1}{2}-\mu\right)}{2^{\mu}\sqrt{\pi}}\sin\frac{\mu\pi}{2}$$ $\left[-1 < \operatorname{Re}\mu < \frac{1}{2}\right]$ BI (55)(12)a

3.628[11] $$\int_0^{\frac{\pi}{2}} \sec^{2p} x\sin^{2p-1} x\,dx = \frac{1}{2\sqrt{\pi}}\,\Gamma(p)\,\Gamma\left(\tfrac{1}{2}-p\right)$$ $\left[0 < p < \frac{1}{2}\right]$ WA 691

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.631

1. $$\int_0^{\pi} \sin^{\nu-1} x \sin ax\, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$
$[\operatorname{Re} \nu > 0]$ LO V 121(67a), WA 337a

2.[7] $$\int_0^{\pi/2} 2 \sin^{\nu-2} x \sin \nu x\, dx = \frac{1}{1-\nu} \cos \frac{\nu\pi}{2}$$ $[\operatorname{Re} \nu > 1]$ GW(332)(16d), FI I 152

3.[6] $$\int_0^{\pi} \sin^{\nu} x \sin \nu x\, dx = 2^{-\nu} \pi \sin \frac{\nu\pi}{2}$$ $[\operatorname{Re} \nu > -1]$ LO V 121(69)

4. $$\int_0^{\pi} \sin^n x \sin 2mx\, dx = 0$$ GW (332)(11a)

5. $$\int_0^{\pi} \sin^{2n} x \sin(2m+1)x\, dx = \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x\, dx$$
$$= \frac{(-1)^m 2^{n+1} n!(2n-1)!!}{(2n-2m-1)!!(2m+2n+1)!!} \qquad [m \le n]^*$$
$$= \frac{(-1)^n 2^{n+1} n!(2m-2n-1)!!(2n-1)!!}{(2m+2n+1)!!} \qquad [m \ge n]^*$$
GW (332)(11b)

6. $$\int_0^{\pi} \sin^{2n+1} x \sin(2m+1)x\, dx = 2\int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x\, dx$$
$$= \frac{(-1)^m \pi}{2^{2n+1}} \binom{2n+1}{n-m} \qquad [n \ge m]$$
$$= 0 \qquad [n < m]$$
BI(40)(12), GW(332)(11c)

7. $$\int_0^{\pi} \sin^n x \cos(2m+1)x\, dx = 0$$ GW (332)(12a)

8. $$\int_0^{\pi} \sin^{\nu-1} x \cos ax\, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$
$[\operatorname{Re} \nu > 0]$ LO V 121(68)a, WA 337a

9. $$\int_0^{\pi/2} \cos^{\nu-1} x \cos ax\, dx = \frac{\pi}{2^{\nu} \nu\, \mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$
$[\operatorname{Re} \nu > 0]$ GW (332)(9c)

10. $$\int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x\, dx = \frac{1}{\nu-1} \sin \frac{\nu\pi}{2}$$ $[\operatorname{Re} \nu > 1]$ GW(332)(16b), FI II 15 2

*In 3.631.5, for $m = n$ we should set $(2n-2m-1)!! = 1$

11. $$\int_0^{\pi} \sin^{\nu} x \cos \nu x\, dx = \frac{\pi}{2^{\nu}} \cos \frac{\nu\pi}{2} \qquad [\operatorname{Re} \nu > -1]$$ LO V 121(70)a

12. $$\int_0^{\pi} \sin^{2n} x \cos 2mx\, dx = 2\int_0^{\pi/2} \sin^{2n} x \cos 2mx\, dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi \qquad [n \geq m]$$
$$= 0 \qquad [n < m]$$
BI(40)(16), GW(332)(12b)

13.[7] $$\int_0^{\pi} \sin^{2n+1} x \cos 2mx\, dx$$
$$= 2\int_0^{\pi/2} \sin^{2n+1} x \cos 2mx\, dx = \frac{(-1)^m 2^{n+1} n!(2n+1)!!}{(2m-2n-3)!!(2m+2n+1)!!} \qquad [n \geq m-1]$$
$$= \frac{(-1)^{n+1} 2^{n+1} n!(2m-2n+3)!!(2n+1)!!}{(2m+2n+1)!!} \qquad [n < m-1]$$
GW (332)(12c)

14. $$\int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x\, dx = \frac{1}{\nu-1} \qquad [\operatorname{Re} \nu > 1]$$ GW(332)(16c), FI II 152

15. $$\int_0^{\pi} \cos^m x \sin nx\, dx = \left[1 - (-1)^{m+n}\right] \int_0^{\pi/2} \cos^m x \sin nx\, dx$$
$$= \left[1 - (-1)^{m+n}\right] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s \frac{m!(n-m-2)!!}{(m+n)!!} \right\}$$
$$\left[r = \begin{cases} m & \text{if } m \leq n \\ n & \text{if } m \geq n \end{cases} \qquad s = \begin{cases} 2 & \text{if } n-m = 4l+2 > 0 \\ 1 & \text{if } n-m = 2l+1 > 0 \\ 0 & \text{if } n-m = 4l \text{ or } n-m < 0 \end{cases} \right]$$
GW (332)(13a)

16. $$\int_0^{\pi/2} \cos^n x \sin nx\, dx = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}$$ FI II 153

17.[11] $$\int_0^{\pi} \cos^n x \sin mx\, dx = \begin{cases} \left[1 + (-1)^{m+n}\right] \dfrac{\pi}{2^{n+1}} \dbinom{n}{k} & \text{if } m \leq n \text{ and } n-m = 2k \\ 0 & \text{otherwise} \end{cases}$$
GW (332)(15a)

18.[6] $$\int_0^{\pi} \cos^m x \cos ax\, dx = \frac{(-1)^m \sin a\pi}{2^m (m+a)}\ {}_2F_1\left(-m, -\frac{a+m}{2}; 1 - \frac{a+m}{2}; -1\right)$$
$$[a \neq 0, \pm 1, \pm 2, \ldots]$$ WA 313

19. $$\int_0^{\pi/2} \cos^{\nu-2} x \cos \nu x\, dx = 0 \qquad [\operatorname{Re} \nu > 1]$$ GW(332)(16a), FI II 152

20.[10] $$\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}} \qquad [\operatorname{Re} n > -1]$$ LO V 122(78), FI II 153

3.632

1. $$\int_0^{\pi} \sin^{p-1} x \cos\left[a\left(\frac{\pi}{2} - x\right)\right] dx = 2^{p-1} \frac{\Gamma\left(\frac{p-a}{2}\right) \Gamma\left(\frac{p+a}{2}\right)}{\Gamma(p-a)\,\Gamma(p+a)}\, \Gamma(p)$$
$$\left[p^2 < a^2\right]$$ BI (62)(11)

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin\left[a\left(x+\frac{\pi}{2}\right)\right] dx = \frac{\pi \sin\frac{a\pi}{2}}{2^{\nu-1}\nu \,\mathrm{B}\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}$$
$[\operatorname{Re}\nu > 0]$ WA 337a

3.[10] $$\int_0^{\pi/2} \cos^p x \sin[(p+2n)x]\, dx = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k 2^k}{p+k+1} \binom{n-1}{k}$$
$[n > 0]$ LI (41)(12)

4. $$\int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)]\, dx = \left[1-(-1)^{n+m}\right] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)]\, dx$$
$$= \frac{\left[1-(-1)^{n+m}\right] \pi \cos ma}{2^{n-1} n \,\mathrm{B}\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}\right)}$$
$[n \geq m]$ LO V 123(80), LO V 139(94a)

5. $$\int_0^{\pi/2} \cos^{p+q-2} x \cos[(p-q)x]\, dx = \frac{\pi}{2^{p+q-1}(p+q-1)\,\mathrm{B}(p,q)}$$
$[p+q > 1]$ WH

3.633

1. $$\int_0^{\pi/2} \cos^{p-1} x \sin ax \sin x\, dx = \frac{a\pi}{2^{p+1} p(p+1)\,\mathrm{B}\left(\frac{p+a}{2}+1, \frac{p-a}{2}+1\right)}$$ LO V 150(110)

2. $$\int_0^{\pi/2} \cos^n x \sin nx \sin 2mx\, dx = \int_0^{\pi/2} \cos^n x \cos nx \cos 2mx\, dx = \frac{\pi}{2^{n+2}} \binom{n}{m}$$ BI (42)(19, 20)

3. $$\int_0^{\pi/2} \cos^{n-1} x \cos[(n+1)x] \cos 2mx\, dx = \frac{\pi}{2^{n+1}} \binom{n-1}{m-1}$$
$[n > m-1]$ BI (42)(21)

4. $$\int_0^{\pi/2} \cos^{p+q} x \cos px \cos qx\, dx = \frac{\pi}{2^{p+q+2}} \left[1 + \frac{1}{(p+q+1)\,\mathrm{B}(p+1,q+1)}\right]$$
$[p+q > -1]$ GW (332)(10c)

5.[6] $$\int_0^{\pi/2} \cos^{p+q} x \sin px \sin qx\, dx = \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k}\binom{q}{k} = \frac{\pi}{2^{p+q+2}} \left[\frac{\Gamma(p+q+1)}{\Gamma(p+1)\,\Gamma(q+1)} - 1\right]$$
$[p+q > -1]$ BI (42)(16)

3.634

1. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x\, dx = \sin\frac{\mu\pi}{2}\,\mathrm{B}(\mu,\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
BI(42)(23), FI II 814a

2. $$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu+\nu)x \, dx = \cos\frac{\mu\pi}{2} \mathrm{B}(\mu,\nu)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI(42)(24), FI II 814a

3. $$\int_0^{\pi/2} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x \, dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n!\,\Gamma(p)}$$

$[p > -n]$ BI (42)(15)

3.635

1. $$\int_0^{\pi/4} \cos^{\mu-1} 2x \tan x \, dx = \frac{1}{4}\left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu}{2}\right)\right]$$ $[\operatorname{Re}\mu > 0]$ BI (34)(7)

2.[7] $$\int_0^{\pi/2} \cos^{p+2n} x \sin px \tan x \, dx = \frac{\pi}{2^{p+2n+1}\,\Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!}$$
$$= \frac{p\pi}{2^{p+2+n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1)\,\Gamma(p+n+1)}$$

$[p > -2n]$ BI (42)(22)

3. $$\int_0^{\pi/2} \cos^{n-1} x \sin[(n+1)x] \cot x \, dx = \frac{\pi}{2}$$ BI (45)(18)

3.636

1. $$\int_0^{\pi/2} \tan^{\pm\mu} x \sin 2x \, dx = \frac{\mu\pi}{2} \operatorname{cosec}\frac{\mu\pi}{2}$$ $[0 < \operatorname{Re}\mu < 2]$ BI (45)(20)a

2. $$\int_0^{\pi/2} \tan^{\pm\mu} x \cos 2x \, dx = \mp\frac{\mu\pi}{2} \sec\frac{\mu\pi}{2}$$ $[|\operatorname{Re}\mu| < 1]$ BI (45)(21)

3.[11] $$\int_0^{\pi/2} \frac{\tan^{2\mu} x}{\cos x} \, dx = \int_0^{\pi/2} \frac{\cot^{2\mu} x}{\sin x} \, dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(-\mu)}{2\sqrt{\pi}}$$

$\left[-\frac{1}{2} < \operatorname{Re}\mu < 1\right]$ (cf. **3.251** 1)
BI (45)(13, 14)

3.637

1. $$\int_0^{\pi/2} \tan^p x \sin^{q-2} x \sin qx \, dx = -\cos\frac{(p+q)\pi}{2} \mathrm{B}(p+q-1, 1-p)$$

$[p+q > 1 > p]$ GW (332)(15d)

2. $$\int_0^{\pi/2} \tan^p x \sin^{q-2} x \cos qx \, dx = \sin\frac{(p+q)\pi}{2} \mathrm{B}(p+q-1, 1-p)$$

$[p+q > 1 > p]$ GW (332)(15b)

3. $$\int_0^{\pi/2} \cot^p x \cos^{q-2} x \sin qx \, dx = \cos\frac{p\pi}{2} \mathrm{B}(p+q-1, 1-p)$$

$[p+q > 1 > p]$ GW (332)(15c)

4. $$\int_0^{\pi/2} \cot^p x \cos^{q-2} x \cos qx\,dx = \sin\frac{p\pi}{2}\,\mathrm{B}(p+q-1, 1-p)$$
$[p+q>1>p]$ GW (332)(15a)

3.638

1. $$\int_0^{\pi/4} \frac{\sin^{2\mu} x\,dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2}\sec\mu\pi$$
$\left[|\mathrm{Re}\,\mu| < \frac{1}{2}\right]$ (cf. **3.192** 2) BI (38)(8)

2. $$\int_0^{\pi/4} \frac{\sin^{\mu-\frac{1}{2}} 2x\,dx}{\cos^\mu 2x \cos x} = \frac{2}{2\mu-1}\cdot\frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(1-\mu)}{\sqrt{\pi}}\sin\left(\frac{2\mu-1}{4}\pi\right)$$
$\left[-\frac{1}{2} < \mathrm{Re}\,\mu < 1\right]$ BI (38)(17)

3. $$\int_0^{\pi/2} \frac{\cos^{p-1} x \sin px}{\sin x}\,dx = \frac{\pi}{2}$$
$[p>0]$ GW(332)(17), BI(45)(5)

3.64–3.65 Powers and rational functions of trigonometric functions

3.641

1. $$\int_0^{\pi/2} \frac{\sin^{p-1} x \cos^{-p} x}{a\cos x + b\sin x}\,dx = \int_0^{\pi/2} \frac{\sin^{-p} x \cos^{p-1} x}{a\sin x + b\cos x}\,dx = \frac{\pi\,\mathrm{cosec}\,p\pi}{a^{1-p}b^p}$$
$[ab>0, \quad 0<p<1]$ GW (331)(62)

2. $$\int_0^{\pi/2} \frac{\sin^{1-p} x \cos^p x}{(\sin x + \cos x)^3}\,dx = \int_0^{\pi/2} \frac{\sin^p x \cos^{1-p} x}{(\sin x + \cos x)^3}\,dx = \frac{(1-p)p}{2}\pi\,\mathrm{cosec}\,p\pi$$
$[-1<p<2]$ BI(48)(5)

3.642

1. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x\,dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{\mu+\nu}} = \frac{1}{2a^{2\mu}b^{2\nu}}\,\mathrm{B}(\mu,\nu)$$
$[\mathrm{Re}\,\mu>0, \quad \mathrm{Re}\,\nu>0]$ BI (48)(28)

2. $$\int_0^{\pi/2} \frac{\sin^{n-1} x \cos^{n-1} x\,dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^n} = \frac{\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)}{2(ab)^n}$$
$[ab>0]$ GW (331)(59a)

3. $$\int_0^{\pi/2} \frac{\sin^{2n} x\,dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}} = \frac{1}{2}\int_0^{\pi} \frac{\sin^{2n} x\,dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}}$$
$$= \int_0^{\pi/2} \frac{\cos^{2n} x\,dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{n+1}} = \frac{1}{2}\int_0^{\pi} \frac{\cos^{2n} x\,dx}{\left(a^2\sin^2 x + b^2\cos^2 x\right)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!ab^{2n+1}}$$
$[ab>0]$ GW (331)(58)

4. $$\int_0^{\pi/2} \frac{\cos^{p+2n} x \cos px\,dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^{n+1}} = \pi\sum_{k=0}^{n}\binom{2n-k}{n}\binom{p+k-1}{k}\frac{b^{p-1}}{(2a)^{2n-k+1}(a+b)^{p+k}}$$
$[a>0, \quad b>0, \quad p>-2n-1]$ GW (332)(30)

3.643

1. $$\int_0^{\pi/2} \frac{\cos^p x \cos px\, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \qquad \left[a^2 < 1, \quad p > -1\right]$$ GW (332)(33c)

2. $$\int_0^{\pi/2} \frac{\sin^{2n} x \cos^\mu x \cos\beta x}{(1 - 2a\cos 2x + a^2)^m}\, dx = \frac{(-1)^n \pi (1-a)^{2n-2m+1}}{2^{2m-\beta-1}(1+a)^{2m+\beta+1}} \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \binom{\beta}{k} \binom{2n}{l} \times \binom{2m-k-l-2}{m-1(-2)^l} (a-1)^k$$
$$\left[a^2 < 1, \quad \beta = 2m - 2n - \mu - 2, \quad \mu > -1\right]$$ GW (332)(33)

3.644

1. $$\int_0^{\pi} \frac{\sin^m x}{p + q\cos x}\, dx = 2^{m-2} \frac{p}{q^2} \sum_{\nu=1}^{k} \left(\frac{p^2 - q^2}{-4q^2}\right)^{\nu-1} \mathrm{B}\left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2}\right) + \left(\frac{p^2-q^2}{-q^2}\right)^k A$$
$$\text{where } A = \begin{cases} \dfrac{\pi p}{q^2}\left(1 - \sqrt{1 - \dfrac{q^2}{p^2}}\right) & \text{if } m = 2k+2 \\ \dfrac{1}{q} \ln \dfrac{p+q}{p-q} & \text{if } m = 2k+1 \end{cases} \qquad \left[k \geq 1, \quad q \neq 0, \quad p^2 - q^2 \geq 0\right]$$

2. $$\int_0^{\pi} \frac{\sin^m x}{1 + \cos x}\, dx = 2^{m-1} \mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \geq 2]$$

3. $$\int_0^{\pi} \frac{\sin^m x}{1 - \cos x}\, dx = 2^{m-1} \mathrm{B}\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \qquad [m \geq 2]$$

4. $$\int_0^{\pi} \frac{\sin^2 x}{p + q\cos x}\, dx = \frac{p\pi}{q^2}\left(1 - \sqrt{1 - \frac{q^2}{p^2}}\right)$$

5. $$\int_0^{\pi} \frac{\sin^3 x}{p + q\cos x}\, dx = 2\frac{p}{q^2} + \frac{1}{q}\left(1 - \frac{p^2}{q^2}\right) \ln \frac{p+q}{p-q}$$

3.645 $$\int_0^{\pi} \frac{\cos^n x\, dx}{(a + b\cos x)^{n+1}} = \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^{n} (-1)^k \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \left(\frac{a+b}{a-b}\right)^k \qquad \left[a^2 > b^2\right]$$ LI (64)(16)

3.646

1. $$\int_0^{\pi/2} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a\cos 2x + a^2}\, dx = \frac{\pi}{4a}\left[\left(\frac{1+a}{2}\right)^n - \frac{1}{2^n}\right] \qquad \left[a^2 < 1\right]$$ BI (50)(6)

2. $$\int_0^{\pi/2} \frac{1 - a\cos 2nx}{1 - 2a\cos 2nx + a^2} \cos^m x \cos mx\, dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + \frac{\pi}{2^{m+1}} \qquad \left[a^2 < 1\right]$$ LI (50)(7)

3.647 $$\int_0^{\pi/2} \frac{\cos^p x \cos px\, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2b} \cdot \frac{a^{p-1}}{(a+b)^p} \qquad [p > -1, \quad a > 0, \quad b > 0]$$ BI (47)(20)

3.648

1. $$\int_0^{\pi/4} \frac{\tan^l x\,dx}{1+\cos\frac{m}{n}\pi\sin 2x}$$
$$= \frac{1}{2n}\operatorname{cosec}\frac{m}{n}\pi\sum_{k=0}^{n-1}(-1)^{k-1}\sin\frac{km}{n}\pi\left[\psi\left(\frac{n+l+k}{2n}\right)-\psi\left(\frac{l+k}{2n}\right)\right] \qquad [m+n \text{ is odd}]$$
$$= \frac{1}{n}\operatorname{cosec}\frac{m}{n}\pi\sum_{k=0}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km}{n}\pi\left[\psi\left(\frac{n+l-k}{n}\right)-\psi\left(\frac{l+k}{n}\right)\right] \qquad [m+n \text{ is even}]$$
$[l \text{ is a natural number}]$ BI (36)(5)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,dx}{1+\cos t\sin 2x} = \pi\operatorname{cosec} t\sin\mu t\operatorname{cosec}(\mu\pi)$$
$\left[|\operatorname{Re}\mu|<1, \quad t^2<\pi^2\right]$ BI (47)(4)

3.649

1. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\sin 2x\,dx}{1\mp 2a\cos 2x+a^2} = \frac{\pi}{4a}\operatorname{cosec}\frac{\mu\pi}{2}\left[1-\left(\frac{1-a}{1+a}\right)^\mu\right] \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{4a}\operatorname{cosec}\frac{\mu\pi}{2}\left[1+\left(\frac{a-1}{a+1}\right)^\mu\right] \qquad \left[a^2>1\right]$$
$[-2<\operatorname{Re}\mu<1]$ BI (50)(3)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\,(1\mp a\cos 2x)}{1\mp 2a\cos 2x+a^2}\,dx = \frac{\pi}{4}\sec\frac{\mu\pi}{2}\left[1+\left(\frac{1-a}{1+a}\right)^\mu\right] \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{4}\sec\frac{\mu\pi}{2}\left[1-\left(\frac{a-1}{a+1}\right)^\mu\right] \qquad \left[a^2>1\right]$$
$[|\operatorname{Re}\mu|<1]$ BI (50)(4)

3.651

1. $$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1+\sin x\cos x} = \frac{1}{3}\left[\psi\left(\frac{\mu+2}{3}\right)-\psi\left(\frac{\mu+1}{3}\right)\right]$$
$[\operatorname{Re}\mu>-1]$ BI (36)(3)

2. $$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin x\cos x} = \frac{1}{3}\left[\beta\left(\frac{\mu+2}{3}\right)+\beta\left(\frac{\mu+1}{3}\right)\right]$$
$[\operatorname{Re}\mu>-1]$ BI (36)(4)a

3.652

1. $$\int_0^{\pi/2} \frac{\tan^\mu x\,dx}{(\sin x+\cos x)\sin x} = \int_0^{\pi/2} \frac{\cot^\mu x\,dx}{(\sin x+\cos x)\cos x} = \pi\operatorname{cosec}\mu\pi$$
$[0<\operatorname{Re}\mu<1]$ BI (49)(1)

2. $$\int_0^{\pi/2} \frac{\tan^\mu x\,dx}{(\sin x-\cos x)\sin x} = \int_0^{\pi/2} \frac{\cot^\mu x\,dx}{(\cos x-\sin x)\cos x} = -\pi\cot\mu\pi$$
$[0<\operatorname{Re}\mu<1]$ BI (49)(2)

3. $$\int_0^{\pi/2} \frac{\cot^{\mu+\frac{1}{2}x}\,dx}{(\sin x+\cos x)\cos x} = \int_0^{\pi/2} \frac{\tan^{\mu-\frac{1}{2}x}\,dx}{(\sin x+\cos x)\cos x} = \pi\sec\mu\pi$$
$\left[|\operatorname{Re}\mu|<\frac{1}{2}\right]$ BI (61)(1, 2)

3.653

1. $$\int_0^{\pi/2} \frac{\tan^{1-2\mu} x\, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \int_0^{\pi/2} \frac{\cot^{1-2\mu} x\, dx}{a^2 \sin^2 x + b^2\cos^2 x} = \frac{\pi}{2a^{2\mu} b^{2-2\mu}\sin\mu\pi}$$
$[0 < \operatorname{Re}\mu < 1]$ GW (331)(59b)

2.[11] $$\int_0^{\pi/2} \frac{\tan^{\mu} x\, dx}{1 - a\sin^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\, dx}{1 - a\cos^2 x} = \frac{\pi \sec\frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}}$$
$[|\operatorname{Re}\mu| < 1, \quad a < 1]$ BI (49)(6)

3. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} t \sec\frac{\mu\pi}{2} \cos\left[\left(\frac{\pi}{2} - t\right)\mu\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(49)(7), BI(47)(21)

4. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x \sin 2x}{1 - \cos^2 t\sin^2 2x}\, dx = \pi \operatorname{cosec} 2t \operatorname{cosec}\frac{\mu\pi}{2}\sin\left[\left(\frac{\pi}{2} - t\right)\mu\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI (47)(22)a

5. $$\int_0^{\pi/2} \frac{\tan^{\mu} x \sin^2 x\, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\cos^2 x\, dx}{1 - \cos^2 t\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} 2t \sec\frac{\mu\pi}{2}\cos\left[\frac{\mu\pi}{2} - (\mu+1)t\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(47)(23)a, BI(49)(10)

6. $$\int_0^{\pi/2} \frac{\tan^{\mu} x \cos^2 x\, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\pi/2} \frac{\cot^{\mu} x\sin^2 x\, dx}{1 - \cos^2 t\sin^2 2x} = \frac{\pi}{2}\operatorname{cosec} 2t \sec\frac{\mu\pi}{2}\cos\left[\frac{\mu\pi}{2} - (\mu-1)t\right]$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$ BI(47)(24)a, BI(49)(9)

3.654

1. $$\int_0^{\pi/2} \frac{\tan^{\mu+1} x\cos^2 x\, dx}{(1 + \cos t\sin 2x)^2} = \int_0^{\pi/2} \frac{\cot^{\mu+1} x \sin^2 x\, dx}{(1 + \cos t \sin 2x)^2} = \frac{\pi\,(\mu\sin t\cos\mu t - \cos t\sin\mu t)}{2\sin\mu\pi\sin^3 t}$$
$\left[|\operatorname{Re}\mu| < 1, \quad t^2 < \pi^2\right]$
BI(48)(3), BI(49)(22)

2. $$\int_0^{\pi/2} \frac{\tan^{\pm\mu} x\, dx}{(\sin x + \cos x)^2} = \frac{\mu\pi}{\sin\mu\pi}$$
$[0 < \operatorname{Re}\mu < 1]$ BI (56)(9)a

3. $$\int_0^{\pi/2} \frac{\tan^{\pm(\mu-1)x}\ dx}{\cos^2 x - \sin^2 x} = \pm\frac{\pi}{2}\cot\frac{\mu\pi}{2}$$
$[0 < \operatorname{Re}\mu < 2]$ BI (45)(27, 29)

3.655 $$\int_0^{\pi/2} \frac{\tan^{2\mu-1} x\, dx}{1 - 2a\left(\cos t_1 \sin^2 x + \cos t_2\cos^2 x\right) + a^2} = \int_0^{\pi/2}\frac{\cot^{2\mu-1} x\, dx}{1 - 2a\left(\cos t_1\cos^2 x + \cos t_2 \sin^2 x\right) + a^2}$$
$$= \frac{\pi\operatorname{cosec}\mu\pi}{(1 - 2a\cos t_2 + a^2)^{\mu}\,(1 - 2a\cos t_1 + a^2)\,1-\mu}$$
$\left[0 < \operatorname{Re}\mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2\right]$ BI (50)(18)

3.656

1. $$\int_0^{\pi/4} \frac{\tan^\mu x\,dx}{1-\sin^2 x\cos^2 x} = \frac{1}{12}\left\{-\psi\left(\frac{\mu+1}{6}\right) - \psi\left(\frac{\mu+2}{6}\right) + \psi\left(\frac{\mu+4}{6}\right) + \psi\left(\frac{\mu+5}{6}\right) + 2\,\psi\left(\frac{\mu+2}{3}\right) - 2\,\psi\left(\frac{\mu+1}{3}\right)\right\}$$
$[\operatorname{Re}\mu > -1]$ (cf. **3.651** 1 and 2) LI (36)(10)

2. $$\int_0^{\pi/2} \frac{\tan^{\mu-1} x\cos^2 x\,dx}{1-\sin^2 x\cos^2 x} = \int_0^{\pi/2} \frac{\cot^{\mu-1} x\sin^2 x\,dx}{1-\sin^2 x\cos^2 x} = \frac{\pi}{4\sqrt{3}}\operatorname{cosec}\frac{\mu\pi}{6}\operatorname{cosec}\left(\frac{2+\mu}{6}\pi\right)$$
$[0 < \operatorname{Re}\mu < 4]$ LI (47)(26)

3.66 Forms containing powers of linear functions of trigonometric functions

3.661

1. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n+1}\,dx = 0$$ BI (68)(9)

2. $$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot 2\pi\left(a^2+b^2\right)^n$$ BI (68)(8)

3. $$\int_0^{\pi} (a+b\cos x)^n\,dx = \frac{1}{2}\int_0^{2\pi} (a+b\cos x)^n\,dx = \pi\left(a^2-b^2\right)^{\frac{n}{2}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) = \frac{\pi}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k}\left(a^2-b^2\right)^k$$
$\left[a^2 > b^2\right]$ GW (332)(37a)

4. $$\int_0^{\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{1}{2}\int_0^{2\pi} \frac{dx}{(a+b\cos x)^{n+1}} = \frac{\pi}{\left(a^2-b^2\right)^{\frac{n+1}{2}}} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right) = \frac{\pi}{2^n(a+b)^n\sqrt{a^2-b^2}}\sum_{k=0}^{n} \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!}\cdot\left(\frac{a+b}{a-b}\right)^k$$
$[a > |b|]$ GW(332)(38), LI(64)(14)

3.662

1. $$\int_0^{\pi/2} (\sec x - 1)^\mu \sin x\,dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cos x\,dx = \mu\pi\operatorname{cosec}\mu\pi$$
$[|\operatorname{Re}\mu| < 1]$ BI (55)(13)

2. $$\int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \sin 2x\,dx = (1-\mu)\mu\pi\operatorname{cosec}\mu\pi$$ $[-1 < \operatorname{Re}\mu < 2]$ BI (48)(7)

3. $$\int_0^{\pi/2} (\sec x - 1)^\mu \tan x\,dx = \int_0^{\pi/2} (\operatorname{cosec} x - 1)^\mu \cot x\,dx = -\pi\operatorname{cosec}\mu\pi$$
$[-1 < \operatorname{Re}\mu < 0]$ BI (46)(4,6)

4. $$\int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2}\operatorname{cosec}\mu\pi$$ $[-1 < \operatorname{Re}\mu < 0]$ BI (38)(22)a

5. $$\int_0^{\pi/4} (\cot x - 1)^\mu \frac{dx}{\cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \qquad [|\operatorname{Re}\mu| < 1]$$ BI (38)(11)a

3.663

1. $$\int_0^u (\cos x - \cos u)^{\nu-\frac{1}{2}} \cos ax\,dx = \sqrt{\frac{\pi}{2}} \sin^\nu u\,\Gamma\left(\nu + \frac{1}{2}\right) P^{-\nu}_{a-\frac{1}{2}}(\cos u)$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}; \quad a > 0, \quad 0 < u < \pi\right]$$ EH I 159(27), ET I 22(28)

2. $$\int_0^u (\cos x - \cos u)^{\nu-1} \cos[(\nu+\beta)x]\,dx = \frac{\sqrt{\pi}\,\Gamma(\beta+1)\,\Gamma(\nu)\,\Gamma(2\nu)\sin^{2\nu-1} u}{2^\nu\,\Gamma(\beta+2\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)} C^\nu_\beta(\cos u)$$
$$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > -1, \quad 0 < u < \pi]$$ EH I 178(23)

3.664

1. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^q dx = \pi P_q(z)$$
$$\left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right]$$ SM 482

2. $$\int_0^\pi \frac{dx}{\left(z + \sqrt{z^2-1}\cos x\right)^q} = \pi P_{q-1}(z)$$
$$\left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}\right]$$ WH

3. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^q \cos nx\,dx = \frac{\pi}{(q+1)(q+2)\cdots(q+n)} P^n_q(z)$$
$$\left[\operatorname{Re} z > 0, \quad \arg\left(z + \sqrt{z^2-1}\cos x\right) = \arg z \text{ for } x = \frac{\pi}{2}, \right.$$
$$\left. z \text{ lies outside the interval } (-1,1) \text{ of the real axis}\right]$$ WH, SM 483(15)

4. $$\int_0^\pi \left(z + \sqrt{z^2-1}\cos x\right)^\mu \sin^{2\nu-1} x\,dx$$
$$= \frac{2^{2\nu-1}\,\Gamma(\mu+1)\,[\Gamma(\nu)]^2}{\Gamma(2\nu+\mu)} C^\nu_\mu(z)$$
$$= \frac{\sqrt{\pi}\,\Gamma(\nu)\,\Gamma(2\nu)\,\Gamma(\mu+1)}{\Gamma(2\nu+\mu)\,\Gamma\left(\nu+\frac{1}{2}\right)} C^\nu_\mu(z) = 2^\nu \sqrt{\frac{\pi}{2}} \left(z^2-1\right)^{\frac{1}{4}-\frac{\nu}{2}} \Gamma(\nu)\, P^{\frac{1}{2}-\nu}_{\mu+\nu-\frac{1}{2}}(z)$$
$$[\operatorname{Re}\nu > 0]$$ EH I 155(6)a, EH I 178(22)

5. $$\int_0^{2\pi} \left[\beta + \sqrt{\beta^2-1}\cos(a-x)\right]^\nu \left(\gamma + \sqrt{\gamma^2-1}\cos x\right)^{\nu-1} dx$$
$$= 2\pi P_\nu\left(\beta\gamma - \sqrt{\beta^2-1}\sqrt{\gamma^2-1}\cos a\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$$ EH I 157(18)

3.665

1. $$\int_0^\pi \frac{\sin^{\mu-1} x\,dx}{(a+b\cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2-b^2)^\mu}} \mathrm{B}\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \qquad [\operatorname{Re}\mu > 0, \quad 0 < b < a]$$ FI II 790a

2. $$\int_0^\pi \frac{\sin^{2\mu-1} x\,dx}{(1+2a\cos x+a^2)^\nu} = \mathrm{B}\left(\mu, \tfrac{1}{2}\right) F\left(\nu, \nu-\mu+\tfrac{1}{2}; \mu+\tfrac{1}{2}; a^2\right)$$
$$[\operatorname{Re}\mu > 0, \quad |a| < 1]$$ EH I 81(9)

3.666

1. $$\int_0^\pi (\beta+\cos x)^{\mu-\nu-\frac{1}{2}} \sin^{2\nu} x\,dx = \frac{2^{\nu+\frac{1}{2}} e^{-i\mu\pi} \left(\beta^2-1\right)^{\frac{\mu}{2}} \Gamma\left(\nu+\frac{1}{2}\right) Q^\mu_{\nu-\frac{1}{2}}(\beta)}{\Gamma\left(\nu+\mu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re}\left(\nu+\mu+\tfrac{1}{2}\right) > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ EH I 155(5)a

2.[6] $$\int_0^\pi (\cosh\beta+\sinh\beta\cos x)^{\mu+\nu} \sin^{-2\nu} x\,dx = \frac{\sqrt{\pi}}{2^\nu} \sinh^\nu(\beta)\,\Gamma\left(\tfrac{1}{2}-\nu\right) P^\nu_\mu(\cosh\beta)$$
$$\left[\operatorname{Re}\nu < \tfrac{1}{2}\right]$$ EH I 156(7)

3. $$\int_0^\pi (\cos t+i\sin t\cos x)^\mu \sin^{2\nu-1} x\,dx = 2^{\nu-\frac{1}{2}} \sqrt{\pi} \sin^{\frac{1}{2}-\nu} t\,\Gamma(\nu)\, P^{\frac{1}{2}-\nu}_{\mu+\nu-\frac{1}{2}}(\cos t)$$
$$\left[\operatorname{Re}\nu > 0, \quad t^2 < \pi^2\right]$$ EH I 158(23)

4. $$\int_0^{2\pi} [\cos t+i\sin t\cos(a-x)]^\nu \cos mx\,dx = \frac{i^{3m} 2\pi\,\Gamma(\nu+1)}{\Gamma(\nu+m+1)} \cos ma\, P^m_\nu(\cos t)$$
$$\left[0 < t < \frac{\pi}{2}\right]$$ EH I 159(25)

5.[10] $$\int_0^{2\pi} [\cos t+i\sin t\cos(a-x)]^\nu \sin mx\,dx = \frac{i^{3m} 2\pi\,\Gamma(\nu+1)}{\Gamma(\nu+m+1)} \sin ma\, P^m_\nu(\cos t)$$
$$\left[0 < t < \frac{\pi}{2}\right]$$ EH I 159(26)

3.667

1. $$\int_0^{\pi/4} \frac{\sin^{\mu-1} 2x\,dx}{(\cos x+\sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{2}\right)} \qquad [\operatorname{Re}\mu > 0]$$ BI (37)(1)

2. $$\int_0^{\pi/4} \frac{\sin^\mu x\,dx}{(\cos x-\sin x)^{\mu+1}\cos x} = -\pi\operatorname{cosec}\mu\pi \qquad [-1 < \operatorname{Re}\mu < 0]$$ (cf. **3.192** 2)
BI (37)(16)

3. $$\int_0^{\pi/4} \frac{(\cos x-\sin x)^\mu}{\sin^\mu x\sin 2x}\,dx = -\frac{\pi}{2}\operatorname{cosec}\mu\pi \qquad [-1 < \operatorname{Re}\mu < 0]$$ BI (35)(27)

4. $$\int_0^{\pi/4} \frac{\sin^\mu x\,dx}{(\cos x-\sin x)^\mu \sin 2x} = \frac{\pi}{2}\operatorname{cosec}\mu\pi \qquad [0 < \operatorname{Re}\mu < 1]$$ LI (37)(20)a

5. $$\int_0^{\pi/4} \frac{\sin^\mu x\,dx}{(\cos x-\sin x)^\mu \cos^2 x} = \mu\pi\operatorname{cosec}\mu\pi \qquad [|\operatorname{Re}\mu| < 1]$$ BI (37)(17)

6. $$\int_0^{\pi/4} \frac{\sin^\mu x\,dx}{(\cos x-\sin x)^{\mu-1}\cos^3 x} = \frac{1-\mu}{2}\mu\pi \operatorname{cosec}\mu\pi$$ $[|\operatorname{Re}\mu|<1]$ BI(35)(24), BI(37)(18)

7. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x\cos^{\nu-1} x}{(\sin x+\cos x)^{\mu+\nu}}\,dx = \mathrm{B}(\mu,\nu)$$ $[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ BI (48)(8)

3.668

1. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)^{\cos 2t} dx = \frac{\pi}{2\sin\left(\pi\cos^2 t\right)}$$ FI II 788

2. $$\int_u^v \frac{(\cos u-\cos x)^{\mu-1}}{(\cos x-\cos v)^\mu}\cdot\frac{\sin x\,dx}{1-2a\cos x+a^2} = \frac{\left(1-2a\cos u+a^2\right)^{\mu-1}}{\left(1-2a\cos v+a^2\right)^\mu}\cdot\frac{\pi}{\sin\mu\pi}$$ $\left[0<\operatorname{Re}\mu<1, \quad a^2<1\right]$ BI (73)(2)

3.669 $$\int_0^{\pi/2} \frac{\sin^{p-1} x\cos^{q-p-1} x\,dx}{(a\cos x+b\sin x)^q} = \int_0^{\pi/2} \frac{\sin^{q-p-1} x\cos^{p-1} x}{(a\sin x+b\cos x)^q}\,dx = \frac{\mathrm{B}(p,q-p)}{a^{q-p}b^p}$$ $[q>p>0, \quad ab>0]$ BI (331)(9)

3.670

1. $$\int_0^\pi \sqrt{a\pm b\cos x}\,dx = \int_{-\pi/2}^{\pi/2} \sqrt{a\pm b\cos x}\,dx = 2\sqrt{a+b}\,\boldsymbol{K}\left(\sqrt{\frac{2b}{a+b}}\right)$$ $[a>b>0]$

2.* $$\int_0^\pi \frac{dx}{\sqrt{a\pm b\cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a\pm b\sin x}} = \frac{2}{\sqrt{a+b}}\,\boldsymbol{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$ $[a>b>0]$

3.67 Square roots of expressions containing trigonometric functions

3.671

1. $$\int_0^{\pi/2} \sin^\alpha x\cos^\beta x\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right)F\left(\frac{\alpha+1}{2},-\frac{1}{2};\frac{\alpha+\beta+2}{2};k^2\right)$$ $[\alpha>-1, \quad \beta>-1, \quad |k|<1]$ GW (331)(93)

2. $$\int_0^{\pi/2} \frac{\sin^\alpha x\cos^\beta x}{\sqrt{1-k^2\sin^2 x}}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right)F\left(\frac{\alpha+1}{2},\frac{1}{2};\frac{\alpha+\beta+2}{2};k^2\right)$$ $[\alpha>-1, \quad \beta>-1, \quad |k|<1]$ GW (331)(92)

3. $$\int_0^\pi \frac{\sin^{2n} x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2^n}\sum_{j=0}^\infty \frac{(2j-1)!!\,(2n+2j-1)!!}{2^{2j}j!(n+j)!}k^{2j} \qquad \left[k^2<1\right]$$
$$= \frac{(2n-1)!!\pi}{2^n\sqrt{1-k^2}}\sum_{j=0}^\infty \frac{\left[(2j-1)!!\right]^2}{2^{2j}j!(n+j)!}\left(\frac{k^2}{k^2-1}\right)^j \qquad \left[k^2<\tfrac{1}{2}\right]$$ LI (67)(2)

4.* $$\int_0^{\pi} \sqrt{a+b\cos x}\,dx = \int_{-\pi/2}^{\pi/2} \sqrt{a+b\sin x}\,dx = 2\sqrt{a+b}\,\boldsymbol{E}\left(\sqrt{\frac{2b}{a+b}}\right)$$

$[a > b]$

5.* $$\int_0^{\pi} \frac{dx}{\sqrt{a \pm b\cos x}} = \int_{-\pi/2}^{\pi/2} \frac{dx}{\sqrt{a \pm b\sin x}} = \frac{2}{a+b}\boldsymbol{K}\left(\sqrt{\frac{2b}{a+b}}\right)$$

$[a > b]$

3.672

1. $$\int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\cos x\,(\cos x - \sin x)}} = 2 \cdot \frac{(2n)!!}{(2n+1)!!}$$ BI (39)(5)

2. $$\int_0^{\pi/4} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\sin x\,(\cos x - \sin x)}} = \frac{(2n-1)!!}{(2n)!!}\pi$$ BI (39)(6)

3.673 $$\int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2}\boldsymbol{K}\left(\sin\frac{\pi - 2u}{4}\right)$$ BI (74)(11)

3.674

1.[8] $$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-(p^2/2)(1-\cos 2x)}} = \boldsymbol{K}(p), \qquad [1 > p > 0]$$ BI (67)(5)

2. $$\int_0^{\pi} \frac{\sin x\,dx}{\sqrt{1-2p\cos x+p^2}} = 2 \qquad [p^2 \le 1]$$
$$= \frac{2}{p} \qquad [p^2 \ge 1]$$
BI (67)(6)

3.[8] $$\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1-2p\cos x+p^2}} = \frac{1}{p}\left[\frac{1+p^2}{1+p}\boldsymbol{K}\left(\frac{2\sqrt{p}}{1+p}\right) - (1+p)\boldsymbol{E}\left(\frac{2\sqrt{p}}{1+p}\right)\right]$$

$[p^2 < 1]$ BI (67)(7)

3.675

1. $$\int_u^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x\,dx}{\sqrt{2(\cos u - \cos x)}} = \frac{\pi}{2}P_n(\cos u)$$ WH

2. $$\int_0^{u} \frac{\cos\left(n+\frac{1}{2}\right)x\,dx}{\sqrt{2(\cos x - \cos u)}} = \frac{\pi}{2}P_n(\cos u)$$ FI II 684, WH

3.676

1. $$\int_0^{\pi/2} \frac{\sin x\,dx}{\sqrt{1+p^2\sin^2 x}} = \frac{1}{p}\arctan p$$ BI (60)(5)

2. $$\int_0^{\pi/2} \tan^2 x\sqrt{1-p^2\sin^2 x}\,dx = \infty$$ BI (53)(8)

3. $$\int_0^{\pi/2} \frac{dx}{\sqrt{p^2\cos^2 x + q^2 \sin^2 x}} = \frac{1}{p}\boldsymbol{K}\left(\frac{\sqrt{p^2-q^2}}{p}\right) \qquad [0 < q < p]$$ FI II 165

3.677

1. $$\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sqrt{1+\sin^2 x}} = \sqrt{2}\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)$$ BI (60)(2)

2. $$\int_0^{\pi/2} \frac{\cos^2 x\, dx}{\sqrt{1+\sin^2 x}} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right]$$ BI (60)(3)

3.678

1. $$\int_0^{\pi/4} \left(\sec^{1/2} 2x - 1\right)\frac{dx}{\tan x} = \ln 2$$ BI (38)(23)

2. $$\int_0^{\pi/4} \frac{\tan^2 x\, dx}{\sqrt{1-k^2\sin^2 2x}} = \sqrt{1-k^2} - \boldsymbol{E}(k) + \frac{1}{2}\boldsymbol{K}(k)$$ BI (39)(2)

3. $$\int_0^{u} \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}}\, dx = \frac{\pi}{2}(1-\cos u) \qquad \left[u^2 < \frac{\pi^2}{4}\right]$$ LI (74)(6)

4. $$\int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x}\sqrt{\operatorname{cosec} x}\, dx = \frac{(2n-1)!!}{(2n)!!}\pi$$ BI (38)(24)

5. $$\int_0^{\pi/4} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x}\tan^m x\sqrt{\operatorname{cosec} x}\, dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!}\pi$$ BI (38)(25)

3.679

1. $$\int_0^{\pi/2} \frac{\cos^2 x}{1-\cos^2\beta\cos^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{\sin\beta\cos\beta\sqrt{1-k'^2\sin^2\beta}}\left\{\frac{\pi}{2} - \boldsymbol{K}\,E(\beta,k') - \boldsymbol{E}\,F(\beta,k') + \boldsymbol{K}\,F(\beta,k')\right\}^*$$ MO 138

2. $$\int_0^{\pi/2} \frac{\sin^2 x}{1-\left(1-k'^2\sin^2\beta\right)\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k'^2\sin\beta\cos\beta\sqrt{1-k'^2\sin^2\beta}}\left\{\frac{\pi}{2} - \boldsymbol{K}\,E(\beta,k') - \boldsymbol{E}\,F(\beta,k') + \boldsymbol{K}\,F(\beta,k')\right\}^*$$ MO 138

3. $$\int_0^{\pi/2} \frac{\sin^2 x}{1-k^2\sin^2\beta\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\boldsymbol{K}\,E(\beta,k) - \boldsymbol{E}\,F(\beta,k)}{k^2\sin\beta\cos\beta\sqrt{1-k^2\sin^2\beta}}$$ MO 138

*In 3.679, $k' = \sqrt{1-k^2}$.

3.68 Various forms of powers of trigonometric functions

3.681

1. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x \, dx}{\left(1 - k^2 \sin^2 x\right)^{\varrho}} = \frac{1}{2} \mathrm{B}(\mu, \nu) \, F\left(\varrho, \mu; \mu + \nu; k^2\right)$$

$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ EH I 115(7)

2. $$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x \, dx}{\left(1 - k^2 \sin^2 x\right)^{\mu+\nu}} = \frac{\mathrm{B}(\mu, \nu)}{2\left(1 - k^2\right)^{\mu}}$$ $[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ EH I 10(20)

3. $$\int_0^{\pi/2} \frac{\sin^{\mu} x \, dx}{\cos^{\mu-3} x \left(1 - k^2 \sin^2 x\right)^{\frac{\mu}{2}-1}} = \frac{\Gamma\left(\frac{\mu+1}{2}\right) \Gamma\left(2 - \frac{\mu}{2}\right)}{k^3 \sqrt{\pi(\mu-1)(\mu-3)(\mu-5)}} \left\{ \frac{1 + (\mu-3)k + k^2}{(1+k)^{\mu-3}} - \frac{1 - (\mu-3)k + k^2}{(1-k)^{\mu-3}} \right\}$$

$[-1 < \operatorname{Re} \mu < 4]$ BI (54)(10)

4.[8] $$\int_0^{\pi/2} \frac{\sin^{\mu+1} x \, dx}{\cos^{\mu} x \left(1 - k^2 \sin^2 x\right)^{\frac{\mu+1}{2}}} = \frac{(1-k)^{-\mu} - (1+k)^{-\mu}}{2k\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right)$$

$[-2 < \operatorname{Re} \mu < 1]$ BI (61)(5)

3.682 $$\int_0^{\pi/2} \frac{\sin^{\mu} x \cos^{\nu} x}{\left(a - b \cos^2 x\right)^{\varrho}} \, dx = \frac{1}{2a^{\varrho}} \mathrm{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2}, \varrho; \frac{\mu+\nu}{2} + 1; \frac{b}{a}\right)$$

$[\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1, \quad a > |b| \geq 0]$ GW (331)(64)

3.683

1. $$\int_0^{\pi/4} \left(\sin^n 2x - 1\right) \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} \left(\cos^n 2x - 1\right) \cot x \, dx = -\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} = -\frac{1}{2}\left[\boldsymbol{C} + \psi(n+1)\right]$$

$[n \geq 0]$ BI(34)(8), BI(35)(11)

2. $$\int_0^{\pi/4} \left(\sin^{\mu} 2x - 1\right) \operatorname{cosec}^{\mu} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} \left(\cos^{\mu} 2x - 1\right) \sec^{\mu} 2x \cot x \, dx = \frac{1}{2}\left[\boldsymbol{C} + \psi(1-\mu)\right]$$

$[\operatorname{Re} \mu < 1]$ BI (35)(20)

3. $$\int_0^{\frac{\pi}{4}} \left(\sin^{2\mu} 2x - 1\right) \operatorname{cosec}^{\mu} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \int_0^{\pi/4} \left(\cos^{2\mu} 2x - 1\right) \sec^{\mu} 2x \cot x \, dx = -\frac{1}{2\mu} + \frac{\pi}{2} \cot \mu\pi$$

BI (35)(21)

4. $$\int_0^{\pi/4} \left(1 - \sec^{\mu} 2x\right) \cot x \, dx = \int_0^{\pi/4} \left(1 - \operatorname{cosec}^{\mu} 2x\right) \tan\left(\frac{\pi}{4} + x\right) dx = \frac{1}{2}\left[\boldsymbol{C} + \psi(1-\mu)\right]$$

$[\operatorname{Re} \mu < 1]$ BI (35)(13)

3.684 $$\int_0^{\pi/4} \frac{(\cot^\mu x - 1)\, dx}{(\cos x - \sin x)\sin x} = \int_0^{\pi/2} \frac{(\tan^\mu x - 1)\, dx}{(\sin x - \cos x)\cos x} = -\boldsymbol{C} - \psi(1-\mu) \qquad [\operatorname{Re}\mu < 1]$$

BI (37)(9)

3.685

1. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x - \sin^{\nu-1} 2x\right) \tan\left(\frac{\pi}{4} + x\right)\, dx = \int_0^{\pi/4} \left(\cos^{\mu-1} 2x - \cos^{\nu-1} 2x\right) \cot x\, dx = \frac{1}{2}\left[\psi(\nu) - \psi(\mu)\right]$$

$[\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0]$ BI(34)(9), BI(35)(12)

2. $$\int_0^{\pi/2} \left(\sin^{\mu-1} x - \sin^{\nu-1} x\right) \frac{dx}{\cos x} = \int_0^{\pi/2} \left(\cos^{\mu-1} x - \cos^{\nu-1} x\right) \frac{dx}{\sin x} = \frac{1}{2}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\mu}{2}\right)\right]$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (46)(2)

3. $$\int_0^{\pi/2} \left(\sin^\mu x - \operatorname{cosec}^\mu x\right) \frac{dx}{\cos x} = \int_0^{\pi/2} \left(\cos^\mu x - \sec^\mu x\right) \frac{dx}{\sin x} = -\frac{\pi}{2}\tan\frac{\mu\pi}{2}$$

$[|\operatorname{Re}\mu| < 1]$ BI (46)(1, 3)

4. $$\int_0^{\pi/4} \left(\sin^\mu 2x - \operatorname{cosec}^\mu 2x\right) \cot\left(\frac{\pi}{4} + x\right)\, dx = \int_0^{\pi/4} \left(\cos^\mu 2x - \sec^\mu 2x\right) \tan x\, dx = \frac{1}{2\mu} - \frac{\pi}{2}\operatorname{cosec}\mu\pi$$

$[|\operatorname{Re}\mu| < 1]$ BI (35)(19, 22)

5. $$\int_0^{\pi/4} \left(\sin^\mu 2x - \operatorname{cosec}^\mu 2x\right) \tan\left(\frac{\pi}{4} + x\right)\, dx = \int_0^{\pi/4} \left(\cos^\mu 2x - \sec^\mu 2x\right) \cot x\, dx = -\frac{1}{2\mu} + \frac{\pi}{2}\cot\mu\pi$$

$[|\operatorname{Re}\mu| < 1]$ BI (35)(14)

6. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x + \operatorname{cosec}^\mu 2x\right) \cot\left(\frac{\pi}{4} + x\right)\, dx = \int_0^{\pi/4} \left(\cos^{\mu-1} 2x + \sec^\mu 2x\right) \tan x\, dx = \frac{\pi}{4}\operatorname{cosec}\mu\pi$$

$[0 < \operatorname{Re}\mu < 1]$ BI (35)(18, 8)

7. $$\int_0^{\pi/4} \left(\sin^{\mu-1} 2x - \operatorname{cosec}^\mu 2x\right) \tan\left(\frac{\pi}{4} + x\right)\, dx = \int_0^{\pi/4} \left(\cos^{\mu-1} 2x - \sec^\mu 2x\right) \cot x\, dx = \frac{\pi}{2}\cot\mu\pi$$

$[0 < \operatorname{Re}\mu < 1]$ BI(35)(7), LI(34)(10)

3.686 $$\int_0^{\pi/2} \frac{\tan x\, dx}{\cos^\mu x + \sec^\mu x} = \int_0^{\pi/2} \frac{\cot x\, dx}{\sin^\mu x + \operatorname{cosec}^\mu x} = \frac{\pi}{4\mu}$$ BI(47)(28), BI(49)(14)

3.687

1. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x}\, dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x}\, dx = \frac{\cos\left(\frac{\nu-\mu}{4}\pi\right)}{2\cos\left(\frac{\nu+\mu}{4}\pi\right)} \mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 2]$

BI (46)(7)

2. $$\int_0^{\pi/2} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x}\, dx = \int_0^{\pi/2} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x}\, dx = \frac{\sin\left(\frac{\nu-\mu}{4}\pi\right)}{2\sin\left(\frac{\nu+\mu}{4}\pi\right)} \mathrm{B}\left(\frac{\mu}{2}, \frac{\nu}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}(\mu+\nu) < 4]$ BI(46)(8)

3. $$\int_0^{\pi/2} \frac{\sin^{\mu} x + \sin^{\nu} x}{\sin^{\mu+\nu} x + 1} \cot x\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x + \cos^{\nu} x}{\cos^{\mu+\nu} x + 1} \tan x\, dx = \frac{\pi}{\mu+\nu} \sec\left(\frac{\mu-\nu}{\mu+\nu} \cdot \frac{\pi}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (49)(15)a, BI (47)(29)

4. $$\int_0^{\pi/2} \frac{\sin^{\mu} x - \sin^{\nu} x}{\sin^{\mu+\nu} x - 1} \cot x\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x - \cos^{\nu} x}{\cos^{\mu+\nu} x - 1} \tan x\, dx = \frac{\pi}{\mu+\nu} \tan\left(\frac{\mu-\nu}{\mu+\nu} \cdot \frac{\pi}{2}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI(149)(16)a, BI(47)(30)

5. $$\int_0^{\pi/2} \frac{\cos^{\mu} x + \sec^{\mu} x}{\cos^{\nu} x + \sec^{\nu} x} \tan x\, dx = \frac{\pi}{2\nu} \sec\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right)$$
$[|\operatorname{Re}\nu| > |\operatorname{Re}\mu|]$ BI (49)(12)

6. $$\int_0^{\pi/2} \frac{\cos^{\mu} x - \sec^{\mu} x}{\cos^{\nu} x - \sec^{\nu} x} \tan x\, dx = \frac{\pi}{2\nu} \tan\left(\frac{\mu}{\nu} \cdot \frac{\pi}{2}\right)$$
$[|\operatorname{Re}\nu| > |\operatorname{Re}\mu|]$ BI (49)(13)

3.688

1. $$\int_0^{\pi/4} \frac{\tan^{\nu} x - \tan^{\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (37)(10)

2. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \tan^{1-\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \pi \cot \mu\pi$$
$[0 < \operatorname{Re}\mu < 1]$ BI (37)(11)

3. $$\int_0^{\pi/4} \left(\tan^{\mu} x + \cot^{\mu} x\right) dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2}$$
$[|\operatorname{Re}\mu| < 1]$ BI (35)(9)

4. $$\int_0^{\pi/4} \left(\tan^{\mu} x - \cot^{\mu} x\right) \tan x\, dx = \frac{1}{\mu} - \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2}$$
$[0 < \operatorname{Re}\mu < 2]$ BI (35)(15)

5. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^{\mu-1} x}{\cos 2x}\, dx = \frac{\pi}{2} \cot \frac{\mu\pi}{2}$$
$[|\operatorname{Re}\mu| < 2]$ BI (35)(10)

6. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{\cos 2x} \tan x\, dx = -\frac{1}{\mu} + \frac{\pi}{2} \cot \frac{\mu\pi}{2}$$
$[-2 < \operatorname{Re}\mu < 0]$ BI (35)(23)

7. $$\int_0^{\pi/4} \frac{\tan^{\mu} x + \cot^{\mu} x}{1 + \cos t \sin 2x}\, dx = \pi \operatorname{cosec} t \operatorname{cosec} \mu\pi \sin \mu t$$
$[t \neq n\pi, \quad |\operatorname{Re}\mu| < 1]$ BI (36)(6)

8. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x + \cot^{\mu} x}{(\sin x + \cos x)\cos x}\, dx = \pi \operatorname{cosec} \mu\pi$$
$[0 < \operatorname{Re}\mu < 1]$ BI (37)(3)

9. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^{\mu} x}{(\sin x + \cos x)\cos x}\, dx = -\pi \operatorname{cosec} \mu\pi + \frac{1}{\mu}$$
$[0 < \operatorname{Re}\mu < 1]$ BI (37)(4)

10. $$\int_0^{\pi/4} \frac{\tan^{\nu} x - \cot^{\mu} x}{(\cos x - \sin x)\cos x}\, dx = \psi(1-\mu) - \psi(1+\nu)$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\nu > -1]$ BI (37)(5)

11. $$\int_0^{\pi/4} \frac{\tan^{\mu-1} x - \cot^\mu x}{(\cos x - \sin x)\cos x}\,dx = \pi \cot \mu\pi \qquad [0 < \operatorname{Re}\mu < 1]$$ BI (37)(7)

12. $$\int_0^{\pi/4} \frac{\tan^{\mu} x - \cot^\mu x}{(\cos x - \sin x)\cos x}\,dx = \pi \cot \mu\pi - \frac{1}{\mu} \qquad [0 < \operatorname{Re}\mu < 1]$$ BI (37)(8)

13. $$\int_0^{\pi/4} \frac{1}{\tan^\mu x + \cot^\mu x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \qquad [\operatorname{Re}\mu \neq 0]$$ BI (37)(12)

14. $$\int_0^{\pi/2} \frac{1}{(\tan^\mu x + \cot^\mu x)^\nu} \cdot \frac{dx}{\tan x} = \int_0^{\pi/2} \frac{1}{(\tan^\mu x + \cot^\mu x)^\nu} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1}\mu} \frac{\Gamma(\nu)}{\Gamma\left(\nu + \frac{1}{2}\right)}$$
$[\nu > 0]$ BI(49)(25), BI(49)(26)

15. $$\int_0^{\pi/4} (\tan^\mu x - \cot^\mu x)(\tan^\nu x - \cot^\nu x)\;dx = \frac{2\pi \sin\frac{\mu\pi}{2} \sin\frac{\nu\pi}{2}}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(17)

16. $$\int_0^{\pi/4} (\tan^\mu x + \cot^\mu x)(\tan^\nu x + \cot^\nu x)\;dx = \frac{2\pi \cos\frac{\mu\pi}{2} \cos\frac{\nu\pi}{2}}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(16)

17. $$\int_0^{\pi/4} \frac{(\tan^\mu x - \cot^\mu x)(\tan^\nu x + \cot^\nu x)}{\cos 2x}\,dx = -\pi \frac{\sin\mu\pi}{\cos\mu\pi + \cos\nu\pi}$$
$[|\operatorname{Re}\mu| < 1, \quad |\operatorname{Re}\nu| < 1]$ BI (35)(25)

18. $$\int_0^{\pi/4} \frac{\tan^\nu x - \cot^\nu x}{\tan^\mu x - \cot^\mu x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \tan\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1]$$ BI (37)(14)

19. $$\int_0^{\pi/4} \frac{\tan^\nu x + \cot^\nu x}{\tan^\mu x + \cot^\mu x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \sec\frac{\nu\pi}{2\mu} \qquad [0 < \operatorname{Re}\nu < 1]$$ BI (37)(13)

20. $$\int_0^{\pi/2} \frac{(1 + \tan x)^\nu - 1}{(1 + \tan x)^{\mu+\nu}} \frac{dx}{\sin x \cos x} = \psi(\mu + \nu) - \psi(\mu) \qquad [\mu > 0, \quad \nu > 0]$$ BI (49)(29)

3.689

1. $$\int_0^{\pi/2} \frac{(\sin^\mu x + \operatorname{cosec}^\mu x)\cot x\,dx}{\sin^\nu x - 2\cos t + \operatorname{cosec}^\nu x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec} \frac{\mu\pi}{\nu} \sin\frac{\mu t}{\nu}$$
$[\mu < \nu]$ LI (50)(14)

2. $$\int_0^{\pi/2} \frac{\sin^\mu x - 2\cos t_1 + \operatorname{cosec}^\mu x}{\sin^\nu x + 2\cos t_2 + \operatorname{cosec}^\nu x} \cdot \cot x\,dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec} \frac{\mu\pi}{\nu} \sin\frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1$$
$[(\nu > \mu > 0)$ or $(\nu < \mu < 0)$ or $(\mu > 0,\ \nu < 0,$ and $\mu + \nu < 0)$ or $(\mu < 0,\ \nu > 0,$ and $\mu + \nu > 0)]$
BI (50)(15)

3.69–3.71 Trigonometric functions of more complicated arguments

3.691

1. $$\int_0^\infty \sin\left(ax^2\right) dx = \int_0^\infty \cos ax^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \qquad [a > 0]$$ FI II 743a, ET I 64(7)a

2. $$\int_0^1 \sin\left(ax^2\right) dx = \sqrt{\frac{\pi}{2a}}\, S\left(\sqrt{a}\right) \qquad [a > 0]$$

3. $$\int_0^1 \cos\left(ax^2\right) dx = \sqrt{\frac{\pi}{2a}}\, C\left(\sqrt{a}\right) \qquad [a > 0]$$ ET I 8(5)a

4. $$\int_0^\infty \sin\left(ax^2\right) \sin 2bx\, dx = \sqrt{\frac{\pi}{2a}} \left\{\cos\frac{b^2}{a}\, C\left(\frac{b}{\sqrt{a}}\right) + \sin\frac{b^2}{a}\, S\left(\frac{b}{\sqrt{a}}\right)\right\}$$ $$[a > 0, \quad b > 0]$$ ET I 82(1)a

5. $$\int_0^\infty \sin\left(ax^2\right) \cos 2bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \left\{\cos\frac{b^2}{a} - \sin\frac{b^2}{a}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cos\left(\frac{b^2}{a} + \frac{\pi}{4}\right)$$ $$[a > 0, \quad b > 0]$$ ET I 82(18), BI(70)(13) GW(334)(5a)

6. $$\int_0^\infty \cos ax^2 \sin 2bx\, dx = \sqrt{\frac{\pi}{2a}} \left\{\sin\frac{b^2}{a}\, C\left(\frac{b}{\sqrt{a}}\right) - \cos\frac{b^2}{a}\, S\left(\frac{b}{\sqrt{a}}\right)\right\}$$ $$[a > 0, \quad b > 0]$$ ET I 83(3)a

7. $$\int_0^\infty \cos ax^2 \cos 2bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \left\{\cos\frac{b^2}{a} + \sin\frac{b^2}{a}\right\} \qquad [a > 0, \quad b > 0]$$ GW(334)(5a), BI(70)(14), ET I 24(7)

8. $$\int_0^\infty (\cos ax + \sin ax) \sin\left(b^2x^2\right) dx$$ $$= \frac{1}{2b}\sqrt{\frac{\pi}{2}} \left\{\left(1 + 2\, C\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) - \left(1 - 2\, S\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right)\right\}$$ $$[a > 0, \quad b > 0]$$ ET I 85(22)

9. $$\int_0^\infty (\cos ax + \sin ax) \cos\left(b^2x^2\right) dx$$ $$= \frac{1}{2b}\sqrt{\frac{\pi}{2}} \left\{\left(1 + 2\, C\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) + \left(1 - 2\, S\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right)\right\}$$ $$[a > 0, \quad b > 0]$$ ET I 25(21)

10. $$\int_0^\infty \sin\left(a^2x^2\right) \sin 2bx \sin 2cx\, dx = \frac{\sqrt{\pi}}{2a} \sin\frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4}\right)$$ $$[a > 0, \quad b > 0, \quad c > 0]$$ ET I 84(15)

11. $$\int_0^\infty \sin\left(a^2x^2\right) \cos 2bx \cos 2cx\, dx = \frac{\sqrt{\pi}}{2a} \cos\frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} + \frac{\pi}{4}\right)$$ $$[a > 0, \quad b > 0, \quad c > 0]$$ ET I 84(21)

12. $$\int_0^\infty \cos\left(a^2x^2\right)\sin 2bx \sin 2cx\, dx = \frac{\sqrt{\pi}}{2a}\sin\frac{2bc}{a^2}\sin\left(\frac{b^2+c^2}{a^2}-\frac{\pi}{4}\right)$$
$[a > 0, \quad b > 0, \quad c > 0]$ ET I 25(19)

13. $$\int_0^\infty \sin\left(ax^2\right)\cos\left(bx^2\right)\, dx = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{a-b}}\right) \qquad [a > b > 0]$$
$$= \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{b+a}}-\frac{1}{\sqrt{b-a}}\right) \qquad [b > a > 0]$$
BI (177)(21)

14. $$\int_0^\infty \left(\sin^2 ax^2 - \sin^2 bx^2\right)\, dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right) \qquad [a > 0, \quad b > 0]$$ BI (178)(1)

15. $$\int_0^\infty \left(\cos^2 ax^2 - \sin^2 bx^2\right)\, dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{b}}+\sqrt{\frac{\pi}{a}}\right) \qquad [a > 0, \quad b > 0]$$ BI (178)(3)

16. $$\int_0^\infty \left(\cos^2 ax^2 - \cos^2 bx^2\right)\, dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right) \qquad [a > 0, \quad b > 0]$$ BI (178)(5)

17. $$\int_0^\infty \left(\sin^4 ax^2 - \sin^4 bx^2\right) x = \frac{1}{64}\left(8-\sqrt{2}\right)\left(\sqrt{\frac{\pi}{b}}-\sqrt{\frac{\pi}{a}}\right)$$
$[a > 0, \quad b > 0]$ BI (178)(2)

18. $$\int_0^\infty \left(\cos^4 ax^2 - \sin^4 bx^2\right)\, dx = \frac{1}{8}\left(\sqrt{\frac{\pi}{a}}+\sqrt{\frac{\pi}{b}}\right)+\frac{1}{32}\left(\sqrt{\frac{\pi}{2a}}-\sqrt{\frac{\pi}{2b}}\right)$$
$[a > 0, \quad b > 0]$ BI (178)(4)

19. $$\int_0^\infty \left(\cos^4 ax^2 - \cos^4 bx^2\right)\, dx = \frac{1}{64}\left(8+\sqrt{2}\right)\left(\sqrt{\frac{\pi}{a}}-\sqrt{\frac{\pi}{b}}\right)$$
$[a > 0, \quad b > 0]$ BI (178)(6)

20. $$\int_0^\infty \sin^{2n} ax^2\, dx = \int_0^\infty \cos^{2n} ax^2\, dx = \infty$$ BI (177)(5, 6)

21. $$\int_0^\infty \sin^{2n+1}\left(ax^2\right)\, dx = \frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^{n+k}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}$$
$[a > 0]$ BI (70)(9)

22. $$\int_0^\infty \cos^{2n+1}\left(ax^2\right)\, dx = \frac{1}{2^{2n+1}}\sum_{k=0}^{n}\binom{2n+1}{k}\sqrt{\frac{\pi}{2(2n-2k+1)a}}$$
$[a > 0]$ BI(177)(7)a, BI(70)(10)

3.692

1. $$\int_0^\infty \left[\sin\left(a-x^2\right)+\cos\left(a-x^2\right)\right]\, dx = \sqrt{\frac{\pi}{a}}\sin a$$ GW(333)(30c), BI(178)(7)a

2. $$\int_0^\infty \cos\left(\frac{x^2}{2}-\frac{\pi}{8}\right)\cos ax\, dx = \sqrt{\frac{\pi}{2}}\cos\left(\frac{a^2}{2}-\frac{\pi}{8}\right) \qquad [a > 0]$$ ET I 24(8)

3. $$\int_0^\infty \sin\left[a\left(1-x^2\right)\right]\cos bx\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{a}}\cos\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right)$$
$[a>0]$ ET I 23(2)

4. $$\int_0^\infty \cos\left[a\left(1-x^2\right)\right]\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\sin\left(a+\frac{b^2}{4a}+\frac{\pi}{4}\right)$$
$[a>0]$ ET I 24(10)

5. $$\int_0^\infty \sin\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \int_0^\infty \cos\left(ax^2+\frac{b^2}{a}\right)\cos 2bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$
$[a>0]$ BI (70)(19, 20)

6.[8] $$\int_{-\infty}^\infty \left[\cos\sqrt{x^2-1}-\cos\sqrt{x^2+1}\right]\,dx = \sum_{n=0}^\infty \frac{\pi}{\left\{2^{4n+1}\left[(2n)!\right]^2\left(n+\frac{1}{2}\right)\right\}}$$

3.693

1. $$\int_0^\infty \sin\left(ax^2+2bx\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)-\sin\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\right\}$$
$[a>0]$ BI (70)(3)

2. $$\int_0^\infty \cos\left(ax^2+2bx\right)\,dx = \sqrt{\frac{\pi}{2a}}\left\{\cos\frac{b^2}{a}\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)+\sin\frac{b^2}{a}\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\right\}$$
$[a>0]$ BI (70)(4)

3.694

1. $$\int_0^\infty \sin\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c+\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c-\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c\right\}$$
$[a>0]$ GW (334)(4a)

2. $$\int_0^\infty \cos\left(ax^2+2bx+c\right)\,dx = \sqrt{\frac{\pi}{2a}}\cos\frac{b^2}{a}\left\{\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\cos c-\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$$+\sqrt{\frac{\pi}{2a}}\sin\frac{b^2}{a}\left\{\left(\frac{1}{2}-S_2\left(\frac{b^2}{a}\right)\right)\cos c+\left(\frac{1}{2}-C_2\left(\frac{b^2}{a}\right)\right)\sin c\right\}$$
$[a>0]$ GW (334)(4b)

3.695

1. $$\int_0^\infty \sin\left(a^3x^3\right)\sin(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)-\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a>0, \quad b>0]$ ET I 83(5)

2. $$\int_0^\infty \cos\left(a^3x^3\right)\cos(bx)\,dx = \frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$
$[a>0, \quad b>0]$ ET I 24(11)

3.696

1. $$\int_0^\infty \sin\left(ax^4\right)\sin\left(bx^2\right)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a > 0, \quad b > 0]$ ET I 83(2)

2. $$\int_0^\infty \sin\left(ax^4\right)\cos\left(bx^2\right)\,dx = -\frac{\pi}{4}\sqrt{\frac{b}{2a}}\sin\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a > 0, \quad b > 0]$ ET I 84(19)

3. $$\int_0^\infty \cos\left(ax^4\right)\sin\left(bx^2\right)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{3}{8}\pi\right)J_{\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a > 0, \quad b > 0]$ ET I 83(4), ET I 25(24)

4. $$\int_0^\infty \cos\left(ax^4\right)\cos\left(bx^2\right)\,dx = \frac{\pi}{4}\sqrt{\frac{b}{2a}}\cos\left(\frac{b^2}{8a}-\frac{\pi}{8}\right)J_{-\frac{1}{4}}\left(\frac{b^2}{8a}\right)$$
$[a > 0, \quad b > 0]$ ET I 25(25)

3.697 $$\int_0^\infty \sin\left(\frac{a^2}{x}\right)\sin(bx)\,dx = \frac{a\pi}{2\sqrt{b}}J_1\left(2a\sqrt{b}\right) \qquad [a > 0, \quad b > 0]$$ ET I 83(6)

3.698

1. $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\sin\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab - \cos 2ab + e^{-2ab}\right]$$
$[a > 0, \quad b > 0]$ ET I 83(9)

2.[8] $$\int_0^\infty \sin\left(\frac{a^2}{x^2}\right)\cos\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab + \cos 2ab - e^{-2ab}\right]$$ ET I 24(13)

3. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\sin\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\sin 2ab + \cos 2ab + e^{-2ab}\right]$$
$[a > 0, \quad b > 0]$ ET I 84(12)

4. $$\int_0^\infty \cos\left(\frac{a^2}{x^2}\right)\cos\left(b^2x^2\right)\,dx = \frac{1}{4b}\sqrt{\frac{\pi}{2}}\left[\cos 2ab - \sin 2ab + e^{-2ab}\right]$$
$[a > 0, \quad b > 0]$ ET I 24(14)

3.699

1. $$\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}\left(\cos 2ab + \sin 2ab\right) \qquad [a > 0, \quad b > 0]$$ BI (70)(27)

2. $$\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}\left(\cos 2ab - \sin 2ab\right) \qquad [a > 0, \quad b > 0]$$ BI (70)(28)

3. $$\int_0^\infty \sin\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)\,dx = \int_0^\infty \cos\left(a^2x^2 - 2ab + \frac{b^2}{x^2}\right)\,dx = \frac{\sqrt{2\pi}}{4a}$$
$[a > 0, \quad b > 0]$
BI(179)(11, 12)a, ET I 83(6)

4. $$\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \qquad [a > 0, \quad b > 0]$$ GW (334)(9b)a

5. $$\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \qquad [a > 0, \quad b > 0]$$ GW (334)(9b)a

3.711 $$\int_0^u \sin\left(a\sqrt{u^2 - x^2}\right) \cos bx \, dx = \frac{\pi a u}{2\sqrt{a^2 + b^2}} J_1\left(u\sqrt{a^2 + b^2}\right) \qquad [a > 0, \quad b > 0, \quad u > 0]$$ ET I 27(37)

3.712

1. $$\int_0^\infty \sin\left(ax^p\right) dx = \frac{\Gamma\left(\frac{1}{p}\right) \sin\frac{\pi}{2p}}{pa^{\frac{1}{p}}} \qquad [a > 0, \quad p > 1]$$ EH I 13(40)

2. $$\int_0^\infty \cos\left(ax^p\right) dx = \frac{\Gamma\left(\frac{1}{p}\right) \cos\frac{\pi}{2p}}{pa^{\frac{1}{p}}} \qquad [a > 0, \quad p > 1]$$ EH I 13(39)

3.713

1. $$\int_0^\infty \sin\left(ax^p + bx^q\right) dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-\frac{kq+1}{p}} \Gamma\left(\frac{kq+1}{p}\right) \sin\left[\frac{k(q-p)+1}{2p}\pi\right]$$
$$[a > 0, \quad b > 0, \quad p > 0, \quad q > 0]$$ BI (70)(7)

2. $$\int_0^\infty \cos\left(ax^p + bx^q\right) dx = \frac{1}{p} \sum_{k=0}^\infty \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \cos\left[\frac{k(q-p)+1}{2p}\pi\right]$$
$$[a > 0, \quad b > 0, \quad p > 0, \quad q > 0]$$ BI (70)(8)

3.714

1. $$\int_0^\infty \cos\left(z \sinh x\right) dx = K_0(z) \qquad [\operatorname{Re} z > 0]$$ WA 202(14)

2. $$\int_0^\infty \sin\left(z \cosh x\right) dx = \frac{\pi}{2} J_0(z) \qquad [\operatorname{Re} z > 0]$$ MO 36

3. $$\int_0^\infty \cos\left(z \cosh x\right) dx = -\frac{\pi}{2} Y_0(z) \qquad [\operatorname{Re} z > 0]$$ MO 37

4. $$\int_0^\infty \cos\left(z \sinh x\right) \cosh \mu x \, dx = \cos\frac{\mu\pi}{2} K_\mu(z) \qquad [\operatorname{Re} z > 0, \quad |\operatorname{Re} \mu| < 1]$$ WA 202(13)

5. $$\int_0^\pi \cos\left(z \cosh x\right) \sin^{2\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{z}\right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) I_\mu(z)$$
$$\left[\operatorname{Re} z > 0, \quad \operatorname{Re} \mu > -\tfrac{1}{2}\right]$$ WH

3.715

1. $$\int_0^\pi \sin\left(z \sin x\right) \sin ax \, dx = \sin a\pi \, s_{0,a}(z) = \sin a\pi \sum_{k=1}^\infty \frac{(-1)^{k-1} z^{2k-1}}{\left(1^2 - a^2\right)\left(3^2 - a^2\right) \ldots \left[(2k-1)^2 - a^2\right]}$$
$$[a > 0]$$ WA 338(13)

2. $$\int_0^\pi \sin(z\sin x)\sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(z\sin x)\sin nx\,dx$$
$$= [1-(-1)^n]\int_0^{\pi/2} \sin(z\sin x)\sin nx\,dx = [1-(-1)^n]\frac{\pi}{2}J_n(z)$$
$[n = 0, \pm 1, \pm 2, \dots]$ WA 30(6), GW(334)(153a)

3. $$\int_0^{\pi/2} \sin(z\sin x)\sin 2x\,dx = \frac{2}{z^2}(\sin z - z\cos z)$$ LI (43)(14)

4. $$\int_0^\pi \sin(z\sin x)\cos ax\,dx = (1+\cos a\pi)\,s_{0,a}(z)$$
$$= (1+\cos a\pi)\sum_{k=1}^\infty \frac{(-1)^{k-1}z^{2k-1}}{(1^2-a^2)(3^2-a^2)\dots[(2k-1)^2-a^2]}$$
$[a > 0]$ WA 338(14)

5. $$\int_0^\pi \sin(z\sin x)\cos[(2n+1)x]\,dx = 0$$ GW (334)(53b)

6. $$\int_0^\pi \cos(z\sin x)\sin ax\,dx = -a(1-\cos a\pi)\,s_{-1,a}(z)$$
$$= -a(1-\cos a\pi)\left\{-\frac{1}{a^2} + \sum_{k=1}^\infty \frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\dots[(2k)^2-a^2]}\right\}$$
$[a > 0]$ WA 338(12)

7. $$\int_0^\pi \cos(z\sin x)\sin 2nx\,dx = 0$$ GW (334)(54a)

8. $$\int_0^\pi \cos(z\sin x)\cos ax\,dx = -a\sin a\pi\,s_{-1,a}(z)$$
$$= -a\sin a\pi\left\{-\frac{1}{a^2} + \sum_{k=1}^\infty \frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\dots[(2k)^2-a^2]}\right\}$$
$[a > 0]$ WA 338(11)

9. $$\int_0^\pi \cos(z\sin x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(z\sin x)\cos nx\,dx$$
$$= [1+(-1)^n]\int_0^{\pi/2} \cos(z\sin x)\cos nx\,dx = [1+(-1)^n]\frac{\pi}{2}J_n(z)$$
GW (334)(54b)

10.[8] $$\int_0^{\pi/2} \cos(z\sin x)\cos^{2n} x\,dx = \frac{\pi}{2}\frac{(2n-1)!!}{z^n}J_n(z)$$ $[n = 0, 1, 2, \dots]$ FI II 486, WA 35a

11. $$\int_0^{\pi/2} \sin(z\cos x)\sin 2x\,dx = \frac{2}{z^2}(\sin z - z\cos z)$$ LI (43)(15)

12.[8] $$\int_0^{\pi/2} \sin(z\cos x)\cos ax\,dx = \cos\frac{a\pi}{2}\, s_{0,a}(z) = \frac{\pi}{4}\operatorname{cosec}\frac{a\pi}{2}\left[\mathbf{J}_a(z) - \mathbf{J}_{-a}(z)\right]$$
$$= -\frac{\pi}{4}\sec\frac{a\pi}{4}\left[\mathbf{E}_a(z) + \mathbf{E}_{-a}(z)\right]$$
$$= \cos\frac{a\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k-1}}{(1^2-a^2)(3^2-a^2)\dots[(2k-1)^2-a^2]}$$
$[a > 0]$ WA 339

13. $$\int_0^{\pi} \sin(z\cos x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(z\cos x)\cos nx\,dx = \pi\sin\frac{n\pi}{2}\,J_n(z)$$ GW (334)(55b)

14. $$\int_0^{\pi/2} \sin(z\cos x)\cos[(2n+1)x]\,dx = (-1)^n\frac{\pi}{2}\,J_{2n+1}(z)$$ WA 30(8)

15.[11] $$\int_0^{\pi/2} \sin(a\cos x)\tan x\,dx = \operatorname{si}(a) + \frac{\pi}{2}$$ $[a > 0]$ BI (43)(17)

16. $$\int_0^{\pi/2} \sin(z\cos x)\sin^{2\nu} x\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{z}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(z)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 358(1)

17.[7] $$\int_0^{\pi/2} \cos(z\cos x)\cos ax\,dx = -a\sin\frac{a\pi}{2}\, s_{-1,a}(z)$$
$$= \frac{\pi}{4}\sec\frac{a\pi}{2}\left[\mathbf{J}_a(z) + \mathbf{J}_{-a}(z)\right] = \frac{\pi}{4}\operatorname{cosec}\frac{a\pi}{2}\left[\mathbf{E}_a(z) - \mathbf{E}_{-a}(z)\right]$$
$$= -a\sin\frac{a\pi}{2}\left\{-\frac{1}{a^2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^{2k}}{a^2(2^2-a^2)(4^2-a^2)\dots[(2k)^2-a^2]}\right\}$$
$[a > 0]$ WA 339

18. $$\int_0^{\pi} \cos(z\cos x)\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(z\cos x)\cos nx\,dx = \pi\cos\frac{n\pi}{2}\,J_n(z)$$ GW (334)(56b)

19. $$\int_0^{\pi/2} \cos(z\cos x)\cos 2nx\,dx = (-1)^n\cdot\frac{\pi}{2}\,J_{2n}(z)$$ WA 30(9)

20. $$\int_0^{\pi/2} \cos(z\cos x)\sin^{2\nu} x\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{z}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(z)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 35, WH

21. $$\int_0^{\pi} \cos(z\cos x)\sin^{2\mu} x\,dx = \sqrt{\pi}\left(\frac{2}{z}\right)^{\mu}\Gamma\left(\mu+\frac{1}{2}\right)J_\mu(z)$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}\right]$ WH

3.716

1. $$\int_0^{\pi/2} \sin(a\tan x)\,dx = \frac{1}{2}\left[e^{-a}\overline{\operatorname{Ei}}(a) - e^{a}\operatorname{Ei}(-a)\right]$$ $[a > 0]$ (cf. **3.723** 1) BI (43)(1)

2. $$\int_0^{\pi/2} \cos(a\tan x)\,dx = \frac{\pi}{2}e^{-a}$$ $[a \geq 0]$ BI (43)(2)

3. $$\int_0^{\pi/2} \sin(a\tan x)\sin 2x\,dx = \frac{a\pi}{2}e^{-a} \qquad [a \ge 0]$$ BI (43)(7)

4. $$\int_0^{\pi/2} \cos(a\tan x)\sin^2 x\,dx = \frac{1-a}{4}\pi e^{-a} \qquad [a \ge 0]$$ BI (43)(8)

5. $$\int_0^{\pi/2} \cos(a\tan x)\cos^2 x\,dx = \frac{1+a}{4}\pi e^{-a} \qquad [a \ge 0]$$ BI (43)(9)

6. $$\int_0^{\pi/2} \sin(a\tan x)\tan x\,dx = \frac{\pi}{2}e^{-a} \qquad [a > 0]$$ BI (43)(5)

7. $$\int_0^{\pi/2} \cos(a\tan x)\tan x\,dx = -\frac{1}{2}\left[e^{-a}\overline{\mathrm{Ei}}(a) + e^{a}\,\mathrm{Ei}(-a)\right] \qquad [a > 0]$$ (cf. **3.723** 5) BI (43)(6)

8. $$\int_0^{\pi/2} \sin(a\tan x)\sin^2 x\tan x\,dx = \frac{2-a}{4}\pi e^{-a} \qquad [a > 0]$$ BI (43)(11)

9. $$\int_0^{\pi/2} \sin^2(a\tan x)\;dx = \frac{\pi}{4}\left(1 - e^{-2a}\right) \qquad [a \ge 0]$$ (cf. **3.742** 1) BI (43)(3)

10. $$\int_0^{\pi/2} \cos^2(a\tan x)\;dx = \frac{\pi}{4}\left(1 + e^{-2a}\right) \qquad [a \ge 0]$$ (cf. **3.742** 3) BI (43)(4)

11. $$\int_0^{\pi/2} \sin^2(a\tan x)\cot^2 x\,dx = \frac{\pi}{4}\left(e^{-2a} + 2a - 1\right) \qquad [a \ge 0]$$ BI (43)(19)

12. $$\int_0^{\pi/2} \left[1 - \sec^2 x\cos(\tan x)\right]\frac{dx}{\tan x} = \boldsymbol{C}$$ BI (51)(14)

13. $$\int_0^{\pi/2} \sin(a\cot x)\sin 2x\,dx = \frac{a\pi}{2}e^{-a} \qquad [a \ge 0]$$ (cf. **3.716** 3)

In general, formulas **3.716** remain valid if we replace $\tan x$ in the argument of the sine or cosine with $\cot x$ if we also replace $\sin x$ with $\cos x$, $\cos x$ with $\sin x$, hence $\tan x$ with $\cot x$, $\cot x$ with $\tan x$, $\sec x$ with $\operatorname{cosec} x$, and $\operatorname{cosec} x$ with $\sec x$ in the factors. Analogously,

3.717 $$\int_0^{\pi/2} \sin(a\operatorname{cosec} x)\sin(a\cot x)\frac{dx}{\cos x} = \int_0^{\pi/2} \sin(a\sec x)\sin(a\tan x)\frac{dx}{\sin x} = \frac{\pi}{2}\sin a \qquad [a \ge 0]$$ BI (52)(11, 12)

3.718

1. $$\int_0^{\pi/2} \sin\left(\frac{\pi}{2}p - a\tan x\right)\tan^{p-1} x\,dx = \int_0^{\pi/2} \cos\left(\frac{\pi}{2}p - a\tan x\right)\tan^{p} x\,dx = \frac{\pi}{2}e^{-a}$$ $$\left[p^2 < 1, \quad p \ne 0, \quad a \ge 0\right]$$ BI (44)(5, 6)

2. $$\int_0^{\pi/2} \sin(a\tan x - \nu x)\sin^{\nu-2} x\,dx = 0 \qquad [\operatorname{Re}\nu > 0, \quad a > 0]$$ NH 157(15)

3. $$\int_0^{\pi/2} \sin(n\tan x + \nu x)\frac{\cos^{\nu-1} x}{\sin x}\,dx = \frac{\pi}{2} \qquad [\operatorname{Re}\nu > 0]$$ BI (51)(15)

4. $$\int_0^{\pi/2} \cos(a\tan x - \nu x)\cos^{\nu-2} x\, dx = \frac{\pi e^{-a} a^{\nu-1}}{\Gamma(\nu)} \qquad [\operatorname{Re}\nu > 1, \quad a > 0]$$
LO V 153(112), NT 157(14)

5. $$\int_0^{\pi/2} \cos(a\tan x + \nu x)\cos^{\nu} x\, dx = 2^{-\nu-1}\pi e^{-a} \qquad [\operatorname{Re}\nu > -1, \quad a \geq 0]$$ BI (44)(4)

6. $$\int_0^{\pi/2} \cos(a\tan x - \gamma x)\cos^{\nu} x\, dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2}, -\frac{\nu+1}{2}}(2a)}{\Gamma\left(1+\frac{\gamma+\nu}{2}\right)}$$
$$\left[a > 0, \quad \operatorname{Re}\nu > -1, \quad \frac{\nu+\gamma}{2} \neq -1, -2, \ldots\right]$$ EH I 274(13)a

7. $$\int_0^{\pi/2} \frac{\sin nx - \sin(nx - a\tan x)}{\sin x}\cos^{n-1} x\, dx = \begin{cases} \pi/2 & [n = 0, \quad a > 0], \\ \pi\left(1 - e^{-a}\right) & [n = 1, \quad a \geq 0] \end{cases}$$
LO V 153(114)

3.719

1.[6] $$\int_0^{\pi} \sin(\nu x - z\sin x)\, dx = \pi\, \mathbf{E}_\nu(z)$$ WA 336(2)

2. $$\int_0^{\pi} \cos(nx - z\sin x)\, dx = \pi\, J_n(z)$$ WH

3. $$\int_0^{\pi} \cos(\nu x - z\sin x)\, dx = \pi\, \mathbf{J}_\nu(z)$$ WA 336(1)

3.72–3.74 Combinations of trigonometric and rational functions

3.721

1. $$\int_0^{\infty} \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2}\operatorname{sign} a$$ FI II 645

2. $$\int_1^{\infty} \frac{\sin(ax)}{x}\, dx = -\operatorname{si}(a)$$ BI 203(1)

3.[8] $$\int_1^{\infty} \frac{\cos(ax)}{x}\, dx = -\operatorname{ci}(a)$$ BI 203(5)

3.722

1. $$\int_0^{\infty} \frac{\sin(ax)}{x+\beta}\, dx = \operatorname{ci}(a\beta)\sin(a\beta) - \cos(a\beta)\operatorname{si}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0]$$
BI(16)(1), FI II 646a

2.[11] $$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x+\beta}\, dx = \pi e^{ia\beta} \qquad [a > 0, \quad \operatorname{Im}\beta > 0]$$

3. $$\int_0^{\infty} \frac{\cos(ax)}{x+\beta}\, dx = -\sin(a\beta)\operatorname{si}(a\beta) - \cos(a\beta)\operatorname{ci}(a\beta) \qquad [|\arg\beta| < \pi, \quad a > 0]$$
ET I 8(7), BI(160)(2)

4.[8] $$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x+\beta}\,dx = -i\pi e^{ia\beta} \qquad [a>0, \quad \operatorname{Im}\beta > 0]$$

5.[10] $$\int_0^{\infty} \frac{\sin(ax)}{\beta - x}\,dx = \sin(\beta a)\operatorname{ci}(\beta a) - \cos(\beta a)\left[\operatorname{si}(\beta a) + \pi\right]$$
$$[a>0, \quad \beta \text{ not real and positive}]$$
FI II 646, BI(161)(1)

6.[8] $$\int_{-\infty}^{\infty} \frac{\sin(ax)}{\beta - x}\,dx = -\pi e^{ia\beta} \qquad [a>0, \quad \operatorname{Im}\beta > 0]$$

7.[10] $$\int_0^{\infty} \frac{\cos(ax)}{\beta - x}\,dx = -\cos(a\beta)\operatorname{ci}(a\beta) + \sin(a\beta)\left[\operatorname{si}(a\beta) + \pi\right]$$
$$[a>0, \quad \beta \text{ not real and positive}]$$
ET I 8(8), BI(161)(2)a

8.[11] $$\int_{-\infty}^{\infty} \frac{\cos(ax)}{\beta - x}\,dx = -i\pi e^{ia\beta} \qquad [a>0, \quad \operatorname{Im}\beta > 0]$$

3.723

1.[11] $$\int_0^{\infty} \frac{\sin(ax)}{\beta^2 + x^2}\,dx = \frac{1}{2\beta}\left[e^{-a\beta}\,\overline{\operatorname{Ei}}(a\beta) - e^{a\beta}\operatorname{Ei}(-a\beta)\right] \qquad [a>0, \quad \beta>0]$$
ET I 65(14), BI(160)(3)

2. $$\int_0^{\infty} \frac{\cos(ax)}{\beta^2 + x^2}\,dx = \frac{\pi}{2\beta} e^{-a\beta} \qquad [a \ge 0, \quad \operatorname{Re}\beta > 0]$$
FI II 741, 750, ET I 8(11), WH

3. $$\int_0^{\infty} \frac{x\sin(ax)}{\beta^2 + x^2}\,dx = \frac{\pi}{2} e^{-a\beta} \qquad [a > 0, \quad \operatorname{Re}\beta > 0]$$
FI II 741, 750, ET I 65(15), WH

4. $$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{\beta^2 + x^2}\,dx = \pi e^{-a\beta} \qquad [a > 0, \quad \operatorname{Re}\beta > 0]$$
BI (202)(10)

5.[11] $$\int_0^{\infty} \frac{x\cos(ax)}{\beta^2 + x^2}\,dx = -\frac{1}{2}\left[e^{-a\beta}\,\overline{\operatorname{Ei}}(a\beta) + e^{a\beta}\operatorname{Ei}(-a\beta)\right] \qquad [a>0, \quad \beta>0]$$
BI (160)(6)

6. $$\int_{-\infty}^{\infty} \frac{\sin[a(b-x)]}{c^2 + x^2}\,dx = \frac{\pi}{c} e^{-ac}\sin(ab) \qquad [a>0, \quad b>0, \quad c>0]$$
LI (202)(9)

7. $$\int_{-\infty}^{\infty} \frac{\cos[a(b-x)]}{c^2 + x^2}\,dx = \frac{\pi}{c} e^{-ac}\cos(ab) \qquad [a>0, \quad b>0, \quad c>0]$$
LI (202)(11)a

8. $$\int_0^{\infty} \frac{\sin(ax)}{\beta^2 - x^2}\,dx = \frac{1}{\beta}\left[\sin(a\beta)\operatorname{ci}(a\beta) - \cos(a\beta)\left(\operatorname{si}(a\beta) + \frac{\pi}{2}\right)\right]$$
$$[|\arg\beta| < \pi, \quad a>0]$$
BI (161)(3)

9. $$\int_0^{\infty} \frac{\cos(ax)}{b^2 - x^2}\,dx = \frac{\pi}{2b}\sin(ab) \qquad [a>0, \quad b>0]$$
BI(161)(5), ET I 9(15)

10. $$\int_0^{\infty} \frac{x\sin(ax)}{b^2 - x^2}\,dx = -\frac{\pi}{2}\cos(ab) \qquad [a>0]$$
FI II 647, ET II 252(45)

11. $$\int_0^{\infty} \frac{x\cos(ax)}{\beta^2 + x^2}\,dx = \cos(a\beta)\operatorname{ci}(a\beta) + \sin(a\beta)\left[\operatorname{si}(a\beta) + \frac{\pi}{2}\right]$$
$$[|\arg\beta| < \pi, \quad a>0]$$
BI (161)(6)

12. $$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x-b)}\,dx = \pi \frac{\cos(ab)-1}{b}$$ $[a>0, \quad b>0]$ ET II 252(44)

3.724

1. $$\int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \sin(ax)\,dx = \left(\frac{cq-b}{\sqrt{p-q^2}} \sin(aq) + c\cos(aq)\right) \pi e^{-a\sqrt{p-q^2}}$$
$\left[a>0, \quad p>q^2\right]$ BI (202)(12)

2. $$\int_{-\infty}^{\infty} \frac{b+cx}{p+2qx+x^2} \cos(ax)\,dx = \left(\frac{b-cq}{\sqrt{p-q^2}} \cos(aq) + c\sin(aq)\right) \pi e^{-a\sqrt{p-q^2}}$$
$\left[a>0, \quad p>q^2\right]$ BI (202)(13)

3. $$\int_{-\infty}^{\infty} \frac{\cos[(b-1)t] - x\cos(bt)}{1-2x\cos t + x^2} \cos(ax)\,dx = \pi e^{-a\sin t} \sin\left(bt + a\cos t\right)$$
$\left[a>0, \quad t^2<\pi^2\right]$ BI (202)(14)

3.725

1. $$\int_0^{\infty} \frac{\sin(ax)\,dx}{x\left(\beta^2+x^2\right)} = \frac{\pi}{2\beta^2}\left(1-e^{-a\beta}\right)$$ $[\operatorname{Re}\beta>0, \quad a>0]$ BI (172)(1)

2. $$\int_0^{\infty} \frac{\sin(ax)\,dx}{x\left(b^2-x^2\right)} = \frac{\pi}{2b^2}\left(1-\cos(ab)\right)$$ $[a>0]$ BI (172)(4)

3. $$\int_0^{\infty} \frac{\sin(ax)\cos(bx)}{x\left(x^2+\beta^2\right)}\,dx = \frac{\pi}{2\beta^2} e^{-\beta b}\sinh(a\beta) \qquad [0<a<b]$$
$$= -\frac{\pi}{2\beta^2} e^{-a\beta}\cosh(b\beta) + \frac{\pi}{2\beta^2} \qquad [a>b>0]$$
ET I 19(4)

3.726

1.[11] $$\int_0^{\infty} \frac{x\sin(ax)\,dx}{b^3 \pm b^2x + bx^2 \pm x^3}$$
$$= \pm\frac{1}{4b}\left[e^{-ab}\overline{\operatorname{Ei}}(ab) - e^{ab}\operatorname{Ei}(-ab) - 2\operatorname{ci}(ab)\sin(ab) + 2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right]$$
$$+\frac{\pi e^{-ab} - \pi\cos(ab)}{4b}$$
[$a>0, \quad b>0;$ if the lower sign is taken, then the integral is a principal value integral]
ET I 65(21)a, BI(176)(10, 13)

2.[7] $$\int_0^{\infty} \frac{x^2\sin(ax)\,dx}{b^3 \pm b^2x + bx^2 \pm x^3}$$
$$= \frac{1}{4}\left[e^{ab}\operatorname{Ei}(-ab) - e^{-ab}\overline{\operatorname{Ei}}(ab) + 2\operatorname{ci}(ab)\sin(ab) - 2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right)\right]$$
$$\pm\pi\left(e^{-ab}+\cos(ab)\right)$$
[$a>0, \quad b>0;$ if the lower sign is taken, then the integral is a principal value integral]
ET I 66(22), BI(176)(11, 14)

3.727

1. $$\int_0^\infty \frac{\cos(ax)}{b^4+x^4}\,dx = \frac{\pi\sqrt{2}}{4b^3}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}}+\sin\frac{ab}{\sqrt{2}}\right)$$
$[a>0, \quad b>0]$ BI(160)(25)a, ET I 9(19)

2.[8] $$\int_0^\infty \frac{\sin(ax)}{b^4-x^4}\,dx = \frac{1}{4b^3}\Big[2\sin(ab)\operatorname{ci}(ab) - 2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right) + e^{-ab}\operatorname{Ei}(ab) - e^{ab}\operatorname{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI (161)(12)

3. $$\int_0^\infty \frac{\cos(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b^3}\left[e^{-ab}+\sin(ab)\right]$$
$[a>0, \quad b>0]$ (cf. **3.723** 2 and **3.723** 9) BI (161)(16)

4. $$\int_0^\infty \frac{x\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2b^2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\sin\frac{ab}{\sqrt{2}}$$
$[a>0, \quad b>0]$ BI (160)(23)a

5. $$\int_0^\infty \frac{x\sin(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b^2}\left[e^{-ab}-\cos(ab)\right]$$
$[a>0, \quad b>0]$ BI (161)(13)

6.[11] $$\int_0^\infty \frac{x\cos(ax)}{b^4-x^4}\,dx = \frac{1}{4b^2}\left[2\cos(ab)\operatorname{ci}(ab) + 2\sin(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right) - e^{-ab}\overline{\operatorname{Ei}}(ab) - e^{ab}\operatorname{Ei}(-ab)\right]$$
$[a>0, \quad b>0]$ (cf. **3.723** 5 and **3.723** 11) BI (161)(17)

7. $$\int_0^\infty \frac{x^2\cos(ax)}{b^4+x^4}\,dx = \frac{\pi\sqrt{2}}{4b}\exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}}-\sin\frac{ab}{\sqrt{2}}\right)$$
$[a>0, \quad b>0]$ BI (160)(26)a

8.[11] $$\int_0^\infty \frac{x^2\sin(ax)\,dx}{b^4-x^4} = \frac{1}{4b}\Big[2\sin(ab)\operatorname{ci}(ab) -2\cos(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right) - e^{-ab}\overline{\operatorname{Ei}}(ab) + e^{ab}\operatorname{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI (161)(14)

9. $$\int_0^\infty \frac{x^2\cos(ax)}{b^4-x^4}\,dx = \frac{\pi}{4b}\left(\sin(ab)-e^{-ab}\right)$$
$[a>0, \quad b>0]$ BI (161)(18)

10. $$\int_0^\infty \frac{x^3\sin(ax)}{b^4+x^4}\,dx = \frac{\pi}{2}\exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}$$
$[a>0, \quad b>0]$ BI (160)(24)

11. $$\int_0^\infty \frac{x^3\sin(ax)}{b^4-x^4}\,dx = \frac{-\pi}{4}\left[e^{-ab}-\cos(ab)\right]$$
$[a>0, \quad b>0]$ BI (161)(15)

12.[7] $$\int_0^\infty \frac{x^3\cos(ax)\,dx}{b^4-x^4} = \frac{1}{4}\Big[2\cos(ab)\operatorname{ci}(ab) + 2\sin(ab)\left(\operatorname{si}(ab)+\frac{\pi}{2}\right) + e^{-ab}\overline{\operatorname{Ei}}(ab) + e^{ab}\operatorname{Ei}(-ab)\Big]$$
$[a>0, \quad b>0]$ BI(161)(19)

13. $$\int_0^\infty \frac{x^3 \sin ax}{\left(x^2+b^2\right)^3}\,dx = \frac{\pi e^{-ab}}{16b}\left(3a - ba^2\right)$$

14. $$\int_0^\infty \frac{x^3 \sin ax}{\left(x^2+b^2\right)^4}\,dx = \frac{\pi e^{-ab} a}{96b^3}\left(3 + 3ab - a^2b^2\right)$$

3.728

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta e^{-a\gamma} - \gamma e^{-a\beta}\right)}{2\beta\gamma\left(\beta^2-\gamma^2\right)}$$ $[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (175)(1)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(e^{-a\beta} - e^{-a\gamma}\right)}{2\left(\gamma^2-\beta^2\right)}$$ $[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (174)(1)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta e^{-a\beta} - \gamma e^{-a\gamma}\right)}{2\left(\beta^2-\gamma^2\right)}$$ $[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (175)(2)

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(\beta^2+x^2\right)\left(\gamma^2+x^2\right)} = \frac{\pi\left(\beta^2 e^{-a\beta} - \gamma^2 e^{-a\gamma}\right)}{2\left(\beta^2-\gamma^2\right)}$$ $[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (174)(2)

5. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(b\sin(ac) - c\sin(ab)\right)}{2bc\left(b^2-c^2\right)}$$ $[a>0, \quad b>0, \quad c>0]$ BI (175)(3)

6. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(\cos(ab) - \cos(ac)\right)}{2\left(b^2-c^2\right)}$$ $[a>0]$ BI (174)(3)

7. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(c\sin(ac) - b\sin(ab)\right)}{2\left(b^2-c^2\right)}$$ $[a>0, \quad b>0, \quad c>0]$ BI (175)(4)

8. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(b^2-x^2\right)\left(c^2-x^2\right)} = \frac{\pi\left(b^2\cos(ab) - c^2\cos(ac)\right)}{2\left(b^2-c^2\right)}$$ $[a>0, \quad b>0, \quad c>0]$ BI (174)(4)

9. $$\int_0^\infty \frac{x\sin ax}{\left(b^2-x^2\right)\left(c^2+x^2\right)}\,dx = \frac{\pi}{2}\,\frac{e^{-ac} - \cos ba}{a^2+c^2}$$ $[a>0, \quad c>0, \quad b \text{ real}]$

3.729

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab}$$ $[a>0, \quad b>0]$ BI (170)(7)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4b}ae^{-ab}$$ $[a>0, \quad b>0]$ BI (170)(3)

3. $$\int_0^\infty \cos(px)\frac{1-x^2}{\left(1+x^2\right)^2}\,dx = \frac{\pi p}{2}e^{-p}$$ BI (43)(10)a

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(b^2+x^2\right)^2} = \frac{\pi}{4}(2-ab)e^{-ab}$$ $[a>0, \quad b>0]$ BI (170)(4)

3.731 **Notation**: $2A^2 = \sqrt{b^4+c^2}+b^2, \quad 2B^2 = \sqrt{b^4+c^2}-b^2,$

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2c}\frac{e^{-aA}\left(B\cos(aB)+A\sin(aB)\right)}{\sqrt{b^4+c^2}}$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(3)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2c}e^{-aA}\sin(aB)$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(1)

3. $$\int_0^\infty \frac{\left(x^2+b^2\right)\cos(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2}\frac{e^{-aA}\left(A\cos(aB)-B\sin(aB)\right)}{\sqrt{b^4+c^2}}$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(4)

4. $$\int_0^\infty \frac{x\left(x^2+b^2\right)\sin(ax)\,dx}{\left(x^2+b^2\right)^2+c^2} = \frac{\pi}{2}e^{-aA}\cos(aB)$$
$[a>0, \quad b>0, \quad c>0]$ BI (176)(2)

3.732

1. $$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} - \frac{1}{\beta^2+(\gamma+x)^2}\right]\sin(ax)\,dx = \frac{\pi}{\beta}e^{-a\beta}\sin(a\gamma)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{ is not real}]$
ET I 65(16)

2. $$\int_0^\infty \left[\frac{1}{\beta^2+(\gamma-x)^2} + \frac{1}{\beta^2+(\gamma+x)^2}\right]\cos(ax)\,dx = \frac{\pi}{\beta}e^{-a\beta}\cos(a\gamma)$$
$[a>0, \quad |\operatorname{Im}\gamma|<\operatorname{Re}\beta]$ ET I 8(13)

3. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2} - \frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right]\sin(ax)\,dx = \pi e^{-a\beta}\cos(a\gamma)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \gamma+i\beta \text{ is not real}]$
LI (175)(17)

4. $$\int_0^\infty \left[\frac{\gamma+x}{\beta^2+(\gamma+x)^2} + \frac{\gamma-x}{\beta^2+(\gamma-x)^2}\right]\cos(ax)\,dx = \pi e^{-a\beta}\sin(a\gamma)$$
$[a>0, \quad |\operatorname{Im} a|<\operatorname{Re}\beta]$ LI (176)(21)

3.733

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b^3}\exp\left(-ab\cos t\right)\frac{\sin\left(t+ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$ BI (176)(7)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b^2}\exp\left(-ab\cos t\right)\frac{\sin\left(ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$
BI(176)(5), ET I 66(23)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{x^4+2b^2x^2\cos 2t+b^4} = \frac{\pi}{2b}\exp\left(-ab\cos t\right)\frac{\sin\left(t-ab\sin t\right)}{\sin 2t}$$
$\left[a>0, \quad b>0, \quad |t|<\frac{\pi}{2}\right]$ BI (176)(8)

4. $$\int_0^\infty \frac{x^3 \sin(ax)\,dx}{x^4 + 2b^2x^2\cos 2t + b^4} = \frac{\pi}{2}\exp(-ab\cos t)\frac{\sin(2t - ab\sin t)}{\sin 2t}$$
$\left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}\right]$ BI (176)(6)

5. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(x^4 + 2b^2x^2\cos 2t + b^4\right)} = \frac{\pi}{2b^4}\left[1 - \exp(-ab\cos t)\frac{\sin(2t + ab\sin t)}{\sin 2t}\right]$$
$\left[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}\right]$ BI (176)(22)

3.734

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(b^4 + x^4\right)} = \frac{\pi}{2b^4}\left[1 - \exp\left(-\frac{ab}{\sqrt{2}}\right)\cos\frac{ab}{\sqrt{2}}\right]$$ $[a > 0, \quad b > 0]$ BI (172)(7)

2. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(b^4 - x^4\right)} = \frac{\pi}{4b^4}\left[2 - e^{-ab} - \cos(ab)\right]$$ $[a > 0, \quad b > 0]$ BI (172)(10)

3.735 $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(b^2 + x^2\right)^2} = \frac{\pi}{2b^4}\left[1 - \frac{1}{2}e^{-ab}(2 + ab)\right]$$ $[a > 0, \quad b > 0]$ WH, BI (172)(22)

3.736

1. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8b^5}\left[\sin(ab) + (2 + ab)e^{-ab}\right]$$
$[a > 0, \quad b > 0]$ BI (176)(5)

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8b^4}\left[(1 + ab)e^{-ab} - \cos(ab)\right]$$
$[a > 0, \quad b > 0]$ BI (174)(5)

3. $$\int_0^\infty \frac{x^2\cos(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8b^3}\left[\sin(ab) - abe^{-ab}\right]$$ $[a > 0, \quad b > 0]$ BI (175)(6)

4. $$\int_0^\infty \frac{x^3\sin(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8b^2}\left[(1 - ab)e^{-ab} - \cos(ab)\right]$$
$[a > 0, \quad b > 0]$ BI (174)(6)

5. $$\int_0^\infty \frac{x^4\cos(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8b}\left[\sin(ab) + (ab - 2)e^{-ab}\right]$$
$[a > 0, \quad b > 0]$ BI (175)(7)

6. $$\int_0^\infty \frac{x^5\sin(ax)\,dx}{\left(b^2 + x^2\right)\left(b^4 - x^4\right)} = \frac{\pi}{8}\left[(ab - 3)e^{-ab} - \cos(ab)\right]$$
$[a > 0, \quad b > 0]$ BI (174)(7)

3.737

1.[8] $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(b^2+x^2\right)^n} = \frac{\pi e^{-ab}}{(2b)^{2n-1}(n-1)!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2ab)^k}{k!(n-k-1)!}$$
$$= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{e^{-ab\sqrt{p}}}{\sqrt{p}}\right)\right]_{p=1}$$
$$= \frac{(-1)^{n-1}\pi}{2b^{2n-1}(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{e^{-abp}}{(1+p)^n}\right)\right]_{p=1}$$
$[a>0, \quad b>0]$ GW(333)(67b), WA 209, WA 192

2. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(x^2+\beta^2\right)^{n+1}} = \frac{\pi a e^{-a\beta}}{2^{2n}n!\beta^{2n-1}}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!}$$
$$= \frac{\pi}{2}e^{-a\beta} \qquad [n=0, \quad \beta \geq 0]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ GW (333)(66c)

3. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(\beta^2+x^2\right)^{n+1}} = \frac{\pi}{2\beta^{2n+2}}\left[1-\frac{e^{-a\beta}}{2^n n!}F_n(a\beta)\right]$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad F_0(z)=1, \quad F_1(z)=z+2,\ldots,F_n(z)=(z+2n)F_{n-1}(z)-zF'_{n-1}(z)\right]$
GW (333)(66e)

4. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2+x^2\right)^3} = \frac{\pi a}{16b^3}(1+ab)e^{-ab}$$
$[a>0, \quad b>0]$ BI(170)(5), ET I 67(35)a

5. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(b^2+x^2\right)^4} = \frac{\pi a}{96b^5}\left(3+3ab+a^2b^2\right)e^{-ab}$$
$[a>0, \quad b>0]$ BI(170)(6), ET I 67(35)a

6. $$\int_0^\infty \frac{x^3\sin ax}{\left(x^2+\beta^2\right)^{n+1}}\,dx = \frac{\pi e^{-a\beta}}{2^{2n}n!\beta^{2n-2}}\left[2^{n-1}(2n-3)!!(2-\beta a)\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{(2n-k-2)!2^k(\beta a)^{k-1}}{k!(n-k-1)!}\left[k(k+1)-2(k+1)\beta a+\beta^2a^2\right]\right]$$

3.738

1. $$\int_0^\infty \frac{x^{m-1}\sin(ax)}{x^{2n}+\beta^{2n}}\,dx = -\frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n}\exp\left[-a\beta\sin\frac{(2k-1)\pi}{2n}\right]$$
$$\times\cos\left\{\frac{(2k-1)m\pi}{2n}+a\beta\cos\frac{(2k-1)\pi}{2n}\right\}$$
$[m \text{ is even}]$, $\left[a>0, \quad |\arg\beta|<\frac{\pi}{2n}, \quad 0<m\leq 2n\right]$ ET I 67(38)

2. $$\int_0^\infty \frac{x^{m-1}\cos(ax)}{x^{2n}+\beta^{2n}}\,dx = \frac{\pi\beta^{m-2n}}{2n}\sum_{k=1}^{n}\exp\left[-a\beta\sin\frac{(2k-1)\pi}{2n}\right]$$
$$\times\sin\left\{\frac{(2k-1)m\pi}{2n}+a\beta\cos\frac{(2k-1)\pi}{2n}\right\}$$
$[m \text{ is odd}]$, $\left[a>0, \quad |\arg\beta|<\frac{\pi}{2n}, \quad 0<m<2n+1\right]$ BI(160)(29)a, ET I 10(29)

3.739

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{x\left(x^2+2^2\right)\left(x^2+4^2\right)\dots\left(x^2+4n^2\right)} = \frac{\pi(-1)^n}{(2n)!2^{2n+1}}\left[2\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}e^{2(k-n)a}+(-1)^n\binom{2n}{n}\right]$$
$[a>0, \quad n\ge 0]$ LI(174)(8)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\left(x^2+1^2\right)\left(x^2+3^2\right)\dots\left[x^2+(2n+1)^2\right]} = \frac{(-1)^n}{(2n+1)!}\frac{\pi}{2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}e^{(2k-2n-1)a} \qquad [a\ge 0, \quad n\ge 0]$$
$$= \frac{\pi 2^{-2n-1}}{(2n+1)\,(n!)^2} \qquad [a=0, \quad n\ge 0]$$
BI(175)(8)

3. $$\int_0^\infty \frac{x\sin(ax)\,dx}{\left(x^2+1^2\right)\left(x^2+3^2\right)\dots\left[x^2+(2n+1)^2\right]} = \frac{\pi(-1)^n}{(2n+1)!2^{2n+1}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}(2n-2k+1)e^{(2k-2n-1)a}$$
$[a>0, \quad n\ge 0]$ LI (174)(9)

4. $$\int_0^\infty \frac{\cos ax\,dx}{\left(x^2+2^2\right)\left(x^2+4^2\right)\dots\left(x^2+4n^2\right)} = \frac{\pi 2^{1-2n}}{(2n)!}\sum_{k=1}^{n}(-1)^k k\binom{2n}{n-k}e^{-2ak}$$
$[n\ge 1, \quad a\ge 0]$

3.741

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x}\,dx = \frac{1}{4}\ln\left(\frac{a+b}{a-b}\right)^2 \qquad [a>0, \quad b>0, \quad a\ne b]$$ FI II 647

2. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{x}\,dx = \frac{\pi}{2} \qquad [a>b\ge 0]$$
$$= \frac{\pi}{4} \qquad [a=b>0]$$
$$= 0 \qquad [b>a\ge 0]$$
FI II 645

3. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2}\,dx = \frac{a\pi}{2} \qquad [0<a\le b]$$
$$= \frac{b\pi}{2} \qquad [0<b\le a]$$
BI (157)(1)

3.742

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}\left(e^{-|a-b|\beta} - e^{-(a+b)\beta}\right) \qquad [a>0, \quad b>0, \quad \operatorname{Re}\beta>0]$$
$$= \frac{\pi}{2\beta}e^{-a\beta}\sinh b\beta \qquad [\beta>0, \quad a\ge b\ge 0]$$
$$= \frac{\pi}{2\beta}e^{-b\beta}\sinh a\beta \qquad [\beta>0, \quad b\ge a\ge 0]$$

BI(162)(1)a, GW(333)(71a)

2. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{\beta^2+x^2}\,dx = \frac{1}{4\beta}e^{-a\beta}\left\{e^{b\beta}\operatorname{Ei}\left[\beta(a-b)\right] + e^{-b\beta}\operatorname{Ei}\left[\beta(a+b)\right]\right\}$$
$$-\frac{1}{4\beta}e^{a\beta}\left\{e^{b\beta}\operatorname{Ei}\left[-\beta(a+b)\right] + e^{-b\beta}\operatorname{Ei}\left[\beta(b-a)\right]\right\}$$

BI (162)(3)

3. $$\int_0^\infty \frac{\cos(ax)\cos(bx)}{\beta^2+x^2}\,dx = \frac{\pi}{4\beta}\left[e^{-|a-b|\beta} + e^{-(a+b)\beta}\right] \qquad [a>0, \quad b>0, \quad \operatorname{Re}\beta>0]$$
$$= \frac{\pi}{2\beta}e^{-a\beta}\cosh b\beta \qquad [\beta>0, \quad a\ge b\ge 0]$$
$$= \frac{\pi}{2\beta}e^{-b\beta}\cosh a\beta \qquad [\beta>0, \quad b\ge a\ge 0]$$

BI(163)(1)a, GW(333)(71c)

4. $$\int_0^\infty \frac{x\cos(ax)\cos(bx)}{\beta^2+x^2}\,dx = -\frac{1}{4}e^{a\beta}\left\{e^{b\beta}\operatorname{Ei}\left[-\beta(a+b)\right] + e^{-b\beta}\operatorname{Ei}\left[\beta(b-a)\right]\right\}$$
$$-\frac{1}{4}e^{-a\beta}\left\{e^{b\beta}\operatorname{Ei}\left[\beta(a-b)\right] + e^{-b\beta}\operatorname{Ei}\left[\beta(a+b)\right]\right\} \qquad [a\ne b]$$
$$= \infty \qquad [a=b]$$

BI (163)(2)

5. $$\int_0^\infty \frac{x\sin(ax)\cos(bx)}{x^2+\beta^2}\,dx = \frac{\pi}{2}e^{-a\beta}\cosh(b\beta) \qquad [0<b<a]$$
$$= \frac{\pi}{4}e^{-2a\beta} \qquad [0<b=a]$$
$$= -\frac{\pi}{2}e^{-b\beta}\sinh(a\beta) \qquad [0<a<b]$$

BI (162)(4)

6. $$\int_0^\infty \frac{\sin(ax)\sin(bx)}{p^2-x^2}\,dx = -\frac{\pi}{2p}\cos(ap)\sin(bp) \qquad [a>b>0]$$
$$= -\frac{\pi}{4p}\sin(2ap) \qquad [a=b>0]$$
$$= -\frac{\pi}{2p}\sin(ap)\cos(bp) \qquad [b>a>0]$$

BI (166)(1)

7. $$\int_0^\infty \frac{\sin(ax)\cos(bx)}{p^2-x^2}x\,dx = -\frac{\pi}{2}\cos(ap)\cos(bp) \qquad [a>b>0]$$
$$= -\frac{\pi}{4}\cos(2ap) \qquad [a=b>0]$$
$$= \frac{\pi}{2}\sin(ap)\sin(bp) \qquad [b>a>0]$$

BI (166)(2)

8. $$\int_0^\infty \frac{\cos(ax)\cos(bx)}{p^2-x^2}\,dx = \frac{\pi}{2p}\sin(ap)\cos(bp) \qquad [a>b>0]$$
$$= \frac{\pi}{4p}\sin(2ap) \qquad [a=b>0]$$
$$= \frac{\pi}{2p}\cos(ap)\sin(bp) \qquad [b>a>0]$$

BI (166)(3)

3.743

1. $$\int_0^\infty \frac{\sin(ax)}{\sin(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\sinh(a\beta)}{\sinh(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0]$$ ET I 80(21)

2. $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = -\frac{\pi}{2}\cdot\frac{\sinh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0]$$ ET I 81(30)

3. $$\int_0^\infty \frac{\cos(ax)}{\sin(bx)}\cdot\frac{x\,dx}{x^2+\beta^2} = \frac{\pi}{2}\cdot\frac{\cosh(a\beta)}{\sinh(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0]$$ ET I 23(37)

4. $$\int_0^\infty \frac{\cos(ax)}{\cos(bx)}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi}{2\beta}\cdot\frac{\cosh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0]$$ ET I 23(36)

5.[6] $$\mathrm{PV}\int_0^\infty \frac{\sin(ax)}{\sin x}\cdot\frac{dx}{b^2-x^2} = 0 \qquad \text{if } 0\le a\le 1$$
$$= \frac{\pi}{b}\sin(a-1)b \qquad \text{if } 1\le a\le 2$$
$$[b \text{ real}, \ b/\pi \notin \mathbb{Z}]$$

3.744[3] $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{dx}{x\left(x^2+\beta^2\right)} = \frac{\pi}{2\beta^2}\cdot\frac{\sinh(a\beta)}{\cosh(b\beta)} \qquad [0<a<b, \quad \operatorname{Re}\beta>0]$$ ET I 82(32)

3.745[3] $$\int_0^\infty \frac{\sin(ax)}{\cos(bx)}\cdot\frac{dx}{x\left(c^2-x^2\right)} = 0 \qquad [0<a<b, \quad c>0]$$ ET I 82(31)

3.746

1. $$\int_0^\infty \frac{dx}{x^{n+1}}\prod_{k=0}^n \sin\left(a_k x\right) = \frac{\pi}{2}\prod_{k=1}^n a_k \qquad \left[a_0>\sum_{k=1}^n a_k, \quad a_k>0\right]$$ FI II 646

2. $$\int_0^\infty \frac{\sin(ax)}{x^{n+1}}\,dx\prod_{k=1}^n \sin\left(a_k x\right)\prod_{j=1}^m \cos\left(b_j x\right) = \frac{\pi}{2}\prod_{k=1}^n a_k \qquad \left[a>\sum_{k=1}^n |a_k|+\sum_{j=1}^m |b_j|\right]$$ WH

3.747

1.[7] $$\int_0^{\pi/2} \frac{x^m}{\sin x}\,dx = \left(\frac{\pi}{2}\right)^m\left[\frac{1}{m}+\sum_{k=1}^\infty \frac{2^{2k-1}-1}{4^{2k-1}(m+2k)}\,\zeta(2k)\right] = 2\pi\boldsymbol{G}-\frac{7}{2}\,\zeta(3)$$
$$[m=2]$$ LI (206)(2)

2. $$\int_0^{\pi/2}\frac{x\,dx}{\sin x} = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\,dx}{\cos x} = 2\boldsymbol{G}$$ BI(204)(18), BI(206)(1), GW(333)(32)

3. $$\int_0^\infty \frac{x\,dx}{\left(x^2+b^2\right)\sin(ax)} = \frac{\pi}{2\sinh(ab)} \qquad [b>0]$$ GW (333)(79c)

4. $$\int_0^\pi x\tan x\,dx = -\pi\ln 2$$ BI (218)(4)

5. $$\int_0^{\pi/2} x \tan x \, dx = \infty$$ BI (205)(2)

6. $$\int_0^{\pi/4} x \tan x \, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G} = 0.1857845358\ldots$$ BI (204)(1)

7. $$\int_0^{\pi/2} x \cot x \, dx = \frac{\pi}{2} \ln 2$$ FI II 623

8. $$\int_0^{\pi/4} x \cot x \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G} = 0.7301810584\ldots$$ BI (204)(2)

9. $$\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \tan x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \tan x \, dx = \frac{\pi}{2} \ln 2$$ GW(333)(33b), BI(218)(12)

10. $$\int_0^{\infty} \tan ax \frac{dx}{x} = \frac{\pi}{2} \qquad [a > 0]$$ LO V 279(5)

11. $$\int_0^{\pi/2} \frac{x \cot x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2$$ BI (206)(12)

3.748

1. $$\int_0^{\pi/4} x^m \tan x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \sum_{k=1}^{\infty} \frac{\left(4^k - 1\right) \zeta(2k)}{4^{2k-1}(m + 2k)}$$ LI (204)(5)

2. $$\int_0^{\pi/2} x^p \cot x \, dx = \left(\frac{\pi}{2}\right)^p \left(\frac{1}{p} - 2 \sum_{k=1}^{\infty} \frac{1}{4^k (p + 2k)} \zeta(2k)\right)$$ LI (205)(7)

3. $$\int_0^{\pi/4} x^m \cot x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \left(\frac{2}{m} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(m + 2k)}\right)$$ LI (204)(6)

3.749

1. $$\int_0^{\infty} \frac{x \tan(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} + 1} \qquad [a > 0, \quad b > 0]$$ GW (333)(79a)

2. $$\int_0^{\infty} \frac{x \cot(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} - 1} \qquad [a > 0, \quad b > 0]$$ GW (333)(79b)

3. $$\int_0^{\infty} \frac{x \tan(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \cot(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \operatorname{cosec}(ax) \, dx}{b^2 - x^2} = \infty$$ BI (161)(7, 8, 9)

3.75 Combinations of trigonometric and algebraic functions

3.751

1. $$\int_0^{\infty} \frac{\sin(ax) \, dx}{\sqrt{x + \beta}} = \sqrt{\frac{\pi}{2a}} \left[\cos(a\beta) - \sin(a\beta) + 2\, C\left(\sqrt{a\beta}\right) \sin(a\beta) - 2\, S\left(\sqrt{a\beta}\right) \cos(a\beta)\right]$$
$[a > 0, \quad |\arg \beta| < \pi]$ ET I 65(12)a

2.[9] $$\int_0^{\infty} \frac{\cos(ax) \, dx}{\sqrt{x + \beta}} = \sqrt{\frac{\pi}{2a}} \left[\cos(a\beta) + \sin(a\beta) - 2\, C\left(\sqrt{a\beta}\right) \cos(a\beta) - 2\, S\left(\sqrt{a\beta}\right) \sin(a\beta)\right]$$
$[a > 0, \quad |\arg \beta| < \pi]$ ET I 8(9)a

3. $$\int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}\,[\sin(au)+\cos(au)] \qquad [a>0, \quad u>0]$$ ET I 65(13)

4. $$\int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}}\,dx = \sqrt{\frac{\pi}{2a}}\,[\cos(au)-\sin(au)] \qquad [a>0, \quad u>0]$$ ET I 8(10)

3.752

1.[8] $$\int_0^1 \sin(ax)\sqrt{1-x^2}\,dx = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a}\,\mathbf{H}_1(a) \qquad [a>0]$$ BI (149)(6)

2. $$\int_0^1 \cos(ax)\sqrt{1-x^2}\,dx = \frac{\pi}{2a}\,J_1(a)$$ KU 65(6)a

3.753

1.[8] $$\int_0^1 \frac{\sin(ax)\,dx}{\sqrt{1-x^2}} = \sum_{k=0}^\infty \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} = \frac{\pi}{2}\,\mathbf{H}_0(a) \qquad [a>0]$$ BI (149)(9)

2. $$\int_0^1 \frac{\cos(ax)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\,J_0(a)$$ WA 30(7)a

3. $$\int_1^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2-1}} = \frac{\pi}{2}\,J_0(a) \qquad [a>0]$$ WA 200(14)

4. $$\int_1^\infty \frac{\cos(ax)}{\sqrt{x^2-1}}\,dx = -\frac{\pi}{2}\,Y_0(a)$$ WA 200(15)

5. $$\int_0^1 \frac{x\sin(ax)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\,J_1(a) \qquad [a>0]$$ WA 30(6)

3.754

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{\beta^2+x^2}} = \frac{\pi}{2}\,[I_0(a\beta)-\mathbf{L}_0(a\beta)] \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ ET I 66(26)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta) \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ WA 191(1), GW(333)(78a)

3. $$\int_0^\infty \frac{x\sin(ax)}{\sqrt{(\beta^2+x^2)^3}}\,dx = a\,K_0(a\beta) \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ ET I 66(27)

3.755

1. $$\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}-\beta}\,\sin(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta} \qquad [a>0]$$ ET I 66(31)

2. $$\int_0^\infty \frac{\sqrt{\sqrt{x^2+\beta^2}+\beta}\,\cos(ax)\,dx}{\sqrt{x^2+\beta^2}} = \sqrt{\frac{\pi}{2a}}e^{-a\beta} \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ ET I 10(25)

3.756

1. $$\int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}} \prod_{k=2}^{n} \sin(a_k x)\, dx = 0 \qquad \left[a_k > 0, \quad a > \sum_{k=2}^{n} a_k\right]$$ ET I 80(22)

2. $$\int_0^\infty x^{\frac{n}{2}-1} \cos(ax) \prod_{k=1}^{n} \cos(a_k x)\, dx = 0 \qquad \left[a_k > 0, \quad a > \sum_{k=1}^{n} a_k\right]$$ ET I 22(26)

3.757

1.[11] $$\int_0^\infty \frac{\sin(ax)}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2a}} \qquad [a > 0]$$ BI (177)(1)

2.[11] $$\int_0^\infty \frac{\cos(ax)}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{2a}} \qquad [a > 0]$$ BI (177)(2)

3.76–3.77 Combinations of trigonometric functions and powers

3.761

1. $$\int_0^1 x^{\mu-1} \sin(ax)\, dx = \frac{-i}{2\mu} \left[\,{}_1F_1(\mu; \mu+1; ia) - {}_1F_1(\mu; \mu+1; -ia)\right]$$
$$[a > 0, \quad \operatorname{Re}\mu > -1, \quad \mu \neq 0]$$ ET I 68(2)a

2.[8] $$\int_u^\infty x^{\mu-1} \sin x\, dx = \frac{i}{2} \left[e^{-\frac{\pi}{2} i\mu}\, \Gamma(\mu, iu) - e^{\frac{\pi}{2} i\mu}\, \Gamma(\mu, -iu)\right]$$
$$[\operatorname{Re}\mu < 1]$$ EH II 149(2)

3. $$\int_1^\infty \frac{\sin(ax)}{x^{2n}}\, dx = \frac{a^{2n-1}}{(2n-1)!} \left[\sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \sin\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^n \operatorname{ci}(a)\right]$$
$$[a > 0]$$ LI (203)(15)

4. $$\int_0^\infty x^{\mu-1} \sin(ax)\, dx = \frac{\Gamma(\mu)}{a^\mu} \sin\frac{\mu\pi}{2} = \frac{\pi \sec\frac{\mu\pi}{2}}{2a^\mu\, \Gamma(1-\mu)} \qquad [a > 0; \quad 0 < |\operatorname{Re}\mu| < 1]$$ FI II 809a, BI(150)(1)

5.[10] $$\int_0^\pi x^m \sin(nx)\, dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m!}{(m-2k)!} (n\pi)^{m-2k} - (-1)^{\lfloor m/2 \rfloor} \frac{m! \left\lfloor m - 2\left\lfloor \frac{m}{2} \right\rfloor - 1 \right\rfloor}{n^{m+1}}$$ GW(333)(6)

6.[8] $$\int_0^1 x^{\mu-1} \cos(ax)\, dx = \frac{1}{2\mu} \left[\,{}_1F_1(\mu; \mu+1; ia) + {}_1F_1(\mu; \mu+1; -ia)\right]$$
$$[a > 0, \quad \operatorname{Re}\mu > 0]$$ ET I 11(2)

7. $$\int_u^\infty x^{\mu-1} \cos x\, dx = \frac{1}{2} \left[e^{-\frac{\pi}{2} i\mu}\, \Gamma(\mu, iu)s + e^{\frac{\pi}{2} i\mu}\, \Gamma(\mu, -iu)\right]$$
$$[\operatorname{Re}\mu < 1]$$ EH II 149(1)

8. $$\int_1^\infty \frac{\cos(ax)}{x^{2n+1}}\,dx = \frac{a^{2n}}{(2n)!}\left[\sum_{k=1}^{2n}\frac{(2n-k)!}{a^{2n-k+1}}\cos\left(a+(k-1)\frac{\pi}{2}\right)+(-1)^{n+1}\operatorname{ci}(a)\right]$$

$[a > 0]$ LI (203)(16)

9.[8] $$\int_0^\infty x^{\mu-1}\cos(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\frac{\mu\pi}{2} = \frac{\pi\operatorname{cosec}\frac{\mu\pi}{2}}{2a^\mu\,\Gamma(1-\mu)} \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1]$$

FI II 809a, BI(150)(2)

10. $$\int_0^\pi x^m\cos(nx)\,dx = \frac{(-1)^n}{n^{m+1}}\sum_{k=0}^{\lfloor(m-1)/2\rfloor}(-1)^k\frac{m!}{(m-2k-1)!}(n\pi)^{m-2k-1} + (-1)^{\lfloor(m+1)/2\rfloor}\frac{2[(m+1)/2]-m}{n^{m+1}}\cdot m!$$

GW (333)(7)

11. $$\int_0^{\pi/2} x^m\cos x\,dx = \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k\frac{m!}{(m-2k)!}\left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{\lfloor m/2\rfloor}\left(2\left\lfloor\frac{m}{2}\right\rfloor-m\right)m!$$

GW (333)(9c)

12. $$\int_0^{2n\pi} x^m\cos kx\,dx = -\sum_{j=0}^{m-1}\frac{j!}{k^{j+1}}\binom{m}{j}(2n\pi)^{m-j}\cos\frac{j+1}{2}\pi$$ BI (226)(2)

3.762

1. $$\int_0^\infty x^{\mu-1}\sin(ax)\sin(bx)\,dx = \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left[|b-a|^{-\mu}-(b+a)^{-\mu}\right]$$

$[a > 0, \quad b > 0, \quad a \neq b, \quad -2 < \operatorname{Re}\mu < 1]$

(for $\mu = 0$, see **3.741** 1, for $\mu = -1$, see **3.741** 3)

BI(149)(7), ET I 321(40)

2. $$\int_0^\infty x^{\mu-1}\sin(ax)\cos(bx)\,dx = \frac{1}{2}\sin\frac{\mu\pi}{2}\,\Gamma(\mu)\left[(a+b)^{-\mu}+|a-b|^{-\mu}\operatorname{sign}(a-b)\right]$$

$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$ (for $\mu = 0$ see **3.741** 2) BI(159)(8)a, ET I 321(41)

3. $$\int_0^\infty x^{\mu-1}\cos(ax)\cos(bx)\,dx = \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left[(a+b)^{-\mu}+|a-b|^{-\mu}\right]$$

$[a > 0, \quad b > 0, \quad 0 < \operatorname{Re}\mu < 1]$

ET I 20(17)

3.763

1. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^\nu}\,dx = \frac{1}{4}\cos\frac{\nu\pi}{2}\,\Gamma(1-\nu)\left\{(c+a-b)^{\nu-1}-(c+a+b)^{\nu-1} - |c-a+b|^{\nu-1}\operatorname{sign}(a-b-c)+|c-a-b|^{\nu-1}\operatorname{sign}(a+b-c)\right\}$$

$[c > 0, \quad 0 < \operatorname{Re}\nu < 4, \quad \nu \neq 1,2,3, \quad a \geq b > 0]$ GW(333)(26a)a, ET I 79(13)

2. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x}\,dx = 0 \qquad [c < a-b \text{ and } c > a+b]$$
$$= \frac{\pi}{8} \qquad [c = a-b \text{ and } c = a+b]$$
$$= \frac{\pi}{4} \qquad [a-b < c < a+b]$$
$$[a \ge b > 0, \quad c > 0]$$ FI II 645

3. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^2}\,dx = \frac{1}{4}(c+a+b)\ln(c+a+b) - \frac{1}{4}(c+a-b)\ln(c+a-b) - \frac{1}{4}|c-a-b|\ln|c-a-b| \times \operatorname{sign}(a+b-c) + \frac{1}{4}|c-a+b|\ln|c-a+b|\operatorname{sign}(a-b-c)$$
$$[a \ge b > 0, \quad c > 0]$$ BI(157)(8)a, ET I 79(11)

4. $$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x^3}\,dx = \frac{\pi bc}{2} \qquad [0 < c < a-b \text{ and } c > a+b]$$
$$= \frac{\pi bc}{2} - \frac{\pi(a-b-c)^2}{8} \qquad [a-b < c < a+b]$$
$$[a \ge b > 0, \quad c > 0]$$ BI(157)(20), ET I 79(12)

3.764

1. $$\int_0^\infty x^p \sin(ax+b)\,dx = \frac{1}{a^{p+1}}\,\Gamma(1+p)\cos\left(b + \frac{p\pi}{2}\right) \qquad [a > 0, \quad -1 < p < 0]$$ GW (333)(30a)

2. $$\int_0^\infty x^p \cos(ax+b)\,dx = -\frac{1}{a^{p+1}}\,\Gamma(1+p)\sin\left(b + \frac{\pi p}{2}\right)$$
$$[a > 0, \quad -1 < p < 0]$$ GW (333)(30b)

3.765

1.[10] $$\int_0^\infty \frac{\sin ax}{x^\nu(x+b)}\,dx$$
$$= a^{1+\nu} b \cos\frac{\pi\nu}{2}\,\Gamma(-1-\nu)\ {}_1F_2\left(1;\ 1+\frac{\nu}{2}, \frac{3}{2}+\frac{\nu}{2};\ -\frac{1}{4}a^2b^2\right)\operatorname{sign}(a) - \frac{\pi \operatorname{cosec}(\pi\nu)\sin(ab)}{b^\nu} - a^\nu\,\Gamma(-\nu)\ {}_1F_2\left(1;\ 1+\frac{\nu}{2}, 1+\frac{\nu}{2};\ -\frac{1}{4}a^2b^2\right)\operatorname{sign}(a)\sin\frac{\pi\nu}{2}$$
$$[\operatorname{Im} a = 0, \quad -1 < \operatorname{Re} b < 2, \quad \arg b \ne \pi]$$ MC

2. $$\int_0^\infty \frac{\cos(ax)}{x^\nu(x+\beta)}\,dx = \frac{\Gamma(1-\nu)}{2\beta^\nu}\left[e^{ia\beta}\,\Gamma(\nu, ia\beta) + e^{-ia\beta}\,\Gamma(\nu, -ia\beta)\right]$$
$$[a > 0, \quad |\operatorname{Re}\nu| < 1, \quad |\arg\beta| < \pi]$$ ET II 221(52)

3.766

1.[10] $$\int_0^\infty \frac{x^{\mu-1}\sin ax}{1+x^2}\,dx$$
$$= -a^{2-\mu}\,\Gamma(\mu-2)\ {}_1F_2\left(1;\ \frac{3-\mu}{2}, \frac{4-\mu}{2};\ \frac{a^2}{4}\right)\operatorname{sign}(a)\sin\frac{\pi\mu}{2} + \frac{\pi}{2}\sec\frac{\pi\mu}{2}\sinh(a)$$
$$[\operatorname{Im} a = 0, \quad -1 < \operatorname{Re}\mu < 3]$$ MC

2. $$\int_0^\infty \frac{x^{\mu-1}\cos(ax)}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}\frac{\mu\pi}{2}\cosh a + \frac{1}{2}\cos\frac{\mu\pi}{2}\,\Gamma(\mu)\left\{\exp\left[-a+i\pi(1-\mu)\right]\gamma(1-\mu,-a) - e^{a}\,\gamma(1-\mu,a)\right\}$$
$[a>0, \quad 0<\operatorname{Re}\mu<3]$ ET I 319(24)

3.[9] $$\int_0^\infty \frac{x^{2\mu+1}\sin(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2\mu}\sec(\mu\pi)\sinh(ab) + \frac{\sin(\mu\pi)}{2a^{2\mu}}\Gamma(2\mu)\left[\,{}_1F_1(1;1-2\mu;ab) + {}_1F_1(1;1-2\mu;-ab)\right]$$
$\left[a>0, \quad -\frac{3}{2}<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 220(39)

4.[9] $$\int_0^\infty \frac{x^{2\mu+1}\cos(ax)\,dx}{x^2+b^2} = -\frac{\pi}{2}b^{2\left(\mu+\frac{1}{2}\right)}\operatorname{cosec}\left[\left(\mu+\frac{1}{2}\right)\pi\right]\cosh(ab) + \frac{\cos\left[\left(\mu+\frac{1}{2}\right)\pi\right]}{2a^{2\left(\mu+\frac{1}{2}\right)}}\Gamma\left[2\left(\mu+\frac{1}{2}\right)\right]\left\{{}_1F_1\left(1;1-2\left(\mu+\frac{1}{2}\right);ab\right) + {}_1F_1\left(1;1-2\left(\mu+\frac{1}{2}\right);-ab\right)\right\}$$
$\left[a>0, \quad -1<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 221(56)

3.767

1. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = -\frac{\pi}{2}\gamma^{\beta-2}e^{-a\gamma}$$
$[a>0, \quad \operatorname{Re}\gamma>0, \quad 0<\operatorname{Re}\beta<2]$ BI (160)(20)

2. $$\int_0^\infty \frac{x^{\beta}\cos\left(ax-\frac{\beta\pi}{2}\right)}{\gamma^2+x^2}\,dx = \frac{\pi}{2}\gamma^{\beta-1}e^{-a\gamma}$$
$[a>0, \quad \operatorname{Re}\gamma>0, \quad |\operatorname{Re}\beta|<1]$ BI (160)(21)

3. $$\int_0^\infty \frac{x^{\beta-1}\sin\left(ax-\frac{\beta\pi}{2}\right)}{x^2-b^2}\,dx = \frac{\pi}{2}b^{\beta-2}\cos\left(ab-\frac{\pi\beta}{2}\right)$$
$[a>0, \quad b>0, \quad 0<\operatorname{Re}\beta<2]$ BI (161)(11)

4. $$\int_0^\infty \frac{x^{\beta}\cos\left(ax-\frac{\beta\pi}{2}\right)}{x^2-b^2}\,dx = -\frac{\pi}{2}b^{\beta-1}\sin\left(ab-\frac{\pi\beta}{2}\right)$$
$[a>0, \quad b>0, \quad |\beta|<1]$ GW (333)(82)

3.768

1. $$\int_u^\infty (x-u)^{\mu-1}\sin(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\left(au+\frac{\mu\pi}{2}\right)$$
$[a>0, \quad 0<\operatorname{Re}\mu<1]$ ET II 203(19)

2. $$\int_u^\infty (x-u)^{\mu-1}\cos(ax)\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\left(au+\frac{\mu\pi}{2}\right)$$
$[a>0, \quad 0<\operatorname{Re}\mu<1]$ ET II 204(24)

3.[11] $$\int_0^1 (1-x)^\nu\sin(ax)\,dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}}C_\nu(a) = a^{-\nu-1/2}\,s_{\nu+1/2,1/2}(a)$$
$[a>0, \quad \operatorname{Re}\nu>-1]$ ET I 11(3)a

Here $C_\nu(a)$ is the Young's function given by:

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)}\left[\,{}_1F_1(1;\nu+1;ia) + {}_1F_1\left(1;\nu+1;-ia\right)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{\nu+2n}}{\Gamma(\nu+2n+1)}.$$

4.[3] $$\int_0^1 (1-x)^\nu \cos(ax)\,dx = \frac{i}{2}a^{-\nu-1}\left\{\exp\left[\frac{i}{2}(\nu\pi-2a)\right]\gamma\left(\nu+1,-ia\right) - \exp\left[-\frac{i}{2}(\nu\pi-2a)\right]\gamma\left(\nu+1,ia\right)\right\} = \Gamma(\nu+1)\sum_{n=0}^{\infty}\frac{\left(-a^2\right)^n}{\Gamma(\nu+2+2n)}$$

$[a>0, \quad \operatorname{Re}\nu > -1]$ ET I 11(3)a

5. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\sin(ax)\,dx = \frac{u^{\mu+\nu-1}}{2i}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\mu+\nu;iau) - {}_1F_1\left(\nu;\mu+\nu;-iau\right)\right]$$

$[a>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1, \quad \nu\neq 0]$ ET II 189(26)

6. $$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\cos(ax)\,dx = \frac{u^{\mu+\nu-1}}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\mu+\nu;iau) + {}_1F_1\left(\nu;\mu+\nu;-iau\right)\right]$$

$[a>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET II 189(32)

7. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\sin(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-1/2}\sin\frac{au}{2}\,\Gamma(\mu)\,J_{\mu-1/2}\left(\frac{au}{2}\right)$$

$[\operatorname{Re}\mu>0]$ ET II 189(25)

8. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-1/2}\Gamma(\mu)\left[\cos\frac{au}{2}\,J_{1/2-\mu}\left(\frac{au}{2}\right) - \sin\frac{au}{2}\,Y_{1/2-\mu}\left(\frac{au}{2}\right)\right]$$

$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 203(20)

9. $$\int_0^u x^{\mu-1}(u-x)^{\mu-1}\cos(ax)\,dx = \sqrt{\pi}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\cos\frac{au}{2}\,\Gamma(\mu)\,J_{\mu-\frac{1}{2}}\left(\frac{au}{2}\right)$$

$[\operatorname{Re}\mu>0]$ ET II 189(31)

10. $$\int_u^\infty x^{\mu-1}(x-u)^{\mu-1}\cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{u}{a}\right)^{\mu-\frac{1}{2}}\Gamma(\mu)\left[\sin\frac{au}{2}\,J_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right) - \cos\frac{au}{2}\,Y_{\frac{1}{2}-\mu}\left(\frac{au}{2}\right)\right]$$

$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\right]$ ET II 204(25)

11.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\sin(ax)\,dx = -\frac{i}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\nu+\mu;ia) - {}_1F_1\left(\nu;\nu+\mu;-ia\right)\right]$$

$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-1, \quad \nu\neq 0]$ ET I 68 (5)a, ET I 317(5)

12.[3] $$\int_0^1 x^{\nu-1}(1-x)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}\,\mathrm{B}(\mu,\nu)\left[\,{}_1F_1(\nu;\nu+\mu;ia) + {}_1F_1\left(\nu;\nu+\mu;-ia\right)\right]$$

$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$ ET I 11(5)

13. $$\int_0^1 x^\mu (1-x)^\mu \sin(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\,\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(a)\sin a$$
$[a>0, \quad \operatorname{Re}\mu > -1]$ ET I 68(4)

14. $$\int_0^1 x^\mu (1-x)^\mu \cos(2ax)\,dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}}\,\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(a)\cos a$$
$[a>0, \quad \operatorname{Re}\mu > -1]$ ET I 11(4)

3.769

1. $$\int_0^\infty \left[(\beta+ix)^{-\nu} - (\beta-ix)^{-\nu}\right]\sin(ax)\,dx = -\frac{\pi i a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 70(15)

2. $$\int_0^\infty \left[(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{\pi a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 13(19)

3. $$\int_0^\infty x\left[(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}\right]\sin(ax)\,dx = -\frac{\pi a^{\nu-2}(\nu-1-a\beta)}{\Gamma(\nu)}e^{-a\beta}$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>0]$ ET I 70(16)

4. $$\int_0^\infty x^{2n}\left[(\beta-ix)^{-\nu} - (\beta+ix)^{-\nu}\right]\sin(ax)\,dx = \frac{(-1)^n i}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}\,L_{2n}^{\nu-2n-1}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu]$ ET I 70(17)

5. $$\int_0^\infty x^{2n}\left[(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{(-1)^n}{\Gamma(\nu)}(2n)!\pi a^{\nu-2n-1}e^{-a\beta}\,L_{2n}^{\nu-2n-1}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu]$ ET I 13(20)

6. $$\int_0^\infty x^{2n+1}\left[(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}\right]\sin(ax)\,dx = \frac{(-1)^{n+1}}{\Gamma(\nu)}(2n+1)!\pi a^{\nu-2n-2}e^{-a\beta}\,L_{2n+1}^{\nu-2n-2}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad -1\le 2n+1<\operatorname{Re}\nu]$ ET I 70(18)

7. $$\int_0^\infty x^{2n+1}\left[(\beta+ix)^{-\nu} - (\beta-ix)^{-\nu}\right]\cos(ax)\,dx = \frac{(-1)^{n+1}}{\Gamma(\nu)}(2n+1)!\pi a^{\nu-2n-2}e^{-a\beta}\,L_{2n+1}^{\nu-2n-2}(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad 0\le 2n<\operatorname{Re}\nu-1]$ ET I 13(21)

3.771

1. $$\int_0^\infty \left(\beta^2+x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\left[I_{-\nu}(a\beta) - \mathbf{L}_\nu(a\beta)\right]$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}, \quad \nu\ne -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots]$ EH II 38a, ET I 68(6)

2. $$\int_0^\infty \left(\beta^2+x^2\right)^{\nu-\frac{1}{2}} \cos(ax)\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\,\Gamma\left(\nu+\frac{1}{2}\right) K_{-\nu}(a\beta)$$
$$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}\right]$$
WA 191(1)a, GW(333)(78)a

3. $$\int_0^u x^{2\nu-1}\left(u^2-x^2\right)^{\mu-1}\sin(ax)\,dx$$
$$= \frac{a}{2}u^{2\mu+2\nu-1}\,\mathrm{B}\left(\mu,\nu+\frac{1}{2}\right)\ {}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\mu+\nu+\frac{1}{2};-\frac{a^2u^2}{4}\right)$$
$$\left[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 189(29)

4. $$\int_0^u x^{2\nu-1}\left(u^2-x^2\right)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}u^{2\mu+2\nu-2}\,\mathrm{B}(\mu,\nu)\ {}_1F_2\left(\nu;\frac{1}{2},\mu+\nu;-\frac{a^2u^2}{4}\right)$$
$$[\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$
ET II 190(35)

5.[7] $$\int_0^\infty x\left(x^2+\beta^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{1}{\sqrt{\pi}}\beta\left(\frac{2\beta}{a}\right)^\nu \cos\nu\pi\,\Gamma\left(\nu+\frac{1}{2}\right)K_{\nu+1}(a\beta)$$
$$= \sqrt{\pi}\beta\left(\frac{2\beta}{a}\right)^\nu \frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)}K_{\nu+1}(a\beta)$$
$$[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<0]$$
ET I 69(11)

6. $$\int_0^u \left(u^2-x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 69(7), WA 358(1)a

7. $$\int_u^\infty \left(x^2-u^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_{-\nu}(au)$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
EH II 81(12)a, ET I 69(8), WA 187(3)a

8. $$\int_0^u \left(u^2-x^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_\nu(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 11(8)

9. $$\int_u^\infty \left(x^2-u^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx = -\frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)Y_{-\nu}(au)$$
$$\left[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
WA 187(4)a, EH II 82(13)a, ET I 11(9)

10. $$\int_0^u x\left(u^2-x^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx = \frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right)J_{\nu+1}(au)$$
$$\left[a>0, \quad u>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 69(9)

11. $$\int_u^\infty x\left(x^2-u^2\right)^{\nu-\frac{1}{2}}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)Y_{-\nu-1}(au)$$
$\left[a>0,\quad u>0,\quad -\frac{1}{2}<\operatorname{Re}\nu<0\right]$ ET I 69(10)

12.[7] $$\int_0^u x\left(u^2-x^2\right)^{\nu-\frac{1}{2}}\cos(ax)\,dx=-\frac{u^{\nu+1}}{a^\nu}s_{(\nu-1)\nu+1}(au)$$
$$=\frac{1}{2}\left(\nu+\frac{1}{2}\right)^{-1}u^{2\nu+1}-\frac{\sqrt{\pi}}{2}u\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_{\nu+1}(au)$$
$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ ET I 12(10)

13. $$\int_u^\infty x\left(x^2-u^2\right)^{\nu-1/2}\cos(ax)\,dx\,\frac{\sqrt{\pi}u}{2}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_{-\nu-1}(au)$$
$\left[a>0,\quad u>0,\quad 0<\operatorname{Re}\nu<\frac{1}{2}\right]$ ET I 12(11)

3.772

1. $$\int_0^\infty\left(x^2+2\beta x\right)^{\nu-1/2}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(a\beta)\cos(a\beta)+Y_{-\nu}(a\beta)\sin(a\beta)\right]$$
$\left[a>0,\quad |\arg\beta|<\pi,\quad \frac{1}{2}>\operatorname{Re}\nu>-\frac{3}{2}\right]$ ET I 69(12)

2. $$\int_0^\infty\left(x^2+2\beta x\right)^{\nu-1/2}\cos(ax)\,dx$$
$$=-\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[Y_{-\nu}(a\beta)\cos(a\beta)-J_{-\nu}(a\beta)\sin(a\beta)\right]$$
$\left[a>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET I 12(13)

3. $$\int_0^{2u}\left(2ux-x^2\right)^{\nu-1/2}\sin(ax)\,dx=\sqrt{\pi}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\sin(au)\,J_\nu(au)$$
$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ ET I 69(13)a

4. $$\int_{2u}^\infty\left(x^2-2ux\right)^{\nu-1/2}\sin(ax)\,dx=\frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(au)\cos(au)-Y_{-\nu}(au)\sin(au)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET I 70(14)

5. $$\int_0^{2u}\left(2ux-x^2\right)^{\nu-1/2}\cos(ax)\,dx=\sqrt{\pi}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(au)\cos(au)$$
$\left[a>0,\quad u>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ ET I 12(4)

6. $$\int_{2u}^\infty\left(x^2-2ux\right)^{\nu-1/2}\cos(ax)\,dx$$
$$=-\frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)\left[J_{-\nu}(au)\sin(au)+Y_{-\nu}(au)\cos(au)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET I 12(12)

3.773

1.[8] $$\int_0^\infty \frac{x^{2\nu}}{(x^2+\beta^2)^{\mu+1}}\sin(ax)\,dx$$
$$= \frac{1}{2}\beta^{2\nu-2\mu}a\,\mathrm{B}\,(1+\nu,\mu-\nu)\ {}_1F_2\left(\nu+1;\ \nu+1-\mu,\frac{3}{2};\ \frac{\beta^2a^2}{4}\right)$$
$$+\frac{\sqrt{\pi}a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}}\frac{\Gamma(\nu-\mu)}{\Gamma\left(\mu-\nu+\frac{3}{2}\right)}\ {}_1F_2\left(\mu+1;\ \mu-\nu+\frac{3}{2},\mu-\nu+1;\ \frac{\beta^2a^2}{4}\right)$$
$$= \frac{\sqrt{\pi}}{2\,\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\,G^{21}_{13}\left(\frac{a^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2},\frac{1}{2},0\end{matrix}\right)$$
$[a>0, \quad \mathrm{Re}\,\beta>0, \quad -1<\mathrm{Re}\,\nu<\mathrm{Re}\,\mu+1]$ ET I 71(28)a, ET II 234(17)

2.[8] $$\int_0^\infty \frac{x^{2m+1}\sin(ax)}{(z+x^2)^{n+1}}\,dx = \frac{(-1)^{n+m}}{n!}\cdot\frac{\pi}{2}\frac{d^n}{dz^n}\left(z^me^{-a\sqrt{z}}\right)$$
$[a>0, \quad 0\le m\le n, \quad |\arg z|<\pi]$
ET I 68(39)

3. $$\int_0^\infty \frac{x^{2m+1}\sin(ax)\,dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^{m+1}\sqrt{\pi}}{2^n\beta^n\,\Gamma\left(n+\frac{1}{2}\right)}\frac{d^{2m+1}}{da^{2m+1}}\left[a^n\,K_n(a\beta)\right]$$
$[a>0, \quad \mathrm{Re}\,\beta>0, \quad -1\le m\le n]$
ET I 67(37)

4. $$\int_0^\infty \frac{x^{2\nu}\cos(ax)\,dx}{(x^2+\beta^2)^{\mu+1}} = \frac{1}{2}\beta^{2\nu-2\mu-1}\,\mathrm{B}\left(\nu+\frac{1}{2},\mu-\nu+\frac{1}{2}\right)\ {}_1F_2\left(\nu+\frac{1}{2};\nu-\mu+\frac{1}{2},\frac{1}{2};\frac{\beta^2a^2}{4}\right)$$
$$+\frac{\sqrt{\pi}a^{2\mu-2\nu+1}}{4^{\mu-\nu+1}}\frac{\Gamma\left(\nu-\mu-\frac{1}{2}\right)}{\Gamma(\mu-\nu+1)}\ {}_1F_2\left(\mu+1;\mu-\nu+1,\mu-\nu+\frac{3}{2};\frac{\beta^2a^2}{4}\right)$$
$$= \frac{\sqrt{\pi}}{2\,\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\,G^{21}_{13}\left(\frac{a^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2},0,\frac{1}{2}\end{matrix}\right)$$
$\left[a>0, \quad \mathrm{Re}\,\beta>0, \quad -\frac{1}{2}<\mathrm{Re}\,\nu<\mathrm{Re}\,\mu+1\right]$ ET I 14(29)a, ET II 235(19)

5. $$\int_0^\infty \frac{x^{2m}\cos(ax)\,dx}{(z+x^2)^{n+1}} = (-1)^{m+n}\frac{\pi}{2\cdot n!}\cdot\frac{d^n}{dz^n}\left(z^{m-\frac{1}{2}}e^{-a\sqrt{z}}\right)$$
$[a>0, \quad n+1>m\ge 0, \quad |\arg z|<\pi]$
ET I 10(28)

6.[7] $$\int_0^\infty \frac{x^{2m}\cos(ax)\,dx}{(\beta^2+x^2)^{n+\frac{1}{2}}} = \frac{(-1)^m\sqrt{\pi}}{2^n\beta^n\,\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{d^{2m}}{da^{2m}}\left\{a^n\,K_n(a\beta)\right\}$$
$\left[a>0, \quad \mathrm{Re}\,\beta>0, \quad 0\le m<n+\frac{1}{2}\right]$
ET I 14(28)

3.774

1. $$\int_0^\infty \frac{\sin(ax)\,dx}{\sqrt{x^2+b^2}\left(x+\sqrt{x^2+b^2}\right)^\nu} = \frac{\pi}{b^\nu\sin(\nu\pi)}\left[\sin\frac{\nu\pi}{2}\,I_\nu(ab)+\frac{i}{2}\,\mathbf{J}_\nu(iab)-\frac{i}{2}\,\mathbf{J}_\nu(-iab)\right]$$
$[a>0, \quad b>0, \quad \mathrm{Re}\,\nu>-1]$
ET I 70(19)

2. $$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{x^2+b^2}\left(x+\sqrt{x^2+b^2}\right)^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)}\left[\frac{1}{2}\mathbf{J}_\nu(iab)+\frac{1}{2}\mathbf{J}_\nu(-iab)-\cos\frac{\nu\pi}{2}\,I_\nu(ab)\right]$$
$[a>0, \quad b>0, \quad \operatorname{Re}\nu>-1]$ ET I 12(15)

3. $$\int_0^\infty \frac{\left(x+\sqrt{x^2+\beta^2}\right)^\nu}{\sqrt{x\left(x^2+\beta^2\right)}}\sin(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu\, I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{3}{2}\right]$ ET I 71(23)

4. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x\left(x^2+\beta^2\right)}}\cos(ax)\,dx = \sqrt{\frac{a\pi}{2}}\beta^\nu\, I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a\beta}{2}\right)$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-\frac{3}{2}\right]$ ET I 12(17)

5. $$\int_0^\infty \frac{\left(\beta+\sqrt{x^2+\beta^2}\right)^\nu}{x^{\nu+\frac{1}{2}}\sqrt{x^2+\beta^2}}\sin(ax)\,dx = \frac{1}{\beta}\sqrt{\frac{2}{a}}\,\Gamma\left(\frac{3}{4}-\frac{\nu}{2}\right)W_{\frac{\nu}{2},\frac{1}{4}}(a\beta)\,M_{-\frac{\nu}{2},\frac{1}{4}}(a\beta)$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{3}{2}\right]$ ET I 71(27)

6. $$\int_0^\infty \frac{\left(\beta+\sqrt{x^2+\beta^2}\right)^\nu}{x^{\nu+\frac{1}{2}}\sqrt{\beta^2+x^2}}\cos(ax)\,dx = \frac{1}{\beta\sqrt{2a}}\,\Gamma\left(\frac{1}{4}-\frac{\nu}{2}\right)W_{\frac{\nu}{2},-\frac{1}{4}}(a\beta)\,M_{-\frac{\nu}{2},-\frac{1}{4}}(a\beta)$$
$\left[a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu<\frac{1}{2}\right]$ ET I 12(18)

3.775

1. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}+x\right)^\nu-\left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x^2+\beta^2}}\sin(ax)\,dx = 2\beta^\nu\sin\frac{\nu\pi}{2}\,K_\nu(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad |\operatorname{Re}\nu|<1]$ ET I 70(20)

2. $$\int_0^\infty \frac{\left(\sqrt{x^2+\beta^2}+x\right)^\nu+\left(\sqrt{x^2+\beta^2}-x\right)^\nu}{\sqrt{x^2+\beta^2}}\cos(ax)\,dx = 2\beta^\nu\cos\frac{\nu\pi}{2}\,K_\nu(a\beta)$$
$[a>0, \quad \operatorname{Re}\beta>0, \quad |\operatorname{Re}\nu|<1]$ ET I 13(22)

3. $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x^2-u^2}}\sin(ax)\,dx = \pi u^\nu\left[J_\nu(au)\cos\frac{\nu\pi}{2}-Y_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<1]$ ET I 70(22)

4. $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x^2-u^2}}\cos(ax)\,dx = -\pi u^\nu\left[Y_\nu(au)\cos\frac{\nu\pi}{2}+J_\nu(au)\sin\frac{\nu\pi}{2}\right]$$
$[a>0, \quad u>0, \quad |\operatorname{Re}\nu|<1]$ ET I 13(25)

5. $$\int_0^u \frac{\left(x+i\sqrt{u^2-x^2}\right)^\nu+\left(x-i\sqrt{u^2-x^2}\right)^\nu}{\sqrt{u^2-x^2}}\sin(ax)\,dx=\frac{\pi}{2}u^\nu\operatorname{cosec}\frac{\nu\pi}{2}\left[\mathbf{J}_\nu(au)-\mathbf{J}_{-\nu}(au)\right]$$
$[a>0,\quad u>0]$ ET I 70(21)

6. $$\int_0^u \frac{\left(x+i\sqrt{u^2-x^2}\right)^\nu+\left(x-i\sqrt{u^2-x^2}\right)^\nu}{\sqrt{u^2-x^2}}\cos(ax)\,dx=\frac{\pi}{2}u^\nu\sec\frac{\nu\pi}{2}\left[\mathbf{J}_\nu(au)+\mathbf{J}_{-\nu}(au)\right]$$
$[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<1]$ ET I 13(24)

7.[6] $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x\left(x^2-u^2\right)}}\sin(ax)\,dx$$
$$=-\sqrt{\left(\frac{\pi}{2}\right)^3 au^\nu}\left[J_{1/4+\nu/2}\left(\frac{au}{2}\right)Y_{1/4-\nu/2}\left(\frac{au}{2}\right)+J_{1/4-\nu/2}\left(\frac{au}{2}\right)Y_{1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{3}{2}\right]$ ET I 71(25)

8.[6] $$\int_u^\infty \frac{\left(x+\sqrt{x^2-u^2}\right)^\nu+\left(x-\sqrt{x^2-u^2}\right)^\nu}{\sqrt{x\left(x^2-u^2\right)}}\cos(ax)\,dx$$
$$=-\sqrt{\left(\frac{\pi}{2}\right)^3 au^\nu}\left[J_{-1/4+\nu/2}\left(\frac{au}{2}\right)Y_{-1/4-\nu/2}\left(\frac{au}{2}\right)+J_{-1/4-\nu/2}\left(\frac{au}{2}\right)Y_{-1/4+\nu/2}\left(\frac{au}{2}\right)\right]$$
$\left[a>0,\quad u>0,\quad |\operatorname{Re}\nu|<\frac{3}{2}\right]$ ET I 13(26)

9. $$\int_0^\infty \frac{\left(x+\beta+\sqrt{x^2+2\beta x}\right)^\nu+\left(x+\beta-\sqrt{x^2+2\beta x}\right)^\nu}{\sqrt{x^2+2\beta x}}\sin(ax)\,dx$$
$$=\pi\beta^\nu\left[Y_\nu(\beta a)\sin\left(\beta a-\frac{\nu\pi}{2}\right)+J_\nu(\beta a)\cos\left(\beta a-\frac{\nu\pi}{2}\right)\right]$$
$[a>0,\quad |\arg\beta|<\pi,\quad |\operatorname{Re}\nu|<1]$ ET I 71(26)

10. $$\int_0^\infty \frac{\left(x+\beta+\sqrt{x^2+2\beta x}\right)^\nu+\left(x+\beta-\sqrt{x^2+2\beta x}\right)^\nu}{\sqrt{x^2+2\beta x}}\cos(ax)\,dx$$
$$=\pi\beta^\nu\left[J_\nu(\beta a)\sin\left(\beta a-\frac{\nu\pi}{2}\right)-Y_\nu(\beta a)\cos\left(\beta a-\frac{\nu\pi}{2}\right)\right]$$
$[a>0,\quad |\arg\beta|<\pi,\quad |\operatorname{Re}\nu|<1]$ ET I 13(23)

11. $$\int_0^{2u} \frac{\left(\sqrt{2u+x}+i\sqrt{2u-x}\right)^{4\nu}+\left(\sqrt{2u+x}-i\sqrt{2u-x}\right)^{4\nu}}{\sqrt{4u^2x-x^3}}\cos(ax)\,dx$$
$$=(4u)^{2\nu}\pi^{3/2}\sqrt{\frac{a}{2}}\,J_{\nu-1/4}(au)\,J_{-\nu-1/4}(au)$$
$[a>0,\quad u>0]$ ET I 14(27)

3.776

1. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\sin(ax)\,dx=\frac{a}{b^p}$$ $[a>0,\quad b>0,\quad p>0]$ BI (170)(1)

2. $$\int_0^\infty \frac{a^2(b+x)^2+p(p+1)}{(b+x)^{p+2}}\cos(ax)\,dx=\frac{p}{b^{p+1}}$$ $[a>0,\quad b>0,\quad p>0]$ BI (170)(2)

3.78–3.81 Rational functions of x and of trigonometric functions

3.781

1. $$\int_0^\infty \left(\frac{\sin x}{x} - \frac{1}{1+x}\right)\frac{dx}{x} = 1 - \boldsymbol{C}$$ (cf. **3.784** 4 and **3.781** 2) BI (173)(7)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x}\right)\frac{dx}{x} = -\boldsymbol{C}$$ BI (173)(8)

3.782

1. $$\int_0^u \frac{1-\cos x}{x}\,dx - \int_u^\infty \frac{\cos x}{x}\,dx = \boldsymbol{C} + \ln u$$ $[u > 0]$ GW (333)(31)

2. $$\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{a\pi}{2}$$ $[a \geq 0]$ BI (158)(1)

3. $$\int_{-\infty}^\infty \frac{1-\cos ax}{x(x-b)}\,dx = \pi\frac{\sin ab}{b}$$ $[a > 0, \quad b \text{ real}, \quad b \neq 0]$ ET II 253(48)

3.783

1. $$\int_0^\infty \left[\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)}\right]\frac{dx}{x} = \frac{1}{2}\boldsymbol{C} - \frac{3}{4}$$ BI (173)(19)

2. $$\int_0^\infty \left(\cos x - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\boldsymbol{C}$$ EH I 17, BI(273)(21)

3.784

1. $$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{b}{a}$$ $[a > 0, \quad b > 0]$ FI II 635, GW(333)(20)

2. $$\int_0^\infty \frac{a\sin bx - b\sin ax}{x^2}\,dx = ab\ln\frac{a}{b}$$ $[a > 0, \quad b > 0]$ FI II 647

3. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{(b-a)\pi}{2}$$ $[a \geq 0, \quad b \geq 0]$ BI(158)(12), FI II 645

4. $$\int_0^\infty \frac{\sin x - x\cos x}{x^2}\,dx = 1$$ BI (158)(3)

5. $$\int_0^\infty \frac{\cos ax - \cos bx}{x(x+\beta)}\,dx = \frac{1}{\beta}\left[\operatorname{ci}(a\beta)\cos a\beta + \operatorname{si}(a\beta)\sin a\beta - \operatorname{ci}(b\beta)\cos b\beta - \operatorname{si}(b\beta)\sin b\beta + \ln\frac{b}{a}\right]$$ $[a > 0, \quad b > 0, \quad |\arg\beta| < \pi]$ ET II 221(49)

6. $$\int_0^\infty \frac{\cos ax + x\sin ax}{1+x^2}\,dx = \pi e^{-a}$$ $[a > 0]$ GW (333)(73)

7. $$\int_0^\infty \frac{\sin ax - ax\cos ax}{x^3}\,dx = \frac{\pi}{4}a^2\operatorname{sign} a$$ LI (158)(5)

8. $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2\left(x^2+\beta^2\right)}\,dx = \frac{\pi\left[(b-a)\beta + e^{-b\beta} - e^{-a\beta}\right]}{2\beta^3}$$ $[a > 0, \quad b > 0, \quad |\arg\beta| < \pi]$ BI(173)(20)a, ET II 222(59)

9.[10] $$\int_0^\infty \frac{\cos mx}{1+a^2\, T_n(x)} = \frac{\pi}{2n\sqrt{1+a^2}} \sum_{k=1}^{n} e^{-m \sin u \sinh \phi} \left(\cos\beta \sin u \cosh\phi + \sin\beta \cos u \sinh\phi\right)$$

$$\left[u = (2k-1)\,\pi/\,(2n)\,, \quad \phi = \operatorname{arcsinh}(1/a), \quad \beta = m\cos u \cosh\phi, \quad 0 < |a| < 1\right]$$

3.785 $$\int_0^\infty \frac{1}{x} \sum_{k=1}^{n} a_k \cos b_k x\, dx = -\sum_{k=1}^{n} a_k \ln b_k \qquad \left[b_k > 0, \quad \sum_{k=1}^{n} a_k = 0\right]$$ FI II 649

3.786

1. $$\int_0^\infty \frac{(1-\cos ax)\sin bx}{x^2}\, dx = \frac{b}{2} \ln \frac{b^2 - a^2}{b^2} + \frac{a}{2} \ln \frac{a+b}{a-b}$$

$[a > 0, \quad b > 0]$ ET I 81(29)

2.[11] $$\int_0^\infty \frac{(1-\cos ax)\cos bx}{x}\, dx = \ln \frac{\sqrt{|a^2 - b^2|}}{b} \qquad [a > 0, \quad b > 0, \quad a \neq b]$$ FI II 647

3.[11] $$\int_0^\infty \frac{(1-\cos ax)\cos bx}{x^2}\, dx = \frac{\pi}{2}(a-b) \qquad [a < b \le 0]$$
$$= 0 \qquad [0 < a \le b]$$

ET I 20(16)

3.787

1. $$\int_0^\infty \frac{(\cos a - \cos nax)\sin mx}{x}\, dx = \frac{\pi}{2}\,(\cos a - 1) \qquad [m > na > 0]$$
$$= \frac{\pi}{2} \cos a \qquad [na > m]$$

BI(155)(7)

2. $$\int_0^\infty \frac{\sin^2 ax - \sin^2 bx}{x}\, dx = \frac{1}{2} \ln \frac{a}{b} \qquad [a > 0, \quad b > 0]$$ GW (333)(20b)

3. $$\int_0^\infty \frac{x^3 - \sin^3 x}{x^5}\, dx = \frac{13}{32}\pi$$ BI (158)(6)

4. $$\int_0^\infty \frac{\left(3 - 4\sin^2 ax\right)\sin^2 ax}{x}\, dx = \frac{1}{2} \ln 2 \qquad [a \text{ real}, \ a \neq 0]$$ HBI (155)(6)

3.788 $$\int_0^{\pi/2} \left(\frac{1}{x} - \cot x\right) dx = \ln \frac{\pi}{2}$$ GW (333)(61)a

3.789 $$\int_0^{\pi/2} \frac{4x^2 \cos x + (\pi - x)x}{\sin x}\, dx = \pi^2 \ln 2$$ LI (206)(10)

3.791

1. $$\int_0^{\pi/2} \frac{x\, dx}{1 + \sin x} = \ln 2$$ GW (333)(55a)

2. $$\int_0^{\pi} \frac{x \cos x}{1 + \sin x}\, dx = \pi \ln 2 - 4\boldsymbol{G}$$ GW (333)(55c)

3. $$\int_0^{\pi/2} \frac{x \cos x}{1 + \sin x}\, dx = \pi \ln 2 - 2\boldsymbol{G}$$ GW (333)(55b)

4. $$\int_0^\pi \frac{\left(\frac{\pi}{2}-x\right)\cos x}{1-\sin x}\,dx = 2\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x}{1-\sin x}\,dx = \pi\ln 2 + 4\boldsymbol{G} = 5.8414484669\ldots$$
BI(207)(3), GW(333)(56c)

5. $$\int_0^{\pi/2} \frac{x^2\,dx}{1-\cos x} = -\frac{\pi^2}{4} + \pi\ln 2 + 4\boldsymbol{G} = 3.3740473667\ldots$$ BI (207)(3)

6. $$\int_0^\pi \frac{x^2\,dx}{1-\cos x} = 4\pi\ln 2$$ BI (219)(1)

7. $$\int_0^{\pi/2} \frac{x^{p+1}\,dx}{1-\cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^p (p+1)\left\{\frac{2}{p} - \sum_{k=1}^\infty \frac{1}{4^{2k-1}(p+2k)}\zeta(2k)\right\}$$
$[p>0]$ LI (207)(4)

8. $$\int_0^{\pi/2} \frac{x\,dx}{1+\cos x} = \frac{\pi}{2} - \ln 2$$ GW (333)(55a)

9. $$\int_0^{\pi/2} \frac{x\sin x\,dx}{1-\cos x} = \frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$$ GW (333)(56a)

10. $$\int_0^\pi \frac{x\sin x\,dx}{1-\cos x} = 2\pi\ln 2$$ GW (333)(56b)

11. $$\int_0^\pi \frac{x-\sin x}{1-\cos x}\,dx = \frac{\pi}{2} + \int_0^{\pi/2} \frac{x-\sin x}{1-\cos x}\,dx = 2$$ GW (333)(57a)

12. $$\int_0^{\pi/2} \frac{x\sin x}{1+\cos x}\,dx = -\frac{\pi}{2}\ln 2 + 2\boldsymbol{G}$$ GW (333)(55b)

3.792

1. $$\int_{-\pi}^\pi \frac{dx}{1-2a\cos x+a^2} = \frac{2\pi}{1-a^2}$$ $[a^2<1]$ FI II 485

2. $$\int_0^{\pi/2} \frac{x\cos x\,dx}{1+2a\sin x+a^2} = \frac{\pi}{2a}\ln(1+a) - \sum_{k=0}^\infty (-1)^k \frac{a^{2k}}{(2k+1)^2}$$
$[a^2<1]$ LI (241)(2)

3. $$\int_0^\pi \frac{x\sin x\,dx}{1-2a\cos x+a^2} = \frac{\pi}{a}\ln(1+a)$$ $[a^2<1, \quad a\neq 0]$
$$= \frac{\pi}{a}\ln\left(1+\frac{1}{a}\right)$$ $[a^2<1]$
BI (221)(2)

4. $$\int_0^{2\pi} \frac{x\sin x\,dx}{1-2a\cos x+a^2} = \frac{2\pi}{a}\ln(1-a)$$ $[a^2<1, \quad a\neq 0]$
$$= \frac{2\pi}{a}\ln\left(1-\frac{1}{a}\right)$$ $[a^2>1]$
BI (223)(4)

5. $$\int_0^{2\pi} \frac{x\sin nx\,dx}{1-2a\cos x+a^2} = \frac{2\pi}{1-a^2}\left[\left(a^{-n}-a^n\right)\ln(1-a) + \sum_{k=1}^{n-1}\frac{a^{-k}-a^k}{n-k}\right]$$
$[a^2<1, \quad a\neq 0]$ BI (223)(5)

6. $$\int_0^\infty \frac{\sin x}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{4a}\left[\left|\frac{1+a}{1-a}\right|-1\right] \qquad [a \text{ real}, \quad a \neq 0, \quad a \neq 1]$$ GW (333)(62b)

7.[8] $$\int_0^\infty \frac{\sin bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{2}\frac{1+a-2a^{[b]+1}}{(1-a^2)(1-a)} \qquad [b \neq 0,1,2,\dots]$$
$$= \frac{\pi}{2}\frac{1+a-a^b-a^{b+1}}{(1-a^2)(1-a)} \qquad [b=1,2,\dots]; \qquad [0<a<1]$$ ET I 81(26)

8. $$\int_0^\infty \frac{\sin x\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{2(1-a)}a^{[b]} \qquad [b \neq 0,1,2,\dots]$$
$$= \frac{\pi}{2(1-a)}a^b+\frac{\pi}{4}a^{b-1} \qquad [b=1,2,3,\dots];$$
$[0<a<1, \quad b>0]$; (for $b=0$, see **3.792** 6) ET I 19(5)

9. $$\int_0^\infty \frac{(1-a\cos x)\sin bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x} = \frac{\pi}{2}\cdot\frac{1-a^{[b]+1}}{1-a} \qquad [b \neq 1,2,3,\dots]$$
$$= \frac{\pi}{2}\cdot\frac{1-a^b}{1-a}+\frac{\pi a^b}{4} \qquad [b=1,2,3,\dots]$$
$[0<a<1, \quad b>0]$ ET I 82(33)

10.[3] $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\frac{dx}{\beta^2+x^2} = \frac{\pi}{2\beta(1-a^2)}\frac{1+ae^{-b\beta}}{1-ae^{-b\beta}}$$
$\left[a^2<1, \quad b \geq 0\right]$ BI (192)(1)

11. $$\int_0^\infty \frac{1}{1-2a\cos bx+a^2}\frac{dx}{\beta^2-x^2} = \frac{a\pi}{\beta(1-a^2)}\frac{\sin b\beta}{1-2a\cos b\beta+a^2}$$
$\left[a^2<1, \quad b>0\right]$ BI (193)(1)

12. $$\int_0^\infty \frac{\sin bcx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2+x^2} = \frac{\pi}{2}\frac{e^{-\beta bc}-a^c}{(1-ae^{-b\beta})(1-ae^{b\beta})}$$
$\left[a^2<1, \quad b>0, \quad c>0\right]$ BI (192)(8)

13. $$\int_0^\infty \frac{\sin bx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2+x^2} = \frac{\pi}{2}\frac{1}{e^{b\beta}-a} \qquad \left[a^2<1, \quad b>0\right]$$
$$= \frac{\pi}{2a}\frac{1}{ae^{b\beta}-1} \qquad \left[a^2>1, \quad b>0\right]$$ BI (192)(2)

14. $$\int_0^\infty \frac{\sin bcx}{1-2a\cos bx+a^2}\frac{x\,dx}{\beta^2-x^2} = \frac{\pi}{2}\frac{a^c-\cos\beta bc}{1-2a\cos\beta b+a^2}$$
$\left[a^2<1, \quad b>0, \quad c>0\right]$ BI (193)(5)

15. $$\int_0^\infty \frac{\cos bcx}{1-2a\cos bx+a^2}\frac{dx}{\beta^2-x^2} = \frac{\pi}{2\beta(1-a^2)}\frac{\left(1-a^2\right)\sin\beta bc+2a^{c+1}\sin\beta b}{1-2a\cos\beta b+a^2}$$
$\left[a^2<1, \quad b>0, \quad c>0\right]$ BI (193)(9)

16. $$\int_0^\infty \frac{1-a\cos bx}{1-2a\cos bx+a^2}\frac{dx}{1+x^2} = \frac{\pi}{2}\frac{e^b}{e^b-a} \qquad \left[a^2<1, \quad b>0\right]$$ FI II 719

17. $$\int_0^\infty \frac{\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi\left(e^{\beta-\beta b}+ae^{\beta b}\right)}{2\beta\left(1-a^2\right)\left(e^{\beta}-a\right)}$$
$[0 \le b < 1, \quad |a| < 1, \quad \operatorname{Re}\beta > 0]$ ET I 21(21)

18. $$\int_0^\infty \frac{\sin bx\sin x}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2}$$
$$= \frac{\pi}{2\beta}\frac{\sinh b\beta}{e^{\beta}-a} \qquad [0 \le b < 1]$$
$$= \frac{\pi}{4\beta\left(ae^{\beta}-1\right)}\left[a^m e^{\beta(m+1-b)} - e^{(1-b)\beta}\right] - \frac{\pi}{4\beta\left(ae^{-\beta}-1\right)}\left[a^m e^{-(m+1-b)\beta} - e^{-(1-b)\beta}\right] \qquad [m \le b \le m+1]$$
$[0 < a < 1, \quad \operatorname{Re}\beta > 0]$ ET I 81(27)

19. $$\int_0^\infty \frac{(\cos x-a)\cos bx}{1-2a\cos x+a^2}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi\cosh\beta b}{2\beta\left(e^{\beta}-a\right)}$$
$[0 \le b < 1, \quad |a| < 1, \quad \operatorname{Re}\beta > 0]$ ET I 21(23)

20. $$\int_0^\infty \frac{\sin x}{\left(1-2a\cos 2x+a^2\right)^{n+1}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\left(1-2a\cos 2x+a^2\right)^{n+1}}\frac{dx}{x}$$
$$= \int_0^\infty \frac{\tan x}{\left(1-2a\cos 4x+a^2\right)^{n+1}}\frac{dx}{x} = \frac{\pi}{2\left(1-a^2\right)^{2n+1}}\sum_{k=0}^{n}\binom{n}{k}^2 a^{2k}$$
BI (187)(14)

3.793

1.[3] $$\int_0^{2\pi} \frac{\sin nx - a\sin[(n+1)x]}{1-2a\cos x+a^2}x\,dx = -2\pi a^n\left[\ln(1-a)+\sum_{k=1}^{n}\frac{1}{ka^k}\right]$$
$[|a| < 1]$ BI (223)(9)

2. $$\int_0^{2\pi} \frac{\cos nx - a\cos[(n+1)x]}{1-2a\cos x+a^2}x\,dx = 2\pi a^n$$
$\left[a^2 < 1\right]$ BI (223)(13)

3.794

1.[3] $$\int_0^{\pi} \frac{x\,dx}{1+a^2+2a\cos x} = \frac{\pi^2}{2\left(1-a^2\right)} + \frac{4}{\left(1-a^2\right)}\sum_{k=0}^{\infty}\frac{a^{2k+1}}{(2k+1)^2}$$
$\left[a^2 < 1\right]$

2. $$\int_0^{2\pi} \frac{x\sin nx}{1\pm a\cos x}\,dx = \frac{2\pi}{\sqrt{1-a^2}}\left[(\mp 1)^n\frac{\left(1+\sqrt{1-a^2}\right)^n-\left(1-\sqrt{1-a^2}\right)^n}{a^n}\right.$$
$$\left.\times\ \ln\frac{2\sqrt{1\pm a}}{\sqrt{1+a}+\sqrt{1-a}} + \sum_{k=0}^{n-1}\frac{(\mp 1)^k}{n-k}\frac{\left(1+\sqrt{1-a^2}\right)^k-\left(1-\sqrt{1-a^2}\right)^k}{a^k}\right]$$
$\left[a^2 < 1\right]$ BI (223)(2)

3.[3] $$\int_0^{2\pi} \frac{x\cos nx}{1\pm a\cos x}\,dx = \frac{2\pi^2}{\sqrt{1-a^2}}\left(\frac{1-\sqrt{1-a^2}}{\mp a}\right)^n$$
$\left[a^2 < 1\right]$ BI (223)(3)

4. $$\int_0^{\pi} \frac{x \sin x\, dx}{a + b\cos x} = \frac{\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a-b)} \qquad [a > |b| > 0]$$ GW (333)(53a)

5. $$\int_0^{2\pi} \frac{x \sin x\, dx}{a + b\cos x} = \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a+b)} \qquad [a > |b| > 0]$$ GW (333)(53b)

6. $$\int_0^{\infty} \frac{\sin x}{a \pm b\cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2-b^2}} \qquad \left[a^2 > b^2\right]$$
$$= 0 \qquad \left[a^2 < b^2\right]$$
BI (181)(1)

3.795 $$\int_{-\infty}^{\infty} \frac{\left(b^2 + c^2 + x^2\right) x \sin ax - \left(b^2 - c^2 - x^2\right) c \sinh ac}{\left[x^2 + (b-c)^2\right]\left[x^2 + (b+c)^2\right](\cos ax + \cosh ac)}\, dx = \pi \qquad [c > b > 0]$$
$$= \frac{2\pi}{e^{ab}+1} \qquad [b > c > 0]$$
$$[a > 0]$$
BI (202)(18)

3.796

1. $$\int_0^{\pi/2} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x\, dx = \mp \frac{\pi}{4} \ln 2 - \boldsymbol{G}$$ BI (207)(8, 9)

2. $$\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x\, dx = \frac{\pi}{4} \ln 2 - \frac{1}{2}\boldsymbol{G}$$ BI (204)(23)

3.797

1. $$\int_0^{\pi/4} \left(\frac{\pi}{4} - x\tan x\right) \tan x\, dx = \frac{1}{2}\ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8}\ln 2$$ BI (204)(8)

2. $$\int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x\right)\tan x\, dx}{\cos 2x} = -\frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (204)(19)

3. $$\int_0^{\pi/4} \frac{\frac{\pi}{4} - x\tan x}{\cos 2x}\, dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (204)(20)

3.798

1.[8] $$\int_0^{\infty} \frac{\tan x}{a + b\cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2-b^2}} \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
BI (181)(2)

2.[8] $$\int_0^{\infty} \frac{\tan x}{a + b\cos 4x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2-b^2}} \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
BI (181)(3)

3.799

1. $$\int_0^{\pi/2} \frac{x\, dx}{(\sin x + a\cos x)^2} = \frac{a}{1+a^2}\frac{\pi}{2} - \frac{\ln a}{1+a^2} \qquad [a > 0]$$ BI (208)(5)

2. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + a\sin x)^2} = \frac{1}{1+a^2}\ln\frac{1+a}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1-a}{(1+a)\left(1+a^2\right)}$$
$[a > 0]$ BI (204)(24)

3. $$\int_0^{\pi} \frac{a\cos x + b}{(a+b\cos x)^2} x^2\,dx = \frac{2\pi}{b}\ln\frac{2(a-b)}{a+\sqrt{a^2-b^2}}$$
$[a > |b| > 0]$ GW (333)(58a)

3.811

1. $$\int_0^{\pi} \frac{\sin x}{1-\cos t_1\cos x}\cdot\frac{x\,dx}{1-\cos t_2\cos x} = \pi\operatorname{cosec}\frac{t_1+t_2}{2}\operatorname{cosec}\frac{t_1-t_2}{2}\ln\frac{1+\tan\frac{t_1}{2}}{1+\tan\frac{t_2}{2}}$$
(cf. **3.794** 4) BI (222)(5)

2. $$\int_0^{\pi/2} \frac{x\,dx}{(\cos x \pm \sin x)\sin x} = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$$
BI (208))(16, 17)

3. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + \sin x)\sin x} = -\frac{\pi}{8}\ln 2 + \boldsymbol{G}$$
BI (204)(29)

4. $$\int_0^{\pi/4} \frac{x\,dx}{(\cos x + \sin x)\cos x} = \frac{\pi}{8}\ln 2$$
BI (204)(28)

5. $$\int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x}\frac{x\,dx}{\cos^2 x} = -\frac{\pi}{8}\ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2$$
BI (204)(30)

3.812

1. $$\int_0^{\pi} \frac{x\sin x\,dx}{a+b\cos^2 x} = \frac{\pi}{\sqrt{ab}}\arctan\sqrt{\frac{b}{a}}$$
$[a > 0, \quad b > 0]$
$$= \frac{\pi}{2\sqrt{-ab}}\ln\frac{\sqrt{a}+\sqrt{-b}}{\sqrt{a}-\sqrt{-b}}$$
$[a > -b > 0]$
GW (333)(60a)

2. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{1+a\cos^2 x} = \frac{\pi}{a}\ln\frac{1+\sqrt{1+a}}{2}$$
$[a > -1, \quad a \neq 0]$ BI (207)(10)

3. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{1+a\sin^2 x} = \frac{\pi}{a}\ln\frac{2\left(1+a-\sqrt{1+a}\right)}{2}$$
$[a > -1, \quad a \neq 0]$ BI (207)(2)

4.[11] $$\int_0^{\pi} \frac{x\,dx}{a^2-\cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}}$$
$\left[a^2 > 1\right]$
$$= 0$$
$\left[\text{principal value for } 0 < a^2 < 1\right]$
$$= \text{divergent}$$
$[a = 0]$
BI (219)(10)

5.[7] $$\int_0^{\pi} \frac{x\sin x\,dx}{a^2-\cos^2 x} = \frac{\pi}{2a}\ln\left|\frac{1+a}{1-a}\right|$$
$[0 < a < 1]$ divergent if $a = 0$
BI (219)(13)

6.[11] $$\int_0^{\pi} \frac{x \sin 2x\, dx}{a^2 - \cos^2 x} = \pi \ln\left\{4\left(1-a^2\right)\right\} \qquad \left[\text{principal value for } 0 \le a^2 < 1\right]$$
$$= 2\pi \ln\left[2\left(1 - a^2 + a\sqrt{a^2-1}\right)\right] \qquad \left[a^2 > 1\right]$$
$$= \text{divergent} \qquad [|a| = 1]$$

BI (219)(19)

7. $$\int_0^{\pi/2} \frac{x \sin x\, dx}{\cos^2 t - \sin^2 x} = -2 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}$$ BI (207)(1)

8. $$\int_0^{\pi} \frac{x \sin x\, dx}{1 - \cos^2 t \sin^2 x} = \pi(\pi - 2t) \operatorname{cosec} 2t$$ BI (219)(12)

9. $$\int_0^{\pi} \frac{x \cos x\, dx}{\cos^2 t - \cos^2 x} = 4 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}$$ BI (219)(17)

10. $$\int_0^{\pi} \frac{x \sin x\, dx}{\tan^2 t + \cos^2 x} = \frac{\pi}{2}(\pi - 2t)\cot t$$ BI (219)(14)

11. $$\int_0^{\infty} \frac{x\left(a \cos x + b\right) \sin x\, dx}{\cot^2 t + \cos^2 x} = 2a\pi \ln \cos\frac{t}{2} + \pi b t \tan t$$ BI (219)(18)

12.* $$\int_0^{\pi} \frac{x \sin x \cos x}{a - \sin^2 x}\, dx = -\pi \ln 2 + \ln\left[1 + \sqrt{\frac{a-1}{a}}\right] \qquad [a > 1]$$

13.* $$\int_0^{\pi/2} \ln\left(a - \sin^2 x\right)\, dx = -\pi \ln 2 + i\pi \ln \arccos\sqrt{a} \qquad [0 < a < 1]$$

14.* $$\mathrm{PV} \int_0^{\pi/2} \ln\left(\left|a - \sin^2 x\right|\right)\, dx = -\pi \ln 2 \qquad [0 < a < 1]$$

15.* $$\mathrm{PV} \int_0^{\pi/2} \ln\left(\left|a - \cos^2 x\right|\right)\, dx = -\pi \ln 2 \qquad [0 < a < 1]$$

3.813

1. $$\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$$
$$[a > 0, \quad b > 0]$$ GW (333)(36)

2. $$\int_0^{\infty} \frac{1}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{dx}{x^2 + \delta^2} = \frac{\pi \sinh(2a\delta)}{4\delta\left(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta)\right)} \left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{2}{\sinh(2a\delta)}\right]$$
$$\left[\left|\arg\frac{\beta}{\gamma}\right| < \pi, \quad \operatorname{Re}\delta > 0, \quad a > 0\right]$$
GW(333)(81), ET II 222(63)

3. $$\int_0^{\infty} \frac{\sin x\, dx}{x\left(a^2 \sin^2 x + b^2 \cos^2 x\right)} = \frac{\pi}{2ab} \qquad [ab > 0]$$ BI (181)(8)

4. $$\int_0^{\infty} \frac{\sin^2 x\, dx}{x\left(a^2 \cos^2 x + b^2 \sin^2 x\right)} = \frac{\pi}{2b(a+b)} \qquad [a > 0, \quad b > 0]$$ BI (181)(11)

5. $$\int_0^{\pi/2} \frac{x \sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b}$$ $[a > 0, \quad b > 0, \quad a \neq b]$ GW (333)(52a)

6. $$\int_0^{\pi} \frac{x \sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a}$$ $[a > 0, \quad b > 0, \quad a \neq b]$ GW (333)(52b)

7. $$\int_0^{\infty} \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{a(a+b)}$$ $[a > 0, \quad b > 0]$ BI (182)(3)

8. $$\int_0^{\infty} \frac{\sin 2ax}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{x \, dx}{x^2 + \delta^2} = \frac{\pi}{2\left(\beta^2 \sinh^2(a\delta) - \gamma^2 \cosh^2(a\delta)\right)} \left[\frac{\beta - \gamma}{\beta + \gamma} - e^{-2a\delta}\right]$$ $\left[a > 0, \quad \left|\arg \frac{\beta}{\gamma}\right| < \pi, \quad \operatorname{Re} \delta > 0\right]$ ET II 222(64), GW(333)(80)

9. $$\int_0^{\infty} \frac{(1 - \cos x) \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$$ $[a > 0, \quad b > 0]$ BI (182)(7)a

10. $$\int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a(a+b)}$$ $[a > 0, \quad b > 0]$ BI (182)(4)

11. $$\int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{2}{a+b}$$ $[a > 0, \quad b > 0]$ BI (182)(1)

3.814

1. $$\int_0^{\pi/2} \frac{(1 - x \cot x) \, dx}{\sin^2 x} = \frac{\pi}{4}$$ BI (206)(9)

2. $$\int_0^{\pi/4} \frac{x \tan x \, dx}{(\sin x + \cos x) \cos x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2$$ BI (204)(30)

3. $$\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \frac{dx}{x} = \frac{\pi}{2ab}$$ $[a > 0, \quad b > 0]$ BI (181)(9)

4. $$\int_0^{\pi/2} \frac{x \cot x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2a^2} \ln \frac{a+b}{b}$$ $[a > 0, \quad b > 0]$ LI (208)(20)

5. $$\int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \tan x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{2} \int_0^{\pi} \frac{\left(\frac{\pi}{2} - x\right) \tan x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2b^2} \ln \frac{a+b}{a}$$ $[a > 0, \quad b > 0]$ GW (333)(59)

6. $$\int_0^{\infty} \frac{\sin^2 x \tan x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)}$$ $[a > 0, \quad b > 0]$ BI (182)(6)

7. $$\int_0^{\infty} \frac{\tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab}$$ $[a > 0, \quad b > 0]$ BI (181)(10)a

8. $$\int_0^{\infty} \frac{\sin^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b}$$ $[a > 0, \quad b > 0]$ BI (182)(2)a

9. $$\int_0^{\infty} \frac{\cos^2 2x \tan x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b}$$ $[a > 0, \quad b > 0]$ BI (182)(5)a

10. $$\int_0^\infty \frac{\sin^2 x \cos x}{a^2\cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x\cos 4x} = -\frac{\pi}{8b}\frac{a}{a^2+b^2} \qquad [a>0, \quad b>0]$$ BI (186)(12)a

11. $$\int_0^\infty \frac{\sin x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x\cos 2x} = \frac{\pi}{2ab}\cdot\frac{b^2-a^2}{b^2+a^2} \qquad [a>0, \quad b>0]$$ BI (186)(4)a

12. $$\int_0^\infty \frac{\sin x \cos x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x\cos 2x} = \frac{\pi}{2a}\cdot\frac{b}{a^2+b^2} \qquad [a>0, \quad b>0]$$ BI (186)(7)a

13. $$\int_0^\infty \frac{\sin x \cos^2 x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x\cos 2x} = \frac{\pi}{2ab}\cdot\frac{b^2}{a^2+b^2} \qquad [a>0, \quad b>0]$$ BI (186)(8)a

14. $$\int_0^\infty \frac{\sin^3 x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x\cos 2x} = -\frac{\pi}{2b}\cdot\frac{a}{a^2+b^2} \qquad [a>0, \quad b>0]$$ BI (186)(10)

15. $$\int_0^\infty \frac{1-\cos x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x\sin x} = \frac{\pi}{2ab} \qquad [a>0, \quad b>0]$$ BI (186)(3)a

3.815

1. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{\left(1+a\sin^2 x\right)\left(1+b\sin^2 x\right)} = \frac{\pi}{a-b}\ln\left\{\frac{1+\sqrt{1+b}}{1+\sqrt{1+a}}\cdot\frac{\sqrt{1+a}}{\sqrt{1+b}}\right\}$$
$[a>0, \quad b>0]$ (cf. **3.812** 3)
BI (208)(22)

2. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{\left(1+a\sin^2 x\right)\left(1+b\cos^2 x\right)} = \frac{\pi}{a+ab+b}\ln\frac{\left(1+\sqrt{1+n}\right)\sqrt{1+a}}{1+\sqrt{1+a}}$$
$[a>0, \quad b>0]$ (cf. **3.812** 2 and 3)
BI (208)(24)

3. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{\left(1+a\cos^2 x\right)\left(1+b\cos^2 x\right)} = \frac{\pi}{a-b}\ln\frac{1+\sqrt{1+a}}{1+\sqrt{1+b}}$$
$[a>0, \quad b>0]$ (cf. **3.812** 2)
BI (208)(23)

4. $$\int_0^{\pi/2} \frac{x\sin 2x\,dx}{\left(1-\sin^2 t_1\cos^2 x\right)\left(1-\sin^2 t_2\cos^2 x\right)} = \frac{2\pi}{\cos^2 t_1 - \cos^2 t_2}\ln\frac{\cos\frac{t_1}{2}}{\cos\frac{t_2}{2}}$$
$[-\pi < t_1 < \pi, \quad -\pi < t_2 < \pi]$
BI (208)(21)

3.816

1. $$\int_0^{\pi} \frac{x^2\sin 2x}{\left(a^2-\cos^2 x\right)^2}\,dx = \pi^2\frac{\sqrt{a^2-1}-a}{a\left(a^2-1\right)} \qquad [a>1]$$ LI (220)(9)

2.[7] $$\int_0^{\pi} \frac{\left(a^2-1-\sin^2 x\right)\cos x}{\left(a^2-\cos^2 x\right)^2}x^2\,dx = \frac{\pi}{2}\ln\left|\frac{1-a}{1+a}\right| \qquad \left[a^2>1\right]$$ (cf. **3.812** 5) BI (220)(12)

3.[11] $$\int_0^{\pi} \frac{a\cos 2x - \sin^2 x}{\left(a+\sin^2 x\right)^2}x^2\,dx = -2\pi\ln\left[2\left(-a+\sqrt{a}\sqrt{a+1}\right)\right]$$
$\left[a<-1 \text{ and } a>0.\ \text{When } a>0, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}.\right]$ LI (220)(10)

4.[11] $$\int_0^\pi \frac{a\cos 2x + \sin^2 x}{\left(a - \sin^2 x\right)^2} x^2\,dx = 2\pi \ln\left[2\left(a - \sqrt{a}\sqrt{a+1}\right)\right]$$
$\left[a < 0 \text{ and } a > 1. \text{ When } a > 1, \text{ can write } \sqrt{a}\sqrt{a+1} \text{ as } \sqrt{a(a+1)}.\right]$ (cf. **3.812** 6)

LI (220)(11)

3.817

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2+b^2}{a^3b^3}$$ $[ab > 0]$ BI (181)(12)

2. $$\int_0^\infty \frac{\sin x \cos x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3b}$$ $[ab > 0]$ BI (182)(8)

3. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3}$$ $[ab > 0]$ BI (181)(15)

4. $$\int_0^\infty \frac{\sin x \cos^2 x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3b}$$ $[ab > 0]$ BI (182)(9)

5. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2+b^2}{a^3b^3}$$ $[ab > 0]$ BI (181)(13)

6. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 2x + b^2 \sin^2 2x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \frac{a^2+b^2}{a^3b^3}$$ $[ab > 0]$ BI (181)(14)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3}$$ $[ab > 0]$ BI (182)(11)

8. $$\int_0^\infty \frac{\tan x \cos^2 2x}{\left(a^2\cos^2 2x + b^2 \sin^2 2x\right)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3b}$$ $[ab > 0]$ BI (182)(10)

3.818

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5}$$
$[ab > 0]$ BI (181)(16)

2. $$\int_0^\infty \frac{\sin x \cos x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2+3b^2}{a^5b^3}$$ $[ab > 0]$ BI (182)(13)

3. $$\int_0^\infty \frac{\sin x \cos^2 x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2+3b^2}{a^5b^3}$$ $[ab > 0]$ BI (182)(14)

4. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x + b^2 \sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2+b^2}{a^3b^5}$$ $[ab > 0]$ LI (181)(19)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\left(a^2\cos^2 2x + b^2 \sin^2 2x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{64} \cdot \frac{3a^2+b^2}{a^3b^5}$$ $[ab > 0]$ BI (182)(17)

6. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16}\frac{3a^4+2a^2b^2+3b^4}{a^5b^5}$$ $[ab>0]$ BI (181)(17)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2+b^2}{a^3b^5}$$ $[ab>0]$ BI (182)(16)

8. $$\int_0^\infty \frac{\tan x}{\left(a^2\cos^2 2x + b^2\sin^2 2x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4+2a^2b^2+3b^4}{a^5b^5}$$ $[ab>0]$ BI (181)(18)

9. $$\int_0^\infty \frac{\tan x \cos^2 2x}{\left(a^2\cos^2 2x + b^2\sin^2 2x\right)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2+3b^2}{a^5b^3}$$ $[ab>0]$ BI (182)(15)

3.819

1. $$\int_0^\infty \frac{\sin x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^6+3a^4b^2+3a^2b^4+5b^6}{a^7b^7}$$ $[ab>0]$ BI (181)(20)

2. $$\int_0^\infty \frac{\sin x \cos x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4+2a^2b^2+5b^4}{a^7b^5}$$ $[ab>0]$ BI (182)(18)

3. $$\int_0^\infty \frac{\sin x \cos^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4+2a^2b^2+5b^4}{a^7b^5}$$ $[ab>0]$ BI (182)(19)

4. $$\int_0^\infty \frac{\sin^3 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4+a^2b^2+b^4}{a^5b^7}$$ $[ab>0]$ BI (181)(23)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2+b^2}{a^5b^5}$$ $[ab>0]$ BI (182)(26)

6. $$\int_0^\infty \frac{\sin x \cos^3 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2+5b^2}{a^7b^3}$$ $[ab>0]$ BI (182)(23)

7. $$\int_0^\infty \frac{\sin^3 x \cos^2 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2+b^2}{a^5b^5}$$ $[ab>0]$ BI (182)(27)

8. $$\int_0^\infty \frac{\sin x \cos^4 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2+5b^2}{a^7b^3}$$ $[ab>0]$ BI (182)(24)

9. $$\int_0^\infty \frac{\sin^5 x}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2+b^2}{a^3b^7}$$ $[ab>0]$ BI (181)(24)

10. $$\int_0^\infty \frac{\sin^3 x \cos x}{\left(a^2 \cos^2 2x + b^2 \sin^2 2x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{128} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5 b^7}$$
$[ab > 0]$ BI (182)(22)

11. $$\int_0^\infty \frac{\sin^5 x \cos^3 x}{\left(a^2 \cos^2 2x + b^2 \sin^2 2x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{512} \cdot \frac{5a^2 + b^2}{a^3 b^7}$$ $[ab > 0]$ BI (182)(30)

12. $$\int_0^\infty \frac{\sin^2 x \tan x}{\left(a^2 \cos^2 x + b^2 \sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + 2a^2b^2 + b^4}{a^5 b^7}$$
$[ab > 0]$ BI (182)(21)

13. $$\int_0^\infty \frac{\sin^4 x \tan x}{\left(a^2 \cos^2 x + b^2 \sin^2 x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3 b^7}$$ $[ab > 0]$ BI (182)(29)

14. $$\int_0^\infty \frac{\cos^2 2x \tan x}{\left(a^2 \cos^2 2x + b^2 \sin^2 2x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2b^2 + 5b^4}{a^7 b^5}$$
$[ab > 0]$ BI (182)(29)

15. $$\int_0^\infty \frac{\sin^3 4x \tan x}{\left(a^2 \cos^2 2x + b^2 \sin^2 2x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{8} \cdot \frac{a^2 + b^2}{a^5 b^5}$$ $[ab > 0]$ BI (182)(28)

16. $$\int_0^\infty \frac{\cos^4 2x \tan x}{\left(a^2 \cos^2 2x + b^2 \sin^2 2x\right)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3}$$ $[ab > 0]$ BI (182)(25)

3.82–3.83 Powers of trigonometric functions combined with other powers

3.821

1. $$\int_0^\pi x \sin^p x \, dx = \frac{\pi^2}{2^{p+1}} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2}$$ $[p > -1]$ BI(218)(7), LO V 121(71)

2. $$\int_0^{r\pi} x \sin^n x \, dx = \frac{\pi^2}{2} \cdot \frac{(2m-1)!!}{(2m)!!} r^2 \qquad [n = 2m]$$
$$= (-1)^{r+1} \pi \frac{(2m)!!}{(2m+1)!!} r \qquad [n = 2m+1]$$
[r is a natural number] GW (333)(8c)

3.[11] $$\int_0^{\pi/2} x \cos^n x \, dx = \frac{\pi^2}{8} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-2}} \sum_{k=0,\, m-k \text{ odd}}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2} \qquad [n = 2m]$$
$$= \frac{\pi}{2} \frac{(n-1)!!}{(n)!!} - \frac{1}{2^{n-1}} \sum_{k=0}^{m-1} \binom{n}{k} \frac{1}{(n-2k)^2} \qquad [n = 2m-1]$$
GW (333)(9b)

4. $$\int_0^\pi x \cos^{2m} x \, dx = \frac{\pi^2}{2} \frac{(2m-1)!!}{(2m)!!}$$ BI (218)(10)

5. $$\int_{r\pi}^{s\pi} x \cos^{2m} x \, dx = \frac{\pi^2}{2} \left(s^2 - r^2\right) \frac{(2m-1)!!}{(2m)!!}$$ BI (226)(3)

6. $$\int_0^\infty \frac{\sin^p x}{x}\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2}\,\mathrm{B}\left(\frac{p}{2},\frac{p}{2}\right)$$
[p is a fraction with odd numerator and denominator] LO V 278, FI II 808

7. $$\int_0^\infty \frac{\sin^{2n+1} x}{x}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}$$ BI (151)(4)

8. $$\int_0^\infty \frac{\sin^{2n} x}{x}\,dx = \infty$$ BI (151)(3)

9. $$\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{a\pi}{2}$$ [$a > 0$] LO V 307, 312, FI II 632

10. $$\int_0^\infty \frac{\sin^{2m} ax}{x^2}\,dx = \frac{(2m-3)!!}{(2m-2)!!}\cdot\frac{a\pi}{2}$$ [$a > 0$] GW (333)(14b)

11. $$\int_0^\infty \frac{\sin^{2m+1} ax}{x^3}\,dx = \frac{(2m-3)!!}{(2m)!!}(2m+1)\frac{a^2\pi}{4}$$ [$a > 0$] GW (333)(14d)

12. $$\int_0^\infty \frac{\sin^p x}{x^m}\,dx$$
$$= \frac{p}{m-1}\int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}}\cos x\,dx$$ [$p > m-1 > 0$]
$$= \frac{p(p-1)}{(m-1)(m-2)}\int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}}\,dx - \frac{p^2}{(m-1)(m-2)}\int_0^\infty \frac{\sin^p x}{x^{m-2}}\,dx$$ [$p > m-1 > 1$]
GW (333)(17)

13. $$\int_0^\infty \frac{\sin^{2n} px}{\sqrt{x}}\,dx = \infty$$ BI (177)(5)

14. $$\int_0^\infty \sin^{2n+1} px\frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$$ BI (177)(7)

3.822

1. $$\int_0^{\pi/2} x^p\cos^m x\,dx = -\frac{p(p-1)}{m^2}\int_0^{\pi/2} x^{p-2}\cos^m x\,dx + \frac{m-1}{m}\int_0^{\pi/2} x^p\cos^{m-2} x\,dx$$
[$m > 1,\quad p > 1$] GW (333)(9a)

2. $$\int_0^\infty x^{-1/2}\cos^{2n+1}(px)\,dx = \frac{1}{2^{2n}}\sqrt{\frac{\pi}{2p}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}\frac{1}{\sqrt{2k+1}}$$ BI (177)(8)

3.823 $$\int_0^\infty x^{\mu-1}\sin^2 ax\,dx = -\frac{\Gamma(\mu)\cos\frac{\mu\pi}{2}}{2^{\mu+1}a^\mu}$$ [$a > 0,\quad -2 < \operatorname{Re}\mu < 0$]
ET I 319(15), GW(333)(19c)a

3.824

1. $$\int_0^\infty \frac{\sin^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}\left(1-e^{-2a\beta}\right)$$ [$a > 0,\quad \operatorname{Re}\beta > 0$] BI (160)(10)

2. $$\int_0^\infty \frac{\cos^2 ax}{x^2+\beta^2}\,dx = \frac{\pi}{4\beta}\left(1+e^{-2a\beta}\right)$$ [$a > 0,\quad \operatorname{Re}\beta > 0$] BI (160)(11)

3.[7] $$\int_0^\infty \sin^{2m} x \frac{dx}{a^2+x^2} = \frac{(-1)^m}{2^{2m+1}} \cdot \frac{\pi}{2} \left\{ 2^{2m} \sinh^{2m} a - 2 \sum_{k=0}^{m} (-1)^k \binom{2m}{k} \sinh[2(m-k)a] \right\}$$

$[a > 0]$ BI (160)(12)

4.[7] $$\int_0^\infty \sin^{2m+1} x \frac{dx}{a^2+x^2} = \frac{(-1)^{m-1}}{2^{2m+2}a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a] \right.$$
$$\left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m+1-2k)a] \right\}$$

$[a > 0]$ BI (160)(14)

5.[7] $$\int_0^\infty \sin^{2m+1} x \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}} e^{-(2m+1)a} \sum_{k=0}^{m} (-1)^{m+k} \binom{2m+1}{k} e^{2ka}$$

$\left[|\arg a| < \frac{\pi}{2}\right], \quad m = 0, 1, 2, \ldots$

6.[7] $$\int_0^\infty \cos^{2m} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \binom{2m}{m} + \frac{\pi}{2^{2m}} \sum_{k=1}^{m} \binom{2m}{m+k} e^{-2ka}$$

$[a > 0]$ BI (160)(16)

7. $$\int_0^\infty \cos^{2m+1} x \frac{dx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a} \sum_{k=1}^{m} \binom{2m+1}{m+k+1} e^{-(2k+1)a}$$

$[a > 0]$ BI (160)(17)

8. $$\int_0^\infty \cos^{2m+1} x \frac{x\,dx}{a^2+x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m-2k+1)a]$$
$$-\frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a]$$

BI (160)(18)

9. $$\int_0^\infty \frac{\cos^2 ax}{b^2-x^2}\,dx = \frac{\pi}{4b} \sin 2ab$$ $[a > 0, \quad b > 0]$ BI (161)(10)

10. $$\int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2+x^2}\,dx = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \qquad [a > b]$$
$$= \frac{\pi}{16\beta} \left[1 - e^{-4a\beta} \right] \qquad [a = b]$$
$$= \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(a-b)\beta} - e^{-2a\beta} \right] \qquad [a < b]$$

$[a > 0, \quad b > 0]$, (cf. **3.824** 1 and 3) BI (162)(6)

11. $$\int_0^\infty \frac{x \sin 2ax \cos^2 bx}{\beta^2 + x^2} dx = \frac{\pi}{8}\left[2e^{-2a\beta} + e^{-2(a+b)\beta} + e^{2(b-a)\beta}\right] \qquad [a > 0]$$
$$= \frac{\pi}{8}\left[e^{-4a\beta} + 2e^{-2a\beta}\right] \qquad [a = b]$$
$$= \frac{\pi}{8}\left[2e^{-2a\beta} + e^{-2(a+b)\beta} - e^{2(a-b)\beta}\right] \qquad [a < b]$$
LI (162)(5)

3.825

1. $$\int_0^\infty \frac{\sin^2 ax\, dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi\left(b - c + ce^{-2ab} - be^{-2ac}\right)}{4bc\left(b^2 - c^2\right)}$$
$[a > 0, \quad b > 0, \quad c > 0]$ BI (174)(15)

2. $$\int_0^\infty \frac{\cos^2 ax\, dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi\left(b - c + be^{-2ac} - ce^{-2ab}\right)}{4bc\left(b^2 - c^2\right)}$$
$[a > 0, \quad b > 0, \quad c > 0]$ BI (175)(14)

3.[3] $$\int_0^\infty \frac{\sin^2 ax\, dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(c \sin 2ab - b \sin 2ac)}{4bc\left(b^2 - c^2\right)} \qquad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c]$$
LI (174)(16)

4.[3] $$\int_0^\infty \frac{\cos^2 ax\, dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(b \sin 2ac - c \sin 2ab)}{4bc\left(b^2 - c^2\right)} \qquad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c]$$
LI (175)(15)

3.826

1. $$\int_0^\infty \frac{\sin^2 ax\, dx}{x^2\left(b^2 + x^2\right)} = \frac{\pi}{4b^2}\left[2a - \frac{1}{b}\left(1 - e^{-2ab}\right)\right] \qquad [a > 0, \quad b > 0]$$ BI (172)(13)

2. $$\int_0^\infty \frac{\sin^2 ax\, dx}{x^2\left(b^2 - x^2\right)} = \frac{\pi}{4b^2}\left(2a - \frac{1}{b}\sin 2ab\right) \qquad [a > 0, \quad b > 0]$$ BII (172)(14)

3.827

1.[8] $$\int_0^\infty \frac{\sin^3 ax}{x^\nu} dx = \frac{3 - 3^{\nu-1}}{4} a^{\nu-1} \cos\frac{\nu\pi}{2}\, \Gamma(1 - \nu) \qquad [a < \operatorname{Re}\nu < 4, \nu \neq 1, 2, 3]$$
GW (333)(19f)

2.[8] $$\int_0^\infty \frac{\sin^3 ax}{x} dx = \frac{\pi}{4}$$ LO V 277

3. $$\int_0^\infty \frac{\sin^3 ax}{x^2} dx = \frac{3}{4} a \ln 3$$ BI (156)(2)

4.[8] $$\int_0^\infty \frac{\sin^3 ax}{x^3} dx = \frac{3}{8} a^2 \pi$$ BI(156)(7)a,LO V 312

5. $$\int_0^\infty \frac{\sin^4 ax}{x^2} dx = \frac{a\pi}{4} \qquad [a > 0]$$ BI (156)(3)

6. $$\int_0^\infty \frac{\sin^4 ax}{x^3} dx = a^2 \ln 2$$ BI (156)(8)

7. $\int_0^\infty \frac{\sin^4 ax}{x^4}\,dx = \frac{a^3\pi}{3}$ $[a>0]$ BI(156)(11), LO V 312

8. $\int_0^\infty \frac{\sin^5 ax}{x^2}\,dx = \frac{5}{16}a\left(3\ln 3 - \ln 5\right)$ BI (156)(4)

9. $\int_0^\infty \frac{\sin^5 ax}{x^3}\,dx = \frac{5}{32}a^2\pi$ $[a>0]$ BI (156)(9)

10. $\int_0^\infty \frac{\sin^5 ax}{x^4}\,dx = \frac{5}{96}a^3\left(25\ln 5 - 27\ln 3\right)$ BI (156)(12)

11. $\int_0^\infty \frac{\sin^5 ax}{x^5}\,dx = \frac{115}{384}a^4\pi$ $[a>0]$ BI(156)(13), LO V 312

12. $\int_0^\infty \frac{\sin^6 ax}{x^2}\,dx = \frac{3}{16}a\pi$ $[a>0]$ BI (156)(5)

13. $\int_0^\infty \frac{\sin^6 ax}{x^3}\,dx = \frac{3}{16}a^2\left(8\ln 2 - 3\ln 3\right)$ BI (156)(10)

14. $\int_0^\infty \frac{\sin^6 ax}{x^5}\,dx = \frac{1}{16}a^4\left(27\ln 3 - 32\ln 2\right)$ BI (156)(14)

15. $\int_0^\infty \frac{\sin^6 ax}{x^6}\,dx = \frac{11}{40}a^5\pi$ $[a>0]$ LO V 312

3.828 In **3.828** 1–21 the restrictions $a>0$, $b>0$, $c>0$ apply.

1.[8] $\int_0^\infty \frac{\sin ax \sin bx}{x}\,dx = \frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|$ $[a \neq b]$ FI II 647

2.[8] $\int_0^\infty \sin ax \sin bx \frac{dx}{x^2} = \frac{1}{2}\pi\min(a,b)$ BI (157)(1)

3.[8] $\int_0^\infty \frac{\sin^2 ax \sin bx}{x}\,dx = \frac{\pi}{4}$ $[b<2a]$

$= \frac{\pi}{8}$ $[b=2a]$

$= 0$ $[b>2a]$

BI (151)(10)

4.[8] $\int_0^\infty \frac{\sin^2 ax \cos bx}{x}\,dx = \frac{1}{4}\ln\frac{4a^2-b^2}{b^2}$ $[2a \neq b]$ BI (151)(12)

5.[8] $\int_0^\infty \frac{\sin^2 ax \cos 2bx}{x^2}\,dx = \frac{1}{2}\pi\max(0, a-b)$

6. $\int_0^\infty \frac{\sin 2ax \cos^2 bx}{x}\,dx = \frac{\pi}{2}$ $[a>b]$

$= \frac{3}{8}\pi$ $[a=b]$

$= \frac{\pi}{4}$ $[a<b]$

BI (151)(9)

7.[8] $$\int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x^2}\,dx = \frac{\pi}{16}\left(|b-2a-c| - |2a-b-c| + 2c\right)$$
$[a > 0, \quad 0 < c \le b]$
BI(157)(9)a, ET I 79(15)

8.[8] $$\int_0^\infty \frac{\sin^2 ax \sin bx \sin cx}{x}\,dx = \frac{1}{8}\ln\left|\frac{(b+c)^2(2a-b+c)(2a+b-c)}{(b-c)^2(2a+b+c)(2a-b-c)}\right|$$
$[b \ne c, \quad 2a+c \ne b, \quad 2a+b \ne c, \quad 2a \ne b+c]$ LI (152)(2)

9. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^2}\,dx = \frac{\pi}{4}a \qquad [0 \le a \le b]$$
$$= \frac{\pi}{4}b \qquad [0 \le b \le a]$$
BI (157)(3)

10.[8] $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4}\,dx = \frac{1}{6}\pi \min\left(a^2, b^2\right)\left[3\max(a,b) - \min(a,b)\right]$$ BI (157)(27)

11.[8] $$\int_0^\infty \frac{\sin^2 ax \cos^2 bx}{x^2}\,dx = \frac{1}{4}\pi\left[a + \max(0, a-b)\right]$$ BI (157)(6)

12. $$\int_0^\infty \frac{\sin^3 ax \sin 3bx}{x^4}\,dx = \frac{a^3\pi}{2} \qquad [b > a]$$
$$= \frac{\pi}{16}\left[8a^3 - 9(a-b)^3\right] \qquad [a \le 3b \le 3a]$$ BI (157)(28)
$$= \frac{9b\pi}{8}\left(a^2 - b^2\right) \qquad [3b \le a]$$ LI (157)(28)

13. $$\int_0^\infty \frac{\sin^3 ax \cos bx}{x}\,dx = 0 \qquad [b > 3a]$$
$$= -\frac{\pi}{16} \qquad [b = 3a]$$
$$= -\frac{\pi}{8} \qquad [3a > b > a]$$
$$= \frac{\pi}{16} \qquad [b = a]$$
$$= \frac{\pi}{4} \qquad [a > b]$$
$[a > 0, \quad b > 0]$ BI (151)(15)

14.[10] $$\int_0^\infty \frac{\sin^3 ax \cos 3bx}{x^2}\,dx = \frac{3}{16}\Big(a\ln 81 - 2(a-3b)\ln(a-3b) + 2(a-b)\ln(a-b) + 2(a+b)\ln(a+b) - 2(a+3b)\ln(a+3)\Big)$$
$[\operatorname{Im} a = 0, \quad \operatorname{Im} b = 0]$ MC

15. $$\int_0^\infty \frac{\sin^3 ax \cos bx}{x^3}\,dx = \frac{\pi}{8}\left(3a^2 - b^2\right) \qquad [b < a]$$
$$= \frac{\pi b^2}{4} \qquad [a = b]$$
$$= \frac{\pi}{16}(3a - b)^2 \qquad [a < b < 3a]$$
$$= 0 \qquad [3a < b]$$
$[a > 0, \quad b > 0]$ BI(157)(19), ET I 19(10)

16. $$\int_0^\infty \frac{\sin^3 ax \sin bx}{x^4}\,dx = \frac{b\pi}{24}\left(9a^2 - b^2\right) \qquad [0 < b \le a]$$
$$= \frac{\pi}{48}\left[24a^3 - (3a - b)^3\right] \qquad [0 < a \le b \le 3a]$$
$$= \frac{\pi a^3}{2} \qquad [0 < 3a \le b]$$
ET I 79(16)

17. $$\int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x}\,dx = \frac{\pi}{8} \qquad [2b > 3a]$$
$$= \frac{5\pi}{32} \qquad [2b = 3a]$$
$$= \frac{3\pi}{16} \qquad [3a > 2b > a]$$
$$= \frac{3\pi}{32} \qquad [2b = a]$$
$$= 0 \qquad [a > 2b]$$
$[a > 0, \quad b > 0]$ BI (151)(14)

18.[8] $$\int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x}\,dx = \frac{1}{16}\ln\left|\frac{(2a+b)^3(b-2a)^3(2a+3b)(3b-2a)}{9b^8}\right|$$
$[2a \ne b, \quad 2a \ne 3b]$ BI (151)(13)

19.[11] $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin^2 cx}{x}\,dx$$
$$= \frac{\pi}{32}\Big(4\operatorname{sign}(c) - 2\operatorname{sign}(2b+c) + 2\operatorname{sign}(2b-c) + \operatorname{sign}(2a-2b+c) - \operatorname{sign}(2a-2b-c)$$
$$+ 2\operatorname{sign}(2a-c) + \operatorname{sign}(2a+2b+c) - \operatorname{sign}(2a+2b-c) - 2\operatorname{sign}(2a+c)\Big)$$
$[\operatorname{Im} a = 0, \quad \operatorname{Im} b = 0, \quad \operatorname{Im} c = 0]$ MC

20. $$\int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx\,dx}{x^2}$$
$$= \frac{a-b-c}{16}\ln 4(a-b-c)^2 - \frac{a+b+c}{16}\ln 4(a+b+c)^2 + \frac{a+b-c}{16}\ln 4(a+b-c)^2$$
$$-\frac{a-b+c}{16}\ln 4(a-b+c)^2 + \frac{a+c}{8}\ln 4(a+c)^2 - \frac{a-c}{8}\ln 4(a-c)^2$$
$$+\frac{b+c}{8}\ln 4(b+c)^2 - \frac{b-c}{8}\ln 4(b-c)^2 - \frac{1}{2}c\ln 2c$$
$[a > 0, \quad b > 0, \quad c > 0]$ BI (157)(10)

21.[8] $$\int_0^\infty \frac{\sin^2 ax \sin^3 bx}{x^3}\,dx = \frac{3b^2\pi}{16} \qquad [2a > 3b]$$
$$= \frac{a^2\pi}{12} \qquad [2a = 3b]$$
$$= \frac{6b^2 - (3b-2a)^2}{32}\pi \qquad [3b > 2a > b]$$
$$= \frac{a^2\pi}{4} \qquad [b \ge 2a]$$
BI (157)(18)

3.829

1. $$\int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}}\,dx = \frac{\pi}{2^n (n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n+1}$$ GW (333)(63)

2. $$\int_0^\infty \left(1 - \cos^{2m-1} x\right) \frac{dx}{x^2} = \int_0^\infty \left(1 - \cos^{2m} x\right) \frac{dx}{x^2} = \frac{m\pi}{2^{2m}} \binom{2m}{m}$$ BI (158)(7, 8)

3.831

1. $$\int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x}\,dx = \frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} \qquad [ab > 0, \quad n = 1, 2, \ldots]$$ FI II 651

2. $$\int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x}\,dx = \left[1 - \frac{(2n-1)!!}{(2n)!!}\right] \ln \frac{b}{a} \qquad [ab > 0, \quad n = 0, 1, \ldots]$$ FI II 651

3. $$\int_0^\infty \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x}\,dx = \ln \frac{b}{a} \qquad [ab > 0, \quad m = 0, 1, \ldots]$$ FI II

4. $$\int_0^\infty \frac{\cos^m ax \cos max - \cos^m bx \cos mbx}{x}\,dx = \left(1 - \frac{1}{2^m}\right) \ln \frac{b}{a}$$
$$[ab > 0, \quad m = 0, 1, \ldots]$$ LI (155)(8)

3.832

1. $$\int_0^{\pi/2} x \cos^{p-1} x \sin ax\,dx = \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right) \Gamma\left(\frac{p-a+1}{2}\right)}$$
$$[p > 0, \quad -(p+1) < a < p+1]$$
BI (205)(6)

2.[3] $$\int_0^\infty \sin^{2m+1} x \sin 2mx \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} \left[\left(1 - e^{-2a}\right)^{2m} - 1\right] \sinh a$$
$$[a > 0, \quad m = 0, 1. \ldots]$$ BI (162)(17)

3. $$\int_0^\infty \sin^{2m-1} x \sin[(2m-1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m+1} \pi}{2^{2m} a} \left(1 - e^{-2a}\right)^{2m-1}$$
$$[a > 0, \quad m = 1, 2, \ldots]$$ BI (162)(11)

4. $$\int_0^\infty \sin^{2m-1} x \sin[(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2^{2m} a} e^{-2a} \left(1 - e^{-2a}\right)^{2m-1}$$
$$[a > 0, \quad m = 1, 2, \ldots]$$ BI (162)(12)

5. $$\int_0^\infty \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \sinh^{2m+1} a$$

$[a > 0]$ BI (162)(18)

6.[3] $$\int_0^\infty \sin^{2m} x \sin[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a \left[\left(1-e^{-2a}\right)^{2m} - \left(1+e^{-2a}\right)\right]$$

$[a \geq 0, \quad m = 0, 1, \ldots]$ BI (162)(13)

7. $$\int_0^\infty \sin^{2m} x \sin(2mx) \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} \left[\left(1-e^{-2a}\right)^{2m} - 1\right]$$

$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(14)

8. $$\int_0^\infty \sin^{2m} x \sin[(2m+2)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} \left(1-e^{-2a}\right)^{2m}$$

$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(15)

9. $$\int_0^\infty \sin^{2m} x \sin 4mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \sinh^{2m} a$$

$[a > 0, \quad m = 1, 2, \ldots]$ BI (162)(16)

10. $$\int_0^\infty \sin^{2m} x \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$ $[m = 1, 2, \ldots]$ GW (333)(15a)

11. $$\int_0^\infty \sin^{2m} x \cos[(2m-1)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m} a} \left[\left(1-e^{-2a}\right)^{2m-1} - 1\right] \sinh a$$

$[a > 0, \quad m = 1, 2, \ldots]$ BI (162)(25)

12. $$\int_0^\infty \sin^{2m} x \cos(2mx) \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} \left(1-e^{-2a}\right)^{2m}$$

$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(26)

13. $$\int_0^\infty \sin^{2m} x \cos[(2m+2)x] \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} \left(1-e^{-2a}\right)^{2m}$$

$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(27)

14. $$\int_0^\infty \sin^{2m} x \cos 4mx \frac{dx}{a^2+x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \sinh^{2m} a$$

$[a > 0, \quad m = 0, 1, \ldots]$ BI (162)(28)

15. $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2}$$ $[m = 0, 1, \ldots]$ GW (333)(15)

16.[3] $$\int_0^\infty \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2}$$ $[m = 1, 2, \ldots]$ GW (333)(15b)

17. $$\int_0^\infty \sin^{2m-1} x \cos[(2m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} \left[\left(1-e^{-2a}\right)^{2m-1} - 1\right]$$

$[m = 1, 2, \ldots, \quad a > 0]$ BI (162)(23)

18.[3] $$\int_0^\infty \sin^{2m+1} x \cos 2mx \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \left\{e^a \left[\left(1-e^{-2a}\right)^{2m+1} - 1\right] - e^{-a}\right\}$$

$[m = 0, 1, \ldots, \quad a \geq 0]$ BI (162)(29)

19. $$\int_0^\infty \sin^{2m-1} x \cos[(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} \left(1-e^{-2a}\right)^{2m-1}$$

$[m = 1, 2, \ldots, \quad a > 0]$ BI (162)(24)

20. $$\int_0^\infty \sin^{2m+1} x \cos[2(2m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{(-1)^{m-1}\pi}{2} e^{-2(2m+1)a} \sinh^{2m+1} a$$

$[m = 0, 1, \ldots, \quad a > 0]$ BI (162)(30)

21. $$\int_0^\infty \cos^m x \sin mx \frac{x\,dx}{a^2+x^2} = \frac{1}{2^{m+1}a} \sum_{k=1}^{m} \binom{m}{k} \left[e^{-2ka} \operatorname{Ei}(2ka) - e^{2ka} \operatorname{Ei}(-2ka)\right]$$

$[a > 0]$ BI (162)(8)

22. $$\int_0^\infty \cos^n sx \sin nsx \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{n+1}} \left[\left(1+e^{-2as}\right)^n - 1\right]$$

$[s > 0, \quad \operatorname{Re} a > 0, \quad n \geq 0]$ BI (163)(9)

23. $$\int_0^\infty \cos^n sx \sin nsx \frac{x\,dx}{a^2-x^2} = \frac{\pi}{2} \left(2^{-n} - \cos^n as \cos nas\right)$$

$[n = 0, 1, \ldots]$ BI (166)(10)

24. $$\int_0^\infty \cos^{m-1} x \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} e^{-2a} \left(1+e^{-2a}\right)^{m-1}$$

$[a > 0, \quad m = 1, 2, \ldots]$ BI (163)(6)

25. $$\int_0^\infty \cos^m x \sin[(m+1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^{m+1}} e^{-a} \left(1+e^{-2a}\right)^m$$

$[m = 0, 1, \ldots, \quad a > 0]$ BI (163)(10)

26.[3] $$\int_0^\infty \cos^m x \sin[(m-1)x] \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2^m} \cosh a \left[\left(1+e^{-2a}\right)^{m-1} - 1\right]$$

$[m = 0, 1, \ldots, \quad a \geq 0]$ BI (163)(7)

27.[11] $$\int_0^\infty \cos^m x \sin(3mx) \frac{x\,dx}{a^2+x^2} = \frac{\pi}{2} e^{-3ma} \cosh^m a$$ $[a > 0, \quad m = 1, 2, \ldots]$ BI (163)(11)

28. $$\int_0^\infty \cos^n sx \cos nxs \frac{dx}{a^2+x^2} = \frac{\pi}{2^{n+1}a} \left(1+e^{-2as}\right)^n$$ $[n = 0, 1, \ldots]$ BI (163)(16)

29. $$\int_0^\infty \cos^n sx \cos nsx \frac{dx}{a^2-x^2} = \frac{\pi}{2a} \cos^n as \sin nas$$ $[n = 0, 1, \ldots]$

30. $$\int_0^\infty \cos^{m-1} x \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^m a} e^{-2a} \left(1+e^{-2a}\right)^{m-1}$$

$[m = 1, 2, \ldots, \quad a > 0]$ BI (163)(14)

31. $$\int_0^\infty \cos^m x \cos[(m-1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1}a} e^{a} \left[\left(1+e^{-2a}\right)^m - \left(1-e^{-2a}\right)\right]$$

$[m = 0, 1, \ldots, \quad a > 0]$ BI (163)(15)

32. $$\int_0^\infty \cos^m x \cos[(m+1)x] \frac{dx}{a^2+x^2} = \frac{\pi}{2^{m+1}a} e^{-a} \left(1+e^{-2a}\right)^m$$

$[m = 0, 1, \ldots, \quad a > 0]$ BI (163)(17)

33. $$\int_0^\infty \sin^p x \cos x \frac{dx}{x^q} = \frac{p}{q-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^\infty \frac{\sin^{p+1} x}{x^{q-1}}\, dx \qquad [p > q-1 > 0]$$
$$= \frac{p(p-1)}{(q-1)(q-2)} \int_0^\infty \sin^{p-2} x \cos x \frac{dx}{x^{q-2}} - \frac{(p+1)^2}{(q-1)(q-2)} \int_0^\infty \sin^p x \cos x \frac{dx}{x^{q-2}} \qquad [p > q-1 > 1]$$

GW (333)(18)

34. $$\int_0^\infty \cos^{2m} x \cos 2nx \sin x \frac{dx}{x} x = \int_0^\infty \cos^{2m-1} x \cos 2nx \sin \frac{dx}{x} x = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}$$

BI (152)(5, 6)

35. $$\int_0^\infty \cos^p ax \sin bx \cos x \frac{dx}{x} = \frac{\pi}{2}$$ $[b > ap, \quad p > -1]$ BI (153)(12)

36. $$\int_0^\infty \cos^p ax \sin pax \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} \left(2^p - 1\right)$$ $[p > -1]$ BI (153)(2)

37. $$\int_0^\infty \frac{dx}{x^2} \left(\prod_{k=1}^n \cos^{p_k} a_k x \right) \sin bx \sin x = \frac{\pi}{2}$$ $\left[b > \sum_{k=1}^n a_k p_k, \quad a_k > 0, \quad p_k > 0 \right]$

BI (157)(15)

3.833

1.[10] $$\int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} = \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!} \pi$$

BI (151)(24, 25)

$$= \frac{1}{2} \mathrm{B} \left(m + \frac{1}{2}, n + \frac{1}{2} \right)$$

GW (333)(24)

2. $$\int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}$$ LI (152)(4)

3.834

1. $$\int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left| \frac{1-a}{1+a} \right|^{2m-1} - \sum_{k=0}^{2m} (-1)^k \binom{m-\frac{1}{2}}{k} \left(\frac{4a}{(1+a)^2} \right)^k \right\}$$

$[|a| \neq 1]$ GW (333)(62a)

2. $$\int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1-2a\cos x+a^2)^p} \cdot \frac{dx}{x} = \frac{n!\pi}{2^{n+1}(2m+n+1)!(1+a)^{2p}} \sum_{k=0}^{n} \frac{(-1)^k(2m+2n-2k+1)!!(2m+2k-1)!!}{k!(n-k)!} \times F\left(m+n-k+\frac{3}{2}, p; 2m+n+2; \frac{4a}{(1+a)^2}\right)$$ $[a \neq \pm 1]$ GW (333)(62)

3.835

1. $$\int_0^\infty \frac{\cos^{2m} x \cos 2mx \sin x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{b^{2m-1}}{a(a+b)^{2m}}$$ $[ab > 0]$ BI (182)(31)a

2. $$\int_0^\infty \frac{\cos^{2m-1} x \cos 2mx \sin x}{a^2\cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \frac{b^{2m-1}}{(a+b)^{2m}}$$ $[ab > 0]$ LI (182)(32)a

3.836

1. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin mx}{x}\, dx = \frac{\pi}{2}$$ $[m \geq n]$ LI (159)(12)

2.[11] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos mx\, dx = \frac{n\pi}{2^n} \sum_{k=0}^{\left\lfloor \frac{1}{2}(m+n)\right\rfloor} \frac{(-1)^k(n+m-2k)^{n-1}}{k!(n-k)!}$$ $[0 \leq m < n]$

$$= 0$$ $[m \geq n \geq 2]$

$$= \frac{\pi}{4}$$ $[m = n = 1]$

GI(159)(14), ET I 20(11)

3. $$\int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin nx \cos x \frac{dx}{x} = \frac{\pi}{2}$$ $[n \geq 1]$ BI (159)(20)

4.[8] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin(anx)}{x}\, dx = \frac{\pi}{2}\left[1 - \frac{1}{2^{n-1}n!} \sum_{k=0}^{\left\lfloor \frac{1}{2}n(1+a)\right\rfloor} (-1)^k \binom{n}{k} (n+an-2k)^n\right]$$ [all real a, $n \geq 1$] ET I 20(11)

5.[10] $$I_n(b) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx\, dx = n\left(2^{n-1}n!\right)^{-1} \sum_{k=0}^{\lfloor r\rfloor} (-1)^k \binom{n}{k} (n-b-2k)^{n-1}$$

where $0 \leq b < n$, $n \geq 1$, $r = (n-b)/2$, and $\lfloor r \rfloor$ is the largest integer contained in r

LO V 340(14)

6.[11] $$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos anx\, dx = 0$$ [$a \leq -1$ or $a \geq 1$, $n \geq 2$; for $n = 1$ see **3.741** 2]

3.837

1. $$\int_0^{\pi/2} \frac{x^2\, dx}{\sin^2 x} = \pi \ln 2$$ BI (206)(9)

2. $$\int_0^{\pi/4} \frac{x^2\, dx}{\sin^2 x} = -\frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 + \boldsymbol{G} = 0.8435118417\ldots$$ BI (204)(10)

3. $$\int_0^{\pi/4} \frac{x^2\,dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4}\ln 2 - \boldsymbol{G}$$ GW (333)(35a)

4. $$\int_0^{\pi/4} \frac{x^{p+1}}{\sin^2 x}\,dx = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1)\left(\frac{\pi}{4}\right)^p \left\{\frac{1}{p} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)}\zeta(2k)\right\}$$ $[p > 0]$ LI (204)(14)

5. $$\int_0^{\pi/2} \frac{x^2\cos x}{\sin^2 x}\,dx = -\frac{\pi^2}{4} + 4\boldsymbol{G} = 1.1964612764\ldots$$ BI (206)(7)

6. $$\int_0^{\pi/2} \frac{x^3\cos x}{\sin^3 x}\,dx = -\frac{\pi^3}{16} + \frac{3}{2}\pi\ln 2$$ BI (206)(8)

7. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\sin^{2n} x\frac{dx}{x^m} = 0$$ $\left[n > \frac{m-1}{2}, \quad m > 0\right]$ BI (180)(16)

8. $$\int_0^{\infty} \frac{\cos 2nx}{\cos x}\sin^{2n+1} x\frac{dx}{x^m} = 0$$ $\left[n > \frac{m-2}{2}, \quad m > 0\right]$ BI (180)(17)

9. $$\int_0^1 \frac{x\,dx}{\cos ax\cos[a(1-x)]} = \frac{1}{a}\operatorname{cosec} a \cdot \ln\sec a$$ $\left[a < \frac{\pi}{2}\right]$ BI (149)(20)

10.[3] $$\int_0^{\pi} \frac{x\sin(2n+1)x}{\sin x}\,dx = \frac{1}{2}\pi^2$$ $[n = 0, 1, 2, \ldots]$

11.[3] $$\int_0^{\pi} \frac{x\sin 2nx}{\sin x}\,dx = -4\sum_{k=1}^{n}(2k-1)^{-2}$$ $[n = 1, 2, 3, \ldots]$

3.838

1. $$\int_0^{\pi/2} \frac{x\cos^{p-1} x}{\sin^{p+1} x}\,dx = \frac{\pi}{2p}\sec\frac{\pi p}{2}$$ $[p < 1]$ BI (206)(13)a

2. $$\int_0^{\pi/4} \frac{x\sin^{p-1} x}{\cos^{p+1} x}\,dx = \frac{\pi}{4p} - \frac{1}{2p}\beta\left(\frac{p+1}{2}\right)$$ $[p > -1]$ LI (204)(15)

3. $$\int_0^{\pi/4} \frac{x\sin^{2m-1} x}{\cos^{2m+1} x}\,dx = \frac{\pi}{8m}(1-\cos m\pi) + \frac{1}{2m}\sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{2m-2k-1}$$ BI (204)(17)

4. $$\int_0^{\pi/4} \frac{x\sin^{2m} x}{\cos^{2m+2} x}\,dx = \frac{1}{2(2m+1)}\left[\frac{\pi}{2} + (-1)^{m-1}\ln 2 + \sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{m-k}\right]$$ BI (204)(16)

3.839

1.[11] $$\int_0^{\pi/4} x\tan^2 x\,dx = \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$ BI (204)(3)

2. $$\int_0^{\pi/4} x\tan^3 x\,dx = \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8}\ln 2 - \frac{1}{2}\boldsymbol{G}$$ BI (204)(7)

3. $$\int_0^{\pi/4} \frac{x^2\tan x}{\cos^2 x}\,dx = \frac{1}{2}\ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16}$$ (cf. **3.839** 1) BI (204)(13)

4. $$\int_0^{\pi/4} \frac{x^2 \tan^2 x}{\cos^2 x}\,dx = \frac{1}{3}\left(1 - \frac{\pi}{4}\ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + \boldsymbol{G}\right)$$ (cf. **3.839** 2) BI (204)(12)

5. $$\int_0^{\pi/2} x \cos^p x \tan x\,dx = \frac{\pi}{2^{p+1}p} \cdot \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right)\right]^2}$$ $[p > -1]$ BI (205)(3)

6. $$\int_0^{\pi/2} x \sin^p x \cot x\,dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p}\,\mathrm{B}\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$ $[p > -1]$ BI (206)(11)

7. $$\int_0^{\infty} \sin^{2n} x \tan x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}$$ GW (333)(16)

8. $$\int_0^{\infty} \cos^s rx \tan qx \frac{dx}{x} = \frac{\pi}{2}$$ $[s > -1]$ BI (151)(26)

9. $$\int_0^{\infty} \frac{\cos[(2n-1)x]}{\cos x} \cdot \left(\frac{\sin x}{x}\right)^{2n} dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1}\pi |B_{2n}|$$ BI (180)(15)

10. $$\int_0^{\infty} \tan^r px \frac{dx}{q^2+x^2} = \frac{\pi}{2q} \sec\frac{r\pi}{2} \tanh^r pq$$ $[r^2 < 1]$ BI (160)(19)

3.84 Integrals containing $\sqrt{1-k^2\sin^2 x}$, $\sqrt{1-k^2\cos^2 x}$, and similar expressions

Notation: $k' = \sqrt{1-k^2}$

3.841

1. $$\int_0^{\infty} \sin x \sqrt{1-k^2\sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(8)

2. $$\int_0^{\infty} \sin x \sqrt{1-k^2\cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(20)

3. $$\int_0^{\infty} \tan x \sqrt{1-k^2\sin^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(9)

4. $$\int_0^{\infty} \tan x \sqrt{1-k^2\cos^2 x}\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(21)

3.842

1.[11] $$\int_0^{\infty} \frac{\sin x}{\sqrt{1+\sin^2 x}}\frac{dx}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^{\infty} \frac{\sin x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \int_0^{\infty} \frac{\tan x}{\sqrt{1+\cos^2 x}}\frac{dx}{x} = \frac{1}{\sqrt{2}}\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771$$ BI (183)(4, 5, 9, 10)

2. $$\int_u^{\frac{\pi}{2}} \frac{x\cos x\,dx}{\sqrt{\sin^2 x - \sin^2 u}} = \frac{\pi}{2}\ln(1+\cos u)$$ BI (226)(4)

3. $$\int_0^\infty \frac{\sin x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \boldsymbol{K}(k)$$ BI (183)(12, 13, 21, 22)

4. $$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}\,dx = \frac{1}{2k^2}\left[-\pi k' + 2\,\boldsymbol{E}(k)\right]$$ BI (211)(1)

5. $$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\,dx = \frac{1}{2k^2}\left[\pi - 2\,\boldsymbol{E}(k)\right]$$ BI (214)(1)

6. $$\int_0^{\alpha} \frac{x\sin x\,dx}{\cos^2 x\sqrt{\sin^2\alpha - \sin^2 x}} = \frac{\pi\sin^2\frac{\alpha}{2}}{\cos^2\alpha}$$ LO III 284

7. $$\int_0^{\beta} \frac{x\sin x\,dx}{\left(1-\sin^2\alpha\sin^2 x\right)\sqrt{\sin^2\beta - \sin^2 x}} = \frac{\pi\ln\dfrac{\cos\alpha + \sqrt{1-\sin^2\alpha\sin^2\beta}}{2\cos\beta\cos^2\frac{\alpha}{2}}}{2\cos\alpha\sqrt{1-\sin^2\alpha\sin^2\beta}}$$ LO III 284

3.843

1. $$\int_0^\infty \tan x\sqrt{1-k^2\sin^2 2x}\,\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(10)

2. $$\int_0^\infty \tan x\sqrt{1-k^2\cos^2 2x}\,\frac{dx}{x} = \boldsymbol{E}(k)$$ BI (154)(22)

3.[11] $$\int_0^\infty \frac{\tan x}{\sqrt{1+\sin^2 2x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1+\cos^2 2x}}\frac{dx}{x} = \frac{1}{\sqrt{2}}\,\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right) \approx 1.3110287771$$ BI (183)(6, 11)

4. $$\int_0^\infty \frac{\tan x}{\sqrt{1-k^2\sin^2 2x}}\frac{dx}{x} = \int_0^\infty \frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x} = \boldsymbol{K}(k)$$ BI (183)(14, 23)

3.844

1. $$\int_0^\infty \frac{\sin x\cos x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (185)(20)

2. $$\int_0^\infty \frac{\sin x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (185)(21)

3. $$\int_0^\infty \frac{\sin x\cos^3 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(22)

4. $$\int_0^\infty \frac{\sin x\cos^4 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(23)

5. $$\int_0^\infty \frac{\sin^3 x\cos x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(24)

6. $$\int_0^\infty \frac{\sin^3 x\cos^2 x}{\sqrt{1-k^2\cos^2 x}}\cdot\frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(25)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2\cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (184)(16)

8. $$\int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2\cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(18)

3.845

1.[11] $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.5990701174$$ BI (185)(6)

2.[11] $$\int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.5990701174$$ BI (185)(7)

3.[11] $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$$ BU (184)(8)

3.846

1. $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(9)

2. $$\int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(10)

3. $$\int_0^\infty \frac{\sin x \cos^3 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(11)

4. $$\int_0^\infty \frac{\sin x \cos^4 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$$ BI (185)(12)

5. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(13)

6. $$\int_0^\infty \frac{\sin^3 x \cos^2 x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$$ BI (185)(14)

7. $$\int_0^\infty \frac{\sin^2 x \tan x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (184)(9)

8. $$\int_0^\infty \frac{\sin^4 x \tan x}{\sqrt{1-k^2\sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$$ BI (184)(11)

3.847[11] $$\int_0^\infty \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^\infty \frac{\sin x \cos^2 x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$$ BI (185)(3, 4)

3.848

1. $$\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$$ BI (185)(15)

2. $$\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BI (184)(12)

3. $\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2-3k^2\right)k'^2\,\boldsymbol{K}(k) - 2\left(k'^2-k^2\right)\boldsymbol{E}(k)\right]$ BI (184)(13)

4. $\int_0^\infty \frac{\sin^2 4x \tan x}{\sqrt{1-k^2\sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4}\left[\left(1+k'^2\right)\boldsymbol{E}(k) - 2k'^2\,\boldsymbol{K}(k)\right]$ BI (184)(17)

5. $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$ BI (185)(26)

6. $\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2}\left[\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]$ BI (184)(19)

7. $\int_0^\infty \frac{\cos^4 2x \tan x}{\sqrt{1-k^2\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4}\left[\left(2+k^2\right)\boldsymbol{K}(k) - 2\left(1+k^2\right)\boldsymbol{E}(k)\right]$ BI (184)(20)

3.849

1.[11] $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.1779896649$ BI (185)(8)

2.[11] $\int_0^\infty \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8}\left[2\,\boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.1497675293$ BI (185)(5)

3.[11] $\int_0^\infty \frac{\cos^2 2x \tan x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \sqrt{2}\left[\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right) - \boldsymbol{E}\left(\frac{\sqrt{2}}{2}\right)\right] \approx 0.7119586598$ BI (184)(7)

3.85–3.88 Trigonometric functions of more complicated arguments combined with powers

3.851

5. $\int_0^\infty \sin\left(ax^2\right)\cos(bx)\frac{dx}{x^2} = \frac{b\pi}{2}\left\{S\left(\frac{b}{2\sqrt{a}}\right) - C\left(\frac{b}{2\sqrt{a}}\right) + \sqrt{a\pi}\sin\left(\frac{b^2}{4a}+\frac{\pi}{4}\right)\right\}$

$[a > 0, \quad b > 0]$, (cf. **3.691** 7)

ET I 23(3)a

3.852

1. $\int_0^\infty \frac{\sin\left(ax^2\right)}{x^2}\,dx = \sqrt{\frac{a\pi}{2}}$ $[a \geq 0]$ BI (177)(10)a

2. $\int_0^\infty \sin\left(ax^2\right)\cos\left(bx^2\right)\frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\sqrt{a+b}+\sqrt{a-b}\right)$ $[a > b > 0]$

$= \frac{1}{2}\sqrt{\pi a}$ $[b = a \geq 0]$

$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\sqrt{a+b}-\sqrt{b-a}\right)$

$[b > a > 0]$, (cf. **3.852** 1) BI (177)(23)

3. $\int_0^\infty \frac{\sin^2\left(a^2x^2\right)}{x^4}\,dx = \frac{2\sqrt{\pi}}{3}a^3$ $[a \geq 0]$ GW (333)(19e)

4.[10] $$\int_0^\infty \frac{\sin^3\left(a^2x^2\right)}{x^2}\,dx = \frac{a}{4}\sqrt{\frac{\pi}{2}}\left(3-\sqrt{3}\right) \qquad \left[\operatorname{Im} a^2 = 0\right]$$ MC

5. $$\int_0^\infty \left(\sin^2 x - x^2\cos x^2\right)\frac{dx}{x^4} = \frac{1}{3}\sqrt{\frac{\pi}{2}}$$ BI (178)(8)

6. $$\int_0^\infty \left(\cos^2 x - \frac{1}{1+x^2}\right)\frac{dx}{x} = -\frac{1}{2}\boldsymbol{C}$$ BI (173)(22)

3.853

1. $$\int_0^\infty \frac{\sin\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right) - \sin\left(a\beta^2\right)\right]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 219(33)a

2. $$\int_0^\infty \frac{\cos\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\pi}{2\beta}\left[\cos\left(a\beta^2\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right]$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 221(51)a

3. $$\int_0^\infty \frac{x^2\sin\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{\beta\pi}{2}\left[\sin\left(a\beta^2\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) + \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right] - \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 219(32)a

4. $$\int_0^\infty \frac{x^2\cos\left(ax^2\right)}{\beta^2+x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2}\left\{\cos\left(a\beta^2\right) - \sqrt{2}\cos\left(a\beta^2+\frac{\pi}{4}\right)C\left(\sqrt{a}\beta\right) - \sqrt{2}\sin\left(a\beta^2+\frac{\pi}{4}\right)S\left(\sqrt{a}\beta\right)\right\}$$
$[a>0, \quad \operatorname{Re}\beta>0]$ ET II 221(50)a

3.854

1. $$\int_0^\infty \left(\cos\left(ax^2\right)-\sin\left(ax^2\right)\right)\frac{dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b^3\sqrt{2}} \qquad [a>0, \quad b>0]$$
LI (178)(11)a, BI (168)(25)

2. $$\int_0^\infty \left(\cos\left(ax^2\right)+\sin\left(ax^2\right)\right)\frac{x^2\,dx}{x^4+b^4} = \frac{\pi e^{-ab^2}}{2b\sqrt{2}} \qquad [a>0, b>0]$$ LI (178)(12)

3. $$\int_0^\infty \left(\cos\left(ax^2\right)+\sin\left(ax^2\right)\right)\frac{x^2\,dx}{\left(x^4+b^4\right)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3}\left(a+\frac{1}{2b^2}\right)$$
$[a>0, \quad b>0]$ LI (178)(14)

4. $$\int_0^\infty \left(\cos\left(ax^2\right)-\sin\left(ax^2\right)\right)\frac{x^4\,dx}{\left(x^4+b^4\right)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b}\left(\frac{1}{2b^2}-a\right)$$
$[a>0, \quad b>0]$ BI (178)(15)

3.855

1. $$\int_0^\infty \frac{\sin\left(ax^2\right)}{\sqrt{\beta^2+x^4}}\,dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}} I_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ ET I 66(28)

2. $$\int_0^\infty \frac{\cos\left(ax^2\right)}{\sqrt{\beta^2+x^4}}\,dx = \frac{1}{2}\sqrt{\frac{a\pi}{2}} I_{-\frac{1}{4}}\left(\frac{a\beta}{2}\right) K_{\frac{1}{4}}\left(\frac{a\beta}{2}\right) \qquad [a>0, \quad \operatorname{Re}\beta>0]$$ ET I 9(22)

3. $$\int_0^u \frac{\sin\left(a^2x^2\right)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{\frac{1}{4}}\left(\frac{a^2}{u^2}2\right)\right]^2 \qquad [a>0]$$ ET I 66(29)

4. $$\int_u^\infty \frac{\sin\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}}\, J_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) Y_{\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) \qquad [a>0]$$ ET I 66(30)

5. $$\int_0^u \frac{\cos\left(a^2x^2\right)}{\sqrt{u^4-x^4}}\,dx = \frac{a}{4}\sqrt{\frac{\pi^3}{2}}\left[J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)\right]^2$$ ET I 9(23)

6. $$\int_u^\infty \frac{\cos\left(a^2x^2\right)}{\sqrt{x^4-u^4}}\,dx = -\frac{a}{4}\sqrt{\frac{\pi^3}{2}}\, J_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right) Y_{-\frac{1}{4}}\left(\frac{a^2u^2}{2}\right)$$ ET I 10(24)

3.856

1. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}} \sin\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\, I_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) \qquad \left[\operatorname{Re}\nu<\frac{3}{2}, \quad |\arg\beta|<\frac{\pi}{4}\right]$$ ET I 71(23)

2. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}+x^2\right)^\nu}{\sqrt{\beta^4+x^4}} \cos\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\, I_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) \qquad \left[\operatorname{Re}\nu<\frac{3}{2}, \quad |\arg\beta|<\frac{\pi}{4}\right]$$ ET I 12(16)

3. $$\int_0^\infty \frac{\left(\sqrt{\beta^4+x^4}-x^2\right)^\nu}{\sqrt{\beta^4+x^4}} \cos\left(a^2x^2\right)\,dx = \frac{a}{2}\sqrt{\frac{\pi}{2}}\beta^{2\nu}\, I_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) K_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2\beta^2}{2}\right) \qquad \left[\operatorname{Re}\nu>-\frac{3}{2}, \quad |\arg\beta|<\frac{\pi}{4}\right]$$ ET I 12(17)

4. $$\int_0^\infty \frac{\sin\left(a^2x^2\right)\,dx}{\sqrt{\beta^4+x^4}\sqrt{x^2+\sqrt{\beta^4+x^4}}} = \frac{\sinh\frac{a^2\beta^2}{2}}{\sqrt{2}\beta^2}\, K_0\left(\frac{a^2\beta^2}{2}\right) \qquad \left[|\arg\beta|<\frac{\pi}{4}\right]$$ ET I 66(32)

5. $$\int_0^\infty \frac{\cos\left(a^2x^2\right)\,dx}{\sqrt{\beta^4+x^4}\sqrt{\left(x^2+\sqrt{\beta^4+x^4}\right)^3}} = \frac{\sinh\frac{a^2\beta^2}{2}}{2\sqrt{2}\beta^4}\, K_1\left(\frac{a^2\beta^2}{2}\right) \qquad \left[|\arg\beta|<\frac{\pi}{4}\right]$$ ET I 10(27)

6. $$\int_0^\infty \frac{\sqrt{\sqrt{\beta^4+x^4}+x^2}}{\sqrt{\beta^4+x^4}} \sin\left(a^2x^2\right)\,dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2\beta^2}{2}}\, I_0\left(\frac{a^2\beta^2}{2}\right)$$
$\left[|\arg\beta| < \frac{\pi}{4}\right]$ ET I 67(33)

3.857

1. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2-R_1}{R_2+R_1}} \sin\left(ax^2\right)\,dx = \frac{1}{2\sqrt{b}}\, K_0(ac)\sin ab$$
$\left[R_1 = \sqrt{c^2+\left(b-x^2\right)^2},\quad R_2 = \sqrt{c^2+\left(b+x^2\right)^2},\quad a>0,\quad c>0\right]$ ET I 67(34)

2. $$\int_0^\infty \frac{x^2}{R_1R_2}\sqrt{\frac{R_2+R_1}{R_2-R_1}} \cos\left(ax^2\right)\,dx = \frac{1}{2\sqrt{b}}\, K_0(ac)\cos ab$$
$\left[R_1 = \sqrt{c^2+\left(b-x^2\right)^2},\quad R_2 = \sqrt{c^2+\left(b+x^2\right)^2},\quad a>0,\quad c>0\right]$ ET I 10(26)

3.858

1. $$\int_u^\infty \frac{\left(x^2+\sqrt{x^4-u^4}\right)^\nu + \left(x^2-\sqrt{x^4-u^4}\right)^\nu}{\sqrt{x^4-u^4}} \sin\left(a^2x^2\right)\,dx$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}}u^{2\nu}\left[J_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) Y_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) + J_{\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) Y_{\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)\right]$$
$\left[\operatorname{Re}\nu < \frac{3}{2}\right]$ ET I 71(25)

2. $$\int_u^\infty \frac{\left(x^2+\sqrt{x^4-u^4}\right)^\nu + \left(x^2-\sqrt{x^4-u^4}\right)^\nu}{\sqrt{x^4-u^4}} \cos\left(a^2x^2\right)\,dx$$
$$= -\frac{a}{4}\sqrt{\frac{\pi^3}{a}}u^{2\nu}\left[J_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) Y_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) + J_{-\frac{1}{4}-\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right) Y_{-\frac{1}{4}+\frac{\nu}{2}}\left(\frac{a^2u^2}{2}\right)\right]$$
$\left[\operatorname{Re}\nu < \frac{3}{2}\right]$ ET I 13(26)

3.859 $$\int_0^\infty \left[\cos\left(x^{2^n}\right) - \frac{1}{1+x^{2^{n+1}}}\right]\frac{dx}{x} = -\frac{1}{2^n}\boldsymbol{C}$$ BI (173)(24)

3.861

1. $$\int_0^\infty \sin^{2n+1}\left(ax^2\right)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac{1}{2}}}{2^{2n-m+\frac{1}{2}}(2m-1)!!}\sum_{k=1}^{n+1}(-1)^{k-1}\binom{2n+1}{n+k}(2k-1)^{m-\frac{1}{2}}$$
[the + sign is taken when $m \equiv 0 \pmod 4$ or $m \equiv 1 \pmod 4$, the − sign is taken when $m \equiv 2 \pmod 4$ or $m \equiv 3 \pmod 4$] BI (177)(19)a

2. $$\int_0^\infty \sin^{2n}\left(ax^2\right)\frac{dx}{x^{2m}} = \pm\frac{\sqrt{\pi}a^{m-\frac{1}{2}}}{2^{2n-2m+1}(2m-1)!!}\sum_{k=1}^{n}(-1)^{k}\binom{2n}{n+k}k^{m-\frac{1}{2}}$$
[the + sign is taken when $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$, the − sign is taken when $m \equiv 2 \pmod 4$ or $m \equiv 1 \pmod 4$] BI (177)(18)a, LI (177)(18)

3.862 $$\int_0^\infty \left[\cos\left(ax^2\sqrt{n}\right) + \sin\left(ax^2\sqrt{n}\right)\right]\left(\frac{\sin^2 x}{x^2}\right)^n dx$$
$$= \frac{\sqrt{\pi}}{(2n-1)!!\sqrt{2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(n-2k+a\sqrt{n}\right)^{n-\frac{1}{2}}$$
$\left[a > \sqrt{n} > 0\right]$ BI (178)(9)

3.863

1. $$\int_0^\infty x^2 \cos\left(ax^4\right) \sin\left(2bx^2\right)\, dx = -\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{\frac{3}{4}}\left(\frac{b^2}{2a}\right)\right]$$
$[a > 0, \quad b > 0]$ ET I 25(22)

2. $$\int_0^\infty x^2 \cos\left(ax^4\right) \cos\left(2bx^2\right)\, dx = -\frac{\pi}{8}\sqrt{\frac{b^3}{a^3}}\left[\sin\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{3}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right)\right]$$
$[a > 0, \quad b > 0]$ ET I 25(23)

3.864

1. $$\int_0^\infty \sin\frac{b}{x} \sin ax \frac{dx}{x} = \frac{\pi}{2}\, Y_0\left(2\sqrt{ab}\right) + K_0\left(2\sqrt{ab}\right)$$ $[a > 0, \quad b > 0]$ WA 204(3)a

2. $$\int_0^\infty \cos\frac{b}{x} \cos ax \frac{dx}{x} = -\frac{\pi}{2}\, Y_0\left(2\sqrt{ab}\right) + K_0\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0]$
WA 204(4)a, ET I 24 (12)

3.865

1. $$\int_0^u \frac{\left(u^2 - x^2\right)^{\mu-1}}{x^{2\mu}} \sin\frac{a}{x}\, dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}}\, \Gamma(\mu)\, J_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$
$[a > 0, \quad u > 0, \quad 0 < \operatorname{Re}\mu < 1]$
ET II 189(30)

2. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \sin\frac{a}{x}\, dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu}\, \Gamma(\mu) \sin\frac{a}{2u}\, J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right)$$
$[a > 0, \quad u > 0, \quad \operatorname{Re}\mu > 0]$
ET II 203(21)

3. $$\int_0^u \frac{\left(u^2 - x^2\right)^{\mu-1}}{x^{2\mu}} \cos\frac{a}{x}\, dx = -\frac{\sqrt{\pi}}{2}\left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) u^{\mu-\frac{3}{2}}\; Y_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$
$[a > 0, \quad u > 0, \quad 0 < \operatorname{Re}\mu < 1]$
ET II 190(36)

4. $$\int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \cos\frac{a}{x}\, dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu}\, \Gamma(\mu) \cos\frac{a}{2u}\, J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right)$$
$[a > 0, \quad u > 0, \quad \operatorname{Re}\mu > 0]$
ET II 204(26)

3.866

1. $$\int_0^\infty x^{\mu-1} \sin\frac{b^2}{x} \sin\left(a^2 x\right)\, dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \operatorname{cosec}\frac{\mu\pi}{2}\left[J_\mu(2ab) - J_{-\mu}(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)\right]$$
$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$
ET I 322(42)

2. $$\int_0^\infty x^{\mu-1} \sin\frac{b^2}{x} \cos\left(a^2 x\right)\, dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \sec\frac{\mu\pi}{2}\left[J_\mu(2ab) + J_{-\mu}(2ab) + I_\mu(2ab) - I_{-\mu}(2ab)\right]$$
$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$
ET I 322(43)

3. $$\int_0^\infty x^{\mu-1} \cos\frac{b^2}{x} \cos\left(a^2 x\right)\,dx = \frac{\pi}{4}\left(\frac{b}{a}\right)^\mu \operatorname{cosec}\frac{\mu\pi}{2}\left[J_{-\mu}(2ab) - J_\mu(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)\right]$$
$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$ ET I 322(44)

3.867

1. $$\int_0^1 \frac{\cos ax - \cos\frac{a}{x}}{1-x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos ax - \cos\frac{a}{x}}{1-x^2}\,dx = \frac{\pi}{2}\sin a$$
$[a > 0]$ GW (334)(7a)

2. $$\int_0^1 \frac{\cos ax + \cos\frac{a}{x}}{1+x^2}\,dx = \frac{1}{2}\int_0^\infty \frac{\cos ax + \cos\frac{a}{x}}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$$
$[a > 0]$ GW (334)(7b)

3.868

1. $$\int_0^\infty \sin\left(a^2 x + \frac{b^2}{x}\right)\frac{dx}{x} = \pi\,J_0(2ab)$$ $[a > 0, \quad b > 0]$
GW (334)(11a), WA 200(16)

2. $$\int_0^\infty \cos\left(a^2 x + \frac{b^2}{x}\right)\frac{dx}{x} = -\pi\,Y_0(2ab)$$ $[a > 0, \quad b > 0]$ GW (334)(11a)

3. $$\int_0^\infty \sin\left(a^2 x - \frac{b^2}{x}\right)\frac{dx}{x} = 0$$ $[a > 0, \quad b > 0]$ GW (334)(11b)

4. $$\int_0^\infty \cos\left(a^2 x - \frac{b^2}{x}\right)\frac{dx}{x} = 2\,K_0(2ab)$$ $[a > 0, \quad b > 0]$ GW (334)(11b)

3.869

1. $$\int_0^\infty \sin\left(ax - \frac{b}{x}\right)\frac{x\,dx}{\beta^2 + x^2} = \frac{\pi}{2}\exp\left(-\alpha\beta - \frac{b}{\beta}\right)$$ $[a > 0, \quad b > 0, \quad \operatorname{Re}\beta > 0]$
ET II 220(42)

2. $$\int_0^\infty \cos\left(ax - \frac{b}{x}\right)\frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta}\exp\left(-a\beta - \frac{b}{\beta}\right)$$ $[a > 0, \quad b > 0, \quad \operatorname{Re}\beta > 0]$
ET II 222(58)

3.871

1. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x + \frac{b^2}{x}\right)\right]dx = \pi b^\mu\left[J_\mu(2ab)\cos\frac{\mu\pi}{2} - Y_\mu(2ab)\sin\frac{\mu\pi}{2}\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\mu < 1]$
ET I 319(17)

2. $$\int_0^\infty x^{\mu-1}\cos\left[a\left(x + \frac{b^2}{x}\right)\right]dx = -\pi b^\mu\left[J_\mu(2ab)\sin\frac{\mu\pi}{2} + Y_\mu(2ab)\cos\frac{\mu\pi}{2}\right]$$
$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$
ET I 321(35)

3. $$\int_0^\infty x^{\mu-1}\sin\left[a\left(x - \frac{b^2}{x}\right)\right]dx = 2b^\mu\,K_\mu(2ab)\sin\frac{\mu\pi}{2}$$ $[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$
ET I 319(16)

4. $$\int_0^\infty x^{\mu-1} \cos\left[a\left(x - \frac{b^2}{x}\right)\right] dx = 2b^\mu K_\mu(2ab) \cos\frac{\mu\pi}{2}$$
$[a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < 1]$ ET I 321(36)

3.872

1. $$\int_0^1 \sin\left[a\left(x+\frac{1}{x}\right)\right] \sin\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1-x^2} = \frac{1}{2}\int_0^\infty \sin\left[a\left(x+\frac{1}{x}\right)\right] \sin\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1-x^2} = -\frac{\pi}{4}\sin 2a$$
$[a \ge 0]$ BI (149)(15), GW (334)(8a)

2. $$\int_0^1 \cos\left[a\left(x+\frac{1}{x}\right)\right] \cos\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1+x^2} = \frac{1}{2}\int_0^\infty \cos\left[a\left(x+\frac{1}{x}\right)\right] \cos\left[a\left(x-\frac{1}{x}\right)\right] \frac{dx}{1+x^2} = \frac{\pi}{4}e^{-2a}$$
$[a \ge 0]$ GW (334)(8b)

3.873

1. $$\int_0^\infty \sin\frac{a^2}{x^2} \cos b^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a}\left[\sin(2ab) + \cos(2ab) + e^{-2ab}\right]$$
$[a > 0, \quad b > 0]$ ET I 24(15)

2. $$\int_0^\infty \cos\frac{a^2}{x^2} \cos b^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2}a}\left[\cos(2ab) - \sin(2ab) + e^{-2ab}\right]$$
$[a > 0, \quad b > 0]$ ET I 24(16)

3.874

1. $$\int_0^\infty \sin\left(a^2x^2 + \frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b}\sin\left(2ab + \frac{\pi}{4}\right)$$ $[a > 0, \quad b > 0]$
BI (179)(6)a, GW(334)(10a)

2. $$\int_0^\infty \cos\left(a^2x^2 + \frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b}\cos\left(2ab + \frac{\pi}{4}\right)$$ $[a > 0, \quad b > 0]$
GI (179)(8)a, GW(334)(10a)

3. $$\int_0^\infty \sin\left(a^2x^2 - \frac{b^2}{x^2}\right)\frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab}$$ $[a \ge 0, \quad b > 0]$ GW (335)(10b)

4. $$\int_0^\infty \cos\left(a^2x^2 - \frac{b^2}{x^2}\right)\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b}e^{-2ab}$$ $[a \ge 0, \quad b > 0]$ GW (334)(10b)

5. $$\int_0^\infty \sin\left(ax - \frac{b}{x}\right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$ $[a > 0, \quad b > 0]$ BI (179)(13)a

6. $$\int_0^\infty \cos\left(ax - \frac{b}{x}\right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b}$$ $[a > 0, \quad b > 0]$ BI (179)(14)a

3.875

1. $$\int_u^\infty \frac{x \sin\left(p\sqrt{x^2-u^2}\right)}{x^2+a^2} \cos bx\, dx = \frac{\pi}{2} \exp\left(-p\sqrt{a^2+u^2}\right) \cosh ab$$ $[0 < b < p]$ ET I 27(39)

2. $$\int_u^\infty \frac{x \sin\left(p\sqrt{x^2-u^2}\right)}{a^2+x^2-u^2} \cos bx\, dx = \frac{\pi}{2} e^{-ap} \cos\left(b\sqrt{u^2-a^2}\right)$$ $[0 < b < p, \quad a > 0]$ ET I 27(38)

3.[6] $$\int_0^\infty \frac{\sin\left(p\sqrt{a^2+x^2}\right)}{\left(a^2+x^2\right)^{3/2}} \cos bx\, dx = \frac{\pi p}{2a} e^{-ab}$$ $[0 < p < b, \quad a > 0]$ ET I 26(29)

3.876

1. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}} \cos bx\, dx = \frac{\pi}{2} J_0\left(a\sqrt{p^2-b^2}\right) \qquad [0 < b < p]$$
$$= 0 \qquad [b > p > 0]$$
$[a > 0]$ ET I 26(30)

2. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}} \cos bx\, dx = -\frac{\pi}{2} Y_0\left(a\sqrt{p^2-b^2}\right) \qquad [0 < b < p]$$
$$= K_0\left(a\sqrt{b^2-p^2}\right) \qquad [b > p > 0]$$
$[a > 0]$ ET I 26(34)

3. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{x^2+c^2} \cos bx\, dx = \frac{\pi}{2c} e^{-bc} \cos\left(p\sqrt{a^2-c^2}\right)$$ $[c > 0, \quad b > p]$ ET I 26(33)

4. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\left(x^2+c^2\right)\sqrt{x^2+a^2}} \cos bx\, dx = \frac{\pi}{2c} \frac{e^{-bc} \sin\left(p\sqrt{a^2-c^2}\right)}{\sqrt{a^2-c^2}} \qquad [c \neq a]$$
$$= \frac{\pi}{2} e^{-ba} \frac{p}{a} \qquad [c = a]$$
$[b > p, \quad c > 0]$ ET I 26(31)a

5.[6] $$\int_0^\infty \frac{\cos\left(p\sqrt{x^2+a^2}\right)}{x^2+a^2} \cos bx\, dx = \frac{\pi}{2a} e^{-ab}$$ $[b > p > 0; \quad a > 0]$ ET I 27(35)a

6.[6] $$\int_0^\infty \frac{x \cos\left(p\sqrt{x^2+a^2}\right)}{x^2+a^2} \sin bx\, dx = \frac{\pi}{2} e^{-ab}$$ $[a > 0, \quad b > p > 0]$ ET I 85(29)a

7. $$\int_0^u \frac{\cos\left(p\sqrt{u^2-x^2}\right)}{\sqrt{u^2-x^2}} \cos bx\, dx = \frac{\pi}{2} J_0\left(u\sqrt{b^2+p^2}\right)$$ ET I 28(42)

8. $$\int_u^\infty \frac{\cos\left(p\sqrt{x^2-u^2}\right)}{\sqrt{x^2-u^2}} \cos bx\, dx = K_0\left(u\sqrt{p^2-b^2}\right) \qquad [0 < b < |p|]$$
$$= -\frac{\pi}{2} Y_0\left(u\sqrt{b^2-p^2}\right) \qquad [b > |p|]$$
ET I 28(43)

3.877

1. $$\int_0^u \frac{\sin\left(p\sqrt{u^2-x^2}\right)}{\sqrt[4]{\left(u^2-x^2\right)^3}} \cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}}\, J_{\frac{1}{4}}\left[\frac{u}{2}\left(\sqrt{b^2+p^2}-b\right)\right] J_{\frac{1}{4}}\left[\frac{u}{2}\left(\sqrt{b^2+p^2}+b\right)\right]$$
$[b>0, \quad p>0]$ ET I 27(40)

2. $$\int_u^\infty \frac{\sin\left(p\sqrt{x^2-u^2}\right)}{\sqrt[4]{\left(x^2-u^2\right)^3}} \cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}}\, J_{\frac{1}{4}}\left[\frac{u}{2}\left(b-\sqrt{b^2-p^2}\right)\right] Z_{\frac{1}{4}}\left[\frac{u}{2}\left(b+\sqrt{b^2-p^2}\right)\right]$$
$[b>p>0]$ ET I 27(41)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^2-x^2}\right)}{\sqrt[4]{\left(u^2-x^2\right)^3}} \cos bx\,dx = \sqrt{\frac{\pi^3 p}{8}}\, J_{-\frac{1}{4}}\left[\frac{u}{2}\left(\sqrt{p^2+b^2}-b\right)\right] J_{-\frac{1}{4}}\left[\frac{u}{2}\left(\sqrt{p^2+b^2}+b\right)\right]$$
$[u>0, \quad p>0]$ ET I 28(44)

4. $$\int_u^\infty \frac{\cos\left(p\sqrt{x^2-u^2}\right)}{\sqrt[4]{\left(x^2-u^2\right)^3}} \cos bx\,dx = -\sqrt{\frac{\pi^3 p}{8}}\, J_{-\frac{1}{4}}\left[\frac{u}{2}\left(b-\sqrt{b^2-p^2}\right)\right] Y_{\frac{1}{4}}\left[\frac{u}{2}\left(b+\sqrt{b^2-p^2}\right)\right]$$
$[b>p>0]$ ET I 28(45)

3.878

1. $$\int_0^\infty \frac{\sin\left(p\sqrt{x^4+a^4}\right)}{\sqrt{x^4+a^4}} \cos bx^2\,dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b}\, J_{-\frac{1}{4}}\left[\frac{a^2}{2}\left(p-\sqrt{p^2-b^2}\right)\right] J_{\frac{1}{4}}\left[\frac{a^2}{2}\left(p+\sqrt{p^2-b^2}\right)\right]$$
$[p>b>0]$ ET I 26(32)

2. $$\int_0^\infty \frac{\cos\left(p\sqrt{x^4+a^4}\right)}{\sqrt{x^4+a^4}} \cos bx^2\,dx$$
$$= -\frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b}\, J_{-\frac{1}{4}}\left[\frac{a^2}{2}\left(p-\sqrt{p^2-b^2}\right)\right] Y_{\frac{1}{4}}\left[\frac{a^2}{2}\left(p+\sqrt{p^2-b^2}\right)\right]$$
$[a>0, \quad p>b>0]$ ET I 27(36)

3. $$\int_0^u \frac{\cos\left(p\sqrt{u^4-x^4}\right)}{\sqrt{u^4-x^4}} \cos bx^2\,dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)^3 b}\, J_{-\frac{1}{4}}\left[\frac{u^2}{2}\left(\sqrt{p^2+b^2}-p\right)\right] J_{-\frac{1}{4}}\left[\frac{u^2}{2}\left(\sqrt{p^2+b^2}+p\right)\right]$$
$[p>0, \quad b>0]$ ET I 28(46)

3.879 $$\int_0^\infty \sin ax^p \frac{dx}{x} = \frac{\pi}{2p}$$ $[a>0, \quad p>0]$ GW (334)(6)

3.881

1. $$\int_0^{\pi/2} x \sin\left(a \tan x\right)\, dx = \frac{\pi}{4} e^{-a}\left[\boldsymbol{C} + \ln 2a - e^{2a}\,\mathrm{Ei}(-2a)\right]$$
$[a>0]$ BI (205)(9)

2. $$\int_0^\infty \sin\left(a\tan x\right) \frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$$ $[a>0]$ BI (151)(6)

3. $$\int_0^\infty \sin\left(a\tan x\right)\cos x \frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$$ $[a>0]$ BI (151)(19)

4. $\int_0^\infty \cos(a\tan x)\sin x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(20)

5. $\int_0^\infty \sin(a\tan x)\sin 2x\frac{dx}{x} = \frac{1+a}{2}\pi e^{-a}$ $[a>0]$ BI (152)(11)

6. $\int_0^\infty \cos(a\tan x)\sin^3 x\frac{dx}{x} = \frac{1-a}{4}\pi e^{-a}$ $[a>0]$ BI (151)(23)

7. $\int_0^\infty \sin(a\tan x)\tan\frac{x}{2}\cos^2 x\frac{dx}{x} = \frac{1+a}{4}\pi e^{-a}$ $[a>0]$ BI (152)(13)

8. $\int_0^{\pi/2} \cos(a\tan x)\frac{x\,dx}{\sin 2x} = -\frac{\pi}{4}\operatorname{Ei}(-a)$ $[a>0]$ BI (206)(15)

9. $\int_0^{\pi/2} \sin(a\cot x)\frac{x\,dx}{\sin^2 x} = \frac{1-e^{-a}}{2a}\pi$ $[a>0]$ LI (206)(14)

10. $\int_0^{\pi/2} x\cos(a\tan x)\tan x\,dx = -\frac{\pi}{4}e^{-a}\left[\boldsymbol{C}+\ln 2a + e^{2a}\operatorname{Ei}(-2a)\right]$

$[a>0]$ BI (205)(10)

11. $\int_0^\infty \cos(a\tan x)\tan x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(21)

12. $\int_0^\infty \cos(a\tan x)\sin^2 x\tan x\frac{dx}{x} = \frac{1-a}{16}\pi e^{-a}$ $[a>0]$ BI (152)(15)

13. $\int_0^\infty \sin(a\tan x)\tan^2 x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (152)(9)

14. $\int_0^\infty \cos(a\tan 2x)\tan x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (151)(22)

15. $\int_0^\infty \sin(a\tan 2x)\cos^2 2x\tan x\frac{dx}{x} = \frac{1+a}{4}\pi e^{-a}$ $[a>0]$ BI (152)(13)

16. $\int_0^\infty \sin(a\tan 2x)\tan x\tan 2x\frac{dx}{x} = \frac{\pi}{2}e^{-a}$ $[a>0]$ BI (152)(10)

17. $\int_0^\infty \sin(a\tan 2x)\tan x\cot 2x\frac{dx}{x} = \frac{\pi}{2}\left(1-e^{-a}\right)$ $[a>0]$ BI (180)(6)

3.882

1. $\int_0^\infty \sin\left(a\tan^2 x\right)\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\exp(-a\tanh b) - e^{-a}\right]$

$[a>0, \quad b>0]$ BI (160)(22)

2. $\int_0^\infty \cos\left(a\tan^2 x\right)\cos x\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\left[\cosh b\exp(-a\tanh b) - e^{-a}\sinh b\right]$

$[a>0, \quad b>0]$ BI (163)(3)

3. $\int_0^\infty \cos\left(a\tan^2 x\right)\operatorname{cosec} 2x\frac{x\,dx}{b^2+x^2} = \frac{\pi}{2\sinh 2b}\exp(-a\tanh b)$

$[a>0, \quad b>0]$ BI (191)(10)

4. $$\int_0^\infty \cos\left(a\tan^2 x\right)\tan x \frac{x\,dx}{b^2+x^2} = \frac{\pi}{2\cosh b}\left[e^{-a}\cosh b - \exp\left(-a\tanh b\right)\sinh b\right]$$
$[a > 0, \quad b > 0]$ BI (163)(4)

5.[11] $$\int_0^\infty \cos\left(a\tan^2 x\right)\cot x \frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\coth b\exp\left(-a\tanh b\right) - e^{-a}\right]$$
$[a > 0, \quad b > 0]$ BI (163)(5)

6. $$\int_0^\infty \cos\left(a\tan^2 x\right)\cot 2x \frac{x\,dx}{b^2+x^2} = \frac{\pi}{2}\left[\coth 2b\exp\left(-a\tanh b\right) - e^{-a}\right]$$
$[a > 0, \quad b > 0]$ BI (191)(11)

3.883

1. $$\int_0^1 \cos\left(a\ln x\right)\frac{dx}{(1+x)^2} = \frac{a\pi}{2\sinh a\pi}$$ BI (404)(4)

2. $$\int_0^1 x^{\mu-1}\sin\left(\beta\ln x\right)\,dx = -\frac{\beta}{\beta^2+\mu^2}$$ $[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$ ET I 319(19)

3. $$\int_0^1 x^{\mu-1}\cos\left(\beta\ln x\right)\,dx = \frac{\mu}{\beta^2+\mu^2}$$ $[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$ ET I 321(38)

3.884[11] $$\int_{-\infty}^\infty \frac{\sin a\sqrt{|x|}}{x-b}\operatorname{sign} x\,dx = \pi\left[\exp\left(-a\sqrt{|-b|}\right) + \exp\left(-a\sqrt{|b|}\right)\right]$$
$[a > 0, \quad \operatorname{Im} b \neq 0]$ ET II 253(46)

3.89–3.91 Trigonometric functions and exponentials

3.891

1. $$\int_0^{2\pi} e^{imx}\sin nx\,dx = 0 \qquad [m \neq n;\ \text{or}\ m = n = 0]$$
$$= \pi i \qquad [m = n \neq 0]$$

2. $$\int_0^{2\pi} e^{imx}\cos nx\,dx = 0 \qquad [m \neq n]$$
$$= \pi \qquad [m = n \neq 0]$$
$$= 2\pi \qquad [m = n = 0]$$

3.892

1.[11] $$\int_0^\pi e^{i\beta x}\sin^{\nu-1} x\,dx = \frac{\pi e^{i\beta\frac{\pi}{2}}}{2^{\nu-1}\nu\,\mathrm{B}\left(\frac{\nu+\beta+1}{2},\frac{\nu-\beta+1}{2}\right)}$$
$[\operatorname{Re}\nu > -1]$ NH 158, EH I 12(29)

2. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x}\cos^{\nu-1} x\,dx = \frac{\pi}{2^{\nu-1}\nu\,\mathrm{B}\left(\frac{\nu+\beta+1}{2},\frac{\nu-\beta+1}{2}\right)}$$
$[\operatorname{Re}\nu > -1]$ GW (335)(19)

3.[6] $$\int_0^{\pi/2} e^{i2\beta x}\sin^{2\mu}x\cos^{2\nu}x\,dx = \frac{1}{2^{2\mu+2\nu+1}}\left\{\exp\left[i\pi\left(\beta-\nu-\tfrac{1}{2}\right)\right]\mathrm{B}\left(\beta-\mu-\nu,2\nu+1\right)\right.$$
$$\times F\left(-2\mu,\beta-\mu-\nu;1+\beta-\mu+\nu;-1\right)+\exp\left[i\pi\left(\mu+\tfrac{1}{2}\right)\right]$$
$$\left.\times\ \mathrm{B}(\beta-\mu-\nu,2\mu+1)\,F\left(-2\nu,\beta-\mu-\nu;1+\beta+\mu-\nu;-1\right)\right\}$$
$\left[\operatorname{Re}\mu > -\tfrac{1}{2},\quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ EH I 80(6)

4. $$\int_0^{\pi} e^{i2\beta x}\sin^{2\mu}x\cos^{2\nu}x\,dx = \frac{\pi\exp\left[i\pi(\beta-\nu)\right]F(-2\nu,\beta-\mu-\nu;1+\beta+\mu-\nu;-1)}{4^{\mu+\nu}(2\mu+1)\,\mathrm{B}(1-\beta+\mu+\nu,1+\beta+\mu-\nu)}$$
EH I 80(8)

5. $$\int_0^{\pi/2} e^{i(\mu+\nu)x}\sin^{\mu-1}x\cos^{\nu-1}x\,dx = e^{i\mu\frac{\pi}{2}}\,\mathrm{B}(\mu,\nu)$$
$$= \frac{1}{2^{\mu+\nu-1}}e^{i\mu\frac{\pi}{2}}\left\{\frac{1}{\mu}F(1-\nu,1;\mu+1;-1)+\frac{1}{\nu}F\left(1-\mu,1;\nu+1;-1\right)\right\}$$
$[\operatorname{Re}\mu > 0,\quad \operatorname{Re}\nu > 0]$ EH I 80(7)

3.893

1.[8] $$\int_0^{\infty} e^{-px}\sin(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left(q\cos\lambda+p\sin\lambda\right)$$ $[\operatorname{Re}p > 0]$ BI (261)(3)

2.[8] $$\int_0^{\infty} e^{-px}\cos(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left(p\cos\lambda-q\sin\lambda\right)$$ $[\operatorname{Re}p > 0]$ BI (261)(4)

3. $$\int_0^{\infty} e^{-x\cos t}\cos\left(t-x\sin t\right)\,dx = 1$$ BI (261)(7)

4.[8] $$\int_0^{\infty}\frac{e^{-\beta x}\sin ax}{\sin bx}\,dx = \operatorname{Re}\left\{\frac{1}{2bi}\left[\psi\left(\frac{a+b}{2b}-i\frac{\beta}{2b}\right)-\psi\left(\frac{b-a}{2b}-i\frac{\beta}{2b}\right)\right]\right\}$$
$[\operatorname{Re}\beta > 0,\quad b \neq 0]$ GW (335)(15)

5.[8] $$\int_0^{\infty}\frac{e^{-2px}\sin[(2n+1)x]}{\sin x}\,dx = \frac{1}{2p}+\sum_{k=1}^{n}\frac{p}{p^2+k^2}$$ $[\operatorname{Re}p > 0]$ BI (267)(15)

6.[8] $$\int_0^{\infty}\frac{e^{-px}\sin 2nx}{\sin x}\,dx = 2p\sum_{k=0}^{n-1}\frac{1}{p^2+(2k+1)^2}$$ $[\operatorname{Re}p > 0]$ GW (335)(15c)

7. $$\int_0^{\infty} e^{-px}\cos[(2n+1)x]\tan x\,dx = \frac{2n+1}{p^2+(2n+1)^2}+(-1)^n 2\sum_{k=0}^{n-1}\frac{(-1)^k(2k+1)}{p^2+(2k+1)^2}$$
$[p > 0]$ LI (267)(16)

3.894 $$\int_{-\pi}^{\pi}\left[\beta+\sqrt{\beta^2-1}\cos x\right]^{\nu} e^{inx}\,dx = \frac{2\pi\,\Gamma(\nu+1)\,P_{\nu}^{m}(\beta)}{\Gamma(\nu+m+1)}$$
$[\operatorname{Re}\beta > 0]$ ET I 157(15)

3.895

1. $$\int_0^{\infty} e^{-\beta x}\sin^{2m}x\,dx = \frac{(2m)!}{\beta(\beta^2+2^2)\,(\beta^2+4^2)\cdots[\beta^2+(2m)^2]}$$
$[\operatorname{Re}\beta > 0]$ FI II 615, WA 620a

2.[10] $$\int_0^{\pi} e^{-px}\sin^{2m} x\,dx = \frac{(2m)!\left(1-e^{-p\pi}\right)}{p\left(p^2+2^2\right)\left(p^2+4^2\right)\cdots\left[p^2+(2m)^2\right]}$$ GW (335)(4a)

3.[10] $$\int_0^{\pi/2} e^{-px}\sin^{2m} x\,dx = \frac{(2m)!}{p\left(p^2+2^2\right)\left(p^2+4^2\right)\cdots\left[p^2+(2m)^2\right]} \times\left\{1-e^{-\frac{p\pi}{2}}\left[1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right]\right\}$$ BI (270)(4)

4. $$\int_0^{\infty} e^{-\beta x}\sin^{2m+1} x\,dx = \frac{(2m+1)!}{\left(\beta^2+1^2\right)\left(\beta^2+3^2\right)\cdots\left[\beta^2+(2m+1)^2\right]}$$ $[\operatorname{Re}\beta>0]$ FI II 615, WA 620a

5.[10] $$\int_0^{\pi} e^{-px}\sin^{2m+1} x\,dx = \frac{(2m+1)!\left(1+e^{-p\pi}\right)}{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m+1)^2\right]}$$ GW (335)(4b)

6.[8] $$\int_0^{\pi/2} e^{-px}\sin^{2m+1} x\,dx = \frac{(2m+1)!}{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m+1)^2\right]} \times\left\{1-pe^{\frac{-p\pi}{2}}\left[1+\frac{p^2+1^2}{3!}+\cdots+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right]\right\}$$ BI (270)(5)

7. $$\int_0^{\infty} e^{-px}\cos^{2m} x\,dx = \frac{(2m)!}{p\left(p^2+2^2\right)\cdots\left[p^2+(2m)^2\right]} \times\left\{1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right\}$$ $[p>0]$ BI (262)(3)

8.[10] $$\int_0^{\pi/2} e^{-px}\cos^{2m} x\,dx = \frac{(2m)!}{p\left(p^2+2^2\right)\cdots\left[p^2+(2m)^2\right]} \times\left\{-e^{-p\frac{\pi}{2}}+1+\frac{p^2}{2!}+\frac{p^2\left(p^2+2^2\right)}{4!}+\cdots+\frac{p^2\left(p^2+2^2\right)\cdots\left[p^2+(2m-2)^2\right]}{(2m)!}\right\}$$ BI (270)(6)

9.[7] $$\int_0^{\infty} e^{-px}\cos^{2m+1} x\,dx = \frac{(2m+1)!p}{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m+1)^2\right]} \times\left\{1+\frac{p^2+1^2}{3!}+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)}{5!}+\cdots+\frac{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right\}$$ $[p>0]$ BI (262)(4)

10.[11] $$\int_0^{\pi/2} e^{-px}\cos^{2m+1} x\,dx = \frac{(2m+1)!}{\left(p^2+1^2\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m+1)^2\right]} \times \left\{ e^{-p\frac{\pi}{2}} + p\left[1 + \frac{p^2+1^2}{3!} + \cdots + \frac{\left(p^2+1\right)\left(p^2+3^2\right)\cdots\left[p^2+(2m-1)^2\right]}{(2m+1)!}\right]\right\}$$ BI (270)(7)

11.[8] $$\int_0^{\infty} e^{-\beta x}\sin^n ax \begin{Bmatrix}\sin bx\\ \cos bx\end{Bmatrix} dx = \frac{2^{-n-2}}{a(n+1)} e^{\frac{1}{4}(1\mp 1+2n)\pi i} \times \left\{ \binom{\frac{b+na+i\beta}{2a}}{n+1}^{-1} \pm (-1)^n \binom{\frac{b+na-i\beta}{2a}}{n+1}^{-1} \right\}$$ $[a>0, \quad b>0, \quad \operatorname{Re}\beta>0]$

12. $$\int_0^{\infty} e^{-ax}\cos^2 mx\,dx = \frac{a^2+2m^2}{a\left(a^2+4m^2\right)}$$ DW61 (861.06)

13. $$\int_0^{\infty} e^{-ax}\cos mx\cos nx\,dx = \frac{a\left(a^2+m^2+n^2\right)}{\left(a^2+(m-n)^2\right)\left(a^2+(m+n)^2\right)}$$ DW61 (861.15)

14. $$\int_0^{\infty} e^{-ax}\sin mx\cos nx\,dx = \frac{m\left(a^2+m^2-n^2\right)}{\left(a^2+(m-n)^2\right)\left(a^2+(m+n)^2\right)}$$ DW61 (861.14)

15. $$\int_0^{\infty} e^{-ax}\sin^2 mx\,dx = \frac{2m}{a\left(a^2+4m^2\right)}$$ $[a>0]$ DW61 (861.10)

16. $$\int_0^{\infty} e^{-ax}\sin mx\sin nx\,dx = \frac{2amn}{\left[a^2+(m-n)^2\right]\left[a^2+(m+n)^2\right]}$$ DW61 (861.13)

3.896

1. $$\int_{-\infty}^{\infty} e^{-q^2x^2}\sin[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}}\sin p\lambda$$ BI (269)(2)

2. $$\int_{-\infty}^{\infty} e^{-q^2x^2}\cos[p(x+\lambda)]\,dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}}\cos p\lambda$$ BI (269)(3)

3. $$\int_0^{\infty} e^{-ax^2}\sin bx\,dx = \frac{b}{2a}\exp\left(-\frac{b^2}{4a}\right)\ {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{b^2}{4a}\right)$$
$$= \frac{b}{2a}\ {}_1F_1\left(1;\frac{3}{2};-\frac{b^2}{4a}\right)$$ ET I 73(18)
$$= \frac{b}{2a}\sum_{k=1}^{\infty}\frac{1}{(2k-1)!!}\left(-\frac{b^2}{2a}\right)^{k-1}$$ $[a>0]$ FI II 720

4. $$\int_0^{\infty} e^{-\beta x^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(-\frac{b^2}{4\beta}\right)$$ $[\operatorname{Re}\beta>0]$ BI (263)(2)

3.897

1.[8] $$\int_0^\infty e^{-\beta x^2-\gamma x}\sin bx\,dx = -\frac{i}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma-ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right] - \exp\frac{(\gamma+ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ib}{2\sqrt{\beta}}\right)\right]\right\}$$
$[\operatorname{Re}\beta>0]$ ET I 74(27)

2. $$\int_0^\infty e^{-\beta x^2-\gamma x}\cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma-ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right] + \exp\frac{(\gamma+ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ib}{2\sqrt{\beta}}\right)\right]\right\}$$
$[\operatorname{Re}\beta>0]$ ET I 15(16)

3.898

1. $$\int_0^\infty e^{-\beta x^2}\sin ax\sin bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}}\right\}$$
$[\operatorname{Re}\beta>0]$ BI (263)(4)

2. $$\int_0^\infty e^{-\beta x^2}\cos ax\cos bx\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}}\right\}$$
$[\operatorname{Re}\beta>0]$ BI (263)(5)

3.[8] $$\int_0^\infty e^{-px^2}\sin^2 ax\,dx = \frac{1}{4}\sqrt{\frac{\pi}{p}}\left(1-e^{-\frac{a^2}{p}}\right)$$ $[\operatorname{Re}p>0]$ BI (263)(6)

3.899

1.[7] $$\int_0^\infty \frac{e^{p^2x^2}\sin[(2n+1)x]}{\sin x}\,dx = \frac{\sqrt{\pi}}{p}\left[\frac{1}{2}+\sum_{k=1}^{n}e^{-\left(\frac{k}{p}\right)^2}\right]$$ $[p>0]$ BI (267)(17)

2. $$\int_0^\infty \frac{e^{-p^2x^2}\cos[(4n+1)x]}{\cos x}\,dx = \frac{\sqrt{\pi}}{p}\left[\frac{1}{2}+\sum_{k=0}^{2n}(-1)^k e^{-\left(\frac{k}{p}\right)^2}\right]$$
$[p>0]$ BI (267)(18)

3. $$\int_0^\infty \frac{e^{-px^2}\,dx}{1-2a\cos x+a^2} = \frac{\sqrt{\frac{\pi}{p}}}{1-a^2}\left\{\frac{1}{2}+\sum_{k=1}^{\infty}a^k\exp\left(-\frac{k^2}{4p}\right)\right\}$$ $[a^2<1,\quad p>0]$ EI (266)(1)
$$= \frac{\sqrt{\frac{\pi}{p}}}{a^2-1}\left\{\frac{1}{2}+\sum_{k=1}^{\infty}a^{-k}\exp\left(-\frac{k^2}{4p}\right)\right\}$$ $[a^2>1,\quad p>0]$ LI (266)(1)

3.911

1. $$\int_0^\infty \frac{\sin ax}{e^{\beta x}+1}\,dx = \frac{1}{2a}-\frac{\pi}{2\beta\sinh\frac{a\pi}{\beta}}$$ $[a>0,\quad \operatorname{Re}\beta>0]$ BI (264)(1)

2. $$\int_0^\infty \frac{\sin ax}{e^{\beta x}-1}\,dx = \frac{\pi}{2\beta}\coth\left(\frac{\pi a}{\beta}\right)-\frac{1}{2a}$$ $[a>0,\quad \operatorname{Re}\beta>0]$ BI (264)(2), WH

3.[11] $$\int_0^\infty \frac{\sin ax}{e^x-1}e^{x/2}\,dx = \frac{1}{2}\pi\tanh(a\pi)$$ $[a>0]$ ET I 73(13)

4. $$\int_0^\infty \frac{\sin ax}{1-e^{-x}} e^{-nx}\,dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a}-1} - \sum_{k=1}^{n-1} \frac{a}{a^2+k^2}$$
$[a > 0]$ BI (264)(8)

5. $$\int_0^\infty \frac{\sin ax}{e^{\beta x} - e^{\gamma x}}\,dx = \frac{1}{2i(\beta-\gamma)} \left[\psi\left(\frac{\beta+ia}{\beta-\gamma}\right) - \psi\left(\frac{\beta-ia}{\beta-\gamma}\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ GW (335)(8)

6. $$\int_0^\infty \frac{\sin ax\,dx}{e^{\beta x}\left(e^{-x}-1\right)} = \frac{i}{2}\left[\psi(\beta+ia) - \psi(\beta-ia)\right]$$
$[\operatorname{Re}\beta > -1]$ ET 73(15)

3.912

1. $$\int_0^\infty e^{-\beta x}\left(1-e^{-\gamma x}\right)^{\nu-1} \sin ax\,dx = -\frac{i}{2\gamma}\left[\mathrm{B}\left(\nu, \frac{\beta-ia}{\gamma}\right) - \mathrm{B}\left(\nu, \frac{\beta+ia}{\gamma}\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > 0, \quad a > 0]$ ET I 73(17)

2. $$\int_0^\infty e^{-\beta x}\left(1-e^{-\gamma x}\right)^{\nu-1} \cos ax\,dx = \frac{1}{2\gamma}\left[\mathrm{B}\left(\nu, \frac{\beta-ia}{\gamma}\right) + \mathrm{B}\left(\nu, \frac{\beta+ia}{\gamma}\right)\right]$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0, \quad \operatorname{Re}\nu > 0, \quad a > 0]$ ET I 15(10)

3.913

1. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^\nu x \left(\beta^2 e^{ix} + \nu^2 e^{-ix}\right)^\mu\,dx = \frac{\pi\ {}_2F_1\left(-\mu, \frac{\beta}{2} - \frac{\nu}{2} - \frac{\mu}{2}; 1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}; \frac{\beta^2}{\nu^2}\right)}{2^\nu(\nu+1)\,\mathrm{B}\left(1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}, 1 - \frac{\beta}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)}$$
$[\operatorname{Re}\nu > -1, \quad |\nu| > |\beta|]$ EH I 81(11)a

2.[11] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-iux} \cos^\mu x \left(a^2 e^{ix} + b^2 e^{-ix}\right)^\nu\,dx$$
$$= \frac{\pi b^{2\nu}\ {}_2F_1\left(-\nu, -\frac{u+\mu+\nu}{2}; 1 + \frac{\mu-\nu-u}{2}; \frac{a^2}{b^2}\right)}{2^\mu(\mu+1)\,\mathrm{B}\left(1 - \frac{u+\nu-\mu}{2}, 1 + \frac{u+\mu+\nu}{2}\right)} \qquad \left[\text{for } a^2 < b^2\right]$$
$$= \frac{\pi a^{2\nu}\ {}_2F_1\left(-\nu, \frac{u-\mu-\nu}{2}; 1 + \frac{\mu-\nu+u}{2}; \frac{b^2}{a^2}\right)}{2^\mu(\mu+1)\,\mathrm{B}\left(1 + \frac{u+\mu-\nu}{2}, 1 + \frac{\mu+\nu-u}{2}\right)} \qquad \left[\text{for } b^2 < a^2\right]$$
$[\operatorname{Re}\mu > -1]$ ET I 122(31)a

3.914

1. $$\int_0^\infty e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\,dx = \frac{\beta\gamma}{\sqrt{\beta^2+b^2}}\, K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ ET I 16(26)

2. $$\int_0^\infty \sqrt{\gamma^2+x^2}\, e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\,dx = \frac{\beta^2\gamma^2}{A^2}\, K_0(\gamma A) + \left(\frac{2\beta^2\gamma}{A^3} - \frac{\gamma}{A}\right) K_1(\gamma A)$$
$\left[A = \sqrt{\beta^2+b^2}\right]$

3. $$\int_0^\infty \left(\gamma^2+x^2\right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\,dx = \left(-\frac{3\beta\gamma^2}{A^2}+\frac{4\beta^3\gamma^2}{A^4}\right) K_0(\gamma A) + \left(-\frac{6\beta\gamma}{A^3}+\frac{8\beta^3\gamma}{A^5}+\frac{\beta^3\gamma^3}{A^3}\right) K_1(\gamma A)$$
$$\left[A=\sqrt{\beta^2+b^2}\right]$$

4. $$\int_0^\infty \frac{e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \cos bx\,dx = K_0\left(\gamma\sqrt{\beta^2+b^2}\right) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad b>0]$$
ET I 16(27)

5. $$\int_0^\infty \left(\frac{1}{\beta\left(\gamma^2+x^2\right)^{3/2}}+\frac{1}{\gamma^2+x^2}\right) e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx\,dx = \frac{1}{\beta\gamma}\sqrt{\beta^2+b^2}\, K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$
(6.726(4))

6. $$\int_0^\infty x e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx\,dx = \frac{b\beta\gamma^2}{\beta^2+b^2} K_2\left(\gamma\sqrt{\beta^2+b^2}\right)$$ ET I 175(35)

7. $$\int_0^\infty x\sqrt{\gamma^2+x^2} e^{-\beta\sqrt{\gamma^2+x^2}} \sin bx\,dx = \left(-\frac{b\gamma^2}{A^2}+\frac{4b\beta^2\gamma^2}{A^4}\right) K_0(\gamma A) + \left(-\frac{2b\gamma}{A^3}+\frac{8b\beta^2\gamma}{A^5}+\frac{b\beta^2\gamma^3}{A^3}\right) K_1(\gamma A)$$
$$\left[A=\sqrt{\beta^2+b^2}\right]$$

8. $$\int_0^\infty \left(\gamma^2+x^2\right) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx\,dx = \left(-\frac{12b\beta\gamma^2}{A^4}+\frac{24b\beta^3\gamma^2}{A^6}+\frac{b\beta^3\gamma^4}{A^4}\right) K_0(\gamma A) + \left(-\frac{24b\beta\gamma}{A^5}+\frac{48b\beta^3\gamma}{A^7}-\frac{3b\beta\gamma^3}{A^3}+\frac{8b\beta^3\gamma^3}{A^5}\right) K_1(\gamma A)$$
$$\left[A=\sqrt{\beta^2+b^2}\right]$$

9. $$\int_0^\infty \frac{x e^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \sin bx\,dx = \frac{\gamma b}{\sqrt{\beta^2+b^2}} K_1\left(\gamma\sqrt{\beta^2+b^2}\right)$$ ET I 75(36)

10. $$\int_0^\infty \left(\frac{1}{\beta\left(\gamma^2+x^2\right)^{3/2}}+\frac{1}{\gamma^2+x^2}\right) e^{-\beta\sqrt{\gamma^2+x^2}} x \sin bx\,dx = \frac{b}{\beta} K_0\left(\gamma\sqrt{\beta^2+b^2}\right)$$ (6.726(3))

3.915

1. $$\int_0^\pi e^{a\cos x} \sin x\,dx = \frac{2}{a}\sinh a$$ GW (337)(15c)

2. $$\int_0^\pi e^{i\beta\cos x} \cos nx\,dx = i^n \pi J_n(\beta)$$ EH II 81(2)

3.[3] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta\sin x} \cos^{2\nu} x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) J_\nu(\beta) \qquad \left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ EH II 81(6)

4. $$\int_0^\pi e^{\pm\beta\cos x} \sin^{2\nu} x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu+\frac{1}{2}\right) I_\nu(\beta) \qquad \left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ GW (337)(15b)

5. $$\int_0^{\pi} e^{i\beta\cos x}\sin^{2\nu} x\,dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ WA 34(2), WA 60(6)

3.916

1. $$\int_0^{\pi/2} e^{-p^2\tan x}\frac{\sin\frac{x}{2}\sqrt{\cos x}}{\sin 2x}\,dx = \left[C(p)-\frac{1}{2}\right]^2 + \left[S(p)-\frac{1}{2}\right]^2$$ NT 33(18)a

2. $$\int_0^{\pi/2} \frac{\exp(-p\tan x)\;dx}{\sin 2x + a\cos 2x + a} = -\frac{1}{2}e^{ap}\operatorname{Ei}(-ap) \qquad [p>0], \qquad (\text{cf. } \mathbf{3552}\ 4 \text{ and } 6)$$ BI (273)(11)

3. $$\int_0^{\pi/2} \frac{\exp(-p\cot x)\;dx}{\sin 2x + a\cos 2x - a} = -\frac{1}{2}e^{-ap}\operatorname{Ei}(ap) \qquad [p>0], \qquad (\text{cf. } \mathbf{3.552}\ 4 \text{ and } 6)$$ BI (273)(12)

4. $$\int_0^{\pi/2} \frac{\exp(-p\tan x)\sin 2x\,dx}{(1-a^2)-2a^2\cos 2x-(1+a^2)\cos^2 2x} = -\frac{1}{4}\left[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)\right] \qquad [p>0]$$ BI (273)(13)

5. $$\int_0^{\pi/2} \frac{\exp(-p\cot x)\sin 2x\,dx}{(1-a^2)+2a^2\cos 2x-(1+a^2)\cos^2 2x} = -\frac{1}{4}\left[e^{-ap}\operatorname{Ei}(ap)+e^{ap}\operatorname{Ei}(-ap)\right] \qquad [p>0]$$ BI (273)(14)

3.917

1. $$\int_0^{\pi/2} e^{-2\beta\cot x}\cos^{\nu-1/2} x\sin^{-(\nu+1)} x\sin\left[\beta-\left(\nu-\frac{1}{2}\right)x\right]dx = \frac{\sqrt{\pi}}{2(2\beta)^{\nu}}\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ WA 186(7)

2. $$\int_0^{\pi/2} e^{-2\beta\cot x}\cos^{\nu-1/2} x\sin^{-(\nu+1)x}\cos\left[\beta-\left(\nu-\frac{1}{2}\right)x\right]dx = \frac{\sqrt{\pi}}{2(2\beta)^{\nu}}\Gamma\left(\nu+\frac{1}{2}\right)Y_\nu(\beta) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ WA 186(8)

3.918

1. $$\int_0^{\pi/2} \frac{\cos^\mu x}{\sin^{2\mu+2} x}e^{i\gamma(\beta-\mu x)-2\beta\cot x}\,dx = \frac{i\gamma}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\,\Gamma(\mu+1)\,H^{(\varepsilon)}_{\mu+\frac{1}{2}}(\beta)$$ $$\left[\varepsilon=1,2, \quad \gamma=(-1)^{\varepsilon+1}, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>-1\right]$$ GW (337)(16)

2. $$\int_0^{\pi/2} \frac{\cos^\mu x\sin(\beta-\mu x)}{\sin^{2\mu+2} x}e^{-2\beta\cot x}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\,\Gamma(\mu+1)\,J_{\mu+\frac{1}{2}}(\beta) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>-1]$$ WH

3. $$\int_0^{\pi/2} \frac{\cos^\mu x\cos(\beta-\mu x)}{\sin^{2\mu+2} x}e^{-2\beta\cot x}\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}(2\beta)^{-\mu}\,\Gamma(\mu+1)\,Y_{\mu+\frac{1}{2}}(\beta) \qquad [\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu>-1]$$ GW (337)(17b)

3.919

1. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp\left(2\pi \cot x\right) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)}$$ BI (275)(6), LI (275)(6)

2. $$\int_0^{\pi/2} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp\left(\pi \cot x\right) - 1} = (-1)^{n-1} \frac{n}{2n+1}$$ BI (275)(7), LI (275)(7)

3.92 Trigonometric functions of more complicated arguments combined with exponentials

3.921[6]

1. $$\int_0^{\infty} e^{-\gamma x} \cos ax^2 \left(\cos \gamma x - \sin \gamma x\right) dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right)$$
$[a > 0, \quad \mathrm{Re}_{\gamma} \geq |\mathrm{Im}\, \gamma|]$ ET I 26(28)

2.[10] $$\int_0^{\pi/4} \prod_{n=1}^{\infty} \exp\left[-\frac{1}{n} \tan^{2n} x\right] = \frac{\pi}{2} - 1$$

3.[10] $$\int_0^{\pi/2} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \sin^{2n} x\right] = \int_0^{\pi/2} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \cos^{2n} x\right] = \frac{\pi}{4}$$

3.922

1. $$\int_0^{\infty} e^{-\beta x^2} \sin ax^2\, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\beta x^2} \sin ax^2\, dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} - \beta}{\beta^2 + a^2}}$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \sin\left(\frac{1}{2} \arctan \frac{a}{\beta}\right)$$
$[\mathrm{Re}\, \beta > 0, \quad a > 0]$ FI II 750, BI (263)(8)

2. $$\int_0^{\infty} e^{-\beta x^2} \cos ax^2\, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\beta x^2} \cos ax^2\, dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} + \beta}{\beta^2 + a^2}}$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \cos\left(\frac{1}{2} \arctan \frac{a}{\beta}\right)$$
$[\mathrm{Re}\, \beta > 0, \quad a > 0]$ FI II 750, BI (263)(9)

[In formulas **3.922** 3 and 4, $a > 0$, $b > 0$, $\mathrm{Re}\, \beta > 0$, and

$$A = \frac{b^2}{4\left(a^2 + \beta^2\right)}, \qquad B = \sqrt{\frac{1}{2}\left(\sqrt{\beta^2 + a^2} + \beta\right)}, \qquad C = \sqrt{\frac{1}{2}\left(\sqrt{\beta^2 + a^2} - \beta\right)}.$$

If a is complex, then $\mathrm{Re}\, \beta > |\mathrm{Im}\, a|$.]

3. $$\int_0^{\infty} e^{-\beta x^2} \sin ax^2 \cos bx\, dx = -\frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} \left(B \sin Aa - C \cos Aa\right)$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4\left(\beta^2 + a^2\right)}\right) \sin\left\{\frac{1}{2} \arctan \frac{a}{\beta} - \frac{ab^2}{4\left(\beta^2 + a^2\right)}\right\}$$
LI (263)(10), GW (337)(5)

4. $$\int_0^\infty e^{-\beta x^2} \cos ax^2 \cos bx \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^2+a^2}} e^{-A\beta} \left(B \cos Aa + C \sin Aa\right)$$
$$= \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+a^2}} \exp\left(-\frac{\beta b^2}{4\left(\beta^2+a^2\right)}\right) \cos\left\{\frac{1}{2}\arctan\frac{a}{\beta} - \frac{ab^2}{4\left(\beta^2+a^2\right)}\right\}$$
LI (263)(11), GW (337)(5)

3.923

1. $$\int_{-\infty}^\infty \exp\left[-\left(ax^2+2bx+c\right)\right] \sin\left(px^2+2qx+r\right) \, dx$$
$$= \frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}} \exp\frac{a\left(b^2-ac\right)-\left(aq^2-2bpq+cp^2\right)}{a^2+p^2}$$
$$\times \sin\left\{\frac{1}{2}\arctan\frac{p}{a} - \frac{p\left(q^2-pr\right)-\left(b^2p-2abq+a^2r\right)}{a^2+p^2}\right\}$$
$[a > 0]$ GW (337)(3), BI (296)(6)

2. $$\int_{-\infty}^\infty \exp\left[-\left(ax^2+2bx+c\right)\right] \cos\left(px^2+2qx+r\right) \, dx$$
$$= \frac{\sqrt{\pi}}{\sqrt[4]{a^2+p^2}} \exp\frac{a\left(b^2-ac\right)-\left(aq^2-2bpq+cp^2\right)}{a^2+p^2}$$
$$\times \cos\left\{\frac{1}{2}\arctan\frac{p}{a} - \frac{p\left(q^2-pr\right)-\left(b^2p-2abq+a^2r\right)}{a^2+p^2}\right\}$$
$[a > 0]$ GW (337)(3), BI (269)(7)

3.924

1. $$\int_0^\infty e^{-\beta x^4} \sin bx^2 \, dx = \frac{\pi}{4}\sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{\frac{1}{4}}\left(\frac{b^2}{8\beta}\right)$$
$[\operatorname{Re}\beta > 0, \quad b > 0]$ ET 73(22)

2. $$\int_0^\infty e^{-\beta x^4} \cos bx^2 \, dx = \frac{\pi}{4}\sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{-\frac{1}{4}}\left(\frac{b^2}{8\beta}\right)$$
$[\operatorname{Re}\beta > 0, \quad b > 0]$ ET I 15(12)

3.925

1. $$\int_0^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2x^2 \, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \sin 2a^2x^2 \, dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} \left(\cos 2ap + \sin 2ap\right)$$
$[a > 0, \quad b > 0]$ BI (268)(12)

2. $$\int_0^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2x^2 \, dx = \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{p^2}{x^2}} \cos 2a^2x^2 \, dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} \left(\cos 2ap - \sin 2ap\right)$$
$[a > 0, \quad b > 0]$ BI (268)(13)

3.926 Notation:

$$u=\sqrt{\frac{\sqrt{a^2+\beta^2}+\beta}{2}}, \qquad v=\sqrt{\frac{\sqrt{a^2+\beta^2}-\beta}{2}}$$

1. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)}\sin ax^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}\left[v\cos\left(2v\sqrt{\gamma}\right)+u\sin\left(2v\sqrt{\gamma}\right)\right]$$

$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (268)(14)

2. $$\int_0^\infty e^{-\left(\beta x^2+\frac{\gamma}{x^2}\right)}\cos ax^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{a^2+\beta^2}}e^{-2u\sqrt{\gamma}}\left[u\cos\left(2v\sqrt{\gamma}\right)-v\sin\left(2v\sqrt{\gamma}\right)\right]$$

$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0]$ BI (268)(15)

3.927 $$\int_0^\infty e^{-\frac{p}{x}}\sin^2\frac{a}{x}\,dx=a\arctan\frac{2a}{p}+\frac{p}{4}\ln\frac{p^2}{p^2+4a^2} \qquad [a>0, \quad p>0]$$ LI (268)(4)

3.928

1. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\sin\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx=\frac{\sqrt{\pi}}{2r}e^{-2rs\cos(A+B)}\sin\left\{A+2rs\sin(A+B)\right\}$$

BI (268)(22)

2. $$\int_0^\infty \exp\left[-\left(p^2x^2+\frac{q^2}{x^2}\right)\right]\cos\left(a^2x^2+\frac{b^2}{x^2}\right)\,dx=\frac{\sqrt{\pi}}{2r}e^{-2rs\cos(A+B)}\cos\left\{A+2rs\sin(A+B)\right\}$$

BI (268)(23)

3.929 $$\int_0^\infty \left[e^{-x}\cos\left(p\sqrt{x}\right)+pe^{-x^2}\sin px\right]\,dx=1$$ LI (268)(3)

Notation: For the formulas in **3.928**: $a^2+p^2>0$, $r=\sqrt[4]{a^4+p^4}$, $s=\sqrt[4]{b^4+q^4}$, $A=\frac{1}{2}\arctan\frac{a^2}{p^2}$, and $B=\frac{1}{2}\arctan\frac{b^2}{q^2}$.

3.93 Trigonometric and exponential functions of trigonometric functions

3.931

1. $$\int_0^{\pi/2} e^{-p\cos x}\sin\left(p\sin x\right)\,dx=\operatorname{Ei}(-p)-\operatorname{ci}(p)$$ NT 13(27)

2. $$\int_0^{\pi} e^{-p\cos x}\sin\left(p\sin x\right)\,dx=-\int_{-\pi}^{0} e^{-p\cos x}\sin\left(p\sin x\right)\,dx=-2\operatorname{shi}(p)$$ GW (337)(11b)

3. $$\int_0^{\pi/2} e^{-p\cos x}\cos\left(p\sin x\right)\,dx=-\operatorname{si}(p)$$ NT 13(26)

4. $$\int_0^{\pi/2} e^{-p\cos x}\cos\left(p\sin x\right)\,dx=\frac{1}{2}\int_0^{2\pi} e^{-p\cos x}\cos\left(p\sin x\right)\,dx=\pi$$ GW (337)(11a)

3.932

1. $$\int_0^{\pi} e^{p\cos x}\sin\left(p\sin x\right)\sin mx\,dx=\frac{1}{2}\int_0^{2\pi} e^{p\cos x}\sin\left(p\sin x\right)\sin mx\,dx=\frac{\pi}{2}\cdot\frac{p^m}{m!}$$

BI (277)(7), GW (337)(13a)

2. $$\int_0^\pi e^{p\cos x}\cos(p\sin x)\cos mx\,dx = \frac{1}{2}\int_0^{2\pi} e^{p\cos x}\cos(p\sin x)\cos mx\,dx = \frac{\pi}{2}\cdot\frac{p^m}{m!}$$ BI (277)(8), GW (337)(13b)

3.933 $$\int_0^\pi e^{p\cos x}\sin(p\sin x)\operatorname{cosec} x\,dx = \pi\sinh p$$ BI (278)(1)

3.934

1. $$\int_0^\pi e^{p\cos x}\sin(p\sin x)\tan\frac{x}{2}\,dx = \pi\left(1-e^p\right)$$ BI (271)(8)

2. $$\int_0^\pi e^{p\cos x}\sin(p\sin x)\cot\frac{x}{2}\,dx = \pi\left(e^p-1\right)$$ BI (272)(5)

3.935 $$\int_0^\pi e^{p\cos x}\cos(p\sin x)\frac{\sin 2nx}{\sin x}\,dx = \pi\sum_{k=0}^{n-1}\frac{p^{2k+1}}{(2k+1)!} \qquad [p>0]$$ LI (278)(3)

3.936

1. $$\int_0^{2\pi} e^{p\cos x}\cos(p\sin x - mx)\,dx = 2\int_0^\pi e^{p\cos x}\cos(p\sin x - mx)\,dx = \frac{2\pi p^m}{m!}$$ BI (277)(9), GW (337)(14a)

2. $$\int_0^{2\pi} e^{p\sin x}\sin(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\sin\frac{m\pi}{2} \qquad [p>0]$$ GW (337)(14b)

3. $$\int_0^{2\pi} e^{p\sin x}\cos(p\cos x + mx)\,dx = \frac{2\pi p^m}{m!}\cos\frac{m\pi}{2} \qquad [p>0]$$ GW (337)(14b)

4. $$\int_0^{2\pi} e^{\cos x}\sin(mx - \sin x)\,dx = 0$$ WH

5. $$\int_0^\pi e^{\beta\cos x}\cos(ax + \beta\sin x)\,dx = \beta^{-a}\sin(a\pi)\,\gamma(a,\beta)$$ EH II 137(2)

3.937 **Notation**: In formulas **3.937** 1 and 2, $(b-p)^2+(a+q)^2>0$, $m=0,1,2,\dots$, $A=p^2-q^2+a^2-b^2$, $B=2(pq+ab)$, $C=p^2+q^2-a^2-b^2$, and $D=2(ap+bq)$.

1.[11] $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\sin(a\cos x + b\sin x - mx)\,dx$$
$$= i\pi\left[(b-p)^2+(a+q)^2\right]^{-\frac{m}{2}}\left\{(A+iB)^{m/2}I_m\left(\sqrt{C-iD}\right) - (A-iB)^{m/2}I_m\left(\sqrt{C+iD}\right)\right\}$$ GW (337)(9b)

2. $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\cos(a\cos x + b\sin x - mx)\,dx$$
$$= \pi\left[(b-p)^2+(a+q)^2\right]^{-\frac{m}{2}}\left\{(A+iB)^{\frac{m}{2}}I_m\left(\sqrt{C-iD}\right) + (A-iB)^{\frac{m}{2}}I_m\left(\sqrt{C+iD}\right)\right\}$$ GW (337)(9a)

3. $$\int_0^{2\pi}\exp(p\cos x + q\sin x)\sin(q\cos x - p\sin x + mx)\,dx = \frac{2\pi}{m!}\left(p^2+q^2\right)\frac{m}{2}\sin\left(m\arctan\frac{q}{p}\right)$$ GW (337)(12)

4. $$\int_0^{2\pi} \exp(p\cos x + q\sin x)\cos(q\cos x - p\sin x + mx)\,dx = \frac{2\pi}{m!}\left(p^2+q^2\right)\frac{m}{2}\cos\left(m\arctan\frac{q}{p}\right)$$ GW (337)(12)

3.938

1. $$\int_0^{\pi} e^{r(\cos px + \cos qx)}\sin(r\sin px)\sin(r\sin qx)\,dx = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{\Gamma(pk+1)\,\Gamma(qk+1)}r^{(p+q)k}$$ BI (277)(14)

2. $$\int_0^{\pi} e^{r(\cos px + \cos qx)}\cos(r\sin px)\cos(r\sin qx)\,dx = \frac{\pi}{2}\left(2+\sum_{k=1}^{\infty}\frac{r^{(p+q)k}}{\Gamma(pk+1)\,\Gamma(qk+1)}\right)$$ BI (277)(15)

3.939

1. $$\int_0^{\pi} e^{q\cos x}\frac{\sin rx}{1-2p^r\cos rx + p^{2r}}\sin(q\sin x)\,dx = \frac{\pi}{2pr}\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)}$$ $[r>0, \quad 0<p<1]$ BI (278)(15)

2.[3] $$\int_0^{\pi} e^{q\cos x}\frac{1-p^r\cos rx}{1-2p^r\cos rx + p^{2r}}\cos(q\sin x)\,dx = \frac{\pi}{2}\left[2+\sum_{k=1}^{\infty}\frac{(pq)^{kr}}{\Gamma(kr+1)}\right]$$ $[r>0, 0<p<1]$ BI (278)(16)

3. $$\int_0^{\pi/2}\frac{e^{p\cos 2x}\cos(p\sin 2x)\,dx}{\cos^2 x + q^2\sin^2 x} = \frac{\pi}{2q}\exp\left(p\frac{q-1}{q+1}\right)$$ BI (273)(8)

3.94–3.97 Combinations involving trigonometric functions, exponentials, and powers

3.941

1. $$\int_0^{\infty} e^{-px}\sin qx\frac{dx}{x} = \arctan\frac{q}{p}$$ $[p>0]$ BI (365)(1)

2. $$\int_0^{\infty} e^{-px}\cos qx\frac{dx}{x} = \infty$$ BI (365)(2)

3.942

1. $$\int_0^{\infty} e^{-px}\cos px\frac{x\,dx}{b^4+x^4} = \frac{\pi}{4b^2}\exp\left(-bp\sqrt{2}\right)$$ $[p>0, \quad b>0]$ BI (386)(6)a

2. $$\int_0^{\infty} e^{-px}\cos px\frac{x\,dx}{b^4-x^4} = \frac{\pi}{4b^2}e^{-bp}\sin bp$$ $[p>0, \quad b>0]$ BI (386)(7)a

3.943 $$\int_0^{\infty} e^{-\beta x}(1-\cos ax)\frac{dx}{x} = \frac{1}{2}\ln\frac{a^2+\beta^2}{\beta^2}$$ $[\operatorname{Re}\beta>0]$ BI (367)(6)

3.944

1. $$\int_0^{u} x^{\mu-1}e^{-\beta x}\sin\delta x\,dx = \frac{i}{2}(\beta+i\delta)^{-\mu}\gamma\left[\mu,(\beta+i\delta)u\right] - \frac{i}{2}(\beta-i\delta)^{-\mu}\gamma\left[\mu,(\beta-i\delta)u\right]$$ $[\operatorname{Re}\mu>-1]$ ET I 318(8)

2. $$\int_u^\infty x^{\mu-1}e^{-\beta x}\sin\delta x\,dx = \frac{i}{2}(\beta+i\delta)^{-\mu}\Gamma\left[\mu,(\beta+i\delta)u\right] - \frac{i}{2}(\beta-i\delta)^{-\mu}\Gamma\left[\mu,(\beta-i\delta)u\right]$$

$[\operatorname{Re}\beta > |\operatorname{Im}\delta|]$ ET I 318(9)

3. $$\int_0^u x^{\mu-1}e^{-\beta x}\cos\delta x\,dx = \frac{1}{2}(\beta+i\delta)^{-\mu}\gamma\left[\mu,(\beta+i\delta)u\right] + \frac{1}{2}(\beta-i\delta)^{-\mu}\gamma\left[\mu,(\beta-i\delta)u\right]$$

$[\operatorname{Re}\mu > 0]$ ET I 320(28)

4. $$\int_u^\infty x^{\mu-1}e^{-\beta x}\cos\delta x\,dx = \frac{1}{2}(\beta+i\delta)^{-\mu}\Gamma\left[\mu,(\beta+i\delta)u\right] + \frac{1}{2}(\beta-i\delta)^{-\mu}\Gamma\left[\mu,(\beta-i\delta)u\right]$$

$[\operatorname{Re}\beta > |\operatorname{Im}\delta|]$ ET I 320(29)

5.[11] $$\int_0^\infty x^{\mu-1}e^{-\beta x}\sin\delta x\,dx = \frac{\Gamma(\mu)}{\left(\beta^2+\delta^2\right)^{\mu/2}}\sin\left(\mu\arctan\frac{\delta}{\beta}\right)$$

$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\beta > |\operatorname{Im}\delta|]$
FI II 812, BI (361)(9)

6. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\cos\delta x\,dx = \frac{\Gamma(\mu)}{\left(\delta^2+\beta^2\right)^{\frac{\mu}{2}}}\cos\left(\mu\arctan\frac{\delta}{\beta}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > |\operatorname{Im}\delta|]$
FI II 812, BI (361)(10)

7. $$\int_0^\infty x^{\mu-1}\exp\left(-ax\cos t\right)\sin\left(ax\sin t\right)\,dx = \Gamma(\mu)a^{-\mu}\sin(\mu t)$$

$\left[\operatorname{Re}\mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}\right]$
EH I 13(36)

8. $$\int_0^\infty x^{\mu-1}\exp\left(-ax\cos t\right)\cos\left(ax\sin t\right)\,dx = \Gamma(\mu)a^{-\mu}\cos(\mu t)$$

$\left[\operatorname{Re}\mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2}\right]$
EH I 13(35)

9. $$\int_0^\infty x^{p-1}e^{-qx}\sin\left(qx\tan t\right)\,dx = \frac{1}{q^p}\Gamma(p)\cos^p t\sin pt$$ $\left[|t| < \frac{\pi}{2}, \quad q > 0\right]$ LO V 288(16)

10. $$\int_0^\infty x^{p-1}e^{-qx}\cos\left(qx\tan t\right)\,dx = \frac{1}{q^p}\Gamma(p)\cos^p(t)\cos pt$$

$\left[|t| < \frac{\pi}{2}, \quad q > 0\right]$ LO V 288(15)

11. $$\int_0^\infty x^n e^{-\beta x}\sin bx\,dx = n!\left(\frac{\beta}{\beta^2+b^2}\right)^{n+1}\sum_{0\le 2k\le n}(-1)^k\binom{n+1}{2k+1}\left(\frac{b}{\beta}\right)^{2k+1}$$
$$= (-1)^n\frac{\partial^n}{\partial\beta^n}\left(\frac{b}{b^2+\beta^2}\right)$$

$[\operatorname{Re}\beta > 0, \quad b > 0]$ GW (336)(3), ET I 72(3)

12. $$\int_0^\infty x^n e^{-\beta x} \cos bx \, dx = n! \left(\frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \le 2k \le n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{\beta} \right)^{2k}$$
$$= (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{\beta}{b^2 + \beta^2} \right)$$
$[\operatorname{Re} \beta > 0, \quad b > 0]$ GW (336)(4), ET I 14(5)

13. $$\int_0^\infty x^{n-1/2} e^{-\beta x} \sin bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left(\frac{\sqrt{\sqrt{\beta^2 + b^2} - \beta}}{\sqrt{\beta^2 + b^2}} \right)$$
$[\operatorname{Re} \beta > 0, \quad b > 0]$ ET I 72(6)

14. $$\int_0^\infty x^{n-1/2} e^{-\beta x} \cos bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \left(\frac{\sqrt{\sqrt{\beta^2 + b^2} + \beta}}{\sqrt{\beta^2 + b^2}} \right)$$
$[\operatorname{Re} \beta > 0, \quad b > 0]$ ET I 15(6)

3.945

1. $$\int_0^\infty \left(e^{-\beta x} \sin ax - e^{-\gamma x} \sin bx \right) \frac{dx}{x^r}$$
$$= \Gamma(1-r) \left\{ \left(b^2 + \gamma^2 \right) \frac{r-1}{2} \sin \left[(r-1) \arctan \frac{b}{\gamma} \right] - \left(a^2 + \beta^2 \right) \frac{r-1}{2} \sin \left[(r-1) \arctan \frac{a}{\beta} \right] \right\}$$
$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1]$ BI(371)(6)

2. $$\int_0^\infty \left(e^{-\beta x} \cos ax - e^{-\gamma x} \cos bx \right) \frac{dx}{x^r}$$
$$= \Gamma(1-r) \left\{ \left(a^2 + \beta^2 \right) \frac{r-1}{2} \cos \left[(r-1) \arctan \frac{a}{\beta} \right] - \left(b^2 + \gamma^2 \right) \frac{r-1}{2} \cos \left[(r-1) \arctan \frac{b}{\gamma} \right] \right\}$$
$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1]$ BI (371)(7)

3. $$\int_0^\infty \left(a e^{-\beta x} \sin bx - b e^{-\gamma x} \sin ax \right) \frac{dx}{x^2} = ab \left[\frac{1}{2} \ln \frac{a^2 + \gamma^2}{b^2 + \beta^2} + \frac{\gamma}{a} \operatorname{arccot} \frac{\gamma}{a} - \frac{\beta}{b} \operatorname{arccot} \frac{\beta}{b} \right]$$
$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$ BI (368)(22)

3.946

1. $$\int_0^\infty e^{-px} \sin^{2m+1} ax \frac{dx}{x} = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \arctan \frac{(2m-2k+1)a}{p}$$
$[m = 0, 1, \ldots, \quad p > 0]$ GW (336)(9a)

2. $$\int_0^\infty e^{-px} \sin^{2m} ax \frac{dx}{x} = \frac{(-1)^{m+1}}{2^{2m}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \ln \left[p^2 + (2m-2k)^2 a^2 \right] - \frac{1}{2^{2m}} \binom{2m}{m} \ln p$$
$[m = 1, 2, \ldots, \quad p > 0]$ GW (336)(9b)

3.947

1. $$\int_0^\infty e^{-\beta x} \sin \gamma x \sin ax \frac{dx}{x} = \frac{1}{4} \ln \frac{\beta^2 + (a+\gamma)^2}{\beta^2 + (a-\gamma)^2}$$ $[\operatorname{Re} \beta > |\operatorname{Im} \gamma|, \quad a > 0]$ BI (365)(5)

2.[11] $$\int_0^\infty e^{-px} \sin ax \sin bx \frac{dx}{x^2} = \frac{|a+b|}{2} \arctan\left(\frac{|a+b|}{p}\right) - \frac{|a-b|}{2} \arctan\left(\frac{|a-b|}{p}\right) + \frac{p}{4} \ln\left(\frac{p^2+(a-b)^2}{p^2+(a+b)^2}\right)$$

$[p > 0,\quad \text{for } p = 0 \text{ see } \mathbf{3.741}\ 3]$ BI (368)(1), FI II 744

3.[11] $$\int_0^\infty e^{-px} \sin ax \cos bx \frac{dx}{x} = \arctan \frac{a+b}{p} + \arctan \frac{a-b}{p}$$

$[a \geq 0, \quad p > 0]$ GW (336)(10b)

3.948

1.[11] $$\int_0^\infty e^{-\beta x} (\sin ax - \sin bx) \frac{dx}{x} = \arctan \frac{a}{\beta} - \arctan \frac{br}{\beta}$$

$[\operatorname{Re} \beta > 0]$, (cf. **3.951** 2) BI (367)(7)

2. $$\int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \ln \frac{b^2+\beta^2}{a^2+\beta^2}$$

$[\operatorname{Re} \beta > 0]$, (cf. **3.951** 3) BI (367)(8), FI II 748a

3. $$\int_0^\infty e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x^2} = \frac{\beta}{2} \ln \frac{a^2+\beta^2}{b^2+\beta^2} + b \arctan \frac{b}{\beta} - a \arctan \frac{a}{\beta}$$

$[\operatorname{Re} p > 0]$ BI (368)(20)

4. $$\int_0^\infty e^{-\beta x} \left(\sin^2 ax - \sin^2 bx\right) \frac{dx}{x^2} = a \arctan \frac{2a}{p} - b \arctan \frac{2b}{p} - \frac{p}{4} \ln \frac{p^2+4a^2}{p^2+4b^2}$$

$[p > 0]$ BI (368)(25)

5. $$\int_0^\infty e^{-\beta x} \left(\cos^2 ax - \cos^2 bx\right) \frac{dx}{x^2} = -a \arctan \frac{2a}{p} + b \arctan \frac{2b}{p} + \frac{p}{4} \ln \frac{p^2+4a^2}{p^2+4b^2}$$

$[p > 0]$ BI (368)(26)

3.949

1. $$\int_0^\infty e^{-px} \sin ax \sin bx \sin cx \frac{dx}{x} = -\frac{1}{4} \arctan \frac{a+b+c}{p} + \frac{1}{4} \arctan \frac{a+b-c}{p} + \frac{1}{4} \arctan \frac{a-b+c}{p} + \frac{1}{4} \arctan \frac{-a+b+c}{p}$$

$[p > 0]$ BI (365)(11)

2.[8] $$\int_0^\infty e^{-px} \sin^2 ax \sin bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{b}{p} - \frac{1}{2} \left[\frac{1}{2} \arctan \frac{2pb}{p^2+4a^2-b^2} + s\frac{\pi}{2}\right]$$

$$\left[s = \begin{cases} 1 & \text{for } p^2+4a^2-b^2 < 0 \\ 0 & \text{for } p^2+4a^2-b^2 \geq 0 \end{cases}\right]$$

BI (365)(8)

3. $$\int_0^\infty e^{-px} \sin^2 ax \cos bx \frac{dx}{x} = \frac{1}{8} \ln \frac{\left[p^2+(2a+b)^2\right]\left[p^2+(2a-b)^2\right]}{\left(p^2+b^2\right)^2}$$

$[p > 0]$ BI (365)(9)

4.[8] $$\int_0^\infty e^{-px} \sin ax \cos^2 bx \frac{dx}{x} = \frac{1}{2} \arctan \frac{a}{p} + \frac{1}{2} \left[\frac{1}{2} \arctan \frac{2pa}{p^2 + 4b^2 - a^2} + s\frac{\pi}{2} \right]$$
$$\left[s = \begin{cases} 1 & \text{for } p^2 + 4b^2 - a^2 < 0 \\ 0 & \text{for } p^2 + 4b^2 - a^2 \geq 0 \end{cases} \right]$$
BI (365)(10)

5. $$\int_0^\infty e^{-px} \sin^2 ax \sin bx \sin cx \frac{dx}{x} = \frac{1}{8} \ln \frac{p^2 + (b+c)^2}{p^2 + (b-c)^2} + \frac{1}{16} \ln \frac{\left[p^2 + (2a - b + c)^2\right] \left[p^2 + (2a + b - c)^2\right]}{\left[p^2 + (2a + b + c)^2\right] \left[p^2 + (2a - b - c)^2\right]}$$
$[p > 0]$ BI (365)(15)

3.951

1. $$\int_0^\infty \left(1 - e^{-x}\right) \cos x \frac{dx}{x} = \ln \sqrt{2}$$ FI II 745

2. $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \arctan \frac{(\beta - \gamma) b}{b^2 + \beta\gamma}$$ $[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma \geq 0]$ BI (367)(3)

3. $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x} \cos bx \, dx = \frac{1}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2}$$ $[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma \geq 0]$ BI (367)(4)

4.[11] $$\int_0^\infty \frac{e^{-\gamma x} - e^{-\beta x}}{x^2} \sin bx \, dx = \frac{b}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} + \beta \arctan \frac{b}{\beta} - \gamma \arctan \frac{b}{\gamma}$$
$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$ BI (368)(21)a

5. $$\int_0^\infty \frac{x}{e^{\beta x} - 1} \cos bx \, dx = \frac{1}{2b^2} - \frac{\pi^2}{2\beta^2} \operatorname{cosech}^2 \frac{b\pi}{\beta}$$ $[\operatorname{Re} \beta > 0]$ ET I 15(18)

6. $$\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx \, dx = \ln b - \frac{1}{2} \left[\psi(ib) + \psi(-ib) \right]$$
$[b > 0]$ ET I 15(9)

7. $$\int_0^\infty \frac{1 - \cos ax}{e^{2\pi x} - 1} \cdot \frac{dx}{x} = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a}$$ $[a > 0]$ BI (387)(10)

8. $$\int_0^\infty \left(e^{-\beta x} - e^{-\gamma x} \cos ax \right) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \gamma^2}{\beta^2}$$ $[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0]$ BI (367)(10)

9. $$\int_0^\infty \frac{\cos px - e^{-px}}{b^4 + x^4} \frac{dx}{x} = \frac{\pi}{2b^4} \exp\left(-\frac{1}{2} bp\sqrt{2} \right) \sin\left(\frac{1}{2} bp\sqrt{2} \right)$$
$[p > 0]$ BI (390)(6)

10. $$\int_0^\infty \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = \boldsymbol{C}$$ NT 65(8)

11. $$\int_0^\infty \left(a e^{-px} - \frac{e^{-qx}}{x} \sin ax \right) \frac{dx}{x} = \frac{a}{2} \ln \frac{a^2 + q^2}{p^2} + q \arctan \frac{a}{q} - a$$
$[p > 0, \quad q > 0]$ BI (368)(24)

12. $$\int_0^\infty \frac{x^{2m}\sin bx}{e^x-1}\,dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{2}\coth b\pi - \frac{1}{2b}\right] \qquad [b>0]$$ GW (336)(15a)

13. $$\int_0^\infty \frac{x^{2m+1}\cos bx}{e^x-1}\,dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{2}\coth b\pi - \frac{1}{2b}\right]$$ $[b>0]$ GW (336)(15b)

14. $$\int_0^\infty \frac{x^{2m}\sin bx\,dx}{e^{(2n+1)cx}-e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{4c}\tanh\frac{b\pi}{2c} - \sum_{k=1}^{n}\frac{b}{b^2+(2k-1)^2c^2}\right]$$ $[b>0]$ GW (336)(14a)

15. $$\int_0^\infty \frac{x^{2m+1}\cos bx\,dx}{e^{(2n+1)cx}-e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{4c}\tanh\frac{b\pi}{2c} - \sum_{k=1}^{n}\frac{b}{b^2+(2k-1)^2c^2}\right]$$ $[b>0]$ GW (336)(14b)

16. $$\int_0^\infty \frac{x^{2m}\sin bx\,dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}}\left[\frac{\pi}{4c}\coth\frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1}\frac{b}{b^2+(2k)^2c^2}\right]$$ $[b>0, \quad c>0]$ GW (336)(14c)

17. $$\int_0^\infty \frac{x^{2m+1}\cos bx\,dx}{e^{2ncx}-e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}}\left[\frac{\pi}{4c}\coth\frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1}\frac{b}{b^2+(2k)^2c^2}\right]$$ $[b>0, \quad c>0]$ GW (336)(14d)

18. $$\int_0^\infty \frac{\cos ax - \cos bx}{e^{(2m+1)px}-e^{(2m-1)px}}\,\frac{dx}{x} = \frac{1}{2}\ln\frac{\cosh\frac{b\pi}{2p}}{\cosh\dfrac{a\pi}{2p}} - \frac{1}{2}\sum_{k=1}^{m}\ln\frac{b^2+(2k-1)^2p^2}{a^2+(2k-1)^2p^2}$$ $[p>0]$ GW (336)(16a)

19. $$\int_0^\infty \frac{\cos ax - \cos bx}{e^{2mpx}-e^{(2m-2)px}}\,\frac{dx}{x} = \frac{1}{2}\ln\frac{a\sinh\frac{b\pi}{2p}}{b\sinh\frac{a\pi}{2p}} - \frac{1}{2}\sum_{k=1}^{m-1}\ln\frac{b^2+4k^2p^2}{a^2+4k^2p^2}$$ $[p>0]$ GW (336)(16b)

20. $$\int_0^\infty \frac{\sin x\sin bx}{1-e^x}\cdot\frac{dx}{x} = \frac{1}{4}\ln\frac{(b+1)\sinh[(b-1)\pi]}{(b-1)\sinh[(b+1)\pi]} \qquad [b^2\neq 1]$$ LO V 305

21. $$\int_0^\infty \frac{\sin^2 ax}{1-e^x}\cdot\frac{dx}{x} = \frac{1}{4}\ln\frac{2a\pi}{\sinh 2a\pi}$$ LO V 306, BI (387)(5)

3.952

1. $$\int_0^\infty xe^{-p^2x^2}\sin ax\,dx = \frac{a\sqrt{\pi}}{4p^3}\exp\left(-\frac{a^2}{4p^2}\right)$$ BI (362)(1)

2. $$\int_0^\infty xe^{-p^2x^2}\cos ax\,dx = \frac{1}{2p^2} - \frac{a}{4p^3}\sum_{k=0}^{\infty}\frac{(-1)^k k!}{(2k+1)!}\left(\frac{a}{p}\right)^{2k+1}$$ $[a>0]$ BI (362)(2)

3. $$\int_0^\infty x^2 e^{-p^2x^2} \sin ax\,dx = \frac{a}{4p^4} + \frac{2p^2 - a^2}{8p^5} \sum_{k=0}^\infty \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1}$$
$[a > 0]$ BI (362)(4)

4. $$\int_0^\infty x^2 e^{-p^2x^2} \cos ax\,dx = \sqrt{\pi} \frac{2p^2 - a^2}{8p^5} \exp\left(-\frac{a^2}{4p^2}\right)$$ BI (362)(5)

5. $$\int_0^\infty x^3 e^{-p^2x^2} \sin ax\,dx = \sqrt{\pi} \frac{6ap^2 - a^3}{16p^7} \exp\left(-\frac{a^2}{4p^2}\right)$$ BI (362)(6)

6.[3] $$\int_0^\infty e^{-p^2x^2} \sin ax \frac{dx}{x} = \frac{a\sqrt{\pi}}{2p} \sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)} \left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2} \Phi\left(\frac{a}{2p}\right)$$ BI (365)(21)

7. $$\int_0^\infty x^{\mu-1} e^{-\beta x^2} \sin \gamma x\,dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}} \Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1 - \frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > -1]$ ET I 318(10)

8.[10] $$\int_0^\infty x^{\mu-1} e^{-\beta x^2} \cos ax\,dx = \frac{1}{2}\beta^{-\mu/2}\, \Gamma\left(\frac{\mu}{2}\right) e^{-a^2/4\beta}\, {}_1F_1\left(-\frac{\mu}{2} + \frac{1}{2}; \frac{1}{2}; \frac{a^2}{4\beta}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad a > 0]$
ET I 320(30)

9. $$\int_0^\infty x^{2n} e^{-\beta^2x^2} \cos ax\,dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+1}\beta^{2n+1}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right)$$
$$= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n}\left(\frac{a}{2\beta}\right)$$
$\left[|\arg\beta| < \frac{\pi}{4}, \quad a > 0\right]$ WH, ET I 15(13)

10. $$\int_0^\infty x^{2n+1} e^{-\beta^2x^2} \sin ax\,dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+\frac{3}{2}}\beta^{2n+2}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right)$$
$$= (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n+1}\left(\frac{a}{2\beta}\right)$$
$\left[|\arg\beta| < \frac{\pi}{4}, \quad a > 0\right]$ WH, ET I 74(23)

3.953

1. $$\int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \sin ax\,dx$$
$$= -\frac{i}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{\exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) - \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right)\right\}$$
$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\beta > 0, \quad a > 0]$ ET I 318(11)

2. $$\int_0^\infty x^{\mu-1} e^{-\gamma x - \beta x^2} \cos ax\,dx$$
$$= \frac{1}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{\exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) + \exp\left(\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right)\right\}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0, \quad a > 0]$ ET I 16(18)

3. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\sin ax\,dx = \frac{i\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\exp\left[-\frac{(\gamma-ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right] - (\gamma+ia)\exp\left[-\frac{(\gamma+ia)^2}{4\beta}\right]\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\}$$
$[\operatorname{Re}\beta>0, \quad a>0]$ ET I 74(28)

4. $$\int_0^\infty xe^{-\gamma x-\beta x^2}\cos ax\,dx = -\frac{\sqrt{\pi}}{8\sqrt{\beta^3}}\left\{(\gamma-ia)\exp\frac{(\gamma-ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ia}{2\sqrt{\beta}}\right)\right] + (\gamma+ia)\exp\frac{(\gamma+ia)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma+ia}{2\sqrt{\beta}}\right)\right]\right\} + \frac{1}{2\beta}$$
$[\operatorname{Re}\beta>0, \quad a>0]$ ET I 16(17)

3.954

1.[11] $$\int_0^\infty e^{-\beta x^2}\sin ax\frac{x\,dx}{\gamma^2+x^2} = -\frac{\pi}{4}e^{\beta\gamma^2}\left[2\sinh a\gamma + e^{-\gamma a}\,\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a}\,\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 74(26)a

2.[11] $$\int_0^\infty e^{-\beta x^2}\cos ax\frac{dx}{\gamma^2+x^2} = \frac{\pi}{4\gamma}e^{\beta\gamma^2}\left[2\cosh a\gamma - e^{-\gamma a}\,\Phi\left(\gamma\sqrt{\beta}-\frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a}\,\Phi\left(\gamma\sqrt{\beta}+\frac{a}{2\sqrt{\beta}}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 15(15)

3.955 $$\int_0^\infty x^\nu e^{-\frac{x^2}{2}}\cos\left(\beta x-\nu\frac{\pi}{2}\right)dx = \sqrt{\frac{\pi}{2}}e^{-\frac{\beta^2}{4}}\,D_\nu(\beta)$$ $[\operatorname{Re}\nu>-1]$ EH II 120(4)

3.956 $$\int_0^\infty e^{-x^2}\left(2x\cos x-\sin x\right)\sin x\frac{dx}{x^2} = \sqrt{\pi}\frac{e-1}{2e}$$ BI (369)(19)

3.957

1. $$\int_0^\infty x^{\mu-1}\exp\left(\frac{-\beta^2}{4x}\right)\sin ax\,dx = \frac{i}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}}\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\frac{\pi i}{4}}\sqrt{a}\right) - \exp\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\pi i/4}\sqrt{a}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu<1, \quad a>0]$ ET I 318(12)

2. $$\int_0^\infty x^{\mu-1}\exp\left(\frac{-\beta^2}{4x}\right)\cos ax\,dx = \frac{1}{2^\mu}\beta^\mu a^{-\frac{\mu}{2}}\left[\exp\left(-\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{\pi i/4}\sqrt{a}\right) + \exp\left(\frac{i}{4}\mu\pi\right)K_\mu\left(\beta e^{-\pi i/4}\sqrt{a}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\mu<1, \quad a>0]$ ET I 320(32)a

3.958

1. $$\int_{-\infty}^\infty x^n e^{-\left(ax^2+bx+c\right)}\sin(px+q)\,dx = -\left(\frac{-1}{2a}\right)^n\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-p^2}{4a}-c\right)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{(n-2k)!k!}a^k \times\sum_{j=0}^{n-2k}\binom{n-2k}{j}b^{n-2k-j}p^j\sin\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$
$[a>0]$ GW (37)(1b)

2. $$\int_{-\infty}^{\infty} x^n e^{-\left(ax^2+bx+c\right)} \cos(px+q)\,dx = \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n!}{(n-2k)!k!} a^k \times \sum_{j=0}^{n-2k} \binom{n-2k}{j} p^j \cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$
$[a>0]$ GW (337)(1a)

3.959 $$\int_0^\infty xe^{-p^2x^2} \tan ax\,dx = \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2k^2}{p^2}\right)$$
$[p>0]$ BI (362)(15)

3.961

1. $$\int_0^\infty \exp\left(-\beta\sqrt{\gamma^2+x^2}\right) \sin ax \frac{x\,dx}{\sqrt{\gamma^2+x^2}} = \frac{a\gamma}{\sqrt{a^2+\beta^2}} K_1\left(\gamma\sqrt{a^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 75(36)

2. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^2+x^2}\right] \cos ax \frac{dx}{\sqrt{\gamma^2+x^2}} = K_0\left(\gamma\sqrt{a^2+\beta^2}\right)$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 17(27)

3.962

1. $$\int_0^\infty \frac{\sqrt{\sqrt{\gamma^2+x^2}-\gamma}\exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}} \sin ax\,dx = \sqrt{\frac{\pi}{2}} \frac{a\exp\left(-\gamma\sqrt{a^2+\beta^2}\right)}{\sqrt{\beta^2+a^2}\sqrt{\beta+\sqrt{a^2+\beta^2}}}$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 75(38)

2. $$\int_0^\infty \frac{x\exp\left(-\beta\sqrt{\gamma^2+x^2}\right)}{\sqrt{\gamma^2+x^2}\sqrt{\sqrt{\gamma^2+x^2}-\gamma}} \cos ax\,dx = \sqrt{\frac{\pi}{2}} \frac{\sqrt{\beta+\sqrt{a^2+\beta^2}}}{\sqrt{a^2+\beta^2}} \exp\left[-\gamma\sqrt{a^2+\beta^2}\right]$$
$[\operatorname{Re}\beta>0, \quad \operatorname{Re}\gamma>0, \quad a>0]$ ET I 17(29)

3.963

1. $$\int_0^\infty e^{-\tan^2 x} \frac{\sin x}{\cos^2 x} \frac{dx}{x} = \frac{\sqrt{\pi}}{2}$$ BI (391)(1)

2. $$\int_0^{\pi/2} e^{-p\tan x} \frac{x\,dx}{\cos^2 x} = \frac{1}{p}\left[\operatorname{ci}(p)\sin p - \cos p \operatorname{si}(p)\right]$$ $[p>0]$ (cf. **3.339**) BI (396)(3)

3.[8] $$\int_0^{\pi/2} xe^{-\tan^2 x} \sin 4x \frac{dx}{\cos^8 x} = -\frac{3}{2}\sqrt{\pi}$$ BI (396)(5)

4.[8] $$\int_0^{\pi/2} xe^{-\tan^2 x} \sin^3 2x \frac{dx}{\cos^8 x} = 2\sqrt{\pi}$$ BI (396)(6)

3.964

1. $$\int_0^{\pi/2} x e^{-p\tan x} \frac{p\sin x - \cos x}{\cos^3 x}\, dx = -\sin p \operatorname{si}(p) - \operatorname{ci}(p)\cos p$$
$[p > 0]$ LI (396)(4)

2. $$\int_0^{\pi/2} x e^{-p\tan^2 x} \frac{p - \cos^2 x}{\cos^4 x \cot x}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{p}}$$ $[p > 0]$ BI (396)(7)

3.[8] $$\int_0^{\pi/2} x e^{-p\tan^2 x} \frac{p - 2\cos^2 x}{\cos^6 x \cot x}\, dx = \frac{1+2p}{8p}\sqrt{\frac{\pi}{p}}$$ $[p > 0]$ BI (396)(8)

3.965

1. $$\int_0^\infty x e^{-\beta x} \sin ax^2 \sin \beta x\, dx = \frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}}$$ $\left[|\arg\beta| < \frac{\pi}{4}, \quad a > 0\right]$ ET I 84(17)

2. $$\int_0^\infty x e^{-\beta x} \cos ax^2 \cos \beta x\, dx = \frac{\beta}{4}\sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}}$$ $[a > 0, \quad \operatorname{Re}\beta > |\operatorname{Im}\beta|]$ ET 26(27)

3.966

1. $$\int_0^\infty x e^{-px} \cos\left(2x^2 + px\right)\, dx = 0$$ $[p > 0]$ BI (361)(16)

2. $$\int_0^\infty x e^{-px} \cos\left(2x^2 - px\right)\, dx = \frac{p\sqrt{\pi}}{8} \exp\left(-\frac{1}{4}p^2\right)$$ $[p > 0]$ BI (361)(17)

3. $$\int_0^\infty x^2 e^{-px} \left[\sin\left(2x^2 + px\right) + \cos\left(2x^2 + px\right)\right]\, dx = 0$$ $[p > 0]$ BI (361)(18)

4. $$\int_0^\infty x^2 e^{-px} \left[\sin\left(2x^2 - px\right) - \cos\left(2x^2 - px\right)\right]\, dx = \frac{\sqrt{\pi}}{16}\left(2 - p^2\right) \exp\left(-\frac{1}{4}p^2\right)$$ BI (361)(19)

5.[3] $$\int_0^\infty x^{\mu-1} e^{-x} \cos\left(x + ax^2\right)\, dx = \frac{e^{\frac{1}{4a}}\,\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \cos\frac{\mu\pi}{4}\, D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$$
$[\operatorname{Re}\mu > 0, \quad a > 0]$ ET I 321(37)

6.[6] $$\int_0^\infty x^{\mu-1} e^{-x} \sin\left(x + ax^2\right)\, dx = \frac{e^{\frac{1}{4a}}\,\Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \sin\frac{\mu\pi}{4}\, D_{-\mu}\left(\frac{1}{\sqrt{a}}\right)$$
$[\operatorname{Re}\mu > -1, \quad a > 0]$ ET I 319(18)

3.967

1. $$\int_0^\infty e^{-\frac{\beta^2}{x^2}} \sin a^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \sin\left(\sqrt{2}a\beta\right)$$ $[\operatorname{Re}\beta > 0, \quad a > 0]$
ET I 75(30)a, BI(369)(3)a

2. $$\int_0^\infty e^{-\frac{\beta^2}{x^2}} \cos a^2x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \cos\left(\sqrt{2}a\beta\right)$$ $[\operatorname{Re}\beta > 0, \quad a > 0]$
BI (369)(4), ET I 16(20)

3. $$\int_0^\infty x^2 e^{-\beta x^2} \cos ax^2\, dx = \frac{\sqrt{\pi}}{4\sqrt[4]{\left(a^2+\beta^2\right)^3}} \cos\left(\frac{3}{2}\arctan\frac{a}{\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ ET I 14(3)a

3.968

1. $$\int_0^\infty e^{-\beta x^2} \sin ax^4\, dx = -\frac{\pi}{8}\sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\cos\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8} + Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\sin\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8}\right]$$

$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 75(34)

2. $$\int_0^\infty e^{-\beta x^2} \cos ax^4\, dx = \frac{\pi}{8}\sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\sin\left(\frac{\beta^2}{8a} + \frac{\pi}{8}\right) - Y_{\frac{1}{4}}\left(\frac{\beta^2}{8a}\right)\cos\left(\frac{\beta^2}{8a}\right) + \frac{\pi}{8}\right]$$

$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 16(24)

3.969

1. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\cos\left(2pqx^3\right) + q\sin\left(2pqx^3\right)\right]\, dx = \frac{\sqrt{\pi}}{2}$$ BI (363)(7)

2. $$\int_0^\infty e^{-p^2x^4+q^2x^2}\left[2px\sin\left(2pqx^3\right) - q\cos\left(2pqx^3\right)\right]\, dx = 0$$ BI (363)(8)

3.971 **Notation**: In formulas **3.971** 1 and 2, $p \ge 0$, $q \ge 0$, $r = \sqrt[4]{a^2+p^2}$, $s = \sqrt[4]{b^2+q^2}$, $A = \arctan\frac{a}{p}$, and $B = \arctan\frac{b}{q}$.

1. $$\int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\sin\left(ax^2 + \frac{b}{x^2}\right)\frac{dx}{x^2} = \frac{1}{2}\int_{-\infty}^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\sin\left(ax^2 + \frac{b}{x^2}\right)\frac{dx}{x^2}$$
$$= \frac{\sqrt{\pi}}{2s}\exp\left[-2rs\cos(A+B)\right]\sin\left[A + 2rs\sin(A+B)\right]$$

BI (369)(16, 17)

2. $$\int_0^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\cos\left(ax^2 + \frac{b}{x^2}\right)\frac{dx}{x^2} = \frac{1}{2}\int_{-\infty}^\infty \exp\left(-px^2 - \frac{q}{x^2}\right)\cos\left(ax^2 + \frac{b}{x^2}\right)\frac{dx}{x^2}$$
$$= \frac{\sqrt{\pi}}{2s}\exp\left[-2rs\cos(A+B)\right]\cos\left[A + 2rs\sin(A+B)\right]$$

BI (369)(15, 18)

3.972

1. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^4+x^4}\right]\sin ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}}$$
$$= \sqrt{\frac{a\pi}{8}}\, I_{1/4}\left[\frac{\gamma^2}{2}\left(\sqrt{\beta^2+a^2} - \beta\right)\right] K_{1/4}\left[\frac{\gamma^2}{4}\left(\sqrt{\beta^2+a^2} + \beta\right)\right]$$

$\left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right]$ ET I 75(37)

2. $$\int_0^\infty \exp\left[-\beta\sqrt{\gamma^4+x^4}\right]\cos ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}}$$
$$= \sqrt{\frac{a\pi}{8}}\, I_{-1/4}\left[\frac{\gamma^2}{2}\left(\sqrt{\beta^2+a^2} - \beta\right)\right] K_{1/4}\left[\frac{\gamma^2}{4}\left(\sqrt{\beta^2+a^2} + \beta\right)\right]$$

$\left[\operatorname{Re}\beta > 0, \quad |\arg\gamma| < \frac{\pi}{4}, \quad a > 0\right]$ ET I 17(28)

3.973

1. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\,\frac{dx}{x} = \frac{\pi}{2}\left(e^p - 1\right) \qquad [p > 0, \quad a > 0]$$ WH, FI II 725

2. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax + bx)\,\frac{x\,dx}{c^2+x^2} = \frac{\pi}{2}\exp\left(-cb + pe^{-ac}\right)$$ $[a > 0, \quad b > 0, \quad c > 0, \quad p > 0]$ BI (372)(3)

3. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax + bx)\,\frac{dx}{c^2+x^2} = \frac{\pi}{2c}\exp\left(-cb + pe^{-ac}\right)$$ $[a > 0, \quad b > 0, \quad c > 0, \quad p > 0]$ BI (372)(4)

4. $$\int_0^\infty \exp(p\cos x)\sin(p\sin x + nx)\,\frac{dx}{x} = \frac{\pi}{2}e^p \qquad [p > 0]$$ BI (366)(2)

5. $$\int_0^\infty \exp(p\cos x)\sin(p\sin x)\cos nx\frac{dx}{x} = \frac{p^n}{n!}\cdot\frac{\pi}{4} + \frac{\pi}{2}\sum_{k=n+1}^{\infty}\frac{p^k}{k!}$$ $[p > 0]$ LI (366)(3)

6. $$\int_0^\infty \exp(p\cos x)\cos(p\sin x)\sin nx\frac{dx}{x} = \frac{\pi}{2}\sum_{k=0}^{n-1}\frac{p^k}{k!} + \frac{p^n}{n!}\frac{\pi}{4}$$ $[p > 0]$ LI (366)(4)

3.974

1. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab}\right)\right]}{2b\sinh ab}$$ $[a > 0, \quad b > 0, \quad p > 0]$ BI (391)(4)

2. $$\int_0^\infty \left[1 - \exp(p\cos ax)\cos(p\sin ax)\right]\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab}\right)\right]}{2\sinh ab}$$ $[a > 0, \quad b > 0, \quad p > 0]$ BI (391)(5)

3. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax + ax)\operatorname{cosec} ax\frac{dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab} - ab\right)\right]}{2b\sinh ab}$$ $[a > 0, \quad b > 0, \quad p > 0]$ BI (391)(6)

4. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax + ax)\operatorname{cosec} ax\frac{x\,dx}{b^2+x^2} = \frac{\pi\left[e^p - \exp\left(pe^{-ab} - ab\right)\right]}{2\sinh ab}$$ $[a > 0, \quad b > 0, \quad p > 0]$ BI (391)(7)

5. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\,\frac{x\,dx}{b^2-x^2} = \frac{\pi}{2}\left[1 - \exp(p\cos ab)\cos(p\sin ab)\right]$$ $[p > 0, \quad a > 0]$ BI (378)(1)

6. $$\int_0^\infty \exp(p\cos ax)\cos(p\sin ax)\,\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\exp(p\cos ab)\sin(p\sin ab)$$ $[a > 0, \quad b > 0, \quad p > 0]$ BI (378)(2)

7. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\tan ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\cdot\tanh ab\left[\exp\left(pe^{-ab}\right)-e^p\right]$$
$[a>0, \quad b>0, \quad p>0]$ BI (372)(14)

8. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\cot ax\frac{dx}{b^2+x^2} = \frac{\pi}{2b}\coth ab\left[e^p-\exp\left(pe^{-ab}\right)\right]$$
$[a>0, \quad b>0, \quad p>0]$ BI (372)(15)

9. $$\int_0^\infty \exp(p\cos ax)\sin(p\sin ax)\operatorname{cosec} ax\frac{dx}{b^2-x^2} = \frac{\pi}{2b}\operatorname{cosec} ab\left[e^p-\exp(p\cos ab)\cos(p\sin ab)\right]$$
$[a>0, \quad b>0, \quad p>0]$ BI (391)(12)

10. $$\int_0^\infty [1-\exp(p\cos ax)\cos(p\sin ax)]\operatorname{cosec} ax\frac{x\,dx}{b^2-x^2} = -\frac{\pi}{2}\exp(p\cos ab)\sin(p\sin ab)\operatorname{cosec} ab$$
$[a>0, \quad b>0, \quad p>0]$ BI (391)(13)

3.975

1. $$\int_0^\infty \frac{\sin\left(\beta\arctan\frac{x}{\gamma}\right)}{\left(\gamma^2+x^2\right)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}-1} = \frac{1}{2}\zeta(\beta,\gamma)-\frac{1}{4\gamma^\beta}-\frac{\gamma^{1-\beta}}{2(\beta-1)}$$
$[\operatorname{Re}\beta>1, \quad \operatorname{Re}\gamma>0]$ WH, ET I 26(7)

2. $$\int_0^\infty \frac{\sin(\beta\arctan x)}{\left(1+x^2\right)^{\frac{\beta}{2}}}\cdot\frac{dx}{e^{2\pi x}+1} = \frac{1}{2(\beta-1)}-\frac{\zeta(\beta)}{2^\beta}$$
$[\operatorname{Re}\beta>1]$ EH I 33(13)

3.976 $$\int_0^\infty \left(1+x^2\right)^{\beta-\frac{1}{2}}e^{-px^2}\cos\left[2px+(2\beta-1)\arctan x\right]dx = \frac{e^{-p}}{2p^\beta}\sin\pi\beta\,\Gamma(\beta)$$
$[\operatorname{Re}\beta>0, \quad p>0]$ WH

3.98–3.99 Combinations of trigonometric and hyperbolic functions

3.981

1. $$\int_0^\infty \frac{\sin ax}{\sinh\beta x}\,dx = \frac{\pi}{2\beta}\tanh\frac{a\pi}{2\beta}$$
$[\operatorname{Re}\beta>0, \quad a>0]$ BI (264)(16)

2. $$\int_0^\infty \frac{\sin ax}{\cosh\beta x}\,dx = -\frac{\pi}{2\beta}\tanh\frac{a\pi}{2\beta}-\frac{i}{2\beta}\left[\psi\left(\frac{\beta+ai}{4\beta}\right)-\psi\left(\frac{\beta-ai}{4\beta}\right)\right]$$
$[\operatorname{Re}\beta>0, \quad a>0]$
GW (335)(12), ET I 88(1)

3. $$\int_0^\infty \frac{\cos ax}{\cosh\beta x}\,dx = \frac{\pi}{2\beta}\operatorname{sech}\frac{a\pi}{2\beta}$$
$[\operatorname{Re}\beta>0, \quad \text{all real } a]$ BI (264)(14)

4. $$\int_0^\infty \sin ax\frac{\sinh\beta x}{\sinh\gamma x}\,dx = \frac{\pi}{2\gamma}\frac{\sinh\frac{a\pi}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}+\frac{i}{2\gamma}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right)-\psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right]$$
$[|\operatorname{Re}\beta|<\operatorname{Re}\gamma, \quad a>0]$ ET I 88(5)

5. $$\int_0^\infty \cos ax\frac{\sinh\beta x}{\sinh\gamma x}\,dx = \frac{\pi}{2\gamma}\frac{\sin\frac{\pi\beta}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$
$[|\operatorname{Re}\beta|<\operatorname{Re}\gamma]$ BI (265)(7)

6. $$\int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh \gamma x}\,dx = \frac{\pi}{\gamma}\frac{\sin\frac{\beta\pi}{2\gamma}\sinh\frac{a\pi}{2\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ BI (265)(2)

7. $$\int_0^\infty \cos ax \frac{\sinh \beta x}{\cosh \gamma x}\,dx = \frac{1}{4\gamma}\left[\left\{\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta-ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ia}{4\gamma}\right)\right\}\right.$$
$$\left.-\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)+\frac{2\pi\sin\frac{\pi\beta}{\gamma}}{\cos\frac{\pi\beta}{\gamma}+\cosh\frac{\pi a}{\gamma}}\right]$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ ET I 31(13)

8. $$\int_0^\infty \sin ax \frac{\cosh \beta x}{\sinh \gamma x}\,dx = \frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{\pi a}{\gamma}}{\cosh\frac{\pi a}{\gamma}+\cos\frac{\pi\beta}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ BI (265)(4)

9. $$\int_0^\infty \sin ax \frac{\cosh \beta x}{\cosh \gamma x}\,dx = \frac{i}{4\gamma}\left[\psi\left(\frac{3\gamma+\beta+ia}{4\gamma}\right)-\psi\left(\frac{3\gamma+\beta-ai}{4\gamma}\right)+\psi\left(\frac{3\gamma-\beta+ia}{4\gamma}\right)\right.$$
$$\left.-\psi\left(\frac{3\gamma-\beta-ai}{4\gamma}\right)-\frac{2\pi i\sinh\frac{\pi a}{\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}\right]$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a>0]$ ET I 88(6)

10. $$\int_0^\infty \cos ax \frac{\cosh \beta x}{\cosh \gamma x}\,dx = \frac{\pi}{\gamma}\frac{\cos\frac{\beta\pi}{2\gamma}\cosh\frac{a\pi}{2\gamma}}{\cosh\frac{a\pi}{\gamma}+\cos\frac{\beta\pi}{\gamma}}$$ $[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad \text{all real } a]$ BI (265)(6)

11.[11] $$\int_0^{\pi/2} \cos^{2m} x \cosh \beta x\,dx = \frac{(2m)!\sinh\frac{\pi\beta}{2}}{\beta\left(\beta^2+2^2\right)\ldots\left[\beta^2+(2m)^2\right]}$$
$[\beta \neq 0]$ WA 620a

12.[11] $$\int_0^{\pi/2} \cos^{2m+1} x \cosh \beta x\,dx = \frac{(2m+1)!\cosh\frac{\pi\beta}{2}}{\left(\beta^2+1^2\right)\left(\beta^2+3^2\right)\ldots\left[\beta^2+(2m+1)^2\right]}$$ WA 620a

3.982

1. $$\int_0^\infty \frac{\cos ax}{\cosh^2 \beta x}\,dx = \frac{a\pi}{2\beta^2\sinh\frac{a\pi}{2\beta}}$$ $[\operatorname{Re}\beta > 0, \quad a>0]$ BI (264)(16)

2. $$\int_0^\infty \sin ax \frac{\sinh \beta x}{\cosh^2 \gamma x}\,dx = \frac{\pi\left(a\sin\frac{\beta\pi}{2\gamma}\cosh\frac{a\pi}{2\gamma}-\beta\cos\frac{\beta\pi}{2\gamma}\sinh\frac{a\pi}{2\gamma}\right)}{\gamma^2\left(\cosh\frac{a\pi}{\gamma}-\cos\frac{\beta\pi}{\gamma}\right)}$$
$[|\operatorname{Re}\beta| < 2\operatorname{Re}\gamma, \quad a>0]$ ET I 88(9)

3.[11] $$\int_0^\infty \frac{\sin^2 x\cos ax}{\sinh^2 hx}\,dx = \frac{\pi}{4}\left\{\frac{a+2}{1-e^{-\pi(a+2)}}-\frac{2a}{1-e^{-\pi a}}+\frac{a-2}{1-e^{-\pi(a-2)}}\right\} = I(a)$$
$$\left[I(0)=\frac{1}{2}\left(\pi\coth\pi-1\right), \quad I(\pm2)=\frac{1}{4}+\frac{\pi}{2}\left(\coth 2\pi-\coth\pi\right)\right]$$

3.983

1.[6] $$\int_0^\infty \frac{\cos ax\,dx}{b\cosh\beta x + c} = \frac{\pi\sin\left(\frac{a}{\beta}\operatorname{arccosh}\frac{c}{b}\right)}{\beta\sqrt{c^2-b^2}\sinh\frac{a\pi}{\beta}}$$ $[c > b > 0]$

$$= \frac{\pi\sinh\left(\frac{a}{\beta}\arccos\frac{c}{b}\right)}{\beta\sqrt{b^2-c^2}\sinh\frac{a\pi}{\beta}}$$ $[b > |c| > 0]$

$[\operatorname{Re}\beta > 0, \quad a > 0]$ GW (335)(13a)

2. $$\int_0^\infty \frac{\cos ax\,dx}{\cosh\beta x + \cos\gamma} = \frac{\pi}{\beta}\frac{\sinh\frac{a\gamma}{\beta}}{\sin\gamma\sinh\frac{a\pi}{\beta}}$$ $\left[\pi\operatorname{Re}\beta < \operatorname{Im}\overline{\beta}\gamma, \quad a > 0\right]$ BI (267)(3)

3.[3] $$\int_0^\infty \frac{\cos ax\,dx}{\cosh x - \cosh b} = -\pi\coth a\pi\frac{\sin ab}{\sinh b}$$ $[a > 0, \quad b > 0]$ ET I 30(8)

4. $$\int_0^\infty \frac{\cos ax\,dx}{1 + 2\cosh\left(\sqrt{\frac{2}{3}\pi}x\right)} = \frac{\sqrt{\frac{\pi}{2}}}{1 + 2\cosh\left(\sqrt{\frac{2}{3}\pi}a\right)}$$ $[a > 0]$ ET I 30(9)

5. $$\int_0^\infty \frac{\sin ax\sinh\beta x}{\cosh\gamma x + \cos\delta}\,dx = \frac{\pi\left\{\sin\left[\frac{\beta}{\gamma}(\pi-\delta)\right]\sinh\left[\frac{a}{\gamma}(\pi+\delta)\right] - \sin\left[\frac{\beta}{\gamma}(\pi+\delta)\right]\sinh\left[\frac{a}{\gamma}(\pi-\delta)\right]\right\}}{\gamma\sin\delta\left(\cosh\frac{2\pi a}{\gamma} - \cos\frac{2\pi\beta}{\gamma}\right)}$$

$[\pi\operatorname{Re}\gamma > |\operatorname{Re}\overline{\gamma}\delta|, \quad |\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a > 0]$ BI (267)(2)

6. $$\int_0^\infty \frac{\cos ax\cosh\beta x}{\cosh\gamma x + \cos b}\,dx = \frac{\pi\left\{\cos\left[\frac{\beta}{\gamma}(\pi-b)\right]\cosh\left[\frac{a}{\gamma}(\pi+b)\right] - \cos\left[\frac{\beta}{\gamma}(\pi+b)\right]\cosh\left[\frac{a}{\gamma}(\pi-b)\right]\right\}}{\gamma\sin b\left(\cosh\frac{2\pi a}{\gamma} - \cos\frac{2\pi\beta}{\gamma}\right)}$$

$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad 0 < b < \pi, \quad a < 0]$
BI (267)(6)

7. $$\int_0^\infty \frac{\cos ax\,dx}{\left(\beta + \sqrt{\beta^2-1}\cosh x\right)^{\nu+1}} = \Gamma(\nu+1-ai)e^{a\pi}\frac{Q_\nu^{ai}(\beta)}{\Gamma(\nu+1)}$$

$[\operatorname{Re}\nu > -1, \quad |\arg(\beta+1)| < \pi, \quad a > 0]$
ET I 30(10)

3.984

1.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\sin ax\sinh cx}{\cosh x + \cos b}\,dx = \pi\frac{\cosh ab}{\sinh a\pi}$$ $[|b| \le \pi, \quad a \text{ real}]$ BI (267)(1)

2.[6] $$\lim_{c\uparrow 1}\int_0^\infty \frac{\cos ax\cosh cx}{\cosh x + \cos b}\,dx = -\pi\cot b\frac{\sinh ab}{\sinh a\pi}$$ $[0 < |b| < \pi, \quad a \text{ real}]$ BI (267)(5)

3.[8] $$\int_0^\infty \frac{\sin ax\sinh\frac{x}{2}}{\cosh x + \cos\beta}\,dx = \frac{\pi\sinh a\beta}{2\sin\frac{\beta}{2}\cosh a\pi}$$ $[Re\beta < \pi, \quad a > 0]$ ET I 80(10)

4. $$\int_0^\infty \frac{\cos ax\cosh\frac{\beta}{2}x}{\cosh\beta x + \cosh\gamma}\,dx = \frac{\pi\cos\frac{a\gamma}{\beta}}{2\beta\cosh\frac{\gamma}{2}\cosh\frac{a\pi}{\beta}}$$ $\left[\pi\operatorname{Re}\beta > \left|\operatorname{Im}\left(\overline{\beta}\gamma\right)\right|\right]$ ET I 31(16)

5. $$\int_0^\infty \frac{\sin ax\sinh\beta x}{\cosh 2\beta x + \cos 2ax}\,dx = \frac{a\pi}{4\left(a^2+\beta^2\right)}$$ $[a > 0, \quad \operatorname{Re}\beta > 0]$ BI (267)(7)

6. $$\int_0^\infty \frac{\cos ax \cosh \beta x}{\cosh 2\beta x + \cos 2ax}\,dx = \frac{\beta\pi}{4\left(a^2+\beta^2\right)}$$ [Re $\beta > 0$, $a > 0$] BI (267)(8)

7.[8] $$\int_0^\infty \frac{\sinh^{2\mu-1} x \cosh^{2\varrho-2\nu+1} x}{\left(\cosh^2 x - \beta \sinh^2 x\right)^{\varrho}}\,dx = \frac{1}{2}\,\mathrm{B}(\mu, \nu-\mu)\ {}_2F_1(\varrho, \mu; \nu; \beta)$$ [Re $\nu >$ Re $\mu > 0$] EH I 115(12)

3.985

1. $$\int_0^\infty \frac{\cos ax\,dx}{\cosh^\nu \beta x} = \frac{2^{\nu-2}}{\beta\,\Gamma(\nu)}\,\Gamma\left(\frac{\nu}{2}+\frac{ai}{2\beta}\right)\Gamma\left(\frac{\nu}{2}-\frac{ai}{2\beta}\right)$$ [Re $\beta > 0$, Re $\nu > 0$, $a > 0$] ET I 30(5)

2. $$\int_0^\infty \frac{\cos ax\,dx}{\cosh^{2n}\beta x} = \frac{4^{n-1}\pi a}{2(2n-1)!\beta^2 \sinh\frac{a\pi}{2\beta}} \prod_{k=1}^{n-1}\left(\frac{a^2}{4\beta^2}+k^2\right) = \frac{\pi a\left(a^2+2^2\beta^2\right)\left(a^2+4^2\beta^2\right)\cdots\left[a^2+(2n-2)^2\beta^2\right]}{2(2n-1)!\beta^{2n}\sinh\frac{a\pi}{2\beta}}$$ [$n \geq 2$, $a > 0$] ET I 30(3)

3. $$\int_0^\infty \frac{\cos ax\,dx}{\cosh^{2n+1}\beta x} = \frac{\pi 2^{2n-1}}{(2n)!\beta\cosh\frac{a\pi}{2\beta}} \prod_{k=1}^{n}\left[\frac{a^2}{4\beta^2}+\left(\frac{2k-1}{2}\right)^2\right] = \frac{\pi\left(a^2+\beta^2\right)\left(a^2+3^2\beta^2\right)\cdots\left[a^2+(2n-1)^2\beta^2\right]}{2(2n)!\beta^{2n+1}\cosh\frac{a\pi}{2\beta}}$$ [Re $\beta > 0$, $n = 0, 1, \ldots,$ all real a] ET I 30(4)

3.986

1. $$\int_0^\infty \frac{\sin\beta x \sin\gamma x}{\cosh\delta x}\,dx = \frac{\pi}{\delta}\cdot\frac{\sinh\frac{\beta\pi}{2\delta}\sinh\frac{\gamma\pi}{2\delta}}{\cosh\frac{\beta}{\delta}\pi + \cosh\frac{\gamma}{\delta}\pi}$$ [$|\mathrm{Im}(\beta+\gamma)| <$ Re δ] BI (264)(19)

2. $$\int_0^\infty \frac{\sin\alpha x \cos\beta x}{\sinh\gamma x}\,dx = \frac{\pi\sinh\frac{\pi\alpha}{\gamma}}{2\gamma\left(\cosh\frac{\alpha\pi}{\gamma}+\cosh\frac{\beta\pi}{\gamma}\right)}$$ [$|\mathrm{Im}(\alpha+\beta)| <$ Re γ] LI (264)(20)

3. $$\int_0^\infty \frac{\cos\beta x \cos\gamma x}{\cosh\delta x}\,dx = \frac{\pi}{\delta}\cdot\frac{\cosh\frac{\beta\pi}{2\delta}\cosh\frac{\gamma\pi}{2\delta}}{\cosh\frac{\beta\pi}{\delta}+\cosh\frac{\gamma\pi}{\delta}}$$ [$|\mathrm{Im}(\beta+\gamma)| <$ Re δ] BI (264)(21)

4.[3] $$\int_0^\infty \frac{\sin^2\beta x}{\sinh^2\pi x}\,dx = \frac{\beta}{\pi\left(e^{2\beta}-1\right)} + \frac{\beta-1}{2\pi} = \frac{\beta\coth\beta - 1}{2\pi}$$ [$|\mathrm{Im}\,\beta| < \pi$] EH I 44(3)

3.987

1. $$\int_0^\infty \sin ax\left(1-\tanh\beta x\right)\,dx = \frac{1}{a} - \frac{\pi}{2\beta\sinh\frac{a\pi}{2\beta}}$$ [Re $\beta > 0$] ET I 88(4)a

2. $$\int_0^\infty \sin ax\left(\coth\beta x - 1\right)\,dx = \frac{\pi}{2\beta}\coth\frac{a\pi}{2\beta} - \frac{1}{a}$$ [Re $\beta > 0$] ET I 88(3)

3.988

1. $$\int_0^{\pi/2} \frac{\cos ax \sinh(2b\cos x)}{\sqrt{\cos x}}\,dx = \frac{\pi}{2}\sqrt{\pi b}\, I_{\frac{a}{2}+\frac{1}{4}}(b)\, I_{-\frac{a}{2}+\frac{1}{4}}(b)$$ $[a > 0]$ ET I 37(66)

2. $$\int_0^{\pi/2} \frac{\cos ax \cosh(2b\cos x)}{\sqrt{\cos x}}\,dx = \frac{\pi}{2}\sqrt{\pi b}\, I_{\frac{a}{2}-\frac{1}{4}}(b)\, I_{-\frac{a}{2}-\frac{1}{4}}(b)$$ $[a > 0]$ ET I 37(67)

3. $$\int_0^{\infty} \frac{\cos ax\,dx}{\sqrt{\cosh x \cos b}} = \frac{\pi\, P_{-\frac{1}{2}+ia}(\cos b)}{\sqrt{2}\cosh a\pi}$$ $[a > 0, \quad b > 0]$ ET I 30(7)

3.989

1. $$\int_0^{\infty} \frac{\sin\frac{a^2x^2}{\pi}\sin bx}{\sinh ax}\,dx = \frac{\pi}{2a}\sin\frac{\pi b^2}{4a^2}\operatorname{cosech}\frac{\pi b}{2a}$$ $[a > 0, \quad b > 0]$ ET I 93(44)

2. $$\int_0^{\infty} \frac{\cos\frac{a^2x^2}{\pi}\sin bx}{\sinh ax}\,dx = \frac{\pi}{2a}\,\frac{\cosh\frac{\pi b}{a} - \cos\frac{\pi b^2}{4a^2}}{\sinh\frac{\pi b}{2a}}$$ $[a > 0, \quad b > 0]$ ET I 93(45)

3. $$\int_0^{\infty} \frac{\sin\frac{x^2}{\pi}\cos ax}{\cosh x}\,dx = \frac{\pi}{2}\,\frac{\cos\frac{a^2\pi}{4} - \frac{1}{\sqrt{2}}}{\cosh\frac{a\pi}{2}}$$ ET I 36(54)

4. $$\int_0^{\infty} \frac{\cos\frac{x^2}{\pi}\cos ax}{\cosh x}\,dx = \frac{\pi}{2}\cdot\frac{\sin\frac{a^2\pi}{4} + \frac{1}{\sqrt{2}}}{\cosh\frac{a\pi}{2}}$$ ET I 36(55)

5. $$\int_0^{\infty} \frac{\sin\left(\pi a x^2\right)\cos bx}{\cosh \pi x}\,dx = -\sum_{k=0}^{\infty}\exp\left[-\left(k+\tfrac{1}{2}\right)b\right]\sin\left[\left(k+\tfrac{1}{2}\right)^2\pi a\right] + \frac{1}{\sqrt{a}}\sum_{k=0}^{\infty}\exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\sin\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right]$$ $[a > 0, \quad b > 0]$ ET I 36(56)

6. $$\int_0^{\infty} \frac{\cos\left(\pi a x^2\right)\cos bx}{\cosh \pi x}\,dx = \sum_{k=0}^{\infty}(-1)^k\exp\left[-\left(k+\frac{1}{2}\right)b\right]\cos\left[\left(k+\frac{1}{2}\right)^2\pi a\right] + \frac{1}{\sqrt{a}}\sum_{k=0}^{\infty}\exp\left[-\frac{b\left(k+\frac{1}{2}\right)}{a}\right]\cos\left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k+\frac{1}{2}\right)^2\pi}{a}\right]$$ $[a > 0, \quad b > 0]$ ET I 36(57)

3.991

1. $$\int_0^{\infty} \sin\pi x^2 \sin ax \coth \pi x\,dx = \frac{1}{2}\tanh\frac{a}{2}\sin\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)$$ ET I 93(42)

2.[11] $$\int_0^{\infty} \cos\pi x^2 \sin ax \coth \pi x\,dx = \frac{1}{2}\tanh\frac{a}{2}\left[1 - \cos\left(\frac{\pi}{4} + \frac{a^2}{4\pi}\right)\right]$$ ET I 93(43)

3.992

1. $$\int_0^\infty \frac{\sin \pi x^2 \cos ax}{1 + 2\cosh\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = -\sqrt{3} + \frac{\cos\left(\frac{\pi}{12} - \frac{a^2}{4\pi}\right)}{4\cosh\frac{a}{\sqrt{3}} - 2}$$ ET I 37(60)

2. $$\int_0^\infty \frac{\cos \pi x^2 \cos ax}{1 + 2\cosh\left(\frac{2}{\sqrt{3}}\pi x\right)}\,dx = 1 - \frac{\sin\left(\frac{\pi}{12} - \frac{a^2}{4\pi}\right)}{4\cosh\frac{a}{\sqrt{3}} - 2}$$ ET I 37(61)

3.993 $$\int_0^\infty \frac{\sin^2 x + \cos x^2}{\cosh\left(\sqrt{\pi}x\right)}\cos ax\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\sin^2 a + \cos a^2}{\cosh\left(\sqrt{\pi}a\right)}$$ ET I 37(58)

3.994

1. $$\int_0^\infty \frac{\sin(2a\cosh x)\cos bx}{\sqrt{\cosh x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{\frac{1}{4}+\frac{ib}{2}}(a)\,Y_{\frac{1}{4}-\frac{ib}{2}}(a) + J_{\frac{1}{4}-\frac{ib}{2}}(a)\,Y_{\frac{1}{4}+\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 37(62)

2. $$\int_0^\infty \frac{\cos(2a\cosh x)\cos bx}{\sqrt{\cosh x}}\,dx = -\frac{\pi}{4}\sqrt{a\pi}\left[J_{-\frac{1}{4}+\frac{ib}{2}}(a)\,Y_{-\frac{1}{4}-\frac{ib}{2}}(a) + J_{-\frac{1}{4}-\frac{ib}{2}}(a)\,Y_{-\frac{1}{4}+\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 37(63)

3. $$\int_0^\infty \frac{\sin(2a\sinh x)\sin bx}{\sqrt{\sinh x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{\frac{1}{4}+\frac{ib}{2}}(a)\,K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 93(47)

4. $$\int_0^\infty \frac{\cos(2a\sinh x)\sin bx}{\sqrt{\sinh x}}\,dx = -\frac{i}{2}\sqrt{\pi a}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{-\frac{1}{4}+\frac{ib}{2}}(a)\,K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 93(48)

5. $$\int_0^\infty \frac{\sin(2a\sinh x)\cos bx}{\sqrt{\sinh x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{\frac{1}{4}-\frac{ib}{2}}(a)\,K_{\frac{1}{4}+\frac{ib}{2}}(a) + I_{\frac{1}{4}+\frac{ib}{2}}(a)\,K_{\frac{1}{4}-\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 37(64)

6. $$\int_0^\infty \frac{\cos(2a\sinh x)\cos bx}{\sqrt{\sinh x}}\,dx = \frac{\sqrt{\pi a}}{2}\left[I_{-\frac{1}{4}-\frac{ib}{2}}(a)\,K_{-\frac{1}{4}+\frac{ib}{2}}(a) + I_{-\frac{1}{4}+\frac{ib}{2}}(a)\,K_{-\frac{1}{4}-\frac{ib}{2}}(a)\right]$$ $[a > 0, \quad b > 0]$ ET I 37(65)

7. $$\int_0^\infty \sin(a\cosh x)\sin(a\sinh x)\,\frac{dx}{\sinh x} = \frac{\pi}{2}\sin a$$ $[a > 0]$ BI (264)(22)

3.995

1. $$\int_0^{\pi/2} \frac{\sin\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\sin\frac{2ac}{b+c}$$ $[b > 0, \quad c > 0]$ BI (273)(9)

2. $$\int_0^{\pi/2} \frac{\cos\left(2a\cos^2 x\right)\cosh(a\sin 2x)}{b^2\cos^2 x + c^2\sin^2 x}\,dx = \frac{\pi}{2bc}\cos\frac{2ac}{b+c}$$ $[b > 0, \quad c > 0]$ BI (273)(10)

3.996

1. $$\int_0^\infty \sin(a\sinh x)\sinh\beta x\,dx = \sin\frac{\beta\pi}{2}\,K_\beta(a) \qquad [|\operatorname{Re}\beta| < 1, \quad a > 0]$$ EH II 82(26)

2. $$\int_0^\infty \cos(a\sinh x)\cosh\beta x\,dx = \cos\frac{\beta\pi}{2}\,K_\beta(a) \qquad [|\operatorname{Re}\beta| < 1, \quad a > 0]$$ WA 202(13)

3. $$\int_0^{\pi/2} \cos(a\sin x)\cosh(\beta\cos x)\;dx = \frac{\pi}{2}\,J_0\left(\sqrt{a^2-\beta^2}\right)$$ MO 40

4. $$\int_0^\infty \sin\left(a\cosh x - \tfrac{1}{2}\beta\pi\right)\cosh\beta x\,dx = \frac{\pi}{2}\,J_\beta(a) \qquad [|\operatorname{Re}\beta| < 1, \quad a > 0]$$ WA 199(12)

5. $$\int_0^\infty \cos\left(a\cosh x - \tfrac{1}{2}\beta\pi\right)\cosh\beta x\,dx = -\frac{\pi}{2}\,Y_\beta(a) \qquad [|\operatorname{Re}\beta| < 1, \quad a > 0]$$ WA 199(13)

3.997

1. $$\int_0^{\pi/2} \sin^\nu x\sinh(\beta\cos x)\;dx = \frac{\sqrt{\pi}}{2}\left(\frac{2}{\beta}\right)^{\frac{\nu}{2}}\Gamma\left(\frac{\nu+1}{2}\right)\mathbf{L}_{\frac{\nu}{2}}(\beta)$$ $[\operatorname{Re}\nu > -1]$ EH II 38(53)

2. $$\int_0^{\pi} \sin^\nu x\cosh(\beta\cos x)\;dx = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\frac{\nu}{2}}\Gamma\left(\frac{\nu+1}{2}\right)I_{\frac{\nu}{2}}(\beta)$$ $[\operatorname{Re}\nu > -1]$ WH

3. $$\int_0^{\pi/2} \frac{dx}{\cosh(\tan x)\cos x\sqrt{\sin 2x}} = \sqrt{2\pi}\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{2k+1}}$$ BI (276)(13)

4. $$\int_0^{\pi/2} \frac{\tan^q x}{\cosh(\tan x)+\cos\lambda}\,\frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin\lambda}\sum_{k=1}^\infty (-1)^{k-1}\frac{\sin k\lambda}{k^q}$$ $[q > 0]$ BI (275)(20)

4.11–4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.111

1. $$\int_0^\infty \frac{\sin ax}{\sinh \beta x} \cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\tanh \frac{a\pi}{2\beta}\right)$$
[$\operatorname{Re}\beta > 0$] (cf. **3.981** 1)
GW (336)(17a)

2. $$\int_0^\infty \frac{\cos ax}{\sinh \beta x} \cdot x^{2m+1}\, dx = (-1)^m \frac{\pi}{2\beta} \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\tanh \frac{a\pi}{2\beta}\right)$$
[$\operatorname{Re}\beta > 0$] (cf. **3.981** 1)
GW (336)(17b)

3. $$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \cdot x^{2m+1}\, dx = (-1)^{m+1} \frac{\pi}{2\beta} \cdot \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}}\right)$$
[$\operatorname{Re}\beta > 0$] (cf. **3.981** 3)
GW (336)(18b)

4. $$\int_0^\infty \frac{\cos ax}{\cosh \beta x} \cdot x^{2m}\, dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\frac{1}{\cosh \frac{a\pi}{2\beta}}\right)$$
[$\operatorname{Re}\beta > 0$] (cf. **3.981** 3)
GW (336)(18a)

5. $$\int_0^\infty x \frac{\sin 2ax}{\cosh \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\sinh \frac{a\pi}{\beta}}{\cosh^2 \dfrac{a\pi}{\beta}}$$
[$\operatorname{Re}\beta > 0, \quad a > 0$] BI (364)(6)a

6. $$\int_0^\infty x \frac{\cos 2ax}{\sinh \beta x}\, dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\cosh^2 \dfrac{a\pi}{\beta}}$$
[$\operatorname{Re}\beta > 0, \quad a > 0$] BI (364)(1)a

7. $$\int_0^\infty \frac{\sin ax}{\cosh \beta x} \frac{dx}{x} = 2 \arctan\left(\exp \frac{\pi a}{2\beta}\right) - \frac{\pi}{2}$$
[$\operatorname{Re}\beta > 0, \quad a > 0$]
BI (387)(1), ET I 89(13), LI (298)(17)

4.112

1. $$\int_0^\infty \left(x^2 + \beta^2\right) \frac{\cos ax}{\cosh \dfrac{\pi x}{2\beta}}\, dx = \frac{2\beta^3}{\cosh^3 a\beta}$$
[$\operatorname{Re}\beta > 0, \quad a > 0$] ET I 32(19)

2. $$\int_0^\infty x \left(x^2 + 4\beta^2\right) \frac{\cos ax}{\sinh \dfrac{\pi x}{2\beta}}\, dx = \frac{6\beta^4}{\cosh^4 a\beta}$$
[$\operatorname{Re}\beta > 0, \quad a > 0$] ET I 32(20)

4.113

1. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2+\beta^2} = -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta \sin \pi\beta} + \frac{1}{2\beta^2}\left[{}_2F_1\left(1,-\beta;1-\beta;-e^{-a}\right) + {}_2F_1\left(1,\beta;1+\beta:-e^{-a}\right)\right]$$
$$= \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta\sin\pi\beta} - \sum_{k=1}^\infty \frac{(-1)^k e^{-ak}}{k^2-\beta^2}$$
$[\operatorname{Re}\beta > 0, \quad \beta \neq 0,1,2,\dots, \quad a > 0]$ ET I 90(18)

2. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x} \cdot \frac{dx}{x^2+m^2} = \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m}\sum_{k=1}^{m-1} \frac{(-1)^k e^{-ka}}{m-k} + \frac{(-1)^m e^{-ma}}{2m} \ln\left(1+e^{-a}\right) + \frac{1}{2m!}\frac{d^{m-1}}{dz^{m-1}}\left[\frac{(1+z)^{m-1}}{z}\ln(1+z)\right]_{z=e^{-a}}$$
$[a > 0]$ ET I 89(17)

3. $$\int_0^\infty \frac{\sin ax}{\sinh \pi x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin ax}{\sinh \pi x}\frac{dx}{1+x^2} = -\frac{a}{2}\cosh a + \sinh a \ln\left(2\cosh\frac{a}{2}\right)$$ GW (336)(21b)

4. $$\int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin ax}{\sinh \frac{\pi}{2}x}\cdot\frac{dx}{1+x^2} = \frac{\pi}{2}\sinh a - \cosh a \arctan\left(\sinh a\right)$$
GW (336)(21a)

5. $$\int_0^\infty \frac{\sin ax}{\sinh \frac{\pi}{4}x}\cdot\frac{dx}{1+x^2} = -\frac{\pi}{\sqrt{2}}e^{-a} + \frac{\sinh a}{\sqrt{2}}\ln\frac{2\cosh a+\sqrt{2}}{2\cosh a-\sqrt{2}} + \sqrt{2}\cosh a \arctan\frac{\sqrt{2}}{2\sinh a}$$
$[a > 0]$ LI (389)(1)

6. $$\int_0^\infty \frac{\sin ax}{\cosh \frac{\pi}{4}x}\cdot\frac{x\,dx}{1+x^2} = \frac{\pi}{\sqrt{2}}e^{-a} + \frac{\sinh a}{\sqrt{2}}\ln\frac{2\cosh a+\sqrt{2}}{2\cosh a-\sqrt{2}} - \sqrt{2}\cosh a \arctan\left(\frac{1}{\sqrt{2}\sinh a}\right)$$
$[a > 0]$ BI (388)(1)

7. $$\int_0^\infty \frac{\cos ax}{\sinh \pi x}\cdot\frac{x\,dx}{1+x^2} = -\frac{1}{2} + \frac{a}{2}e^{-a} + \cosh a \ln\left(1+e^{-a}\right)$$
$[a > 0]$ BI (389)(14), ET I 32(24)

8. $$\int_0^\infty \frac{\cos ax}{\sinh \frac{\pi}{2}x}\cdot\frac{x\,dx}{1+x^2} = 2\sinh a \arctan\left(e^{-a}\right) + \frac{\pi}{2}e^{-a} - 1$$
$[a > 0]$ BI (389)(11)

9.[11] $$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\cdot\frac{dx}{x^2+\beta^2} = \frac{\pi e^{-a\beta}}{2\beta\cos(\beta\pi)} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{\left(k+\frac{1}{2}\right)^2-\beta^2}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 32(26)

10.[11] $$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\cdot\frac{dx}{x^2+\left(m+\frac{1}{2}\right)^2} = \frac{(-1)^m e^{-a\beta}\left(a\beta+\frac{1}{2}\right)}{2\beta^2} - \sum_{k=0}^\infty \frac{(-1)^k e^{-(k+1/2)a}}{\left(k+\frac{1}{2}\right)^2-\beta^2}$$
$[\operatorname{Re}\beta > 0, \quad a > 0]$ ET I 32(25)

11. $$\int_0^\infty \frac{\cos ax}{\cosh \pi x} \cdot \frac{dx}{1+x^2} = 2\cosh\frac{a}{2} - \left[e^a \arctan\left(e^{-\frac{a}{2}}\right) + e^{-a}\arctan\left(e^{\frac{a}{2}}\right)\right]$$
$[a > 0]$ ET I 32(21)

12. $$\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = ae^{-a} + \cosh a \ln\left(1 + e^{-2a}\right)$$
$[a > 0]$ BI (388)(6)

13. $$\int_0^\infty \frac{\cos ax}{\cosh \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}} e^{-a} + \frac{2\sinh a}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}\sinh a}\right) - \frac{\cosh a}{\sqrt{2}} \ln \frac{2\cosh a + \sqrt{2}}{2\cosh a - \sqrt{2}}$$
$[a > 0]$ BI (388)(5)

4.114

1. $$\int_0^\infty \frac{\sin ax}{x} \frac{\sinh \beta x}{\sinh \gamma x}\, dx = \arctan\left(\tan\frac{\beta\pi}{2\gamma} \tanh\frac{a\pi}{2\gamma}\right)$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma, \quad a > 0]$ BI (387)(6)a

2. $$\int_0^\infty \frac{\cos ax}{x} \frac{\sinh \beta x}{\cosh \gamma x}\, dx = \frac{1}{2} \ln \frac{\cosh \frac{a\pi}{2\gamma} + \sin \frac{\beta\pi}{2\gamma}}{\cosh \frac{a\pi}{2\gamma} - \sin \frac{\beta\pi}{2\gamma}}$$
$[|\operatorname{Re}\beta| < \operatorname{Re}\gamma]$ ET I 33(34)

4.115

1. $$\int_0^\infty \frac{x \sin ax}{x^2+b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x}\, dx = \frac{\pi}{2} \frac{e^{-ab}\sin b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{ke^{-ak}\sin k\beta}{k^2-b^2}$$
$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0]$
BI (389)(23)

2. $$\int_0^\infty \frac{x \sin ax}{x^2+1} \cdot \frac{\sinh \beta x}{\sinh \pi x}\, dx = \frac{1}{2} e^{-a}\left(a\sin\beta - \beta\cos\beta\right) - \frac{1}{2}\sinh a \sin\beta \ln\left[1 + 2e^{-a}\cos\beta + e^{-2a}\right] + \cosh a \cos\beta \arctan\frac{\sin\beta}{e^a + \cos\beta}$$
$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$ LI (389)(10)

3. $$\int_0^\infty \frac{x \sin ax}{x^2+1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2} x}\, dx = \frac{\pi}{2} e^{-a}\sin\beta + \frac{1}{2}\cos\beta \sinh a \ln\frac{\cosh a + \sin\beta}{\cosh a - \sin\beta} - \sin\beta\cosh a \arctan\left(\frac{\cos\beta}{\sinh a}\right)$$
$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0\right]$ BI (389)(8)

4. $$\int_0^\infty \frac{\cos ax}{x^2+b^2} \cdot \frac{\sinh \beta x}{\sinh \pi x}\, dx = \frac{\pi}{2b} \cdot \frac{e^{-ab}\sin b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{e^{-ak}\sin k\beta}{k^2-b^2}$$
$[0 < \operatorname{Re}\beta < \pi, \quad a > 0, \quad b > 0]$
BI (389)(22)

5. $$\int_0^\infty \frac{\cos ax}{x^2+1} \cdot \frac{\sinh \beta x}{\sinh \pi x}\, dx = \frac{1}{2} e^{-a}\left(a\sin\beta - \beta\cos\beta\right) + \frac{1}{2}\cosh a \sin\beta \ln\left(1 + 2e^{-a}\cos\beta + e^{-2a}\right) - \sinh a \cos\beta \arctan\frac{\sin\beta}{e^a + \cos\beta}$$
$[|\operatorname{Re}\beta| < \pi, \quad a > 0, \quad b > 0]$ BI (389)(20)a

6. $$\int_0^\infty \frac{\cos ax}{x^2+1} \cdot \frac{\sinh \beta x}{\sinh \frac{\pi}{2} x}\, dx = \frac{\pi}{2} e^{-a} \sin\beta - \frac{1}{2}\cosh a \cos\beta \ln \frac{\cosh a + \sin\beta}{\cosh a - \sin\beta} + \sinh a \sin\beta \arctan \frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0, \quad b > 0\right]$$
BI (389)(18)

7. $$\int_0^\infty \frac{\sin ax}{x^2+\frac{1}{4}} \cdot \frac{\sinh \beta x}{\cosh \pi x}\, dx = e^{-\frac{a}{2}} \left(a \sin\frac{\beta}{2} - \beta\cos\frac{\beta}{2}\right) - \sinh\frac{a}{2}\sin\frac{\beta}{2}\ln\left(1 + 2e^{-a}\cos\beta + e^{-2a}\right) + \cosh\frac{a}{2}\cos\frac{\beta}{2}\arctan\frac{\sin\beta}{1+e^{-a}\cos\beta}$$
$$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$$
ET I 91(26)

8. $$\int_0^\infty \frac{\sin ax}{x^2+\beta^2} \cdot \frac{\cosh \gamma x}{\sinh \pi x}\, dx = \frac{1}{2\beta^2} - \frac{\pi}{2\beta} \cdot \frac{e^{-a\beta}\cos\beta\gamma}{\sin\beta\pi} + \sum_{k=1}^\infty (-1)^{k-1} \frac{e^{-ak}\cos k\gamma}{k^2-\beta^2}$$
$$[0 \le \operatorname{Re}\beta, \quad |\operatorname{Re}\gamma| < \pi, \quad a > 0]$$
BI (389)(21)

9. $$\int_0^\infty \frac{\sin ax}{x^2+1} \cdot \frac{\cosh \beta x}{\sinh \pi x}\, dx = -\frac{1}{2} e^{-a}\left(a\cos\beta + \beta\sin\beta\right) + \frac{1}{2}\sinh a \cos\beta \ln\left(1 + 2e^{-a}\cos\beta + e^{-2a}\right) + \cosh a \sin\beta \arctan\frac{\sin\beta}{e^a + \cos\beta}$$
$$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$$
ET I 91(25), LI (389)(9)

10. $$\int_0^\infty \frac{\sin ax}{x^2+1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x}\, dx = -\frac{\pi}{2} e^{-a}\cos\beta + \frac{1}{2}\sinh a \sin\beta \ln\frac{\cosh a + \sin\beta}{\cosh a - \sin\beta} + \cosh a \cos\beta \arctan\frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0\right]$$
BI (389)(7)

11. $$\int_0^\infty \frac{x\cos ax}{x^2+b^2} \cdot \frac{\cosh \beta x}{\sinh \pi x}\, dx = \frac{\pi}{2} \cdot \frac{e^{-ab}\cos b\beta}{\sin b\pi} + \sum_{k=1}^\infty (-1)^k \frac{ke^{-ak}\cos k\beta}{k^2-b^2}$$
$$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$$
BI (389)(24)

12. $$\int_0^\infty \frac{x\cos ax}{x^2+1} \cdot \frac{\cosh \beta x}{\sinh \pi x}\, dx = \frac{1}{2} e^{-a}\left(a\cos\beta + \beta\sin\beta\right) - \frac{1}{2} + \frac{1}{2}\cosh a \cos\beta \ln\left[1 + 2e^{-a}\cos\beta + e^{-2a}\right] + \sinh a \sin\beta \arctan\frac{\sin\beta}{e^a + \cos\beta}$$
$$[|\operatorname{Re}\beta| < \pi, \quad a > 0]$$
BI (389)(19)

13. $$\int_0^\infty \frac{x\cos ax}{x^2+1} \cdot \frac{\cosh \beta x}{\sinh \frac{\pi}{2} x}\, dx = -1 + \frac{\pi}{2} e^{-a}\cos\beta + \frac{1}{2}\cosh a \sin\beta \ln\frac{\cosh a + \sin\beta}{\cosh a - \sin\beta} + \sinh a \cos\beta \arctan\frac{\cos\beta}{\sinh a}$$
$$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, \quad a > 0\right]$$
BI (389)(17)

14. $$\int_0^\infty \frac{\cos ax}{x^2+1} \cdot \frac{\cosh \beta x}{\cosh \frac{\pi}{2} x}\, dx = ae^{-a} \cos\beta + \beta e^{-a} \sin\beta + \sinh a \sin\beta \arctan \frac{e^{-2a}\sin 2\beta}{1+e^{-2a}\cos 2\beta} + \frac{1}{2}\cosh a \cos\beta \ln\left(1 + 2e^{-2a}\cos 2\beta + e^{-4a}\right)$$
$\left[|\operatorname{Re}\beta| < \frac{\pi}{2}, a > 0\right]$ ET I 34(37)

4.116

1.[6] $$\int_0^\infty x \cos 2ax \tanh x\, dx$$ the integral is divergent BI (364)(2)

2. $$\int_0^\infty \cos ax \tanh \beta x \frac{dx}{x} = \ln\coth\frac{a\pi}{4\beta}$$ $[\operatorname{Re}\beta > 0, \quad a > 0]$ BI (387)(8)

4.117

1. $$\int_0^\infty \frac{\sin ax}{1+x^2} \tanh\frac{\pi x}{2}\, dx = a\cosh a - \sinh a \ln(2\sinh a)$$
$[a > 0]$ BI (388)(3)

2. $$\int_0^\infty \frac{\sin ax}{1+x^2} \tanh\frac{\pi x}{4}\, dx = -\frac{\pi}{2}e^{a} + \sinh a \ln\coth\frac{a}{2} + 2\cosh a \arctan\left(e^{a}\right)$$ BI (388)(4)

3. $$\int_0^\infty \frac{\sin ax}{1+x^2} \coth \pi x\, dx = \frac{a}{2}e^{-a} - \sinh a \ln\left(1 - e^{-a}\right)$$ $[a > 0]$ BI (389)(5)

4. $$\int_0^\infty \frac{\sin ax}{1+x^2} \coth\frac{\pi}{2}x\, dx = \sinh a \ln\coth\frac{a}{2}$$ $[a > 0]$ BI (389)(6)

5. $$\int_0^\infty \frac{x\cos ax}{1+x^2} \tanh\frac{\pi}{2}x\, dx = -ae^{-a} - \cosh a \ln\left(1 - e^{-2a}\right)$$
$[a > 0]$ BI (388)(7)

6. $$\int_0^\infty \frac{x\cos ax}{1+x^2} \tanh\frac{\pi}{4}x\, dx - \frac{\pi}{2}e^{a} + \cosh a \ln\coth\frac{a}{2} + 2\sinh a \arctan\left(e^{a}\right)$$
$[a > 0]$ BI (388)(8)

7. $$\int_0^\infty \frac{x\cos ax}{1+x^2} \coth \pi x\, dx = -\frac{a}{2}e^{-a} - \frac{1}{2} - \cosh a \ln\left(1 - e^{-a}\right)$$ BI (389)(15)a, ET I 33(31)a

8. $$\int_0^\infty \frac{x\cos ax}{1+x^2} \coth\frac{\pi}{2}x\, dx = -1 + \cosh a \ln\coth\frac{a}{2}$$ $[a > 0]$ BI (389)(12)

9. $$\int_0^\infty \frac{x\cos ax}{1+x^2} \coth\frac{\pi}{4}x\, dx = -2 + \frac{\pi}{2}e^{-a} + \cosh a \ln\coth\frac{a}{2} + 2\sinh a \arctan\left(e^{-a}\right)$$
$[a > 0]$ BI (389)(13)

4.118[8] $$\int_0^\infty \frac{x\sin ax}{\cosh^2 x}\, dx = \frac{\pi}{2}\frac{1}{\sinh\frac{1}{2}\pi a}\left(\frac{1}{2}\pi a \coth\frac{1}{2}\pi a - 1\right)$$ ET I 89(14)

4.119 $$\int_0^\infty \frac{1 - \cos px}{\sinh qx} \cdot \frac{dx}{x} = \ln\left(\cosh\frac{p\pi}{2q}\right)$$ BI (387)(2)a

4.121

1. $$\int_0^\infty \frac{\sin ax - \sin bx}{\cosh \beta x} \cdot \frac{dx}{x} = 2 \arctan \frac{\exp \frac{a\pi}{2\beta} - \exp \frac{b\pi}{2\beta}}{1 + \exp \frac{(a+b)\pi}{2\beta}}$$
$[\operatorname{Re} \beta > 0]$ GW (336)(19b)

2. $$\int_0^\infty \frac{\cos ax - \cos bx}{\sinh \beta x} \cdot \frac{dx}{x} = \ln \frac{\cosh \frac{b\pi}{2\beta}}{\cosh \frac{a\pi}{2\beta}}$$
$[\operatorname{Re} \beta > 0]$ GW (336)(19a)

4.122

1.[6] $$\int_0^\infty \frac{\cos \beta x \sin \gamma x}{\cosh \delta x} \cdot \frac{dx}{x} = \arctan \frac{\sinh \frac{\gamma\pi}{2\delta}}{\cosh \frac{\beta\pi}{2\delta}}$$
$[\operatorname{Re} \delta > |\operatorname{Im} \beta| + |\operatorname{Im} \gamma|]$ ET I 93(46)a

2. $$\int_0^\infty \sin^2 ax \frac{\cosh \beta x}{\sinh x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\cosh 2a\pi + \cos \beta\pi}{1 + \cos \beta\pi}$$
$[|\operatorname{Re} \beta| < 1]$ BI (387)(7)

4.123

1. $$\int_0^\infty \frac{\sin x}{\cosh ax + \cos x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \arctan \frac{1}{a} - \frac{1}{a}$$
BI (390)(1)

2. $$\int_0^\infty \frac{\sin x}{\cosh ax - \cos x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{a}{1+a^2} - \arctan \frac{1}{a}$$
BI (390)(2)

3. $$\int_0^\infty \frac{\sin 2x}{\cosh 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1 + 2a^2}{1 + a^2} - \arctan \frac{1}{a}$$
BI (390)(4)

4. $$\int_0^\infty \frac{\cosh ax \sin x}{\cosh 2ax - \cos 2x} \cdot \frac{x\,dx}{x^2 - \pi^2} = \frac{-1}{2a\left(1 + a^2\right)}$$
LI (390)(3)

5. $$\int_0^\infty \frac{\cos ax}{\cosh \pi x + \cos \pi\beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma\left(\cos \gamma\pi + \cos \beta\pi\right)} + \frac{1}{\sinh \beta\pi} \sum_{k=0}^\infty \left\{ \frac{e^{-(2k+1-\beta)a}}{\gamma^2 - (2k+1-\beta)^2} - \frac{e^{-(2k+1+\beta)a}}{\gamma^2 - (2k+1+\beta)^2} \right\}$$
$[0 < \operatorname{Re} \beta < 1, \quad \operatorname{Re} \gamma > 0, \quad a > 0]$ ET I 33(27)

6. $$\int_0^\infty \frac{\sin ax \sinh bx}{\cos 2ax + \cosh 2bx} x^{p-1}\,dx = \frac{\Gamma(p)}{\left(a^2 + b^2\right)^{\frac{p}{2}}} \sin\left(p \arctan \frac{a}{b}\right) \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^p}$$
$[p > 0]$ BI (364)(8)

7. $$\int_0^\infty \sin ax^2 \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi x}{2}}{\cos \pi x + \cosh \pi x} \cdot x\,dx = \frac{1}{4} \left[\frac{\partial\, \vartheta_1(z \mid q)}{\partial z}\right]_{z=0, q=e^{-2a}}$$
$[a > 0]$ ET I 93(49)

4.124

1. $$\int_0^1 \frac{\cos px \cosh\left(q\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} J_0\left(\sqrt{p^2-q^2}\right)$$ MO (40)

2. $$\int_u^\infty \cos ax \cosh\sqrt{\beta\left(u^2-x^2\right)} \cdot \frac{dx}{\sqrt{u^2-x^2}} = \frac{\pi}{2} J_0\left(\frac{u}{\sqrt{a^2-\beta^2}}\right)$$ ET I 34(38)

4.125

1. $$\int_0^\infty \sinh(a\sin x)\cos(a\cos x)\sin x \sin 2nx \frac{dx}{x} = \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\frac{\pi}{8}\left[1+\frac{a^2}{2n(2n+1)}\right]$$ LI (367)(14)

2. $$\int_0^\infty \cosh(a\sin x)\cos(a\cos x)\sin x \cos(2n-1)x \frac{dx}{x} = \frac{(-1)^{n-1}a^{2(n-1)}}{[2(n-1)]!}\frac{\pi}{8}\left[1-\frac{a^2}{2n(2n-1)}\right]$$ LI (367)(15)

3. $$\int_0^\infty \sinh(a\sin x)\cos(a\cos x)\cos x \cos 2nx \frac{dx}{x} = \frac{\pi}{2}\sum_{k=n+1}^\infty \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1}}{(2n+1)!}\frac{3\pi}{8} + \frac{(-1)^{n-1}a^{2n-1}}{(2n-1)!}\frac{\pi}{8}$$ LI (367)(21)

4.126

1. $$\int_0^\infty \sin(a\cos bx)\sinh(a\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}\left[\cos(a\cos bc)\cosh(a\sin bc)-1\right]$$ $[b>0]$ BI (381)(2)

2. $$\int_0^\infty \sin(a\cos bx)\cosh(a\sin bx)\frac{dx}{c^2-x^2} = \frac{\pi}{2c}\cos(a\cos bc)\sinh(a\sin bc)$$ $[b>0, \quad c>0]$ BI (381)(1)

3. $$\int_0^\infty \cos(a\cos bx)\sinh(a\sin bx)\frac{x\,dx}{c^2-x^2} = \frac{\pi}{2}\left[a\cos bc - \sin(a\cos bc)\cosh(a\sin bc)\right]$$ $[b>0]$ BI (381)(4)

4. $$\int_0^\infty \cos(a\cos bx)\cosh(a\sin bx)\frac{dx}{c^2-x^2} = -\frac{\pi}{2c}\sin(a\cos bc)\sinh(a\sin bc)$$ $[b>0]$ BI (381)(3)

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

1. $$\int_0^\infty \sin ax \sinh^\nu \gamma x e^{-\beta x}\,dx = -\frac{i\,\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu+ai}{2\gamma}+1\right)}\right\}$$ $[\operatorname{Re}\nu > -2, \quad \operatorname{Re}\gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re}\beta]$ ET I 91(30)a

2. $$\int_0^\infty \cos ax \sinh^\nu \gamma x e^{-\beta x}\,dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma}\left\{\frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\gamma\nu-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)}\right\}$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\gamma > 0, \quad |\operatorname{Re}(\gamma\nu)| < \operatorname{Re}\beta]$ ET I 34(40)a

3. $$\int_0^\infty e^{-\beta x}\frac{\sin ax}{\sinh \gamma x}\,dx = \sum_{k=1}^\infty \frac{2a}{a^2+[\beta+(2k-1)\gamma]^2}$$ BI (264)(9)a
$$= \frac{1}{2\gamma i}\left[\psi\left(\frac{\beta+\gamma+ia}{2\gamma}\right) - \psi\left(\frac{\beta+\gamma-ia}{2\gamma}\right)\right]$$
$[\operatorname{Re}\beta > |\operatorname{Re}\gamma|]$ ET I 91(28)

4. $$\int_0^\infty e^{-x}\frac{\sin ax}{\sinh x}\,dx = \frac{\pi}{2}\coth\frac{a\pi}{2} - \frac{1}{a}$$ ET I 91(29)

4.132

1. $$\int_0^\infty \frac{\sin ax \sinh \beta x}{e^{\gamma x}-1}\,dx = -\frac{a}{2\left(a^2+\beta^2\right)} + \frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{2\pi a}{\gamma}}{\cosh\frac{2\pi a}{\gamma} - \cos\frac{2\pi\beta}{\gamma}} + \frac{i}{2\gamma}\left[\psi\left(\frac{\beta}{\gamma}+i\frac{a}{\gamma}+1\right) - \psi\left(\frac{\beta}{\gamma}-i\frac{a}{\gamma}+1\right)\right]$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|, a > 0]$ ET I 92(33)

2. $$\int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x}-1}\,dx = -\frac{a}{2\left(a^2+\beta^2\right)} + \frac{\pi}{2\gamma}\cdot\frac{\sinh\frac{2\pi a}{\gamma}}{\cosh\frac{2\pi a}{\gamma} - \cos\frac{2\pi\beta}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ BI (265)(5)a, ET I 92(34)

3. $$\int_0^\infty \frac{\sin ax \cosh \beta x}{e^{\gamma x}+1}\,dx = \frac{a}{2\left(a^2+\beta^2\right)} - \frac{\pi}{\gamma}\cdot\frac{\sinh\frac{a\pi}{\gamma}\cos\frac{\beta\pi}{\gamma}}{\cosh\frac{2a\pi}{\gamma} - \cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ ET I 92(35)

4. $$\int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x}-1}\,dx = \frac{\beta}{2\left(a^2+\beta^2\right)} - \frac{\pi}{2\gamma}\cdot\frac{\sin\frac{2\pi\beta}{\gamma}}{\cosh\frac{2a\pi}{\gamma} - \cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ LI (265)(8)

5. $$\int_0^\infty \frac{\cos ax \sinh \beta x}{e^{\gamma x}+1}\,dx = -\frac{\beta}{2\left(a^2+\beta^2\right)} + \frac{\pi}{\gamma}\frac{\sin\frac{\pi\beta}{\gamma}\cosh\frac{\pi a}{\gamma}}{\cosh\frac{2a\pi}{\gamma} - \cos\frac{2\beta\pi}{\gamma}}$$
$[\operatorname{Re}\gamma > |\operatorname{Re}\beta|]$ ET I 34(39)

4.133

1.[11] $$\int_0^\infty \sin ax \sinh \beta x \exp\left(-\frac{x^2}{4\gamma}\right)\,dx = \sqrt{\pi\gamma}\exp\left[\gamma\left(\beta^2-a^2\right)\right]\sin(2a\beta\gamma)$$
$[\operatorname{Re}\gamma > 0]$ ET I 92(37)

2.[11] $$\int_0^\infty \cos ax \cosh \beta x \exp\left(-\frac{x^2}{4\gamma}\right)\,dx = \sqrt{\pi\gamma}\exp\left[\gamma\left(\beta^2-a^2\right)\right]\cos(2a\beta\gamma)$$
$[\operatorname{Re}\gamma > 0]$ ET I 35(41)

4.134

1. $$\int_0^\infty e^{-\beta x^2} (\cosh x - \cos x)\, dx = \sqrt{\frac{\pi}{\beta}} \cosh \frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0] \qquad \text{ME 24}$$

2. $$\int_0^\infty e^{-\beta x^2} (\cosh x - \cos x)\, dx = \sqrt{\frac{\pi}{\beta}} \sinh \frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0] \qquad \text{ME 24}$$

4.135

1. $$\int_0^\infty \sin ax^2 \cosh 2\gamma x e^{-\beta x^2}\, dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2+\beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right) \sin\left(\frac{a\gamma^2}{a^2+\beta^2} + \frac{1}{2}\arctan\frac{a}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ LI (268)(7)

2. $$\int_0^\infty \cos ax^2 \cosh 2\gamma x e^{-\beta x^2}\, dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2+\beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2+\beta^2}\right) \cos\left(\frac{a\gamma^2}{a^2+\beta^2} + \frac{1}{2}\arctan\frac{a}{\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ LI (268)(8)

4.136

1. $$\int_0^\infty \left(\sinh^2 x + \sin x^2\right) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh\frac{1}{8\beta}$$
$[\operatorname{Re}\beta > 0]$ ME 24

2. $$\int_0^\infty \left(\sinh^2 x - \sin x^2\right) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh\frac{1}{8\beta}$$
$[\operatorname{Re}\beta > 0]$ ME 24

3. $$\int_0^\infty \left(\cosh^2 x + \cos x^2\right) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \cosh\frac{1}{8\beta}$$
$[\operatorname{Re}\beta > 0]$ ME 24

4. $$\int_0^\infty \left(\cosh^2 x - \cos x^2\right) e^{-\beta x^4}\, dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \sinh\frac{1}{8\beta}$$
$[\operatorname{Re}\beta > 0]$ ME 24

4.137

1. $$\int_0^\infty \sin 2x^2 \sinh 2x^2 e^{-\beta x^4}\, dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} + \frac{\pi}{4}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty \sin 2x^2 \cosh 2x^2 e^{-\beta x^4}\, dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{\pi}{4}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

3. $$\int_0^\infty \cos 2x^2 \sinh 2x^2 e^{-\beta x^4}\, dx = \frac{-\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{\pi}{4}\right)$$
$[\operatorname{Re}\beta > 0]$ MI 32

4. $$\int_0^\infty \cos 2x^2 \cosh 2x^2 e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} + \frac{\pi}{4}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

4.138

1. $$\int_0^\infty \left(\sin^2 2x \cosh 2x^2 + \cos 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty \left(\sin^2 2x \cosh 2x^2 - \cos 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

3. $$\int_0^\infty \left(\cos^2 2x \cosh 2x^2 + \sin 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

4. $$\int_0^\infty \left(\cos^2 2x \cosh 2x^2 - \sin 2x^2 \sinh 2x^2\right) e^{-\beta x^4}\,dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.141

1. $$\int_0^\infty x e^{-\beta x^2} \cosh x \sin x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}} \left(\cos\frac{1}{2\beta} + \sin\frac{1}{2\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty x e^{-\beta x^2} \sinh x \cos x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}} \left(\cos\frac{1}{2\beta} - \sin\frac{1}{2\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

3. $$\int_0^\infty x^2 e^{-\beta x^2} \cosh x \cos x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}} \left(\cos\frac{1}{2\beta} - \frac{1}{\beta}\sin\frac{1}{2\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

4. $$\int_0^\infty x^2 e^{-\beta x^2} \sinh x \sin x\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}} \left(\sin\frac{1}{2\beta} + \frac{1}{\beta}\cos\frac{1}{2\beta}\right)$$

$[\operatorname{Re}\beta > 0]$ MI 32

4.142

1. $$\int_0^\infty xe^{-\beta x^2} (\sinh x + \sin x)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \cosh\frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0]$$ ME 24

2. $$\int_0^\infty xe^{-\beta x^2} (\sinh x - \sin x)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \sinh\frac{1}{4\beta} \qquad [\operatorname{Re}\beta > 0]$$ ME 24

3. $$\int_0^\infty x^2e^{-\beta x^2} (\cosh x + \cos x)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \left(\cosh\frac{1}{4\beta} + \frac{1}{2\beta}\sinh\frac{1}{4\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ ME 24

4. $$\int_0^\infty x^2e^{-\beta x^2} (\cosh x - \cos x)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} \left(\sinh\frac{1}{4\beta} + \frac{1}{2\beta}\cosh\frac{1}{4\beta}\right)$$
$[\operatorname{Re}\beta > 0]$ ME 24

4.143

1. $$\int_0^\infty xe^{-\beta x^2} (\cosh x \sin x + \sinh x \cos x)\, dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}} \cos\frac{1}{2\beta}$$
$[\operatorname{Re}\beta > 0]$ MI 32

2. $$\int_0^\infty xe^{-\beta x^2} (\cosh x \sin x - \sinh x \cos x)\, dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}} \sin\frac{1}{2\beta}$$
$[\operatorname{Re}\beta > 0]$ MI 32

4.144 $$\int_0^\infty e^{-x^2} \sinh x^2 \cos ax \frac{dx}{x^2} = \sqrt{\frac{\pi}{2}} e^{-\frac{a^2}{8}} - \frac{\pi a}{4}\left[1 - \Phi\left(\frac{a}{\sqrt{8}}\right)\right]$$
$[a > 0]$ ET I 35(44)

4.145

1. $$\int_0^\infty xe^{-\beta x^2} \cosh(2ax\sin t)\sin(2ax\cos t)\, dx = \frac{a}{2}\sqrt{\frac{\pi}{\beta^3}} \exp\left(-\frac{a^2}{\beta}\cos 2t\right)\cos\left(t - \frac{a^2}{\beta}\sin 2t\right)$$
$[\operatorname{Re}\beta > 0]$ BI (363)(5)

2. $$\int_0^\infty xe^{-\beta x^2} \sinh(2ax\sin t)\cos(2ax\cos t)\, dx = \frac{a}{2}\sqrt{\frac{\pi}{\beta^3}} \exp\left(-\frac{a^2}{\beta}\cos 2t\right)\sin\left(t - \frac{a^2}{\beta}\sin 2t\right)$$
$[\operatorname{Re}\beta > 0]$ BI (363)(6)

4.146[10]

1.[8] $$\int_0^\infty e^{-\beta x^2} \sinh ax \sin bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\left(\frac{a^2 - b^2}{4\beta}\right)\sin\frac{ab}{2\beta}$$
$[\operatorname{Re}\beta > 0]$

2.[8] $$\int_0^\infty e^{-\beta x^2} \cosh ax \cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\left(\frac{a^2 - b^2}{4\beta}\right)\cos\frac{ab}{2\beta}$$
$[\operatorname{Re}\beta > 0]$

3. $$\int_0^\infty xe^{-\beta x^2} \cosh ax \sin ax \, dx = \frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta} + \sin\frac{a^2}{2\beta}\right)$$ $[\operatorname{Re}\beta > 0]$

4. $$\int_0^\infty xe^{-\beta x^2} \sinh ax \cos ax \, dx = \frac{a}{4\beta}\sqrt{\frac{\pi}{\beta}}\left(\cos\frac{a^2}{2\beta} - \sin\frac{a^2}{2\beta}\right)$$ $[\operatorname{Re}\beta > 0]$

5.[8] $$\int_0^\infty x^2 e^{-\beta x^2} \cosh ax \sin ax \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\sin\frac{a^2}{2\beta} + \frac{a^2}{\beta}\cos\frac{a^2}{2\beta}\right)$$ $[\operatorname{Re}\beta > 0]$

6.[8] $$\int_0^\infty x^2 e^{-\beta x^2} \cosh ax \cos ax \, dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta^3}}\left(\cos\frac{a^2}{2\beta} - \frac{a^2}{\beta}\sin\frac{a^2}{2\beta}\right)$$ $[\operatorname{Re}\beta > 0]$

4.2–4.4 Logarithmic Functions

4.21 Logarithmic functions

4.211

1. $$\int_e^\infty \frac{dx}{\ln\frac{1}{x}} = -\infty$$ BI (33)(9)

2. $$\int_0^u \frac{dx}{\ln x} = \operatorname{li} u$$ FI III 653, FI II 606

4.212

1.[7] $$\int_0^1 \frac{dx}{a + \ln x} = e^{-a}\operatorname{Ei}(a)$$ $[a > 0]$ BI (31)(4)

2. $$\int_0^1 \frac{dx}{a - \ln x} = -e^{a}\operatorname{Ei}(-a)$$ $[a > 0]$ BI (31)(5)

3.[7] $$\int_0^1 \frac{dx}{(a + \ln x)^2} = -\frac{1}{a} + e^{-a}\operatorname{Ei}(a)$$ $[a \geq 0]$ BI (31)(14)

4. $$\int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^{a}\operatorname{Ei}(-a)$$ $[a > 0]$ BI (31)(16)

5.[8] $$\int_0^1 \frac{\ln x \, dx}{(a + \ln x)^2} = 1 + (1-a)e^{-a}\operatorname{Ei}(a)$$ $[a \geq 0]$ BI (31)(15)

6. $$\int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1+a)e^{a}\operatorname{Ei}(-a)$$ $[a > 0]$ BI (31)(17)

7. $$\int_1^e \frac{\ln x \, dx}{(1 + \ln x)^2} = \frac{e}{2} - 1$$ BI (33)(10)

8.[7] $$\int_0^1 \frac{dx}{(a+\ln x)^n} = \frac{1}{(n-1)!} e^{-a} \operatorname{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! a^{k-n}$$
$[a \geq 0]$ BI (31))(22)

9. $$\int_0^1 \frac{dx}{(a-\ln x)^n} = \frac{(-1)^n}{(n-1)!} e^{a} \operatorname{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)!(-a)^{k-n}$$
$[a > 0, \quad n \text{ odd}]$ BI (31)(23)

In integrals of the form $\int \frac{(\ln x)^m}{[a^n + (\ln x)^n]^l} dx$, it is convenient to make the substitution $x = e^{-t}$.

Results **4.212** 3, **4.212** 5, and **4.212** 8 [for $n > 1$] and **4.213** 6, **4.213** 8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z)\,dz}{(z-z_0)^n} = \frac{1}{(n-1)!} \left(\frac{d}{dz_0}\right)^{n-1} \left[\mathrm{PV} \int_0^a \frac{f(z)}{z-z_0}\,dz\right]$$

where $a > z_0 > 0, n = 1, 2, 3, \ldots$ and PV indicates the Cauchy principal value.

4.213

1. $$\int_0^1 \frac{dx}{a^2 + (\ln x)^2} = \frac{1}{a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a]$$
$[a > 0]$ BI (31)(6)

2.[7] $$\int_0^1 \frac{dx}{a^2 - (\ln x)^2} = \frac{1}{2a} \left[e^{-a} \overline{\operatorname{Ei}}(a) - e^{a} \operatorname{Ei}(-a)\right]$$
$[a > 0]$, (cf. **4.212** 1 and 2) BI (31)(8)

3. $$\int_0^1 \frac{\ln x\,dx}{a^2 + (\ln x)^2} = \operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a$$
$[a > 0]$ BI (31)(7)

4.[7] $$\int_0^1 \frac{\ln x\,dx}{a^2 - (\ln x)^2} = -\frac{1}{2} \left[e^{-a} \overline{\operatorname{Ei}}(a) + e^{a} \operatorname{Ei}(-a)\right]$$
$[a > 0]$, (cf. **4.212** 1 and 2) BI (31)(9)

5. $$\int_0^1 \frac{dx}{\left[a^2 + (\ln x)^2\right]^2} = \frac{1}{2a^3} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2} [\operatorname{ci}(a) \cos a + \operatorname{si}(a) \sin a]$$
$[a > 0]$ LI (31)(18)

6.[8] $$\int_0^1 \frac{dx}{\left[a^2 - (\ln x)^2\right]^2}$$ is divergent

7. $$\int_0^1 \frac{\ln x\,dx}{\left[a^2 + (\ln x)^2\right]^2} = \frac{1}{2a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2}$$
$[a > 0]$ BI (31)(19)

8.[8] $$\int_0^1 \frac{\ln x\,dx}{\left[a^2 - (\ln x)^2\right]^2}$$ is divergent

4.214

1. $$\int_0^1 \frac{dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^3}\left[e^a \operatorname{Ei}(-a) - e^{-a}\overline{\operatorname{Ei}}(a) - 2\operatorname{ci}(a)\sin a + 2\operatorname{si}(a)\cos a\right]$$
$[a > 0]$ BI (31)(10)

2. $$\int_0^1 \frac{\ln x\, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^2}\left[e^a \operatorname{Ei}(-a) + e^{-a}\overline{\operatorname{Ei}}(a) - 2\operatorname{ci}(a)\cos a - 2\operatorname{si}(a)\sin a\right]$$
$[a > 0]$ BI (31)(11)

3. $$\int_0^1 \frac{(\ln x)^2\, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a}\left[e^a \operatorname{Ei}(-a) - e^{-a}\overline{\operatorname{Ei}}(a) + 2\operatorname{ci}(a)\sin a - 2\operatorname{si}(a)\cos a\right]$$
$[a > 0]$ BI (31)(12)

4.[7] $$\int_0^1 \frac{(\ln x)^3\, dx}{a^4 - (\ln x)^4} = -\frac{1}{4}\left[e^a \operatorname{Ei}(-a) + e^{-a}\overline{\operatorname{Ei}}(a) + 2\operatorname{ci}(a)\cos a + 2\operatorname{si}(a)\sin a\right]$$
$[a > 0]$ BI (31)(13)

4.215

1. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} dx = \Gamma(\mu) \qquad [\operatorname{Re}\mu > 0]$$ FI II 778

2. $$\int_0^1 \frac{dx}{\left(\ln\frac{1}{x}\right)^{\mu}} = \frac{\pi}{\Gamma(\mu)}\operatorname{cosec}\mu\pi \qquad [\operatorname{Re}\mu < 1]$$ BI (31)(1)

3. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\, dx = \frac{\sqrt{\pi}}{2}$$ BI (32)(1)

4. $$\int_0^1 \frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}$$ BI (32)(3)

4.216

1. $$\int_0^{1/e} \frac{dx}{\sqrt{(\ln x)^2 - 1}} = K_0(1)$$ GW (32)(2)

2.* $$\int_0^{1/e} \frac{dx}{\sqrt{-\ln x - 1}} = \frac{\sqrt{\pi}}{e}$$

4.22 Logarithms of more complicated arguments

4.221

1. $$\int_0^1 \ln x \ln(1-x)\, dx = 2 - \frac{\pi^2}{6}$$ BI (30)(7)

2. $$\int_0^1 \ln x \ln(1+x)\, dx = 2 - \frac{\pi^2}{12} - 2\ln 2$$ BI (30)(8)

3. $$\int_0^1 \ln \frac{1-ax}{1-a} \frac{dx}{\ln x} = -\sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \qquad [a<1] \qquad \text{BI (31)(3)}$$

4.222

1. $$\int_0^\infty \ln \frac{a^2+x^2}{b^2+x^2}\, dx = (a-b)\pi \qquad [a>0, \quad b>0] \qquad \text{GW (322)(20)}$$

2. $$\int_0^\infty \ln x \ln \frac{a^2+x^2}{b^2+x^2}\, dx = \pi(b-a) + \pi \ln \frac{a^a}{b^b} \qquad [a>0, \quad b>0] \qquad \text{BI (33)(1)}$$

3. $$\int_0^\infty \ln x \ln\left(1+\frac{b^2}{x^2}\right) dx = \pi b\,(\ln b - 1) \qquad [b>0] \qquad \text{BI (33)(2)}$$

4. $$\int_0^\infty \ln\left(1+a^2x^2\right) \ln\left(1+\frac{b^2}{x^2}\right) dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab) - b\right]$$
$[a>0, \quad b>0]$ BI (33)(3)

5. $$\int_0^\infty \ln\left(a^2+x^2\right) \ln\left(1+\frac{b^2}{x^2}\right) dx = 2\pi\left[(a+b)\ln(a+b) - a\ln a - b\right]$$
$[a>0, \quad b>0]$ BI (33)(4)

6. $$\int_0^\infty \ln\left(1+\frac{a^2}{x^2}\right) \ln\left(1+\frac{b^2}{x^2}\right) dx = 2\pi\left[(a+b)\ln(a+b) - a\ln a - b\ln b\right]$$
$[a>0, \quad b>0]$ BI (33)(5)

7. $$\int_0^\infty \ln\left(a^2+\frac{1}{x^2}\right) \ln\left(1+\frac{b^2}{x^2}\right) dx = 2\pi\left[\frac{1+ab}{a}\ln(1+ab) - b\ln b\right]$$
$[a>0, \quad b>0]$ BI (33)(7)

8.* $$\int_0^\infty \ln(1+ax)\, x^b e^{-x}\, dx = \sum_{m=0}^{b} \frac{b!}{(b-m)!}\left[\frac{(-1)^{b-m-1}}{a^{b-m}} e^{1/a}\, \mathrm{Ei}\left(-\frac{1}{a}\right) + \sum_{k=1}^{b-m} \frac{(k-1)!}{(-a)^{b-m-k}}\right]$$
$[b>0, \quad$ an integer$]$

4.223

1. $$\int_0^\infty \ln\left(1+e^{-x}\right) dx = \frac{\pi^2}{12} \qquad \text{BI (256)(10)}$$

2. $$\int_0^\infty \ln\left(1-e^{-x}\right) dx = -\frac{\pi^2}{6} \qquad \text{BI (256)(11)}$$

3. $$\int_0^\infty \ln\left(1+2e^{-x}\cos t + e^{-2x}\right) dx = \frac{\pi^2}{6} - \frac{t^2}{2} \qquad [|t|<\pi] \qquad \text{BI (256)(18)}$$

4.224

1. $$\int_0^u \ln \sin x\, dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right) \qquad \text{LO III 186(15)}$$

2. $$\int_0^{\pi/4} \ln \sin x\, dx = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\boldsymbol{G} \qquad \text{BI (285)(1)}$$

3. $\int_0^{\pi/2} \ln \sin x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$ FI II 629,643

4. $\int_0^{u} \ln \cos x \, dx = -L(u)$ LO III 184(10)

5. $\int_0^{\pi/4} \ln \cos x \, dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} \boldsymbol{G}$ BI (286)(1)

6. $\int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2$ BI 306(1)

7. $\int_0^{\pi/2} (\ln \sin x)^2 \, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]$ BI (305)(19)

8. $\int_0^{\pi/2} (\ln \cos x)^2 \, dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right]$ BI (306)(14)

9.[8] $\int_0^{\pi} \ln (a + b \cos x) \, dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2}$ $[a \geq |b| > 0]$ GW (322)(15)

10. $\int_0^{\pi} \ln (1 \pm \sin x) \, dx = -\pi \ln 2 \pm 4 \boldsymbol{G}$ GW (322)(16a)

11.[7] $\int_0^{\pi/2} \ln (1 + a \sin x) \, dx = \frac{\pi}{2} \ln \frac{a}{2} + 2\boldsymbol{G} + 2 \sum_{k=1}^{\infty} \frac{b^k}{k} \sum_{n=1}^{k} \frac{(-1)^{n+1}}{2n-1}$ $[a > 0]$ $b = \frac{1-a}{1+a}$

$= -\frac{\pi}{2} \ln 2 + 2\boldsymbol{G}$ $[a = 1]$

12. $\int_0^{\pi} \ln (1 + a \cos x) \, dx = \pi \ln \left(\frac{1 + \sqrt{1 - a^2}}{2} \right)$ $[a^2 \leq 1]$ BI (330)(1)

12 (1) $\int_0^{\pi} \ln (1 + a \cos x)^2 \, dx = \begin{cases} 2\pi \ln \left(\frac{1 + \sqrt{1 - a^2}}{2} \right) & \text{for } a^2 \leq 1 \\ \frac{\pi}{2} \ln \frac{a^2}{4} & \text{for } a^2 \geq 1 \end{cases}$

13. $\int_0^{\pi/2} \ln \left(1 + 2a \sin x + a^2\right) \, dx = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2}{(2k+1) \cdot (2k+1)!!} \left(\frac{2a}{1+a^2} \right)^{2k+1}$

$[a^2 \leq 1]$ BI (308)(24)

14.[11] $\int_0^{n\pi} \ln \left(a^2 - 2ab \cos x + b^2\right) \, dx = 2n\pi \ln [\max (|a|, |b|)]$

$[ab > 0]$ FI II 142, 163, 688

15.[8] $\int_0^{n\pi} \ln \left(1 - 2a \cos x + a^2\right) \, dx = 0$ $[a^2 \leq 1]$

$= n\pi \ln a^2$ $[a^2 \geq 1]$

4.225

1. $\int_0^{\pi/4} \ln (\cos x - \sin x) \, dx = -\frac{\pi}{8} \ln 2 - \frac{1}{2} \boldsymbol{G}$ GW (322)(9b)

2. $$\int_0^{\pi/4} \ln(\cos x + \sin x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln(\cos x + \sin x)\, dx = -\frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ GW (322)(9a)

3. $$\int_0^{2\pi} \ln(1 + a\sin x + b\cos x)\, dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2 - b^2}}{2} \qquad \left[a^2 + b^2 < 1\right]$$ BI (332)(2)

4. $$\int_0^{2\pi} \ln\left(1 + a^2 + b^2 + 2a\sin x + 2b\cos x\right) dx = 0 \qquad \left[a^2 + b^2 \le 1\right]$$
$$= 2\pi \ln\left(a^2 + b^2\right) \qquad \left[a^2 + b^2 \ge 1\right]$$ BI (322)(3)

4.226

1. $$\int_0^{\pi/2} \ln\left(a^2 - \sin^2 x\right)^2 dx = -2\pi \ln 2 \qquad \left[a^2 \le 1\right]$$
$$= 2\pi \ln \frac{a + \sqrt{a^2 - 1}}{2} = 2\pi\left(\operatorname{arccosh} a - \ln 2\right) \qquad [a > 1]$$ FI II 644, 687

2. $$\int_0^{\pi/2} \ln\left(1 + a\sin^2 x\right) dx = \frac{1}{2}\int_0^{\pi} \ln\left(1 + a\sin^2 x\right) dx = \int_0^{\pi/2} \ln\left(1 + a\cos^2 x\right) dx$$
$$= \frac{1}{2}\int_0^{\pi} \ln\left(1 + a\cos^2 x\right) dx = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \qquad [a \ge -1]$$ BI (308)(15), GW(322)(12)

3. $$\int_0^{u} \ln\left(1 - \sin^2\alpha \sin^2 x\right) dx = (\pi - 2\theta)\ln\cot\frac{\alpha}{2} + 2u\ln\left(\frac{1}{2}\sin\alpha\right) - \frac{\pi}{2}\ln 2$$
$$+ L(\theta + u) - L(\theta - u) + L\left(\frac{\pi}{2} - 2u\right)$$
$$\left[\cot\theta = \cos\alpha\tan u; \quad -\pi \le \alpha \le \pi, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}\right]$$ LO III 287

4. $$\int_0^{\pi/2} \ln\left[1 - \cos^2 x\left(\sin^2\alpha - \sin^2\beta\sin^2 x\right)\right] dx = \pi \ln\left[\frac{1}{2}\left(\cos^2\frac{\alpha}{2} + \sqrt{\cos^4\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}}\right)\right]$$
$$[\alpha > \beta > 0]$$ LO III 283

5. $$\int_0^{u} \ln\left(1 - \frac{\sin^2 x}{\sin^2\alpha}\right) dx = -u\ln\sin^2\alpha - L\left(\frac{\pi}{2} - \alpha + u\right) + L\left(\frac{\pi}{2} - \alpha - u\right)$$
$$\left[-\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad |\sin u| \le |\sin\alpha|\right]$$ LO III 287

6. $$\int_0^{\pi/2} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right) dx = \frac{1}{2}\int_0^{\pi} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right) dx = \pi \ln \frac{a + b}{2}$$
$$[a > 0, \quad b > 0]$$ GW (322)(13)

7. $$\int_0^{\pi/2} \ln \frac{1+\sin t \cos^2 x}{1-\sin t \cos^2 x}\, dx = \pi \ln \frac{1+\sin \frac{t}{2}}{\cos \frac{t}{2}} = \pi \ln \cot \frac{\pi - t}{4}$$ $\left[|t| < \frac{\pi}{2}\right]$ LO III 283

4.227

1. $$\int_0^u \ln \tan x \, dx = L(u) + L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right)$$ LO III 186(16)

2. $$\int_0^{\pi/4} \ln \tan x \, dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \tan x \, dx = -\boldsymbol{G}$$ BI (286)(11)

3. $$\int_0^{\pi/2} \ln (a \tan x) \, dx = \frac{\pi}{2} \ln a$$ $[a > 0]$ BI (307)(2)

4.[7] $$\int_0^{\pi/4} (\ln \tan x)^n \, dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{n+1} |E_n|$$ $[n \text{ even}]$ BI (286)(21)

5.[7] $$\int_0^{\pi/2} (\ln \tan x)^{2n} \, dx = 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|$$ BI (307)(15)

6. $$\int_0^{\pi/2} (\ln \tan x)^{2n+1} \, dx = 0$$ BI (307)(14)

7. $$\int_0^{\pi/4} (\ln \tan x)^2 \, dx = \frac{\pi^3}{16}$$ BI (286)(16)

8. $$\int_0^{\pi/4} (\ln \tan x)^4 \, dx = \frac{5}{64}\pi^5$$ BI (286)(19)

9. $$\int_0^{\pi/4} \ln (1 + \tan x) \, dx = \frac{\pi}{8} \ln 2$$ BI (287)(1)

10. $$\int_0^{\pi/2} \ln (1 + \tan x) \, dx = \frac{\pi}{4} \ln 2 + \boldsymbol{G}$$ BI (308)(9)

11. $$\int_0^{\pi/4} \ln (1 - \tan x) \, dx = \frac{\pi}{8} \ln 2 - \boldsymbol{G}$$ BI (287)(2)

12.[11] $$\int_0^{\pi/2} (\ln (1 - \tan x))^2 \, dx = \frac{\pi}{2} \ln 2 - 2\boldsymbol{G}$$ BI (308)(10)

13. $$\int_0^{\pi/4} \ln (1 + \cot x) \, dx = \frac{\pi}{8} \ln 2 + \boldsymbol{G}$$ BI (287)(3)

14. $$\int_0^{\pi/4} \ln (\cot x - 1) \, dx = \frac{\pi}{8} \ln 2$$ BI (287)(4)

15. $$\int_0^{\pi/4} \ln(\tan x + \cot x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln(\tan x + \cot x)\, dx = \frac{\pi}{2}\ln 2$$ BI (287)(5), BI (308)(11)

16.[11] $$\int_0^{\pi/4} \left(\ln(\cot x - \tan x)\right)^2\, dx = \frac{1}{2}\int_0^{\pi/2} \left(\ln(\cot x - \tan x)\right)^2\, dx = \frac{\pi}{2}\ln 2$$ BI (287)(6), BI (308)(12)

17. $$\int_0^{\pi/2} \ln\left(a^2 + b^2\tan^2 x\right)\, dx = \frac{1}{2}\int_0^{\pi} \ln\left(a^2 + b^2\tan^2 x\right)\, dx = \pi\ln(a+b)$$ $[a > 0, \quad b > 0]$ GW (322)(17)

4.228

1. $$\int_0^{\pi/2} \ln\left(\sin t \sin x + \sqrt{1 - \cos^2 t \sin^2 x}\right) dx = \frac{\pi}{2}\ln 2 - 2\,L\left(\frac{t}{2}\right) - 2\,L\left(\frac{\pi - t}{2}\right)$$ LO III 290

2. $$\int_0^{u} \ln\left(\cos x + \sqrt{\cos^2 x - \cos^2 t}\right) dx = -\left(\frac{\pi}{2} - t - \varphi\right)\ln\cos t + \frac{1}{2}\,L(u+\varphi) - \frac{1}{2}\,L(u-\varphi) - L(\varphi)$$ $\left[\cos\varphi = \frac{\sin u}{\sin t} \quad 0 \le u \le t \le \frac{\pi}{2}\right]$ LO III 290

3. $$\int_0^{t} \ln\left(\cos x + \sqrt{\cos^2 x - \cos^2 t}\right) dx = -\left(\frac{\pi}{2} - t\right)\ln\cos t$$ LO III 285

4. $$\int_0^{u} \ln\frac{\sin u + \sin t\cos x\sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t\cos x\sqrt{\sin^2 u - \sin^2 x}}\, dx = \pi\ln\left[\tan\frac{t}{2}\sin u + \sqrt{\tan^2\frac{t}{2}\sin^2 u + 1}\right]$$ $[t > 0, \quad u > 0]$ LO III 283

5. $$\int_0^{\pi/4} \sqrt{\ln\cot x}\, dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{(2k+1)^3}}$$ BI (297)(9)

6. $$\int_0^{\pi/4} \frac{dx}{\sqrt{\ln\cot x}} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2k+1}}$$ BI (304)(24)

7. $$\int_0^{\pi/4} \ln\left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \frac{1}{2}\int_0^{\pi/2} \ln\left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (287)(7), BI (308)(22)

8. $$\int_0^{\pi/4} \ln^2\left(\sqrt{\cot x} - \sqrt{\tan x}\right) dx = \frac{1}{2}\int_0^{\pi/2} \ln^2\left(\sqrt{\cot x} - \sqrt{\tan x}\right) dx = \frac{\pi}{4}\ln 2 - \boldsymbol{G}$$ BI (287)(8), BI (308)(23)

4.229

1. $$\int_0^{1} \ln\left(\ln\frac{1}{x}\right) dx = -\boldsymbol{C}$$ FI II 807

2.[11] $$\mathrm{PV}\int_0^{1} \frac{dx}{\ln\left(\ln\frac{1}{x}\right)} = \mathrm{PV}\int_0^{\infty}\frac{e^{-u}}{\ln u}\, du \approx -0.154479$$ BI (31)(2)

3. $$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{\sqrt{\ln\frac{1}{x}}} = -\left(\boldsymbol{C} + 2\ln 2\right)\sqrt{\pi}$$ BI (32)(4)

4.[11] $$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} dx = \psi(\mu)\,\Gamma(\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (30)(10)

If the integrand contains $\left(\ln\ln\frac{1}{x}\right)$, it is convenient to make the substitution $\ln\frac{1}{x} = u$ so that $x = e^{-u}$.

5.[7] $$\int_0^1 \ln\left(a + \ln x\right)\, dx = \ln a - e^{-a}\operatorname{Ei}(a) \qquad [a > 0]$$ BI (30)(5)

6. $$\int_0^1 \ln\left(a - \ln x\right)\, dx = \ln a - e^{a}\operatorname{Ei}(-a) \qquad [a > 0]$$ BI (30)(6)

7. $$\int_{\pi/4}^{\pi/2} \ln\ln\tan x\, dx = \frac{\pi}{2}\ln\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right)$$ BI (308)(28)

4.23 Combinations of logarithms and rational functions

4.231

1. $$\int_0^1 \frac{\ln x}{1+x}\, dx = -\frac{\pi^2}{12}$$ FI II 483a

2. $$\int_0^1 \frac{\ln x}{1-x}\, dx = -\frac{\pi^2}{6}$$ FI II 714

3. $$\int_0^1 \frac{x\ln x}{1-x}\, dx = 1 - \frac{\pi^2}{6}$$ BI (108)(7)

4. $$\int_0^1 \frac{1+x}{1-x}\ln x\, dx = 1 - \frac{\pi^2}{3}$$ BI (108)(9)

5.[11] $$\int_0^\infty \frac{\ln x\, dx}{(x+a)^2} = \frac{\ln a}{a} \qquad [0 < a]$$ BI (139)(1)

6. $$\int_0^1 \frac{\ln x}{(1+x)^2}\, dx = -\ln 2$$ BI (111)(1)

7.[7] $$\int_0^\infty \ln x \frac{dx}{\left(a^2 + b^2x^2\right)^n} = \frac{\Gamma\left(n - \frac{1}{2}\right)\sqrt{\pi}}{4(n-1)!a^{2n-1}b}\left[2\ln\frac{a}{2b} - \boldsymbol{C} - \psi\left(n - \frac{1}{2}\right)\right]$$
$$[a > 0, \quad b > 0]$$ LI (139)(3)

8. $$\int_0^\infty \frac{\ln x\, dx}{a^2 + b^2x^2} = \frac{\pi}{2ab}\ln\frac{a}{b} \qquad [ab > 0]$$ BI (135)(6)

9. $$\int_0^\infty \frac{\ln px}{q^2 + x^2}\, dx = \frac{\pi}{2q}\ln pq \qquad [p > 0, \quad q > 0]$$ BI (135)(4)

10. $$\int_0^\infty \frac{\ln x\, dx}{a^2 - b^2x^2} = -\frac{\pi^2}{4ab} \qquad [ab > 0]$$

11. $$\int_0^a \frac{\ln x\, dx}{x^2 + a^2} = \frac{\pi \ln a}{4a} - \frac{\boldsymbol{G}}{a} \qquad [a > 0]$$ GW (324)(7b)

12. $$\int_0^1 \frac{\ln x}{1 + x^2}\, dx = -\int_1^\infty \frac{\ln x}{1 + x^2}\, dx = -\boldsymbol{G}$$ FI II 482, 614

13. $$\int_0^1 \frac{\ln x\, dx}{1 - x^2} = -\frac{\pi^2}{8}$$ BI (108)(11)

14. $$\int_0^1 \frac{x \ln x}{1 + x^2}\, dx = -\frac{\pi^2}{48}$$ GW (324)(7b)

15. $$\int_0^1 \frac{x \ln x}{1 - x^2}\, dx = -\frac{\pi^2}{24}$$

16. $$\int_0^1 \ln x \frac{1 - x^{2n+2}}{\left(1 - x^2\right)^2}\, dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^n \frac{n - k + 1}{(2k-1)^2}$$ BI (111)(5)

17. $$\int_0^1 \ln x \frac{1 + (-1)^n x^{n+1}}{(1 + x)^2}\, dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^n (-1)^k \frac{n - k + 1}{k^2}$$ BI (111)(2)

18. $$\int_0^1 \ln x \frac{1 - x^{n+1}}{(1 - x)^2}\, dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^n \frac{n - k + 1}{k^2}$$ BI (111)(3)

19.* $$\int_0^1 \frac{x \ln x}{1 + x}\, dx = -1 + \frac{\pi^2}{2}$$

20.* $$\int_0^1 \frac{(1 - x) \ln x}{1 + x}\, dx = 1 - \frac{\pi^2}{6}$$

4.232

1. $$\int_u^v \frac{\ln x\, dx}{(x + u)(x + v)} = \frac{\ln uv}{2(v - u)} \ln \frac{(u + v)^2}{4uv}$$ BI (145)(32)

2. $$\int_0^\infty \frac{\ln x\, dx}{(x + \beta)(x + \gamma)} = \frac{(\ln \beta)^2 - (\ln \gamma)^2}{2(\beta - \gamma)} \qquad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi]$$ ET II 218(24)

3. $$\int_0^\infty \frac{\ln x}{x + a} \frac{dx}{x - 1} = \frac{\pi^2 + (\ln a)^2}{2(a + 1)} \qquad [a > 0]$$ BI (140)(10)

4.233

1.[3] $$\int_0^1 \frac{\ln x\, dx}{1 + x + x^2} = \frac{2}{9} \left[\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right] = -0.7813024129\ldots$$ LI (113)(1)

2.[3] $$\int_0^1 \frac{\ln x\, dx}{1 - x + x^2} = \frac{1}{3} \left[\frac{2\pi^2}{3} - \psi'\left(\frac{1}{3}\right)\right] = -1.17195361934\ldots$$ LI (113)(2)

3.[11] $$\int_0^1 \frac{x \ln x\, dx}{1 + x + x^2} = -\frac{1}{9} \left[\frac{7\pi^2}{6} - \psi'\left(\frac{1}{3}\right)\right] = -0.15766014917\ldots$$ LI (113)(2)

4.[3] $$\int_0^1 \frac{x \ln x\, dx}{1 - x + x^2} = \frac{1}{6} \left[\frac{5\pi^2}{6} - \psi'\left(\frac{1}{3}\right)\right] = -0.3118211319\ldots$$ LI (113)(4)

5. $$\int_0^\infty \frac{\ln x\,dx}{x^2+2xa\cos t+a^2} = \frac{t\ln a}{a\sin t} \qquad [a>0, \quad 0<t<\pi]$$ GW (324)(13c)

4.234

1.[11] $$\int_1^\infty \frac{\ln x\,dx}{\left(1+x^2\right)^2} = \frac{\boldsymbol{G}}{2} - \frac{\pi}{8}$$ BI (144)(18)a

2. $$\int_0^1 \frac{x\ln x\,dx}{\left(1+x^2\right)^2} = -\frac{1}{4}\ln 2$$ BI (111)(4)

3. $$\int_0^\infty \frac{1+x^2}{\left(1-x^2\right)^2}\ln x\,dx = 0$$ BI (142)(2)a

4. $$\int_0^\infty \frac{1-x^2}{\left(1+x^2\right)^2}\ln x\,dx = -\frac{\pi}{2}$$ BI (142)(1)a

5. $$\int_0^1 \frac{x^2\ln x\,dx}{\left(1-x^2\right)\left(1+x^4\right)} = -\frac{\pi^2}{16\left(2+\sqrt{2}\right)}$$ BI (112)(21)

6. $$\int_0^\infty \frac{\ln x\,dx}{\left(a^2+b^2x^2\right)\left(1+x^2\right)} = \frac{b\pi}{2a\left(b^2-a^2\right)}\ln\frac{a}{b} \qquad [ab>0]$$ BI (317)(16)a

7. $$\int_0^\infty \frac{\ln x}{x^2+a^2}\cdot\frac{dx}{1+b^2x^2} = \frac{\pi}{2\left(1-a^2b^2\right)}\left(\frac{1}{a}\ln a + b\ln b\right) \qquad [a>0, \quad b>0]$$ LI (140)(12)

8. $$\int_0^\infty \frac{x^2\ln x\,dx}{\left(a^2+b^2x^2\right)\left(1+x^2\right)} = \frac{a\pi}{2b\left(b^2-a^2\right)}\ln\frac{b}{a} \qquad [ab>0]$$ LI (140)(12), BI (317)(15)a

4.235

1. $$\int_0^\infty \ln x\frac{(1-x)x^{n-2}}{1-x^{2n}}\,dx = -\frac{\pi^2}{4n^2}\tan^2\frac{\pi}{2n} \qquad [n>1]$$ BI (135)(10)

2. $$\int_0^\infty \ln x\frac{\left(1-x^2\right)x^{m-1}}{1-x^{2n}}\,dx = -\frac{\pi^2\sin\left(\frac{m+1}{n}\right)\pi\sin\left(\frac{\pi}{n}\right)}{4n^2\sin^2\left(\frac{m\pi}{2n}\right)\sin^2\left(\frac{m+2}{2n}\pi\right)}$$ LI (135)(12)

3.[11] $$\int_0^\infty \ln x\frac{\left(1-x^2\right)x^{n-3}}{1-x^{2n}}\,dx = -\frac{\pi^2}{4n^2}\tan^2\left(\frac{\pi}{n}\right) \qquad [n>2]$$ BI (135)(11)

4. $$\int_0^1 \ln x\frac{x^{m-1}+x^{n-m-1}}{1-x^n}\,dx = -\frac{\pi^2}{n^2\sin^2\left(\frac{m}{n}\pi\right)} \qquad [n>m]$$ BI (108)(15)

4.236

1. $$\int_0^1 \left\{\frac{1+(p-1)\ln x}{1-x} + \frac{x\ln x}{(1-x)^2}\right\}x^{p-1}\,dx = -1+\psi'(p) \qquad [p>0]$$ BI (111)(6)a, GW (326)(13)

2. $$\int_0^1 \left[\frac{1}{1-x} + \frac{x\ln x}{(1-x)^2}\right]dx = \frac{\pi^2}{6} - 1$$ GW (326)(13a)

4.24 Combinations of logarithms and algebraic functions

4.241

1. $$\int_0^1 \frac{x^{2n}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right)$$ BI (118)(5)a

2. $$\int_0^1 \frac{x^{2n+1}\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n)!!}{(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right)$$ BI (118)(5)a

3. $$\int_0^1 x^{2n}\sqrt{1-x^2}\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2\right)$$ LI (117)(4), GW (324)(53a)

4. $$\int_0^1 x^{2n+1}\sqrt{1-x^2}\ln x\,dx = \frac{(2n)!!}{(2n+3)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k} - \frac{1}{2n+3}\right)$$ BI (117)(5), GW (324)(53b)

5. $$\int_0^1 \ln x\cdot\sqrt{\left(1-x^2\right)^{2n-1}}\,dx = -\frac{(2n-1)!!}{4\cdot(2n)!!}\pi\left[\psi(n+1) + \boldsymbol{C} + \ln 4\right]$$ BI (117)(3)

6. $$\int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\boldsymbol{G}$$ BI (145)(1)

7. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\ln 2$$ FI II 614, 643

8. $$\int_1^\infty \frac{\ln x\,dx}{x^2\sqrt{x^2-1}} = 1 - \ln 2$$ BI (144)(17)

9. $$\int_0^1 \sqrt{1-x^2}\ln x\,dx = -\frac{\pi}{8} - \frac{\pi}{4}\ln 2$$ BI (117)(1), GW (324)(53c)

10. $$\int_0^1 x\sqrt{1-x^2}\ln x\,dx = \tfrac{1}{3}\ln 2 - \tfrac{4}{9}$$ BI (117)(2)

11. $$\int_0^1 \frac{\ln x\,dx}{\sqrt{x\left(1-x^2\right)}} = -\frac{\sqrt{2\pi}}{8}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$ GW (324)(54a)

4.242

1. $$\int_0^\infty \frac{\ln x\,dx}{\sqrt{\left(a^2+x^2\right)\left(x^2+b^2\right)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab$$ $[a > b > 0]$ BY (800.04)

2. $$\int_0^b \frac{\ln x\,dx}{\sqrt{\left(a^2+x^2\right)\left(b^2-x^2\right)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\ln ab - \frac{\pi}{2}\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\right]$$ $[a > 0, \quad b > 0]$ BY (800.02)

3. $$\int_b^\infty \frac{\ln x\,dx}{\sqrt{(x^2+a^2)(x^2-b^2)}} = \frac{1}{2\sqrt{a^2+b^2}}\left[\boldsymbol{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right]$$

$[a > 0, \quad b > 0]$ BY (800.06)

4. $$\int_0^b \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab - \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$

$[a > b > 0]$ BY (800.01)

5. $$\int_b^a \frac{\ln x\,dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{2a}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\ln ab$$ BY (800.03)

6. $$\int_a^\infty \frac{\ln x\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{2a}\left[\boldsymbol{K}\left(\frac{b}{a}\right)\ln ab + \frac{\pi}{2}\boldsymbol{K}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]$$

$[a > b > 0]$ BY (800.05)

4.243 $$\int_0^1 \frac{x\ln x}{\sqrt{1-x^4}}\,dx = -\frac{\pi}{8}\ln 2$$ GW (324)(56b)

4.244

1. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{x\left(1-x^2\right)^2}} = -\frac{1}{8}\left[\Gamma\left(\frac{1}{3}\right)\right]^3$$ GW (324)(54b)

2. $$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 + \frac{\pi}{3\sqrt{3}}\right)$$ BI (118)(7)

3. $$\int_0^1 \frac{x\ln x\,dx}{\sqrt[3]{\left(1-x^3\right)^2}} = \frac{\pi}{3\sqrt{3}}\left(\frac{\pi}{3\sqrt{3}} - \ln 3\right)$$ BI (118)(8)

4.245

1. $$\int_0^1 \frac{x^{4n+1}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{8}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \ln 2\right)$$ GW (324)(56a)

2. $$\int_0^1 \frac{x^{4n+3}\ln x}{\sqrt{1-x^4}}\,dx = \frac{(2n)!!}{4\cdot(2n+1)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right)$$ GW (324)(56c)

4.246 $$\int_0^1 \left(1-x^2\right)^{n-\frac{1}{2}}\ln x\,dx = -\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{4}\left[2\ln 2 + \sum_{k=1}^{n}\frac{1}{k}\right]$$ GW (324)(55)

4.247

1.[6] $$\int_0^1 \frac{\ln x}{\sqrt[n]{1-x^{2n}}}\,dx = -\frac{\pi\,\mathrm{B}\left(\frac{1}{2n},\frac{1}{2n}\right)}{8n^2\sin\frac{\pi}{2n}}$$ $[n > 1]$ GW (324)(54c)a

2.[6] $$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}\left(1-x^2\right)}} = -\frac{\pi\,\mathrm{B}\left(\frac{1}{2n},\frac{1}{2n}\right)}{8\sin\frac{\pi}{2n}}$$ GW (324)(54)

4.25 Combinations of logarithms and powers

4.251

1. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{\beta+x}\,dx = \frac{\pi\beta^{\mu-1}}{\sin\mu\pi}\left(\ln\beta - \pi\cot\mu\pi\right) \qquad [|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 1]$$ BI (135)(1)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{a-x}\,dx = \pi a^{\mu-1}\left(\cot\mu\pi\ln a - \frac{\pi}{\sin^2\mu\pi}\right) \qquad [a > 0, \quad 0 < \operatorname{Re}\mu < 1]$$ ET I 314(5)

3.[10] $$\int_0^1 \frac{x^{\mu-1}\ln x}{x+1}\,dx = \beta'(\mu) \qquad [\operatorname{Re}\mu > 0]$$ GW (324)(6), ET I 314(3)

4. $$\int_0^1 \frac{x^{\mu-1}\ln x}{1-x}\,dx = -\psi'(\mu) = -\zeta(2,\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (108)(8)

5.[11] $$\int_0^1 \ln x\frac{x^{2n}}{1+x}\,dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k^2}$$ BI (108)(4)

6.[11] $$\int_0^1 \ln x\frac{x^{2n-1}}{1+x}\,dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1}\frac{(-1)^{k}}{k^2}$$ BI (108)(5)

4.252

1. $$\int_0^\infty \frac{x^{\mu-1}\ln x}{(x+\beta)(x+\gamma)}\,dx = \frac{\pi}{(\gamma-\beta)\sin\mu\pi}\left[\beta^{\mu-1}\ln\beta - \gamma^{\mu-1}\ln\gamma - \pi\cot\mu\pi\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\right]$$
$$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad 0 < \operatorname{Re}\mu < 2, \quad \mu \neq 1]$$ BI (140)(9)a, ET 314(6)

2. $$\int_0^\infty \frac{x^{\mu-1}\ln x\,dx}{(x+\beta)(x-1)} = \frac{\pi}{(\beta+1)\sin^2\mu\pi}\left[\pi - \beta^{\mu-1}\left(\sin\mu\pi\ln\beta - \pi\cos\mu\pi\right)\right]$$
$$[|\arg\beta| < \pi, \quad 0 < \operatorname{Re}\mu < 2, \quad \mu \neq 1]$$ BI (140)(11)

3. $$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^2}\,dx = -\frac{\pi^2}{4}\operatorname{cosec}^2\frac{p\pi}{2} \qquad [0 < p < 2]$$ (see also **4.254** 2)

4.[6] $$\int_0^\infty \frac{x^{\mu-1}\ln x}{(x+a)^2}\,dx = \frac{(1-\mu)a^{\mu-2}\pi}{\sin\mu\pi}\left(\ln a - \pi\cot\mu\pi + \frac{1}{\mu-1}\right)$$
$$[|\arg a| < \pi \quad 0 < \operatorname{Re}\mu < 2 \quad (\mu \neq 1)]$$ GW (324)(13b)

4.253

1.[8] $$\int_0^1 x^{\mu-1}\left(1-x^r\right)^{\nu-1}\ln x\,dx = \frac{1}{r^2}\,\mathrm{B}\left(\frac{\mu}{r},\nu\right)\left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r}+\nu\right)\right]$$
$$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad r > 0]$$ GW (324)(3b)a, BI (107)(5)a

2. $$\int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}}\ln x\,dx = -\frac{\pi}{p}\operatorname{cosec}p\pi \qquad [0 < p < 1]$$ bi (319)(10)a

3. $$\int_u^\infty \frac{(x-u)^{\mu-1}\ln x\,dx}{x^\lambda} = u^{\mu-\lambda}\,\mathrm{B}(\lambda-\mu,\mu)\left[\ln u+\psi(\lambda)-\psi(\lambda-\mu)\right]$$
$[0 < \operatorname{Re}\mu < \operatorname{Re}\lambda]$ ET II 203(18)

4.[11] $$\int_0^\infty \ln x \left(\frac{x}{a^2+x^2}\right)^p \frac{dx}{x} = \frac{\ln a}{2a^p}\,\mathrm{B}\left(\frac{p}{2},\frac{p}{2}\right)$$ $[a > 0, \quad p > 0]$ BI (140)(6)

5. $$\int_1^\infty (x-1)^{p-1}\ln x\,dx = \frac{\pi}{p}\operatorname{cosec}\pi p$$ $[-1 < p < 0]$ BI (289)(12)a

6.[7] $$\int_0^\infty \ln x \frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^\mu}\left(\ln a - \boldsymbol{C} - \psi(\mu)\right)$$ $[\operatorname{Re}\mu > 0, \quad a \neq 0, \quad |\arg a| < \pi]$
NT 68(7)

7.[7] $$\int_0^\infty \ln x \frac{dx}{(a+x)^{n+\frac{1}{2}}} = \frac{2}{(2n-1)a^{n-\frac{1}{2}}}\left(\ln a + 2\ln 2 - 2\sum_{k=1}^{n-1}\frac{1}{2k-1}\right)$$
$[|\arg a| < \pi, \quad n = 1, 2, \ldots]$ BI (142)(5)

4.254

1. $$\int_0^1 \frac{x^{p-1}\ln x}{1-x^q}\,dx = -\frac{1}{q^2}\,\psi'\left(\frac{p}{q}\right)$$ $[p > 0, \quad q > 0]$ GW (324)(5)

2. $$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^q}\,dx = -\frac{\pi^2}{q^2\sin^2\dfrac{p\pi}{q}}$$ $[0 < p < q]$ BI (135)(8)

3. $$\int_0^\infty \frac{\ln x}{x^q-1}\frac{dx}{x^p} = \frac{\pi^2}{q^2\sin^2\dfrac{p-1}{q}\pi}$$ $[p < 1, \quad p+q > 1]$ BI (140)(2)

4.[3] $$\int_0^1 \frac{x^{p-1}\ln x}{1+x^q}\,dx = \frac{1}{q^2}\beta'\left(\frac{p}{q}\right)$$ $[p > 0, \quad q > 0]$ GW (324)(7)

5. $$\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\,dx = -\frac{\pi^2}{q^2}\frac{\cos\dfrac{p\pi}{q}}{\sin^2\dfrac{p\pi}{q}}$$ $[0 < p < q]$ BI (135)(7)

6. $$\int_0^1 \frac{x^{q-1}\ln x}{1-x^{2q}}\,dx = -\frac{\pi^2}{8q^2}$$ $[q > 0]$ BI (108)(12)

4.255

1. $$\int_0^1 \ln x \frac{\left(1-x^2\right)x^{p-2}}{1+x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2 \frac{\sin\frac{\pi}{2p}}{\cos^2\left(\frac{\pi}{2p}\right)}$$ $[p > 1]$ BI (108)(13)

2. $$\int_0^1 \ln x \frac{\left(1+x^2\right)x^{p-2}}{1-x^{2p}}\,dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2\left(\frac{\pi}{2p}\right)$$ $[p > 1]$ BI (108)(14)

3. $$\int_0^\infty \ln x \frac{1-x^p}{1-x^2}\,dx = \frac{\pi^2}{4}\tan^2\left(\frac{p\pi}{2}\right)$$ $[p < 1]$ BI (140)(3)

4.256 $$\int_0^1 \ln\frac{1}{x}\frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2}\,\mathrm{B}\left(\frac{\mu}{n},\frac{m}{n}\right)\left[\psi\left(\frac{\mu+m}{n}\right)-\psi\left(\frac{\mu}{n}\right)\right]$$

$[\operatorname{Re}\mu > 0]$ LI (118)(12)

4.257

1. $$\int_0^\infty \frac{x^\nu \ln\frac{x}{\beta}\,dx}{(x+\beta)(x+\gamma)} = \frac{\pi\left[\gamma^\nu \ln\frac{\gamma}{\beta} + \pi\left(\beta^\nu - \gamma^\nu\right)\cot\nu\pi\right]}{\sin\nu\pi(\gamma-\beta)}$$

$[|\arg\beta| < \pi, \quad |\arg\gamma| < \pi, \quad |\operatorname{Re}\nu| < 1]$ ET II 219(30)

2. $$\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)\frac{dx}{x} = 0$$ $[q > 0]$ BI (140)(4)a

3. $$\int_0^\infty \ln\frac{x}{q}\left(\frac{x^p}{q^{2p}+x^{2p}}\right)^r \frac{dx}{q^2+x^2} = 0$$ $[q > 0]$ BI (140)(4)a

4. $$\int_0^\infty \ln x \ln\frac{x}{a}\frac{dx}{(x-1)(x-a)} = \frac{\left[4\pi^2 + (\ln a)^2\right]\ln a}{6(a-1)}$$ $[a > 0]$ (for $a = 1$ see **4.261** 5)

BI (141)(5)

5. $$\int_0^\infty \ln x \ln\frac{x}{a}\frac{x^p\,dx}{(x-1)(x-a)} = \frac{\pi^2\left[(a^p+1)\ln a - 2\pi\left(a^p-1\right)\cot p\pi\right]}{(a-1)\sin^2 p\pi}$$

$[p^2 < 1, \quad a > 0]$ BI (141)(6)

4.26–4.27 Combinations involving powers of the logarithm and other powers

4.261

1.[7] $$\int_0^1 (\ln x)^2 \frac{dx}{1+2x\cos t + x^2} = \frac{t\left(\pi^2 - t^2\right)}{6\sin t}$$ $[0 \le t \le \pi]$ BI (113)(7)

2. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2-x+1} = \frac{10\pi^3}{81\sqrt{3}}$$ GW (324)(16c)

3. $$\int_0^1 \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{1}{2}\int_0^\infty \frac{(\ln x)^2\,dx}{x^2+x+1} = \frac{8\pi^3}{81\sqrt{3}}$$ GW (324)(16b)

4. $$\int_0^\infty (\ln x)^2 \frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2 + (\ln a)^2\right]\ln a}{3(1+a)}$$ $[a > 0]$ BI (141)(1)

5. $$\int_0^\infty (\ln x)^2 \frac{dx}{(1-x)^2} = \frac{2}{3}\pi^2$$ BI (139)(4)

6. $$\int_0^1 (\ln x)^2 \frac{dx}{1+x^2} = \frac{\pi^3}{16}$$ BI (109)(3)

7. $$\int_0^1 (\ln x)^2 \frac{1+x^2}{1+x^4}\,dx = \frac{1}{2}\int_0^\infty (\ln x)^2 \frac{1+x^2}{1+x^4}\,dx = \frac{3\sqrt{2}}{64}\pi^3$$ BI (109)(5), BI (135)(13)

8.[11] $$\int_0^1 (\ln x)^2 \frac{1-x}{1-x^6}\,dx = \frac{8\sqrt{3}\pi^3 + 351\,\zeta(3)}{486}$$

9. $$\int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left[(\ln 2)^2 + \frac{\pi^2}{12}\right]$$ BI (118)(13)

10. $$\int_0^\infty (\ln x)^2 \frac{x^{\mu-1}}{1+x}\,dx = \frac{\pi^3\left(2-\sin^2\mu\pi\right)}{\sin^3\mu\pi} \qquad [0 < \operatorname{Re}\mu < 1]$$ ET I 315(10)

11.[7] $$\int_0^1 (\ln x)^2 \frac{x^n\,dx}{1+x} = 2\sum_{k=n}^\infty \frac{(-1)^{n+k}}{(k+1)^3} = (-1)^n\left(\frac{3}{2}\zeta(3) + 2\sum_{k=1}^n \frac{(-1)^k}{k^3}\right)$$
$[n = 0, 1, \ldots]$ BI (109)(1)

12.[7] $$\int_0^1 (\ln x)^2 \frac{x^n\,dx}{1-x} = 2\sum_{k=n}^\infty \frac{1}{(k+1)^3} = 2\left(\zeta(3) - \sum_{k=1}^n \frac{1}{k^3}\right)$$
$[n = 0, 1, \ldots]$ BI (109)(2)

13.[11] $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{1-x^2} = 2\sum_{k=n}^\infty \frac{1}{(2k+1)^3} = \frac{7}{4}\zeta(3) - 2\sum_{k=1}^n \frac{1}{(2k-1)^3}$$
$[n = 0, 1, \ldots]$ BI (109)(4)

14. $$\int_0^\infty (\ln x)^2 \frac{x^{p-1}\,dx}{x^2+2x\cos t+1} = \frac{\pi\sin(1-p)t}{\sin t\sin p\pi}\left\{\pi^2 - t^2 + 2\pi\cot p\pi\left[\pi\cot p\pi + t\cot(1-p)t\right]\right\}$$
$[0 < t < \pi, \quad 0 < p < 2, \quad p \neq 1]$
GW (324)(17)

15. $$\int_0^1 (\ln x)^2 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2\cdot(2n)!!}\pi\left\{\frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}$$ GW (324)(60a)

16. $$\int_0^1 (\ln x)^2 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}\left\{-\frac{\pi^2}{12} - \sum_{k=1}^{2n+1}\frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}$$
GW (324)(60b)

17.[7] $$\int_0^1 (\ln x)^2 x^{\mu-1}(1-x)^{\nu-1}\,dx = \mathrm{B}(\mu,\nu)\left\{[\psi(\mu) - \psi(\nu+\mu)]^2 + \psi'(\mu) - \psi'(\mu+\nu)\right\}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET I 315(11)

18. $$\int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2}\,dx = 2(n+1)\,\zeta(3) - 2\sum_{k=1}^n \frac{n-k+1}{k^3}$$ LI (111)(8)

19. $$\int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\,dx = \frac{3}{2}(n+1)\,\zeta(3) - 2\sum_{k=1}^n (-1)^{k-1}\frac{n-k+1}{k^3}$$ LI (111)(7)

20.[7] $$\int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{\left(1-x^2\right)^2}\,dx = \frac{7}{4}(n+1)\,\zeta(3) - 2\sum_{k=1}^n \frac{n-k+1}{(2k-1)^3}$$
$[n = 0, 1, \ldots]$ LI (111)(9)

21. $$\int_0^1 (\ln x)^2 x^{p-1}\left(1-x^r\right)^{q-1}\,dx = \frac{1}{r^3}\mathrm{B}\left(\frac{p}{r}, q\right)\left\{\psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r}+q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r}+q\right)\right]^2\right\}$$
$[p > 0, \quad q > 0, \quad r > 0]$ GW (324)(8a)

4.262

1. $$\int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120}\pi^4$$ BI (109)(9)

2. $$\int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15}$$ BI (109)(11)

3. $$\int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{\left[\pi^2 + (\ln a)^2\right]^2}{4(a+1)} \qquad [a > 0]$$ BI (141)(2)

4. $$\int_0^1 (\ln x)^3 \frac{x^n\, dx}{1+x} = (-1)^{n+1}\left[\frac{7\pi^4}{120} - 6\sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4}\right] \qquad [n = 1, 2, \ldots]$$ BI (109)(10)

5. $$\int_0^1 (\ln x)^3 \frac{x^n\, dx}{1-x} = -\frac{\pi^4}{15} + 6\sum_{k=0}^{n-1} \frac{1}{(k+1)^4} \qquad [n = 1, 2, \ldots]$$ BI (109)(12)

6. $$\int_0^1 (\ln x)^3 \frac{x^{2n}\, dx}{1-x^2} = -\frac{\pi^4}{16} + 6\sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} \qquad [n = 1, 2, \ldots]$$ BI (109)(14)

7. $$\int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2}\, dx = -\frac{(n+1)\pi^4}{15} + 6\sum_{k=1}^{n} \frac{n-k+1}{k^4}$$ BI (111)(11)

8. $$\int_0^1 (\ln x)^3 \frac{1+(-1)^n x^{n+1}}{(1+x)^2}\, dx = -\frac{7(n+1)\pi^4}{120} + 6\sum_{k=1}^{n} (-1)^{k-1} \frac{n-k+1}{k^4}$$ BI (111)(10)

9. $$\int_0^1 (\ln x)^3 \frac{1-x^{2n+2}}{\left(1-x^2\right)^2}\, dx = -\frac{(n+1)\pi^4}{16} + 6\sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^4}$$ BI (111)(12)

4.263

1.[8] $$\int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a \left[\pi^2 + (\ln a)^2\right]\left[7\pi^2 + 3(\ln a)^2\right]}{15(1+a)}$$ $[a > 0]$ BI (141)(3)

2. $$\int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64}$$ BI (109)(17)

3. $$\int_0^1 (\ln x)^4 \frac{dx}{1+2x\cos t + x^2} = \frac{t\left(\pi^2 - t^2\right)\left(7\pi^2 - 3t^2\right)}{30 \sin t}$$ $[|t| < \pi]$ BI (113)(8)

4.264

1. $$\int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252}$$ BI (109)(20)

2. $$\int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63}$$ BI (109)(21)

3. $$\int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{\left[\pi^2 + (\ln a)^2\right]^2 \left[3\pi^2 + (\ln a)^2\right]}{6(1+a)}$$ $[a > 0]$ BI (141)(4)

4.265 $$\int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256}$$ BI (109)(25)

4.266

1. $$\int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240}$$ BI (109)(28)

2. $$\int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15}$$ BI (109)(29)

4.267

1. $$\int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = \ln \frac{2}{\pi}$$ BI (127)(3)

2. $$\int_0^1 \frac{(1-x)^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{\pi}{4}$$ BI (128)(2)

3.[8] $$\int_0^1 \frac{(1-x)^2}{1 + 2x\cos\frac{mx}{n} + x^2} \cdot \frac{dx}{\ln x}$$
$$= \frac{1}{\sin\left(\frac{m\pi}{n}\right)} \sum_{k=1}^{n-1} (-1)^k \sin\left(\frac{km\pi}{n}\right) \ln \frac{\left\{\Gamma\left(\frac{n+k+1}{2n}\right)\right\}^2 \Gamma\left(\frac{k+2}{2n}\right) \Gamma\left(\frac{k}{2n}\right)}{\left\{\Gamma\left(\frac{k+1}{2n}\right)\right\}^2 \Gamma\left(\frac{n+k}{2n}\right) \Gamma\left(\frac{n+k+2}{2n}\right)}$$ $[m+n \text{ is odd}]$
$$= \frac{1}{\sin\left(\frac{m\pi}{n}\right)} \sum_{k=1}^{\left\lfloor \frac{1}{2}(n-1) \right\rfloor} (-1)^k \sin\left(\frac{km\pi}{n}\right) \ln \frac{\left\{\Gamma\left(\frac{n-k+1}{n}\right)\right\}^2 \Gamma\left(\frac{k+2}{n}\right) \Gamma\left(\frac{k}{n}\right)}{\left\{\Gamma\left(\frac{k+1}{n}\right)\right\}^2 \Gamma\left(\frac{n-k}{n}\right) \Gamma\left(\frac{n-k+2}{n}\right)}$$ $[m+n \text{ is even}]$

$[m < n]$ BI (130)(3)

4. $$\int_0^1 \frac{1-x}{1+x} \cdot \frac{1}{1+x^2} \cdot \frac{dx}{\ln x} = -\frac{\ln 2}{2}$$ BI (130)(16)

5. $$\int_0^1 \frac{1-x}{1+x} \cdot \frac{x^2}{1+x^2} \cdot \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi}$$ BI (130)(17)

6.[11] $$\int_0^1 (1-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \ln(1+k)$$ $[p \geq 1]$ BI (123)(2)

7. $$\int_0^1 \left(\frac{1-x^p}{1-x} - p\right) \frac{dx}{\ln x} = \ln \Gamma(p+1)$$ GW (326)(10)

8. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x}\, dx = \ln \frac{p}{q}$$ $[p > 0, \quad q > 0]$ FI II 647

9. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \cdot \frac{dx}{1+x} = \ln \frac{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}$$ $[p > 0, \quad q > 0]$ FI II 186

10. $$\int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x)\ln x}\, dx = \frac{1}{2} \int_0^\infty \frac{x^{p-1} - x^{-p}}{(1+x)\ln x}\, dx = \ln\left(\tan \frac{p\pi}{2}\right)$$

$[0 < p < 1]$ FI II 816

11. $$\int_0^1 (x^p - x^q)\, x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q} \qquad [r > 0, \quad p > 0, \quad q > 0]$$ LI (123)(5)

12. $$\int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k} \qquad [p > 0, \quad q > 0, \quad a^2 < 1]$$ BI (130)(15)

13. $$\int_0^1 (x^p - 1)(x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)} \qquad [p > -1, \quad q > -1, \quad p+q > -1]$$ GW (324)(19b)

14. $$\int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x}\, dx = \ln \frac{\Gamma\left(\frac{p}{2}+n+1\right) \Gamma\left(\frac{q+1}{2}+n\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}+n+1\right) \Gamma\left(\frac{p+1}{2}+n\right) \Gamma\left(\frac{q+1}{2}\right) \Gamma\left(\frac{p}{2}\right)}$$
$[p > 0, \quad q > 0]$ BI (127)(7)

15. $$\int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x}\, dx = \ln \frac{\Gamma(q+1)\, \Gamma(p+r+1)}{\Gamma(p+1)\, \Gamma(q+r+1)}$$
$[p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1]$ GW (324)(23)

16. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right) \Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right) \Gamma\left(\frac{p}{2r}\right)} \qquad [p > 0, \quad q > 0, \quad r > 0]$$ GW (324)(21)

17. $$\int_0^1 \frac{1 - x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1}\, dx}{\ln x} = \ln \tan \frac{q\pi}{4p} \qquad [0 < q < p]$$ BI (128)(6)

18. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x}\, dx = \ln \left(\tan \frac{p\pi}{2r} \cot \frac{q\pi}{2r} \right) \qquad [0 < p < r, \quad 0 < q < r]$$ GW (324)(22), BI (143)(2)

19. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1-x^r) \ln x}\, dx = \ln \left(\frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}} \right) \qquad [0 < p < r, \quad 0 < q < r]$$ BI (143)(4)

20. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right) \Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right) \Gamma\left(\frac{p}{2n}\right)} \qquad [p > 0, \quad q > 0]$$ BI (128)(11)

21. $$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \frac{1+x^2}{\ln x}\, dx = \ln \frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{q+2}{4(2n+1)}\right) \Gamma\left(\frac{p+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{p+2}{4(2n+1)}\right) \Gamma\left(\frac{q+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{p}{4(2n+1)}\right)}$$
$[p > 0, \quad q > 0]$ BI (128)(7)

22. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1+x^{2(2n+1)}} \cdot \frac{1+x^2}{\ln x}\, dx = \ln \left\{ \tan \frac{p\pi}{4(2n+1)} \cdot \tan \frac{(p+2)\pi}{4(2n+1)} \cdot \cot \frac{q\pi}{4(2n+1)} \cdot \cot \frac{(q+2)\pi}{4(2n+1)} \right\}$$
$[0 < p < 4n, \quad 0 < q < 4n]$ BI (143)(5)

23. $$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1 - x^{2n}} \frac{1 - x^2}{\ln x}\, dx = \ln \frac{\sin \dfrac{p\pi}{2n} \cdot \sin \frac{(q+2)\pi}{2n}}{\sin \dfrac{q\pi}{2n} \cdot \sin \dfrac{(p+2)\pi}{2n}}$$
$[0 < p < 2n, \quad 0 < q < 2n]$ BI (143)(6)

24. $$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{r-1}\, dx}{\ln x} = \ln \frac{(p+q+r)r}{(p+r)(q+r)}$$ $[p > 0, \quad q > 0, \quad r > 0]$ BI (123)(8)

25. $$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{r-1}\, dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+r)\,\Gamma(q+r)}{\Gamma(p+q+r)\,\Gamma(r)}$$
$[r > 0, \quad r + p > 0, \quad r + q > 0, \quad r + p + q > 0]$ FI II 815a

26. $$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{dx}{\ln x} = \ln \frac{(p+q+1)(q+r+1)(r+p+1)}{(p+q+r+1)(p+1)(q+1)(r+1)}$$
$[p > -1, \quad q > -1, \quad r > -1, \quad p + q > -1, \quad p + r > -1, \quad q + r > -1, \quad p + q + r > -1]$
GW (324)(19c)

27. $$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(r+1)\,\Gamma(p+q+r+1)}{\Gamma(p+q+1)\,\Gamma(p+r+1)\,\Gamma(q+r+1)}$$
$[p > -1, \quad q > -1, \quad r > -1, \quad p + q > -1, \quad p + r > -1, \quad q + r > -1, \quad p + q + r > -1]$
FI II 815

28. $$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1}\, dx}{\ln x} = \ln \frac{(p+q+s)(p+r+s)(q+r+s)s}{(p+s)(q+s)(r+s)(p+q+r+s)}$$
$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$
BI (123)(10)

29. $$\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{s-1}\, dx}{(1-x^r)\ln x} = \ln \frac{\Gamma\left(\frac{p+s}{r}\right)\Gamma\left(\frac{q+s}{r}\right)}{\Gamma\left(\frac{s}{r}\right)\Gamma\left(\frac{p+q+s}{r}\right)}$$
$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$
GW (324)(23a)

30. $$\int_0^\infty (1 - x^p)(1 - x^q) \frac{x^{s-1}\, dx}{(1-x^{p+q+2s})\ln x} = 2\int_0^1 (1 - x^p)(1 - x^q) \frac{x^{s-1}\, dx}{(1-x^{p+q+2s})\ln x}$$
$$= 2\ln\left(\sin \frac{s\pi}{p+q+2s} \operatorname{cosec} \frac{(p+s)\pi}{p+q+2s}\right)$$
$[s > 0, \quad s + p > 0, \quad s + p + q > 0]$ GW (324)(23b)a

31. $$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1}\, dx}{(1-x)\ln x} = \ln \frac{\Gamma(p+s)\,\Gamma(q+s)\,\Gamma(r+s)\,\Gamma(p+q+r+s)}{\Gamma(p+q+s)4\,\Gamma(p+r+s)\,\Gamma(q+r+s)\,\Gamma(s)}$$
$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$* BI (127)(11)

32. $$\int_0^1 (1 - x^p)(1 - x^q)(1 - x^r) \frac{x^{s-1}\, dx}{(1-x^t)\ln x} = \ln \frac{\Gamma\left(\frac{p+s}{t}\right)\Gamma\left(\frac{q+s}{t}\right)\Gamma\left(\frac{r+s}{t}\right)\Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right)\Gamma\left(\frac{q+r+s}{t}\right)\Gamma\left(\frac{p+r+s}{t}\right)\Gamma\left(\frac{s}{t}\right)}$$
$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0, \quad t > 0]$* GW (324)(23b)

*In 4.267.31 the restrictions can be somewhat weakened by writing, for example, $s > 0$, $p + s > 0$, $q + s > 0$, $r + s > 0$, $p + q + s > 0$, $p + r + s > 0$, $q + r + s > 0$, $p + q + r + s > 0$, in **4.267** 31 and 32.

33. $$\int_0^1 \left\{\frac{x^p - x^{p+q}}{1-x} - q\right\} \frac{dx}{\ln x} = \ln \frac{\Gamma(p+q+1)}{\Gamma(p+1)}$$ $[p > -1, \quad p+q > -1]$ BI (127)(19)

34. $$\int_0^1 \left\{\frac{x^\mu - x}{x-1} - x(\mu-1)\right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu)$$ $[\operatorname{Re}\mu > 0]$ WH, BI (127)(18)

35. $$\int_0^1 \left\{1 - x - \frac{(1-x^p)(1-x^q)}{1-x}\right\} \frac{dx}{x \ln x} = -\ln\{\mathrm{B}(p,q)\}$$ $[p > 0, \quad q > 0]$ BI (130)(18)

36. $$\int_0^1 \left\{\frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)}\right\} \frac{dx}{\ln x} = q \ln p$$ $[p > 0]$ BI (130)(20)

37. $$\int_0^1 \left\{\frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1}\right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left(pq - \frac{1}{2}\right) \ln p$$ $[p > 0, \quad q > 0]$ BI (130)(22)

38. $$\int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x)\ln x} dx = \ln \mathrm{B}(p,q)$$ $[p > 0, \quad q > 0]$ GW (324)(24)

39.[6] $$\int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^{n} \binom{n}{n-k} (-1)^{n-k} \ln(pk+1)$$ $[n > 0, \quad pn > -1]$ GW (324)(19d), BI (123)(12)a

40.[6] $$\int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln \Gamma[(n-k)p+1]$$ $[n > 1, \quad pn > -1]$ BI (127)(12)

41. $$\int_0^1 (x^p - 1)^n x^{q-1} \frac{dx}{\ln x} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln[q + (n-k)p]$$ $[n > 0, \quad q > 0, \quad pn > -q]$ BI (123)(12)

42.[6] $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x)\ln x} = \sum_{k=0}^{n} (-1)^{k-1} \ln \Gamma[(n-k)p+q]$$ $[n > 1, \quad q > 0, \quad pn > -q]$ BI (127)(13)

43.[10] $$\int_0^1 (x^p - 1)^n (x^q - 1)^m \frac{x^{r-1}\, dx}{\ln x} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \ln[r + (m-k)q + (n-j)p]$$ $[n \geq 0, \quad m \geq 0, \quad n+m > 0, \quad r > 0, \quad pn + qm + r > 0]$ BI (123)(16)

4.268

1. $$\int_0^1 \frac{(x^p - x^q)(1-x^r)}{(\ln x)^2} dx = (p+1)\ln(p+1) - (q+1)\ln(q+1) - (p+r+1)\ln(p+r+1) + (q+r+1)\ln(q+r+1)$$ $[p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1]$ GW (324)(26)

2. $$\int_0^1 (x^p - x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1)\ln(2p+1) + (2q+1)\ln(2q+1) - 2(p+q+1)\ln(p+q+1)$$
$$\left[p > -\tfrac{1}{2}, \quad q > -\tfrac{1}{2}\right]$$ GW (324)(26a)

3. $$\int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{(\ln x)^2}$$
$$= (p+q+1)\ln(p+q+1) + (q+r+1)\ln(q+r+1) + (p+r+1)\ln(p+r+1)$$
$$-(p+1)\ln(p+1) - (q+1)\ln(q+1) - (r+1)\ln(r+1) - (p+q+r)\ln(p+q+r)$$
$$[p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > 0]$$ BI (124)(4)

4. $$\int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q)$$
$$\left[q > 0, \quad p > -\frac{q}{n}\right]$$ BI (124)(14)

5. $$\int_0^1 (1-x^p)^n (1-x^q)^m x^{r-1} \frac{dx}{(\ln x)^2} = \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j}\right) \left(\sum_{k=0}^{m} (-1)^k \binom{m}{k}\right)$$
$$\times[(m-k)q + (n-j)p + r]\ln[(m-k)q + (n-j)p + r]$$
$$[r > 0, \quad mq + r > 0, \quad np + r > 0, \quad mq + np + r > 0]$$ BI (124)(8)

6. $$\int_0^1 \left[(q-r)x^{p-1} + (r-p)x^{q-1} + (p-q)x^{r-1}\right] \frac{dx}{(\ln x)^2}$$
$$= (q-r)p\ln p + (r-p)q\ln q + (p-q)r\ln r$$
$$[p > 0, \quad q > 0, \quad r > 0]$$ BI (124)(9)

7. $$\int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right.$$
$$\left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)}\right] \frac{dx}{(\ln x)^2} = \frac{1}{2}\left[\frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)}\right.$$
$$\left. + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)}\right]$$
$$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]$$ BI (124)(16)

4.269

1. $$\int_0^1 \sqrt{\ln \frac{1}{x}} \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}$$ BI (115)(33)

2.[11] $$\int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}\,(1+x^2)} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$$ BI (133)(2)

3. $$\int_0^1 \sqrt{\ln\frac{1}{x}}\, x^{p-1}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{p^3}}$$ $[p > 0]$ GW (324)(1c)

4. $$\int_0^1 \frac{x^{p-1}}{\sqrt{\ln\frac{1}{x}}}\, dx = \sqrt{\frac{\pi}{p}}$$ $[p > 0]$ BI (133)(1)

5. $$\int_0^1 \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1}\sin nt}{1 - 2x\cos t + x^2} \cdot \frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}\sum_{k=1}^{n} \frac{\sin kt}{\sqrt{k}}$$ $[|t| < \pi]$ BI (133)(5)

6. $$\int_0^1 \frac{\cos t - x - x^{n-1}\cos nt + x^n \cos[(n-1)t]}{1 - 2x\cos t + x^2} \cdot \frac{dx}{\sqrt{\ln\frac{1}{x}}} = \sqrt{\pi}\sum_{k=1}^{n-1} \frac{\cos kt}{\sqrt{k}}$$ $[|t| < \pi]$ BI (133)(6)

7. $$\int_u^v \frac{dx}{x\cdot\sqrt{\ln\frac{x}{u}\ln\frac{v}{x}}} = \pi$$ $[uv > 0]$ BI (145)(37)

4.271

1. $$\int_0^1 (\ln x)^{2n}\, \frac{dx}{1+x} = \frac{2^{2n}-1}{2^{2n}} \cdot (2n)!\, \zeta(2n+1)$$ BI (110)(1)

2. $$\int_0^1 (\ln x)^{2n-1}\, \frac{dx}{1+x} = \frac{1-2^{2n-1}}{2n}\pi^{2n}|B_{2n}|$$ $[n = 1, 2, \ldots]$ BI (110)(2)

3. $$\int_0^1 (\ln x)^{2n-1}\, \frac{dx}{1-x} = -\frac{1}{n}2^{2n-2}\pi^{2n}|B_{2n}|$$ $[n = 1, 2, \ldots]$ BI (110)(5), GW(324)(9a)

4. $$\int_0^1 (\ln x)^{p-1}\, \frac{dx}{1-x} = e^{i(p-1)\pi}\, \Gamma(p)\, \zeta(p)$$ $[p > 1]$ GW (324)(9b)

5. $$\int_0^1 (\ln x)^{n}\, \frac{dx}{1+x^2} = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}$$ BI (110)(11)

6. $$\int_0^1 (\ln x)^{2n}\, \frac{dx}{1+x^2} = \frac{1}{2}\int_0^\infty (\ln x)^{2n}\, \frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}}|E_{2n}|$$ GW (324)(10)a

7. $$\int_0^\infty \frac{(\ln x)^{2n+1}}{1+bx+x^2}\, dx = 0$$ $[|b| < 2]$ BI (135)(2)

8. $$\int_0^1 (\ln x)^{2n}\, \frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}} \cdot (2n)!\, \zeta(2n+1)$$ $[n = 1, 2, \ldots]$ BI (110)(12)

9. $$\int_0^\infty (\ln x)^{2n}\, \frac{dx}{1-x^2} = 0$$ BI (312)(7)a

10. $$\int_0^1 (\ln x)^{2n-1}\, \frac{dx}{1-x^2} = \frac{1}{2}\int_0^\infty (\ln x)^{2n-1}\, \frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n}\pi^{2n}|B_{2n}|$$ $[n = 1, 2, \ldots]$ BI (290)(17)a, BI(312)(6)a

11. $$\int_0^1 (\ln x)^{2n-1} \frac{x\,dx}{1-x^2} = -\frac{1}{4n}\pi^{2n}|B_{2n}| \qquad [n=1,2,\ldots]$$ BI (290)(19)a

12. $$\int_0^1 (\ln x)^{2n} \frac{1+x^2}{(1-x^2)^2}\,dx = \frac{2^{2n}-1}{2}\pi^{2n}|B_{2n}| \qquad [n=1,2,\ldots]$$ BI (296)(17)a

13. $$\int_0^1 (\ln x)^{2n+1} \frac{(\cos 2a\pi - x)\;dx}{1-2x\cos 2a\pi + x^2} = -(2n+1)!\sum_{k=1}^{\infty}\frac{\cos 2ak\pi}{k^{2n+2}}$$
$[a \text{ is not an integer}]$ LI (113)(10)

14.[6] $$\int_0^\infty (\ln x)^n \frac{x^{\nu-1}\,dx}{a^2+2ax\cos t+x^2} = -\pi\operatorname{cosec} t \frac{d^n}{d\nu^n}\left[a^{\nu-2}\frac{\sin(\nu-1)t}{\sin\nu\pi}\right]$$
$[a>0, \quad 0<\operatorname{Re}\nu<2, \quad 0<|t|<\pi]$ ET I 315(12)

15. $$\int_0^1 (\ln x)^n \frac{x^{p-1}}{1-x^q}\,dx = -\frac{1}{q^{n+1}}\psi^{(n)}\left(\frac{p}{q}\right) \qquad [p>0, \quad q>0]$$ GW (324)(9)

16.[3] $$\int_0^1 (\ln x)^n \frac{x^{p-1}}{1+x^q}\,dx = \frac{1}{q^{n+1}}\beta^{(n)}\left(\frac{p}{q}\right) \qquad [p>0, \quad q>0]$$ GW (324)(10)

4.272

1. $$\int_0^1 \frac{\left[\ln\left(\frac{1}{x}\right)\right]^{q-1}\,dx}{1+2x\cos t+x^2} = \operatorname{cosec} t\,\Gamma(q)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kt}{k^q} \qquad [|t|<\pi, q<1]$$ LI (130)(1)

2. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{q-1} \frac{(1+x)\,dx}{1+2x\cos t+x^2} = \sec\frac{t}{2}\cdot\Gamma(q)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\left[\left(k-\frac{1}{2}\right)t\right]}{k^q}$$
$\left[|t|<\pi, \quad q<\frac{1}{2}\right]$ LI (130)(5)

3.[9] $$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\mu} \frac{x^{\nu-1}\,dx}{1-2ax\cos t+x^2a^2} = \frac{\Gamma(\mu+1)}{a\sin t}\sum_{k=1}^{\infty}\frac{a^k\sin kt}{(\nu+k-1)^{\mu+1}}$$
$[a>0, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0, \quad -\pi<t<\pi]$ BI (140)(14)a

4. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1} \frac{\cos\lambda - px}{1+p^2x^2-2px\cos\lambda}x^{q-1}\,dx = \Gamma(r)\sum_{k=1}^{\infty}\frac{p^{k-1}\cos k\lambda}{(q+k-1)^r}$$
$[r>0, \quad q>0]$ BI (113)(11)

5. $$\int_1^\infty (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p) \qquad [p>-1]$$ BI (149)(1)

6. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\Gamma(\mu) \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ BI (107)(3)

7. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-\frac{1}{2}} x^{\nu-1}\,dx = \frac{(2n-1)!!}{(2\nu)^n}\sqrt{\frac{\pi}{\nu}} \qquad [\operatorname{Re}\nu>0]$$ BI (107)(2)

8.[11] $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1+x}\,dx = \Gamma\left(3-\frac{1}{n}\right)\left(p^{\frac{1}{n}-3} - q^{\frac{1}{n}-3}\right)$$

$[\operatorname{Re}\nu > 0]$ BI (110)(4)

9. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1-x}\,dx = (n-1)!\,\zeta(n,\nu)$$ $[\operatorname{Re}\nu > 0]$ BI (110)(7)

10. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} (x-1)^n \left(a+\frac{nx}{x-1}\right) x^{a-1}\,dx = \Gamma(\mu)\sum_{k=0}^{n}\frac{(-1)^k n(n-1)\dots(n-k+1)}{(a+n-k)^{\mu-1}k!}$$

$[\operatorname{Re}\mu > 0]$ LI (110)(10)

11. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n-1} \frac{1-x^m}{1-x}\,dx = (n-1)!\sum_{k=1}^{m}\frac{1}{k^n}$$ LI (110)(9)

12. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{\mu-1} \frac{x^{\nu-1}\,dx}{1-x^2} = \Gamma(\mu)\sum_{k=0}^{\infty}\frac{1}{(\nu+2k)^\mu} = \frac{1}{2^\mu}\,\Gamma(\mu)\,\zeta\left(\mu,\frac{\nu}{2}\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (110)(13)

13. $$\int_0^1 \frac{x^q - x^{-q}}{1-x^2}\left(\ln\frac{1}{x}\right)^p dx = \Gamma(p+1)\sum_{k=1}^{\infty}\left\{\frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}}\right\}$$

$\left[p > -1, \quad q^2 < 1\right]$ LI (326)(12)a

14. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{r-1} \frac{x^{p-1}\,dx}{(1+x^q)^s} = \Gamma(r)\sum_{k=0}^{\infty}\binom{-s}{k}\frac{1}{(p+kq)^r}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad 0 < s < r+2]$ GW (324)(11)

15. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n} (1+x^q)^m\, x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{1}{(p+kq)^{n+1}}$$

$[p > 0, \quad q > 0]$ BI (107)(6)

16. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{n} (1-x^q)^m\, x^{p-1}\,dx = n!\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(p+kq)^{n+1}}$$

$[p > 0, \quad q > 0]$ BI (107)(7)

17. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \frac{1}{aq^p}\,\Gamma(p)\sum_{k=1}^{\infty}\frac{a^k}{k^p}$$ $[p > 0, \quad q > 0, \quad a < 1]$ LI (110)(8)

18. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{2-\frac{1}{n}} \left(x^{p-1} - x^{q-1}\right)\,dx = \frac{n}{n-1}\,\Gamma\left(\frac{1}{n}\right)\left(q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right)$$

$[q > p > 0]$ BI (133)(4)

19. $$\int_0^1 \left(\ln\frac{1}{x}\right)^{2n-1} \frac{x^p - x^{-p}}{1-x^q} x^{q-1}\,dx = \frac{1}{p^{2n}}\sum_{k=n}^{\infty}\left(\frac{2p\pi}{q}\right)^k \frac{|B_{2k}|}{2k\cdot(2k-2n)!}$$

$\left[p < \frac{q}{2}\right]$ LI (110)(16)

4.273 $$\int_u^v \left(\ln\frac{x}{u}\right)^{p-1}\left(\ln\frac{v}{x}\right)^{q-1}\frac{dx}{x} = \mathrm{B}(p,q)\left(\ln\frac{v}{u}\right)^{p+q-1}$$ $[p>0, \quad q>0, \quad uv>0]$ BI (145)(36)

4.274 $$\int_0^{\frac{1}{e}} \frac{\sqrt[q]{x}\,dx}{x\sqrt{-(1+\ln x)}} = \frac{\sqrt{q\pi}}{\sqrt[q]{e}}$$ $[q>0]$ BI (145)(4)

4.275

1. $$\int_0^1 \left[\left(\ln\frac{1}{x}\right)^{q-1} - x^{p-1}(1-x)^{q-1}\right]dx = \frac{\Gamma(q)}{\Gamma(p+q)}\left[\Gamma(p+q)-\Gamma(p)\right]$$ $[p>0, \quad q>0]$ BI (107)(8)

2. $$\int_0^1 \left[x - \left(\frac{1}{1-\ln x}\right)^q\right]\frac{dx}{x\ln x} = -\psi(q)$$ $[q>0]$ BI (126)(5)

4.28 Combinations of rational functions of $\ln x$ and powers

4.281

1. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1}{1-x}\right]dx = \boldsymbol{C}$$ BI (127)(15)

2. $$\int_1^\infty \frac{dx}{x^2(\ln p - \ln x)} = \frac{1}{p}\operatorname{li}(p)$$ LA 281(30)

3. $$\int_0^1 \frac{x^{p-1}\,dx}{q\pm\ln x} = \pm e^{\mp pq}\operatorname{Ei}(\pm pq)$$ $[p>0, \quad q>0]$ LI (144)(11,12)

4. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x}\right]dx = -\psi(\mu)$$ $[\operatorname{Re}\mu>0]$ WH

5. $$\int_0^1 \left[\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x}\right]dx = \ln p - \psi(q)$$ $[p>0, \quad q>0]$ BI (127)(17)

6. $$\int_0^1 \left[\frac{1}{1-x^2} + \frac{1}{2x\ln x}\right]\frac{dx}{\ln x} = \frac{\ln 2}{2}$$ LI (130)(19)

7. $$\int_0^1 \left[q - \frac{1}{2} + \frac{(1-x)(1+q\ln x) + x\ln x}{(1-x)^2}x^{q-1}\right]\frac{dx}{\ln x} = \frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2}$$ $[q>0]$ BI (128)(15)

4.282

1. $$\int_0^1 \frac{\ln x}{4\pi^2+(\ln x)^2}\cdot\frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2}\boldsymbol{C}$$ BI (129)(1)

2. $$\int_0^1 \frac{1}{a^2+(\ln x)^2}\cdot\frac{dx}{1+x^2} = \frac{1}{2a}\beta\left(\frac{2a+\pi}{4\pi}\right)$$ $\left[a>-\frac{\pi}{2}\right]$ BI (129)(9)

3. $$\int_0^1 \frac{1}{\pi^2+(\ln x)^2}\frac{dx}{1+x^2} = \frac{4-\pi}{4\pi}$$ BI (129)(6)

4. $$\int_0^1 \frac{\ln x}{\pi^2+(\ln x)^2}\cdot\frac{dx}{1-x^2} = \frac{1}{2}\left(\frac{1}{2} - \ln 2\right)$$ BI (129)(10)

5. $$\int_0^1 \frac{\ln x}{a^2 + (\ln x)^2} \cdot \frac{x\,dx}{1 - x^2} = \frac{1}{2}\left[\frac{\pi}{2a} + \ln\frac{\pi}{a} + \psi\left(\frac{a}{\pi}\right)\right] \qquad [a > 0]$$ BI (129)(14)

6. $$\int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{x\,dx}{1 - x^2} = \frac{1}{2}\left(\frac{1}{2} - \boldsymbol{C}\right)$$ BI (129)(13)

7. $$\int_0^1 \frac{1}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1 + x^2} = \frac{\ln 2}{4\pi}$$ BI (129)(7)

8. $$\int_0^1 \frac{\ln x}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1 - x^2} = \frac{2 - \pi}{16}$$ BI (129)(11)

9.[10] $$\int_0^1 \frac{1}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1 + x^2} = \frac{1}{8\pi\sqrt{2}}\left[\pi + 2\ln\left(\sqrt{2} - 1\right)\right]$$ BI (129)(8)

10. $$\int_0^1 \frac{\ln x}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1 - x^2} = -\frac{\pi}{32\sqrt{2}} + \frac{1}{16} + \frac{1}{16\sqrt{2}}\ln\left(\sqrt{2} - 1\right)$$ BI (129)(12)

11. $$\int_0^1 \frac{\ln x}{\left[a^2 + (\ln x)^2\right]^2} \frac{dx}{1 - x} = -\frac{\pi^2}{a^4}\sum_{k=1}^{\infty} |B_{2k}| \left(\frac{2\pi}{a}\right)^{2k-2}$$ BI (129)(4)

12. $$\int_0^1 \frac{\ln x}{\left[a^2 + (\ln x)^2\right]^2} \frac{x\,dx}{1 - x^2} = -\frac{\pi^2}{4a^4}\sum_{k=1}^{\infty} |B_{2k}| \left(\frac{\pi}{a}\right)^{2k-2}$$ BI (129)(16)

13. $$\int_0^1 \frac{x^p - x^{-p}}{x^2 - 1} \frac{dx}{q^2 + (\ln x)^2} = \frac{2\pi}{q}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kp\pi}{2q + k\pi} \qquad \left[p^2 < 1\right]$$ BI (132)(13)a

4.283

1. $$\int_0^1 \left(\frac{x - 1}{\ln x} - x\right)\frac{dx}{\ln x} = \ln 2 - 1$$ BI (132)(17)a

2. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1 - x} - \frac{1}{2}\right)\frac{dx}{\ln x} = \frac{\ln 2\pi}{2} - 1$$ BI (127)(20)

3. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{x}{1 - x} + \frac{x}{2}\right)\frac{dx}{x \ln x} = \frac{\ln 2\pi}{2}$$ BI (127)(23)

4. $$\int_0^1 \left[\frac{1}{(\ln x)^2} - \frac{x}{(1 - x)^2}\right] dx = \boldsymbol{C} - \frac{1}{2}$$ GW (326)(8a)

5. $$\int_0^1 \left(\frac{1}{1 - x^2} + \frac{1}{2\ln x} - \frac{1}{2}\right)\frac{dx}{\ln x} = \frac{\ln 2 - 1}{2}$$ BI (128)(14)

6. $$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{2} \cdot \frac{1 + x}{1 - x} - \ln x\right)\frac{dx}{\ln x} = \frac{\ln 2\pi}{2}$$ BI (127)(22)

7. $$\int_0^1 \left[\frac{1}{1 - \ln x} - x\right]\frac{dx}{x \ln x} = -\boldsymbol{C}$$ GW (326)(11a)

8. $$\int_0^1 \left[\frac{x^q - 1}{x(\ln x)^2} - \frac{q}{\ln x}\right] dx = q \ln q - q \qquad [q > 0]$$ BI (126)(2)

9. $$\int_0^1 \left[x + \frac{1}{a\ln x - 1}\right]\frac{dx}{x\ln x} = \ln\frac{a}{q} + \boldsymbol{C}$$ $[a > 0, \quad q > 0]$ BI (126)(8)

10. $$\int_0^1 \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)}\right]\frac{x^{p-1}}{\ln x}\,dx = -\ln\Gamma(p) + \left(p - \frac{1}{2}\right)\ln p - p + \frac{\ln 2\pi}{2}$$ $[p > 0]$ GW (326)(9)

11. $$\int_0^1 \left[p - 1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x}\right)x^{p-1}\right]\frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)\ln p + p - \frac{\ln 2\pi}{2}$$ $[p > 0]$ BI (127)(25)

12. $$\int_0^1 \left[-\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2}\right]dx = -\psi(p) + p - \frac{3}{2}$$ $[p > 0]$ GW (326)(8)

13. $$\int_0^1 \left[\left(p - \frac{1}{2}\right)x^3 + \frac{1}{2}\left(1 - \frac{1}{\ln x}\right)\left(x^{2p-1} - 1\right)\right]\frac{dx}{\ln x} = \left(\frac{1}{2} - p\right)(\ln p - 1)$$ $[p > 0]$ BI (132)(23)a

14. $$\int_0^1 \left[\left(q - \frac{1}{2}\right)\frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r}\right]\frac{dx}{\ln x} = (p-r)\left[\frac{1}{2} - q - \ln\Gamma(q) + \frac{\ln 2\pi}{2}\right]$$ $[q > 0]$ BI (132)(13)

4.284

1. $$\int_0^1 \left[\frac{x^q - 1}{x(\ln x)^3} - \frac{q}{x(\ln x)^2} - \frac{q^2}{2\ln x}\right]dx = \frac{q^2}{2}\ln q - \frac{3}{4}q^2$$ $[q > 0]$ BI (126)(3)

2. $$\int_0^1 \left[\frac{x^q - 1}{x(\ln x)^4} - \frac{q}{x(\ln x)^3} - \frac{q^2}{2x(\ln x)^2} - \frac{q^3}{6\ln x}\right]dx = \frac{q^3}{6}\ln q - \frac{11}{36}q^3$$ $[q > 0]$ BI (126)(4)

4.285 $$\int_0^1 \frac{x^{p-1}\,dx}{(q + \ln x)^n} = \frac{p^{n-1}}{(n-1)!}e^{-pq}\,\mathrm{Ei}(pq) - \frac{1}{(n-1)!q^{n-1}}\sum_{k=1}^{n-1}(n-k-1)!(pq)^{k-1}$$ $[p > 0, \quad q < 0]$ BI (125)(21)

In integrals of the form $\int \frac{x^a (\ln x)^n\,dx}{\left[b \pm (\ln x)^m\right]^l}$, we should make the substitution $x = e^t$ or $x = e^{-t}$ and then seek the resulting integrals in **3.351–3.356**.

4.29–4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.291

1. $$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$ FI II 483

2. $$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$ FI II 714

3. $$\int_0^{1/2} \frac{\ln(1-x)}{x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (145)(2)

4. $$\int_0^1 \ln\left(1-\frac{x}{2}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (114)(18)

5. $$\int_0^1 \frac{\ln\frac{1+x}{2}}{1-x}\,dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$$ BI (115)(1)

6. $$\int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \frac{1}{2}(\ln 2)^2$$ BI (114)(14)a

7.[7] $$\int_0^\infty \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{\pi}{4}\ln\left(1+a^2\right) - \int_0^a \frac{\ln u\,du}{1+u^2} \qquad [a>0]$$ GI II (2209)

8. $$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$ FI II 157

9. $$\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{4}\ln 2 + \boldsymbol{G}$$ BI (136)(1)

10. $$\int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2 - \boldsymbol{G}$$ BI (114)(17)

11. $$\int_1^\infty \frac{\ln(x-1)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$ BI (144)(4)

12. $$\int_0^1 \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2$$ BI (144)(4)

13. $$\int_0^\infty \frac{\ln(1+x)}{x(1+x)}\,dx = \frac{\pi^2}{6}.$$ BI (141)(9)a

14. $$\int_0^1 \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{1}{a(a-b)}\ln\frac{a+b}{b} + \frac{2\ln 2}{b^2-a^2} \qquad [a \neq b, \quad ab>0]$$
$$= \frac{1}{2a^2}(1-\ln 2) \qquad [a=b]$$ LI (114)(5)a

15. $$\int_0^\infty \frac{\ln(1+x)}{(ax+b)^2}\,dx = \frac{\ln\frac{a}{b}}{a(a-b)} \qquad [ab>0]$$ BI (139)(5)

16. $$\int_0^1 \ln(a+x)\frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}}\operatorname{arccot}\sqrt{a}\ln[(1+a)a] \qquad [a>0]$$ BI (114)(20)

17. $$\int_0^\infty \ln(a+x)\frac{dx}{(b+x)^2} = \frac{a\ln a - b\ln b}{b(a-b)} \qquad [a>0, \quad b>0, \quad a \neq b]$$ LI (139)(6)

18. $$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx = \frac{1}{2}\arctan a\ln\left(1+a^2\right)$$ GI II (2195)

19. $$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\,dx = \frac{1}{2\sqrt{a}} \arctan\sqrt{a}\ln(1+a) \qquad [a>0]$$ BI (114)(21)

20. $$\int_0^1 \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[\frac{1}{2}(a+b)\ln(a+b) - b\ln b - a\ln 2\right]$$
$[a>0, \quad b>0, \quad a \neq b]$ BI (114)(22)

21. $$\int_0^\infty \frac{\ln(ax+b)}{(1+x)^2}\,dx = \frac{1}{a-b}\left[a\ln a - b\ln b\right] \qquad [a>0, \quad b>0]$$ BI (139)(8)

22. $$\int_0^\infty \ln(a+x)\frac{x\,dx}{\left(b^2+x^2\right)^2} = \frac{1}{2\left(a^2+b^2\right)}\left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right)$$
$[a>0, \quad b>0]$ BI (139)(9)

23. $$\int_0^1 \ln(1+x)\frac{1+x^2}{(1+x)^4}\,dx = -\frac{1}{3}\ln 2 + \frac{23}{72}$$ LI (114)(12)

24. $$\int_0^1 \ln(1+x)\frac{1+x^2}{a^2+x^2}\cdot\frac{dx}{1+a^2x^2} = \frac{1}{2a\left(1+a^2\right)}\left[\frac{\pi}{2}\ln\left(1+a^2\right) - 2\arctan a\cdot\ln a\right]$$
$[a>0]$ LI (114)(11)

25. $$\int_0^1 \ln(1+x)\frac{1-x^2}{(ax+b)^2}\frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2}\left\{\frac{1}{a-b}\left[\frac{a+b}{ab}\ln(a+b) - \frac{1}{a}\ln b - \frac{1}{b}\ln a\right] + \frac{4\ln 2}{b^2-a^2}\right\}$$
$\left[a>0, \quad b>0, \quad a^2 \neq b^2\right]$ LI (114)(13)

26. $$\int_0^\infty \ln(1+x)\frac{1-x^2}{(ax+b)^2}\cdot\frac{dx}{(bx+a)^2} = \frac{1}{ab\left(a^2-b^2\right)}\ln\frac{b}{a}$$
$[a>0, \quad b>0]$ LI (139)(14)

27. $$\int_0^1 \ln(1+ax)\frac{1-x^2}{\left(1+x^2\right)^2}\,dx = \frac{1}{2}\frac{(1+a)^2}{1+a^2}\ln(1+a) - \frac{1}{2}\cdot\frac{a}{1+a^2}\ln 2 - \frac{\pi}{4}\cdot\frac{a^2}{1+a^2}$$
$[a>-1]$ BI (114)(23)

28. $$\int_0^\infty \ln(a+x)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx = \frac{1}{a^2+b^2}\left(a\ln\frac{b}{a} - \frac{b\pi}{2}\right)$$
$[a>0, \quad b>0]$ BI (139)(11)

29. $$\int_0^\infty \ln^2(a-x)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx = \frac{2}{a^2+b^2}\left(a\ln\frac{a}{b} - \frac{b\pi}{2}\right)$$
$[a>0, \quad b>0]$ BI (139)(12)

30. $$\int_0^\infty \ln^2(a-x)\frac{x\,dx}{\left(b^2+x^2\right)^2} = \frac{1}{a^2+b^2}\left(\ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2}\ln a\right)$$
$[a>0, \quad b>0]$ BI (139)(10)

4.292

1. $$\int_0^1 \frac{\ln(1 \pm x)}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\ln 2 \pm 2\boldsymbol{G}$$ GW (325)(20)

2. $$\int_0^1 \frac{x\ln(1 \pm x)}{\sqrt{1-x^2}}\,dx = -1 \pm \frac{\pi}{2}$$ GW (325)(22c)

3. $$\int_{-a}^a \frac{\ln(1+bx)}{\sqrt{a^2-x^2}}\,dx = \pi \ln \frac{1+\sqrt{1-a^2b^2}}{2} \qquad \left[0 \le |b| \le \frac{1}{a}\right]$$ BI (145)(16, 17)a, GW (325)(21e)

4. $$\int_0^1 \frac{x\ln(1+ax)}{\sqrt{1-x^2}}\,dx = -1 + \frac{\pi}{2}\cdot\frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a}\arcsin a \qquad [|a| \le 1]$$
$$= -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a}\ln\left(a+\sqrt{a^2-1}\right) \qquad [a \ge 1]$$ GW (325)(22)

5. $$\int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin a\,(\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2}(\arccos a)^2$$
$[|a| \le 1]$ BI (120)(4), GW (325)(21a)

4.293

1. $$\int_0^1 x^{\mu-1}\ln(1+x)\,dx = \frac{1}{\mu}\left[\ln 2 - \beta(\mu+1)\right] \qquad [\operatorname{Re}\mu > -1]$$ BI (106)(4)a

2.[6] $$\int_1^\infty x^{\mu-1}\ln(1+x)\,dx = \frac{-1}{\mu}\left[\beta(-\mu) + \ln 2\right] \qquad [\operatorname{Re}\mu < 0]$$ ET I 315(17)

3. $$\int_0^\infty x^{\mu-1}\ln(1+x)\,dx = \frac{\pi}{\mu\sin\mu\pi} \qquad [-1 < \operatorname{Re}\mu < 0]$$ GW (325)(3)a

4. $$\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}$$ GW (325)(2b)

5. $$\int_0^1 x^{2n}\ln(1+x)\,dx = \frac{1}{2n+1}\left[\ln 4 + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right]$$ GW (325)(2c)

6.[11] $$\int_0^1 x^{n-\frac{1}{2}}\ln(1+x)\,dx = \frac{2\ln 2}{2n+1} + \frac{(-1)^n\cdot 4}{2n+1}\left[\frac{\pi}{4} - \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\right]$$ GW (325)(2f)

7. $$\int_0^\infty x^{\mu-1}\ln|1-x|\,dx = \frac{\pi}{\mu}\cot(\mu\pi) \qquad [-1 < \operatorname{Re}\mu < 0]$$ BI (134)(4), ET I 315(18)

8. $$\int_0^1 x^{\mu-1}\ln(1-x)\,dx = -\frac{1}{\mu}\left[\psi(\mu+1) - \psi(1)\right] = -\frac{1}{\mu}\left[\psi(\mu+1) + \boldsymbol{C}\right]$$
$[\operatorname{Re}\mu > -1]$ ET I 316(19)

9.[7] $$\int_1^\infty x^{\mu-1}\ln(x-1)\,dx = \frac{1}{\mu}\left[\pi\cot(\mu\pi) + \psi(\mu+1) + \boldsymbol{C}\right]$$
$[\operatorname{Re}\mu < 0]$ ET I 316(20)

10. $$\int_0^\infty x^{\mu-1}\ln(1+\gamma x)\,dx = \frac{\pi}{\mu\gamma^\mu\sin\mu\pi} \qquad [-1<\operatorname{Re}\mu<0, \quad |\arg\gamma|<\pi]$$ BI (134)(3)

11.[11] $$\int_0^\infty \frac{x^{\mu-1}\ln(1+x)}{1+x}\,dx = -\frac{\pi}{\sin\mu\pi}\left[\boldsymbol{C}+\psi(1-\mu)\right] \qquad [-1<\operatorname{Re}\mu<1]$$ ET I 316(21)

12. $$\int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}}\,dx = -\frac{\ln 2}{2^\mu\mu}+\frac{2^\mu-1}{2^\mu\mu^2}$$ BI (114)(6)

13. $$\int_0^1 \frac{x^{\mu-1}\ln(1-x)}{(1-x)^{1-\nu}}\,dx = \mathrm{B}(\mu,\nu)\left[\psi(\nu)-\psi(\mu+\nu)\right] \qquad [\operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0]$$ ET I 316(122)

14. $$\int_0^\infty \frac{x^{\mu-1}\ln(\gamma+x)}{(\gamma+x)^\nu}\,dx = \gamma^{\mu-\nu}\,\mathrm{B}(\mu,\nu-\mu)\left[\psi(\nu)-\psi(\nu-\mu)+\ln\gamma\right]$$
$[0<\operatorname{Re}\mu<\operatorname{Re}\nu]$ ET I 316(23)

4.294

1. $$\int_0^1 \ln(1+x)\frac{(p-1)x^{p-1}-px^{-p}}{x}\,dx = 2\ln 2-\frac{\pi}{\sin p\pi}$$
$[0<p<1]$ BI (114)(2)

2. $$\int_0^1 \ln(1+x)\frac{1+x^{2n+1}}{1+x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1}-\sum_{j=1}^{2n+1}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(7)

3. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1+x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1}-\sum_{j=1}^{2n}\frac{1}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(8)

4. $$\int_0^1 \ln(1+x)\frac{1-x^{2n}}{1-x}\,dx = 2\ln 2\cdot\sum_{k=0}^{n-1}\frac{1}{2k+1}+\sum_{i=1}^{2n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(9)

5. $$\int_0^1 \ln(1+x)\frac{1-x^{2n+1}}{1-x}\,dx = 2\ln 2\sum_{k=0}^{n}\frac{1}{2k+1}+\sum_{j=1}^{2n+1}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}$$ BI (114)(10)

6. $$\int_0^1 \ln(1-x)\frac{1-(-1)^n x^n}{1-x}\,dx = \sum_{j=1}^{n}\frac{(-1)^j}{j}\sum_{k=1}^{j}\frac{1}{k}$$ BI (114)(15)

7. $$\int_0^1 \ln(1-x)\frac{1-x^n}{1-x}\,dx = -\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j}\frac{1}{k}$$ BI (114)(16)

8. $$\int_0^\infty \ln^2(1-x)x^p\,dx = \frac{2\pi}{p+1}\cot p\pi \qquad [-2<p<-1]$$ BI (134)(13)a

9. $$\int_0^1 \left[\ln(1+x)\right]^n(1+x)^r\,dx = (-1)^{n-1}\frac{n!}{(r+1)^{n+1}}+2^{r+1}\sum_{k=0}^{n}\frac{(-1)^k n!\,(\ln 2)^{n-k}}{(n-k)!(r+1)^{k+1}}$$ LI (106)(34)a

10. $$\int_0^1 \left[\ln(1-x)\right]^n(1-x)^r\,dx = (-1)^n\frac{n!}{(r+1)^{n+1}} \qquad [r>-1]$$ BI (106)(35)a

11. $\int_0^1 \left(\ln \frac{1}{1-x^2}\right)^n x^{2q-1}\,dx = \frac{n!}{2}\,\zeta(n+1, q+1)$ $[-1 < q < 0]$ BI (311)(15)a

12. $\int_0^1 (\ln x)^{2n} \ln\left(1-x^2\right) \frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)} |B_{2n+2}|$ BI (309)(5)a

13.[6] $\int_0^1 \left[\ln \frac{1}{x}\right]^m \ln\left(1-x^2\right)\,dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}}$ $[m+1 > 0, \quad n+1 > 0]$

4.295

1. $\int_0^\infty \ln\left(\mu x^2 + \beta\right) \frac{dx}{\gamma + x^2} = \frac{\pi}{\sqrt{\gamma}} \ln\left(\sqrt{\mu\gamma} + \sqrt{\beta}\right)$ $[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg\gamma| < \pi]$ ET II 218(27)

2. $\int_0^1 \ln\left(1+x^2\right) \frac{dx}{x^2} = \frac{\pi}{2} - \ln 2$ GW (325)(2g)

3. $\int_0^\infty \ln\left(1+x^2\right) \frac{dx}{x^2} = \pi$ GW (325)(4c)

4. $\int_0^\infty \ln\left(1+x^2\right) \frac{dx}{(a+x)^2} = \frac{2a}{1+a^2}\left(\frac{\pi}{2a} + \ln a\right)$ $[a > 0]$ BI (319)(6)a

5. $\int_0^1 \ln\left(1+x^2\right) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 - \boldsymbol{G}$ BI (114)(24)

6. $\int_1^\infty \ln\left(1+x^2\right) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 + \boldsymbol{G}$ BI (114)(5)

7. $\int_0^\infty \ln\left(a^2 + b^2x^2\right) \frac{dx}{c^2 + g^2x^2} = \frac{\pi}{cg} \ln \frac{ag + bc}{g}$ $[a > 0, \quad b > 0, \quad c > 0, \quad g > 0]$ BI (136)(11-14)a

8. $\int_0^\infty \ln\left(a^2 + b^2x^2\right) \frac{dx}{c^2 - g^2x^2} = -\frac{\pi}{cg} \arctan \frac{bc}{ag}$ $[a > 0, \quad b > 0, \quad c > 0, \quad g > 0]$ BI (136)(15)a

9. $\int_0^\infty \frac{\ln\left(1+p^2x^2\right) - \ln\left(1+q^2x^2\right)}{x^2}\,dx = \pi(p-q)$ $[p > 0, \quad q > 0]$ FI II 645

10. $\int_0^1 \ln \frac{1+a^2x^2}{1+a^2} \frac{dx}{1-x^2} = -\left(\arctan a\right)^2$ BI (115)(2)

11. $\int_0^1 \ln\left(1-x^2\right) \frac{dx}{x} = -\frac{\pi^2}{12}$

12. $\int_0^\infty \ln^2\left(1-x^2\right) \frac{dx}{x^2} = 0$ BI (142)(9)a

13. $\int_0^1 \ln\left(1-x^2\right) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 - \boldsymbol{G}$ GW (325)(17)

14. $\int_1^\infty \ln\left(x^2-1\right) \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2 + \boldsymbol{G}$ BI (144)(6)

15. $$\int_0^\infty \ln^2\left(a^2-x^2\right)\frac{dx}{b^2+x^2}=\frac{\pi}{b}\ln\left(a^2+b^2\right) \qquad [b>0]$$ BI (136)(16)

16. $$\int_0^\infty \ln^2\left(a^2-x^2\right)\frac{b^2-x^2}{\left(b^2+x^2\right)^2}\,dx=-\frac{2b\pi}{a^2+b^2} \qquad [b>0]$$ BI (136)(20)

17. $$\int_0^1 \ln\left(1+x^2\right)\frac{dx}{x\left(1+x^2\right)}=\frac{1}{2}\left[\frac{\pi^2}{12}-\frac{1}{2}\left(\ln 2\right)^2\right]$$ BI (114)(25)

18. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{x\left(1+x^2\right)}=\frac{\pi^2}{12}$$ BI (141)(9)

19. $$\int_0^1 \ln\left(\cos^2 t+x^2\sin^2 t\right)\frac{dx}{1-x^2}=-t^2$$ BI (114)(27)a

20. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{dx}{(c+gx)^2}=\frac{2\ln b}{cg}+\frac{b^2}{a^2g^2+b^2c^2}\left(\frac{a}{b}\pi+2\frac{c}{g}\ln\frac{c}{g}+2\frac{a^2g}{b^2c}\ln\frac{a}{b}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0]$$
BI (139)(16)a

21. $$\int_0^1 \ln\left(a^2+b^2x^2\right)\frac{dx}{(c+gx)^2}$$
$$=\frac{2}{c(c+g)}\ln a+\frac{b^2}{a^2g^2+b^2c^2}\left[\frac{2a}{b}\operatorname{arccot}\frac{a}{b}+\frac{cb^2-ga^2}{b^2(c+g)}\ln\frac{a^2+b^2}{a^2}-2\frac{c}{g}\ln\frac{c+g}{c}\right]$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0]$$ BI (114)(28)a

22.[11] $$\int_0^\infty \frac{\ln\left(1+p^2x^2\right)}{r^2+q^2x^2}\,dx=\int_0^\infty \frac{\ln\left(p^2+x^2\right)}{q^2+r^2x^2}\,dx=\frac{\pi}{qr}\ln\frac{q+pr}{r}$$
$$[qr>0, \quad p>0]$$
FI II 745a, BI (318)(1)a, BI (318)(4)a

23. $$\int_0^\infty \frac{\ln\left(1+a^2x^2\right)}{b^2+c^2x^2}\frac{dx}{d^2+g^2x^2}=\frac{\pi}{b^2g^2-c^2d^2}\left[\frac{g}{d}\ln\left(1+\frac{ad}{g}\right)-\frac{c}{b}\ln\left(1+\frac{ab}{c}\right)\right]$$
$$\left[a>0, \quad b>0, \quad c>0, \quad d>0, \quad g>0, \quad b^2g^2\neq c^2d^2\right]$$ BI (141)(10)

24. $$\int_0^\infty \frac{\ln\left(1+a^2x^2\right)}{b^2+c^2x^2}\frac{x^2\,dx}{d^2+g^2x^2}=\frac{\pi}{b^2g^2-c^2d^2}\left[\frac{b}{c}\ln\left(1+\frac{ab}{c}\right)-\frac{d}{g}\ln\left(1+\frac{ad}{g}\right)\right]$$
$$\left[a>0, \quad b>0, \quad c>0, \quad d>0, \quad g>0, \quad b^2g^2\neq c^2d^2\right]$$ BI (141)(11)

25. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{dx}{\left(c^2+g^2x^2\right)^2}=\frac{\pi}{2c^3g}\left(\ln\frac{ag+bc}{g}-\frac{bc}{ag+bc}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0]$$
GW (325)(18a)

26. $$\int_0^\infty \ln\left(a^2+b^2x^2\right)\frac{x^2\,dx}{\left(c^2+g^2x^2\right)^2}=\frac{\pi}{2cg^3}\left(\ln\frac{ag+bc}{g}+\frac{bc}{ag+bc}\right)$$
$$[a>0, \quad b>0, \quad c>0, \quad g>0]$$
GW (325)(18b)

27. $$\int_0^1 \ln\left(1+ax^2\right)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}$$ $[a > 0]$ BI (117)(6)

28. $$\int_0^1 \ln\left(1+a-ax^2\right)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\left\{\ln\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right\}$$ $[a > 0]$ BI (117)(7)

29. $$\int_0^1 \ln\left(1-a^2x^2\right)\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1-a^2}}{2}$$ $\left[a^2 < 1\right]$ BI (119)(1)

30.[6] $$\int_0^1 \ln\left(1-a^2x^2\right)\frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos|a| - \frac{\pi}{2}\right)^2$$ LI (120)(11)

31. $$\int_0^1 \ln\left(1-x^2\right)\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \ln\frac{k'}{k}\boldsymbol{K}(k) - \frac{\pi}{2}\boldsymbol{K}\left(k'\right)$$ BI (120)(12)

32. $$\int_0^1 \ln\left(1\pm kx^2\right)\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \frac{1}{2}\ln\frac{2\pm 2k}{\sqrt{k}}\boldsymbol{K}(k) - \frac{\pi}{8}\boldsymbol{K}\left(k'\right)$$ BI (120)(8), BI (120)(14)

33. $$\int_0^1 \frac{\ln\left(1-k^2x^2\right)}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}\,dx = \ln k'\,\boldsymbol{K}(k)$$ BI (119)(27)

34. $$\int_0^1 \ln\left(1-k^2x^2\right)\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = \left(2-k^2\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$ BI (119)(3)

35. $$\int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}}\ln\left(1-k^2x^2\right)\,dx = \frac{1}{k^2}\left(1+k'^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$ BI (119)(7)

36. $$\int_{-1}^1 \ln\left(1-x^2\right)\frac{dx}{(a+bx)\sqrt{1-x^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$$ $[a > 0, \quad b > 0, \quad a \neq b]$ BI (145)(15)

37.[8] $$\int_0^1 \ln\left(1-x^2\right)\left(px^{p-1}-qx^{q-1}\right)\,dx = \psi\left(\frac{q}{2}+1\right) + \psi\left(\frac{p}{2}+1\right)$$ $[p > -2, \quad q > -2]$ BI (106)(15)

38. $$\int_0^1 \ln\left(1+ax^2\right)\frac{dx}{\sqrt{1-x^2}} = \pi\ln\frac{1+\sqrt{1+a}}{2}$$ $[a \geq -1]$ GW (325)(21b)

39. $$\int_0^1 \ln\left(1+x^2\right)x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln 2 - \beta\left(\frac{\mu}{2}+1\right)\right]$$ $[\operatorname{Re}\mu > -2]$ BI (106)(12)

40. $$\int_0^\infty \ln\left(1+x^2\right)x^{\mu-1}\,dx = \frac{\pi}{\mu\sin\frac{\mu\pi}{2}}$$ $[-2 < \operatorname{Re}\mu < 0]$ BI (311)(4)a, ET I 315(15)

41. $$\int_0^\infty \ln\left(1+x^2\right) \frac{x^{\mu-1}\,dx}{1+x} = \frac{\pi}{\sin\mu\pi}\left\{\ln 2 - (1-\mu)\sin\frac{\mu\pi}{2}\,\beta\left(\frac{1-\mu}{2}\right) - (2-\mu)\cos\frac{\mu\pi}{2}\,\beta\left(\frac{2-\mu}{2}\right)\right\}$$
$[-2 < \operatorname{Re}\mu < 1]$ ET I 316(25)

4.296

1. $$\int_0^1 \ln\left(1+2x\cos t+x^2\right)\frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2}$$ BI (114)(34)

2. $$\int_{-\infty}^\infty \ln\left(a^2-2ax\cos t+x^2\right)\frac{dx}{1+x^2} = \pi\ln\left(1+2a|\sin t|+a^2\right)$$ BI (145)(28)

3. $$\int_0^\infty \ln\left(1+2x\cos t+x^2\right)x^{\mu-1}\,dx = \frac{2\pi}{\mu}\frac{\cos\mu t}{\sin\mu\pi}$$ $[|t| < \pi, \quad -1 < \operatorname{Re}\mu < 0]$ ET I 316(27)

4. $$\int_0^\infty \ln\left(\frac{x^2+2ax\cos t+a^2}{x^2-2ax\cos t+a^2}\right)\frac{x\,dx}{x^2+b^2} = \frac{1}{2}\pi^2 - \pi t + \pi\arctan\frac{\left(a^2-b^2\right)\cos t}{\left(a^2+b^2\right)\sin t+2ab}$$
$[a > 0, \quad b > 0, \quad 0 < t < \pi]$

4.297

1. $$\int_0^1 \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = \frac{1}{a-b}\left[(a+b)\ln\frac{a+b}{2} - a\ln a - b\ln b\right]$$
$[a > 0, \quad b > 0]$ BI (115)(16)

2. $$\int_0^\infty \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = 0$$ $[ab > 0]$ BI (139)(23)

3. $$\int_0^1 \ln\frac{1-x}{x}\frac{dx}{1+x^2} = \frac{\pi}{8}\ln 2$$ BI (115)(5)

4. $$\int_0^1 \ln\frac{1+x}{1-x}\frac{dx}{1+x^2} = \boldsymbol{G}$$ BI (115)(17)

5.[11] $$\int_0^\infty \ln\left(\frac{1+x}{1-x}\right)^2\frac{dx}{x\left(1+x^2\right)} = \frac{\pi^2}{2}$$ BI (141)(13)

6. $$\int_u^v \ln\frac{v+x}{u+x}\frac{dx}{x} = \frac{1}{2}\left(\ln\frac{v}{u}\right)^2$$ $[uv > 0]$ BI (145)(33)

7. $$\int_0^\infty \frac{b\ln(1+ax) - a\ln(1+bx)}{x^2}\,dx = ab\ln\frac{b}{a}$$ $[a > 0, \quad b > 0]$ FI II 647

8. $$\int_0^1 \ln\frac{1+ax}{1-ax}\frac{dx}{x\sqrt{1-x^2}} = \pi\arcsin a$$ $[|a| \le 1]$ GW (325)(21c), BI (122)(2)

9. $$\int_u^v \ln\left(\frac{1+ax}{1-ax}\right)\frac{dx}{\sqrt{\left(x^2-u^2\right)\left(v^2-x^2\right)}} = \frac{\pi}{v}F\left(\arcsin av, \frac{u}{v}\right)$$
$[|av| < 1]$ BI (145)(35)

10.[8] $$\text{PV}\int_0^1 \ln\left|\frac{a+y}{a-y}\right|\frac{dy}{y\sqrt{1-y^2}} = \frac{\pi^2}{2}$$ $[0 < a \le 1]$

4.298

1. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1+x}\,dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1)$$ BI (137)(1)

2. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n}}{1+x}\,dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1)$$ BI (137)(3)

3. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1-x}\,dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1)$$ BI (137)(2)

4. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n}}{1-x}\,dx = -\frac{\ln 2}{2n} - \frac{1}{4n^2} + \frac{1}{2n}\beta(2n+1)$$ BI (137)(4)

5. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{x^{2n-1}}{1+x^2}\,dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n}\beta(2n+1)$$ BI (137)(10)

6. $$\int_0^1 \ln\frac{1+x^2}{x}x^{2n}\,dx = \frac{1}{2n+1}\left\{(-1)^n\frac{\pi}{2} + \ln 2 - \frac{1}{2n+1} + 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right\}$$ BI (294)(8)

7. $$\int_0^1 \ln\frac{1+x^2}{x}x^{2n-1}\,dx = \frac{1}{2n}\left\{(-1)^{n+1}\ln 2 + \ln 2 - \frac{1}{2n} + (-1)^{n+1}\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right\}$$ BI (294)(9)a

8. $$\int_0^1 \ln\frac{1+x^2}{x}\frac{dx}{1+x^2} = \frac{\pi}{2}\ln 2$$ BI (115)(7)

9. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{dx}{1+x^2} = \pi\ln 2$$ BI (137)(8)

10. $$\int_0^\infty \ln\frac{1+x^2}{x}\frac{dx}{1-x^2} = 0$$ BI (137)(9)

11. $$\int_0^1 \ln\frac{1-x^2}{x}\frac{dx}{1+x^2} = \frac{\pi}{4}\ln 2$$ BI (115)(9)

12. $$\int_1^\infty \ln\frac{1+x^2}{x+1}\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln 2$$ BI (144)(8)

13. $$\int_0^1 \ln\frac{1+x^2}{x+1}\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln 2 - \boldsymbol{G}$$ BI (115)(18)

14. $$\int_1^\infty \ln\frac{1+x^2}{x-1}\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln 2 + \boldsymbol{G}$$ BI (144)(9)

15. $$\int_0^1 \ln\frac{1+x^2}{1-x}\frac{dx}{1+x^2} = \frac{3\pi}{8}\ln 2$$ BI (115)(19)

16. $$\int_0^\infty \ln\frac{1+x^2}{x^2}\frac{x\,dx}{1+x^2} = \frac{\pi^2}{12}$$ BI (138)(3)

17. $$\int_0^\infty \ln\frac{a^2+b^2x^2}{x^2}\frac{dx}{c^2+g^2x^2} = \frac{\pi}{cg}\ln\frac{ag+bc}{c} \qquad [a>0, \quad b>0, \quad c>0, \quad g>0]$$ BI (138)(6, 7, 9, 10)a

18. $$\int_0^\infty \ln\frac{a^2+b^2x^2}{x^2}\frac{dx}{c^2-g^2x^2} = \frac{1}{cg}\arctan\frac{ag}{bc} \qquad [a>0, b>0, c>0, g>0]$$ BI (138)(8, 11)a

19. $$\int_0^\infty \ln\frac{1+x^2}{x^2}\frac{x^2\,dx}{\left(1+x^2\right)^2} = \frac{\pi}{4}(\ln 4 - 1)$$ BI (139)(21)

20. $$\int_0^1 \ln^2\left(\frac{1-x^2}{x^2}\right)\sqrt{1-x^2}\,dx = \pi$$ FI II 643a

21. $$\int_0^1 \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\frac{dx}{x} = \frac{1}{2}\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}\frac{dx}{x} = -\frac{t^2}{2}$$ $[|t| < \pi]$ BI (115)(23), BI (134)(15)

22. $$\int_0^\infty \ln\frac{1+2x\cos t+x^2}{(1+x)^2}x^{p-1}\,dx = -\frac{2\pi\,(1-\cos pt)}{p\sin p\pi}$$ $[0 < |p| < 1, \quad |t| < \pi]$ BI (134)(17)

23. $$\int_0^1 \ln\frac{1+x^2\sin t}{1-x^2\sin t}\frac{dx}{\sqrt{1-x^2}} = \pi\ln\cot\left(\frac{\pi-t}{4}\right)$$ $[|t| < \pi]$ GW (325)(21d)

4.299

1. $$\int_0^\infty \ln\frac{(x+1)\left(x+a^2\right)}{(x+a)^2}\frac{dx}{x} = (\ln a)^2$$ $[a > 0]$ BI (134)(14)

2. $$\int_0^1 \ln\frac{(1-ax)\left(1+ax^2\right)}{\left(1-ax^2\right)^2}\frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}}\arctan\sqrt{a}\ln(1+a)$$ $[a > 0]$ BI (115)(25)

3. $$\int_0^1 \ln\frac{\left(1-a^2x^2\right)\left(1+ax^2\right)}{\left(1-ax^2\right)^2}\frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}}\arctan\sqrt{a}\ln(1+a)$$ $[a > 0]$ BI (115)(26)

4. $$\int_0^1 \ln\frac{(x+1)\left(x+a^2\right)}{(x+a)^2}x^{\mu-1}\,dx = \frac{\pi\left(a^\mu-1\right)^2}{\mu\sin\mu\pi}$$ $[a > 0, \quad \operatorname{Re}\mu > 0]$ BI (134)(16)

4.311

1.[11] $$\int_0^\infty \frac{\ln\left(1+x^n\right)}{x^n}\,dx = \frac{\pi\operatorname{cosec}\left(\frac{\pi}{n}\right)}{n-1}$$ $n = 2, 3, \ldots$

2. $$\int_0^\infty \ln\left(1+x^3\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\ln 3$$ LI (136)(8)

3. $$\int_0^\infty \ln\left(1+x^3\right)\frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 - \frac{\pi^2}{9}$$ LI (136)(6)

4. $$\int_0^\infty \ln\left(1+x^3\right)\frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}}\ln 3 + \frac{\pi^2}{9}$$ LI (136)(7)

5. $$\int_0^\infty \ln\left(1+x^3\right)\frac{1-x}{1+x^3}\,dx = -\frac{2}{9}\pi^2$$ BI (136)(9)

6.[8] $$\int_0^\infty \left|1-\frac{x^3}{a^3}\right|\frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$$

4.312

1. $$\int_0^\infty \ln \frac{1+x^3}{x^3} \frac{dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9}$$ BI (138)(12)

2. $$\int_0^\infty \ln \frac{1+x^3}{x^3} \frac{x\,dx}{1+x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9}$$ BI (138)(13)

4.313

1. $$\int_0^\infty \ln x \ln\left(1+a^2x^2\right) \frac{dx}{x^2} = \pi a\left(1-\ln a\right) \qquad [a>0]$$ BI (134)(18)

2. $$\int_0^\infty \ln\left(1+c^2x^2\right) \ln\left(a^2+b^2x^2\right) \frac{dx}{x^2} = 2\pi\left[\left(c+\frac{b}{a}\right)\ln(b+ac) - \frac{b}{a}\ln b - c\ln c\right]$$
$$[a>0, \quad b>0, \quad c>0]$$ BI (134)(20, 21)a

3. $$\int_0^\infty \ln\left(1+c^2x^2\right) \ln\left(a^2+\frac{b^2}{x^2}\right) \frac{dx}{x^2} = 2\pi\left[\frac{a+bc}{b}\ln(a+bc) - \frac{a}{b}\ln a - c\right]$$
$$[a>0, \quad a+bc>0]$$ BI (134)(22, 23)a

4. $$\int_0^\infty \ln x \ln\frac{1+a^2x^2}{1+b^2x^2} \frac{dx}{x^2} = \pi(a-b) + \pi\ln\frac{b^b}{a^a} \qquad [a>0, \quad b>0]$$ BI (134)(24)

5. $$\int_0^\infty \ln x \ln\frac{a^2+2bx+x^2}{a^2-2bx+x^2} \frac{dx}{x} = 2\pi\ln a \arcsin\frac{b}{a} \qquad [a \ge |b|]$$ BI (134)(25)

6. $$\int_0^\infty \ln(1+x)\frac{x\ln x - x - a}{(x+a)^2}\frac{dx}{x} = \frac{(\ln a)^2}{2(a-1)} \qquad [a>0]$$ BI (141)(7)

7. $$\int_0^\infty \ln^2(1-x)\frac{x\ln x - x - a}{(x+a)^2}\frac{dx}{x} = \frac{\pi^2+(\ln a)^2}{1+a} \qquad [a>0]$$ LI (141)(8)

4.314

1.[11] $$\int_0^1 \ln(1+ax)\frac{x^{p-1}-x^{q-1}}{\ln x}\,dx = \sum_{k=1}^\infty (-1)^{k+1}\frac{a^k}{k}\ln\frac{p+k}{q+k}$$
$$[|a|<1, \quad p>0, \quad q>0]$$ BI (123)(18)

2. $$\int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q}\right]\frac{dx}{x\ln(1+x)} = \ln\Gamma(q)$$
$$[q>0]$$ BI (143)(7)

3. $$\int_0^1 \frac{x\ln x + 1 - x}{x\left(\ln x\right)^2}\ln(1+x)dx = \ln\frac{4}{\pi}$$ BI (126)(12)

4. $$\int_0^1 \frac{\ln\left(1-x^2\right)dx}{x\left(q^2+(\ln x)^2\right)} = -\frac{\pi}{q}\ln\Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q}\ln 2q + \ln\frac{q}{\pi} - 1$$
$$[q>0]$$ LI (327)(12)a

4.315

1. $$\int_0^1 \ln(1+x)\,(\ln x)^{n-1}\,\frac{dx}{x} = (-1)^{n-1}(n-1)!\left(1-\frac{1}{2^n}\right)\zeta(n+1)$$ BI (116)(3)

2. $$\int_0^1 \ln(1+x)\,(\ln x)^{2n}\,\frac{dx}{x} = \frac{2^{2n+1}-1}{(2n+1)(2n+2)}\pi^{2n+2}|B_{2n+2}|$$ BI (116)(1)

3. $$\int_0^1 \ln(1-x)\,(\ln x)^{n-1}\,\frac{dx}{x} = (-1)^n(n-1)!\,\zeta(n+1)$$ BI (116)(4)

4. $$\int_0^1 \ln(1-x)\,(\ln x)^{2n}\,\frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)}\pi^{2n+2}\;|B_{2n+2}|$$ BI (116)(2)

4.316

1. $$\int_0^1 \ln\left(1-ax^r\right)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -\frac{1}{r^{p+1}}\,\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k}{k^{p+2}}$$ $[p>-1, \quad a<1, \quad r>0]$ BI (116)(7)

2. $$\int_0^1 \ln\left(1-2ax\cos t+a^2x^2\right)\left(\ln\frac{1}{x}\right)^p\frac{dx}{x} = -2\,\Gamma(p+1)\sum_{k=1}^{\infty}\frac{a^k\cos kt}{k^{p+2}}$$ LI (116)(8)

4.317

1. $$\int_0^\infty \ln\frac{\sqrt{1+x^2}+a}{\sqrt{1+x^2}-a}\frac{dx}{\sqrt{1+x^2}} = \pi\arcsin a$$ $[|a|<1]$ BI (142)(11)

2. $$\int_0^1 \ln\frac{\sqrt{1-a^2x^2}-x\sqrt{1-a^2}}{1-x}\frac{dx}{x} = \frac{1}{2}\left(\arcsin a\right)^2$$ BI (115)(32)

3. $$\int_0^1 \ln\frac{1+\cos t\sqrt{1-x^2}}{1-\cos t\sqrt{1-x^2}}\frac{dx}{x^2+\tan^2 v} = \pi\cot t\frac{\cos\dfrac{v-t}{2}}{\sin\dfrac{v+t}{2}}$$ BI (115)(30)

4. $$\int_0^1 \ln^2\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)\frac{x\,dx}{1-x^2} = \frac{\pi^2}{2}$$ BI (115)(31)

5. $$\int_0^1 \ln\left\{\sqrt{1+kx}+\sqrt{1-kx}\right\}\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \frac{1}{4}\ln(4k)\,\boldsymbol{K}(k)+\frac{\pi}{8}\,\boldsymbol{K}\left(k'\right)$$ BI (121)(8)

6. $$\int_0^1 \ln\left\{\sqrt{1+kx}-\sqrt{1-kx}\right\}\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \frac{1}{4}\ln(4k)\,\boldsymbol{K}(k)+\frac{3}{8}\pi\,\boldsymbol{K}\left(k'\right)$$ BI (121)(9)

7. $$\int_0^1 \ln\left\{1+\sqrt{1-k^2x^2}\right\}\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \frac{1}{2}\ln k\,\boldsymbol{K}(k)+\frac{\pi}{4}\,\boldsymbol{K}\left(k'\right)$$ BI (121)(6)

8. $$\int_0^1 \ln\left\{1-\sqrt{1-k^2x^2}\right\}\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}} = \frac{1}{2}\ln k\,\boldsymbol{K}(k)-\frac{3}{4}\pi\,\boldsymbol{K}\left(k'\right)$$ BI (121)(7)

9. $$\int_0^1 \ln\frac{1+p\sqrt{1-x^2}}{1-p\sqrt{1-x^2}}\frac{dx}{1-x} = \pi\arcsin p$$ $[p^2<1]$ BI (115)(29)

10. $$\int_0^1 \ln \frac{1+q\sqrt{1-k^2x^2}}{1-q\sqrt{1-k^2x^2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \pi F(\arcsin q, k')$$

$[q^2 < 1]$ BI (122)(15)

11.[10] $$\int_{-\infty}^{\infty} \ln \left| \frac{1+2\sqrt{1+x^2}}{1-2\sqrt{1+x^2}} \right| \frac{dx}{\sqrt{1+x^2}} = \frac{\pi^2}{3}$$

4.318

1. $$\int_0^1 \frac{\ln(1-x^q)}{1+(\ln x)^2} \frac{dx}{x} = \pi \left[\ln \Gamma\left(\frac{q}{2\pi}+1\right) - \frac{\ln q}{2} + \frac{q}{2\pi}\left(\ln \frac{q}{2\pi} - 1\right) \right]$$

$[q > 0]$ BI (126)(11)

2. $$\int_0^\infty \ln(1+x^r) \left[\frac{(p-r)x^p - (q-r)x^q}{\ln x} + \frac{x^q - x^p}{(\ln x)^2} \right] \frac{dx}{x^{r+1}} = r \ln\left(\tan \frac{q\pi}{2r} \cot \frac{p\pi}{2r}\right)$$

$[p < r, \quad q < r]$ BI (143)(9)

In integrals containing $\ln(a+bx^r)$, it is useful to make the substitution $x^r = t$ and then to seek the resulting integral in the tables. For example,

$$\int_0^\infty x^{p-1} \ln(1+x^r)\, dx = \frac{1}{r} \int_0^\infty t^{\frac{p}{r}-1} \ln(1+t)\, dt = \frac{\pi}{p \sin \frac{p\pi}{r}}$$ (see **4.293** 3)

4.319

1. $$\int_0^\infty \ln\left(1-e^{-2a\pi x}\right) \frac{dx}{1+x^2} = -\pi \left[\frac{1}{2} \ln 2a\pi + a(\ln a - 1) - \ln \Gamma(a+1) \right]$$

$[a > 0]$ BI (354)(6)

2. $$\int_0^\infty \ln\left(1+e^{-2a\pi x}\right) \frac{dx}{1+x^2} = \pi \left[\ln \Gamma(2a) - \ln \Gamma(a) + a(1-\ln a) - \left(2a - \frac{1}{2}\right) \ln 2 \right]$$

$[a > 0]$ BI (354)(7)

3. $$\int_0^\infty \ln \frac{a+be^{-px}}{a+be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q}$$ $\left[\frac{b}{a} > -1, \quad pq > 0\right]$

FI II 635, BI (354)(1)

4.321

1. $$\int_{-\infty}^{\infty} x \ln \cosh x\, dx = 0$$ BI (358)(2)a

2. $$\int_{-\infty}^{\infty} \ln \cosh x \frac{dx}{1-x^2} = 0$$ BI (138)(20)a

4.322

1.[11] $$\int_0^\pi x \ln \sin x\, dx = \frac{1}{2} \int_0^\pi x \ln \cos^2 x\, dx = -\frac{\pi^2}{2} \ln 2$$ BI (432)(1, 2) FI II 643

2. $$\int_0^\infty \frac{\ln \sin^2 ax}{b^2+x^2}\, dx = \frac{\pi}{b} \ln \frac{1-e^{-2ab}}{2}$$ $[a > 0, \quad b > 0]$ GW (338)(28b)

3. $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 + x^2}\,dx = \frac{\pi}{b} \ln \frac{1 + e^{-2ab}}{2}$ $[a > 0, \quad b > 0]$ GW (338)(28a)

4. $\int_0^\infty \frac{\ln \sin^2 ax}{b^2 - x^2}\,dx = -\frac{\pi^2}{2b} + a\pi$ $[a > 0, \quad b > 0]$ BI (418)(1)

5.[11] $\int_0^\infty \frac{\ln \cos^2 ax}{b^2 - x^2}\,dx = \infty$ BI (418)(2)

6. $\int_0^\infty \frac{\ln \cos^2 x}{x^2}\,dx = -\pi$ FI II 686

7.[7] $\int_0^{\pi/4} \ln \sin x x^{\mu-1}\,dx = -\frac{1}{2\mu}\left(\frac{\pi}{4}\right)^\mu \left[\ln 2 + \frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)}\right]$

$[\operatorname{Re}\mu > 0]$ LI (425)(1)

8.[7] $\int_0^{\pi/2} \ln \sin x x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\frac{\pi}{2}\right)^\mu \left[\frac{1}{\mu} - 2\sum_{k=1}^\infty \frac{\zeta(2k)}{4^k(\mu + 2k)}\right]$

$[\operatorname{Re}\mu > 0]$ LI (430)(1)

9. $\int_0^{\pi/2} \ln(1 - \cos x)\, x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\frac{\pi}{2}\right)^\mu \left[\frac{2}{\mu} - \sum_{k=1}^\infty \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)}\right]$

$[\operatorname{Re}\mu > 0]$ LI (430)(2)

10. $\int_0^\infty \ln\left(1 \pm 2p\cos\beta x + p^2\right) \frac{dx}{q^2 + x^2} = \frac{\pi}{q} \ln\left(1 \pm pe^{-\beta q}\right)$ $[p^2 < 1]$

$= \frac{\pi}{q} \ln\left(p \pm e^{-\beta q}\right)$ $[p^2 > 1]$

FI II 718a

4.323

1.[11] $\int_0^\pi x \ln \tan^2 x\,dx = 0$ BI (432)(3)

2. $\int_0^\infty \frac{\ln \tan^2 ax}{b^2 + x^2}\,dx = \frac{\pi}{b} \ln \tanh ab$ $[a > 0, \quad b > 0]$ GW (338)(28c)

3. $\int_0^\infty \ln\left(\frac{1 + \tan x}{1 - \tan x}\right)^2 \frac{dx}{x} = \frac{\pi^2}{2}$ GW (338)(26)

4.324

1. $\int_0^\infty \ln\left(\frac{1 + \sin x}{1 - \sin x}\right)^2 \frac{dx}{x} = \pi^2$ GW (338)(25)

2. $\int_0^\infty \ln \frac{1 + 2a\cos px + a^2}{1 + 2a\cos qx + a^2} \frac{dx}{x} = \ln(1 + a) \ln \frac{q^2}{p^2}$ $[-1 < a \le 1]$

$= \ln\left(1 + \frac{1}{a}\right) \ln \frac{q^2}{p^2}$ $[a < -1 \text{ or } a \ge 1]$

GW (338)(27)

3. $$\int_0^\infty \ln\left(a^2\sin^2 px + b^2\cos^2 px\right)\frac{dx}{c^2+x^2} = \frac{\pi}{c}\left[\ln\left(a\sinh cp + b\cosh cp\right) - cp\right]$$
$[a > 0, \quad b > 0, \quad c > 0, \quad p > 0]$
GW (338)(29)

4.325

1.[3] $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x} = -\boldsymbol{C}\ln 2 + \sum_{k=2}^\infty (-1)^k\frac{\ln k}{k} = -\boldsymbol{C}\ln 2 + 0.159868905\cdots = -\frac{1}{2}(\ln 2)^2$$
GW (325)(25a)

2. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{x+e^{i\lambda}} = \sum_{k=1}^\infty \frac{(-1)^k}{k}e^{-ik\lambda}\left(\boldsymbol{C} + \ln k\right)$$ GW (325)(26)

3. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{(1+x)^2} = \int_1^\infty \ln\ln x\frac{dx}{(1+x)^2} = \frac{1}{2}\left[\psi\left(\frac{1}{2}\right) + \ln 2\pi\right] = \frac{1}{2}\left(\ln\frac{\pi}{2} - \boldsymbol{C}\right)$$
BI (147)(7)

4. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$ BI (148)(1)

5. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\frac{\sqrt[3]{2\pi}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}$$ BI (148)(2)

6. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right]$$ BI (148)(5)

7. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+2x\cos t+x^2} = \int_1^\infty \ln\ln x\frac{dx}{1+2x\cos t+x^2} = \frac{\pi}{2\sin t}\ln\frac{(2\pi)^{t/\pi}\,\Gamma\left(\frac{1}{2}+\frac{t}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{t}{2\pi}\right)}$$
BI (147)(9)

8. $$\int_0^1 \ln\ln\frac{1}{x}x^{\mu-1}\,dx = -\frac{1}{\mu}\left(\boldsymbol{C} + \ln\mu\right)$$ $[\operatorname{Re}\mu > 0]$ BI (147)(1)

9. $$\int_1^\infty \ln\ln x\frac{x^{n-2}\,dx}{1+x^2+x^4+\cdots+x^{2n-2}}$$
$$= \frac{\pi}{2n}\tan\frac{\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)}$$ $[n \text{ is even}]$
$$= \frac{\pi}{2n}\tan\frac{\pi}{2n}\ln\pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{k\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)}$$ $[n \text{ is odd}]$
BI (148)(4)

10. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right) \frac{dx}{(1+x^2)\sqrt{\ln\frac{1}{x}}} = \int_1^\infty \ln\ln x \frac{dx}{(1+x^2)\sqrt{\ln x}}$$
$$= \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} \left[\ln(2k+1) + 2\ln 2 + \boldsymbol{C}\right]$$
BI (147)(4)

11. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right) \frac{x^{\mu-1}\,dx}{\sqrt{\ln\frac{1}{x}}} = -\left(\boldsymbol{C} + \ln 4\mu\right)\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0]$$
BI (147)(3)

12. $$\int_0^1 \ln\ln\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{\mu-1} x^{\nu-1}\,dx = \frac{1}{\nu^\mu}\Gamma(\mu)\left[\psi(\mu) - \ln(\nu)\right]$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ BI (147)(2)

4.326

1. $$\int_0^1 \ln(a - \ln x)\, x^{\mu-1}\,dx = \frac{1}{\mu}\left[\ln a - e^{a\mu}\operatorname{Ei}(-a\mu)\right] \qquad [\operatorname{Re}\mu > 0, \quad a > 0]$$
BI (107)(23)

2. $$\int_0^{\frac{1}{e}} \ln\left(2\ln\frac{1}{x} - 1\right)\frac{x^{2\mu-1}}{\ln x}\,dx = -\frac{1}{2}\left[\operatorname{Ei}(-\mu)\right]^2 \qquad [\operatorname{Re}\mu > 0]$$
BI (145)(5)

4.327

1. $$\int_0^1 \ln\left[a^2 + (\ln x)^2\right]\frac{dx}{1+x^2} = \pi \ln\frac{2\,\Gamma\left(\frac{2a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2a+\pi}{4\pi}\right)} + \frac{\pi}{2}\ln\frac{\pi}{2}$$
$\left[a > -\frac{\pi}{2}\right]$ BI (147)(10)

2. $$\int_0^1 \ln\left[a^2 + 4(\ln x)^2\right]\frac{dx}{1+x^2} = \pi \ln\frac{2\,\Gamma\left(\frac{a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{a+\pi}{4\pi}\right)} + \frac{\pi}{2}\ln\pi$$
$[a > -\pi]$ BI (147)(16)a

3. $$\int_0^\infty \ln\left[a^2 + (\ln x)^2\right] x^{\mu-1}\,dx = \frac{2}{\mu}\left[-\cos a\mu \operatorname{ci}(a\mu) - \sin a\mu \operatorname{si}(a\mu) + \ln a\right]$$
$[a > 0, \quad \operatorname{Re}\mu > 0]$ GW (325)(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains $\ln\ln\frac{1}{x}$, it is useful to make the substitution $\ln x = t$ and then seek the transformed integral in the tables.

4.33–4.34 Combinations of logarithms and exponentials

4.331

1. $$\int_0^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}\left(\boldsymbol{C} + \ln\mu\right) \qquad [\operatorname{Re}\mu > 0]$$
BI (256)(2)

2. $$\int_1^\infty e^{-\mu x}\ln x\,dx = -\frac{1}{\mu}\operatorname{Ei}(-\mu) \qquad [\operatorname{Re}\mu > 0]$$
BI (260)(5)

3. $$\int_0^1 e^{\mu x} \ln x \, dx = -\frac{1}{\mu} \int_0^1 \frac{e^{\mu x} - 1}{x} \, dx \qquad [\mu \neq 0]$$ GW (324)(81a)

4.332

1. $$\int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} - 1} = \frac{2\pi}{\sqrt{3}} \left[\frac{5}{6} \ln 2\pi - \ln \Gamma\left(\frac{1}{6}\right) \right] \qquad (\text{cf. } \mathbf{4.325}\ 6)$$ BI (257)(6)

2. $$\int_0^\infty \frac{\ln x \, dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}} \ln \left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \sqrt{2\pi} \right] \qquad (\text{cf. } \mathbf{4.325}\ 5)$$ BI (257)(7)a, LI (260)(3)

4.333 $$\int_0^\infty e^{-\mu x^2} \ln x \, dx = -\frac{1}{4} (\boldsymbol{C} + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re} \mu > 0]$$ BI (256)(8), FI II 807a

4.334 $$\int_0^\infty \frac{\ln x \, dx}{e^{x^2} + 1 + e^{-x^2}} = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^\infty (-1)^k \frac{\boldsymbol{C} + \ln 4k}{\sqrt{k}} \sin \frac{k\pi}{3}$$ BI (357)(13)

4.335

1. $$\int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + (\boldsymbol{C} + \ln \mu)^2 \right] \qquad [\operatorname{Re} \mu > 0]$$ ET I 149(13)

2. $$\int_0^\infty e^{-x^2} (\ln x)^2 \, dx = \frac{\sqrt{\pi}}{8} \left[(\boldsymbol{C} + 2 \ln 2)^2 + \frac{\pi^2}{2} \right]$$ FI II 808

3.[7] $$\int_0^\infty e^{-\mu x} (\ln x)^3 \, dx = -\frac{1}{\mu} \left[(\boldsymbol{C} + \ln \mu)^3 + \frac{\pi^2}{2} (\boldsymbol{C} + \ln \mu) - \psi''(1) \right]$$ MI 26

4.336

1.[7] $$\mathrm{PV} \int_0^\infty \frac{e^{-x}}{\ln x} \, dx = -0.154479567$$ BI (260)(9)

2. $$\int_0^\infty \frac{e^{-\mu x} \, dx}{\pi^2 + (\ln x)^2} = \nu'(\mu) - e^\mu \qquad [\operatorname{Re} \mu > 0]$$ MI 26

4.337

1. $$\int_0^\infty e^{-\mu x} \ln(\beta + x) dx = \frac{1}{\mu} \left[\ln \beta - e^{\mu\beta} \operatorname{Ei}(-\beta\mu) \right] \qquad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0]$$ BI (256)(3)

2. $$\int_0^\infty e^{-\mu x} \ln(1 + \beta x) dx = -\frac{1}{\mu} e^{\frac{\mu}{\beta}} \operatorname{Ei}\left(-\frac{\mu}{\beta}\right) \qquad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0]$$ ET I 148(4)

3. $$\int_0^\infty e^{-\mu x} \ln |a - x| \, dx = \frac{1}{\mu} \left[\ln a - e^{-a\mu} \operatorname{Ei}(a\mu) \right] \qquad [a > 0, \quad \operatorname{Re} \mu > 0]$$ BI (256)(4)

4.[7] $$\int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| dx = \frac{1}{\mu} \left[e^{-\beta\mu} \operatorname{Ei}(\beta\mu) \right] \qquad [\operatorname{Re} \mu > 0]$$ MI 26

5.* $$\int_0^\infty \ln(1 + ax) x^\zeta e^{-x} \, dx = \sum_{\mu=0}^{\zeta} \frac{\zeta!}{(\zeta - \mu)!} \left[\frac{(-1)^{\zeta-\mu-1}}{a^{\zeta-\mu}} e^{1/a} \operatorname{Ei}\left(-\frac{1}{a}\right) + \sum_{k=1}^{\zeta-\mu} (k-1)! \left(-\frac{1}{a}\right)^{\zeta-\mu-k} \right]$$

4.338

1. $$\int_0^\infty e^{-\mu x} \ln\left(\beta^2 + x^2\right) \, dx = \frac{2}{\mu} \left[\ln \beta - \operatorname{ci}(\beta\mu) \cos(\beta\mu) - \operatorname{si}(\beta\mu) \sin(\beta\mu) \right]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0]$$ BI (256)(6)

2. $$\int_0^\infty e^{-\mu x} \ln^2\left(x^2-\beta^2\right)\, dx = \frac{2}{\mu}\left[\ln^2\beta - e^{\beta\mu}\operatorname{Ei}(-\beta\mu) - e^{\beta\mu}\operatorname{Ei}(\beta\mu)\right]$$

$[\operatorname{Im}\beta > 0, \quad \operatorname{Re}\mu > 0]$ BI (256)(5)

4.339 $$\int_0^\infty e^{-\mu x} \ln\left|\frac{x+1}{x-1}\right| dx = \frac{1}{\mu}\left[e^{-\mu}\left(\ln 2\mu + \gamma\right) - e^{\mu}\operatorname{Ei}(-2\mu)\right]$$

$[\operatorname{Re}\mu > 0]$ MI 27

4.341 $$\int_0^\infty e^{-\mu x} \ln\frac{\sqrt{x+ai}+\sqrt{x-ai}}{\sqrt{2a}}\, dx = \frac{\pi}{4\mu}\left[\mathbf{H}_0(a\mu) - Y_0(a\mu)\right]$$

$[a > 0, \quad \operatorname{Re}\mu > 0]$ ET I 149(20)

4.342

1. $$\int_0^\infty e^{-2nx} \ln\left(\sinh x\right)\, dx = \frac{1}{2n}\left[\frac{1}{n} + \ln 2 - 2\,\beta(2n+1)\right]$$ BI (256)(17)

2. $$\int_0^\infty e^{-\mu x} \ln\left(\cosh x\right)\, dx = \frac{1}{\mu}\left[\beta\left(\frac{\mu}{2}\right) - \frac{1}{\mu}\right]$$ $[\operatorname{Re}\mu > 0]$ ET I 165(32)

3.[11] $$\int_0^\infty e^{-\mu x}\left[\ln\left(\sinh x\right) - \ln x\right] dx = \frac{1}{\mu}\left[\ln\frac{\mu}{2} - \frac{1}{\mu} - \psi\left(\frac{\mu}{2}\right)\right]$$

$[\operatorname{Re}\mu > 0]$ ET I 165(33)

4.343 $$\int_0^\pi e^{\mu\cos x}\left[\ln\left(2\mu\sin^2 x\right) + \boldsymbol{C}\right] dx = -\pi\, K_0(\mu)$$ WA 95(16)

4.35–4.36 Combinations of logarithms, exponentials, and powers

4.351

1. $$\int_0^1 (1-x)e^{-x}\ln x\, dx = \frac{1-e}{e}$$ BI (352)(1)

2. $$\int_0^1 e^{\mu x}\left(\mu x^2 + 2x\right)\ln x\, dx = \frac{1}{\mu^2}\left[(1-\mu)e^{\mu} - 1\right]$$ BI (352)(2)

3. $$\int_1^\infty \frac{e^{-\mu x}\ln x}{1+x}\, dx = \frac{1}{2}e^{\mu}\left[\operatorname{Ei}(-\mu)\right]^2$$ $[\operatorname{Re}\mu > 0]$ NT 32(10)

4.352

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\ln x\, dx = \frac{1}{\mu^\nu}\Gamma(\nu)\left[\psi(\nu) - \ln\mu\right]$$ $[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$

BI (353)(3), ET I 315(10)a

2. $$\int_0^\infty x^n e^{-\mu x}\ln x\, dx = \frac{n!}{\mu^{n+1}}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \boldsymbol{C} - \ln\mu\right]$$

$[\operatorname{Re}\mu > 0]$ ET I 148(7)

3. $$\int_0^\infty x^{n-\frac{1}{2}}e^{-\mu x}\ln x\, dx = \sqrt{\pi}\frac{(2n-1)!!}{2^n\mu^{n+\frac{1}{2}}}\left[2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) - \boldsymbol{C} - \ln 4\mu\right]$$

$[\operatorname{Re}\mu > 0]$ ET I 148(10)

4. $$\int_0^\infty x^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu) \qquad [\operatorname{Re}\mu > 0]$$ GW (324)(83a)

4.353

1. $$\int_0^\infty (x-\nu) x^{\nu-1} e^{-x} \ln x \, dx = \Gamma(\nu) \qquad [\operatorname{Re}\nu > 0]$$ GW (324)(84)

2. $$\int_0^\infty \left(\mu x - n - \frac{1}{2}\right) x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n}\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0]$$ BI (357)(2)

3. $$\int_0^1 (\mu x + n + 1) x^n e^{\mu x} \ln x \, dx = e^\mu \sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(n-k)!\mu^{k+1}} + (-1)^n \frac{n!}{\mu^{n+1}} \qquad [\mu \neq 0]$$ GW (324)(82)

4.354

1.[6] $$\int_0^\infty \frac{x^{\nu-1}\ln x}{e^x+1}\, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} \left[\psi(\nu) - \ln k\right] \qquad [\operatorname{Re}\nu > 0]$$
$$= -\frac{1}{2}(\ln 2)^2 \qquad [\text{for } \nu = 1]$$ GW (324)(86a)

2.[7] $$\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^2}\, dx = \Gamma(\nu) \sum_{k=2}^\infty \frac{(-1)^{k}(k-1)}{k^\nu} \left[\psi(\nu) - \ln k\right] \qquad [\operatorname{Re}\nu > 1]$$ GW (324)(86b)

3. $$\int_0^\infty \frac{(x-\nu)e^x - \nu}{(e^x+1)^2} x^{\nu-1} \ln x \, dx = \Gamma(\nu) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\nu} \qquad [\operatorname{Re}\nu > 0]$$ GW (324)(87a)

4. $$\int_0^\infty \frac{(x-2n)e^x - 2n}{(e^x+1)^2} x^{2n-1} \ln x \, dx = \frac{2^{2n-1}-1}{2n} \pi^{2n} |B_{2n}| \qquad [n = 1, 2, \ldots]$$ GW (324)(87b)

5. $$\int_0^\infty \frac{x^{\nu-1}\ln x}{(e^x+1)^n}\, dx = (-1)^n \frac{\Gamma(\nu)}{(n-1)!} \sum_{k=n}^\infty \frac{(-1)^k (k-1)!}{(k-n)!k^\nu} \left[\psi(\nu) - \ln k\right] \qquad [\operatorname{Re}\nu > 0]$$ GW (324)(86c)

4.355

1. $$\int_0^\infty x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu}\left(2 - \ln 4\mu - \boldsymbol{C}\right)\sqrt{\frac{\pi}{\mu}} \qquad [\operatorname{Re}\mu > 0]$$ BI (357)(1)a

2. $$\int_0^\infty x\left(\mu x^2 - \nu x - 1\right) e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu}{4\mu}\sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\nu^2}{\mu}\right)\left[1 + \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \qquad [\operatorname{Re}\mu > 0]$$ BI (358)(1)

3. $$\int_0^\infty \left(\mu x^2 - n\right) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n} \qquad [\operatorname{Re}\mu > 0]$$ BI (353)(4)

4. $$\int_0^\infty \left(2\mu x^2 - 2n - 1\right) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}}$$

$[\operatorname{Re} \mu > 0]$ BI (353)(5)

4.356

1. $$\int_0^\infty \exp\left[-\mu\left(\frac{x}{a} + \frac{a}{x}\right)\right] \ln x \frac{dx}{x} = 2 \ln a \, K_0(2\mu)$$ $[a > 0, \quad \operatorname{Re} \mu > 0]$ GW (324)(91)

2. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[2ax^2 - (2n+1)x - 2b\right] x^{n-\frac{1}{2}} \, dx = 2\left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k)!}{(n-k)!(2k)!!\left(2\sqrt{ab}\right)^k}$$

$[a > 0, \quad b > 0]$ BI (357)(4)

3. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[2ax^2 + (2n-1)x - 2b\right] \frac{dx}{x^{n+\frac{3}{2}}} = 2\left(\frac{a}{b}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-k-1)!(2k)!!\left(2\sqrt{ab}\right)^k}$$

$[a > 0, \quad b > 0]$ BI (357)(11)

For $n = \frac{1}{2}$:

4. $$\int_0^\infty \exp\left(-ax - \frac{a}{x}\right) \ln x \frac{ax^2 - b}{x^2} \, dx = 2 \, K_0\left(2\sqrt{ab}\right)$$ $[a > 0, \quad b > 0]$ GW (324)(92c)

For $n = 0$:

5. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}} \, dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}$$

$[a > 0, \quad b > 0]$ BI (357)(7), GW(324)(92a)

For $n = -1$:

6. $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}} \, dx = \frac{1 + 2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}$$

$[a > 0, \quad b > 0]$ LI (357)(6), GW (324)(92b)

7.[9] $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left(a - \frac{b}{x^2}\right) \, dx = K_0\left(2\sqrt{ab}\right)$$

$[a > 0, \quad b > 0]$

8.[9] $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[2ax^2 - (2n+1)x - 2b\right] x^{n-\frac{3}{2}}\, dx$$
$$= 4\left(\frac{b}{a}\right)^{(2n+1)/4} K_{n+\frac{1}{2}}\left(2\sqrt{ab}\right)$$
$$= 2\left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!(2k)!!\left(2\sqrt{ab}\right)^k}$$
$[n = 0, 1, \ldots, a > 0, \quad b > 0]$

9.[9] $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln\left[\left(ax^2 - b\right)\cos\left(\alpha \ln x\right) + \alpha x \sin\left(\alpha \ln x\right)\right] \frac{dx}{x^2}$$
$$= 2\cos\left(\alpha \ln \sqrt{b/a}\right) K_{i\alpha}\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]$

10.[9] $$\int_0^\infty \exp\left(-ax - \frac{b}{x}\right) \ln x \left[\left(ax^2 - b\right)\sin\left(\alpha \ln x\right) - \alpha x \cos\left(\alpha \ln x\right)\right] \frac{dx}{x^2}$$
$$= 2\sin\left(\alpha \ln \sqrt{b/a}\right) K_{i\alpha}\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]$

11.[9] $$q\int_0^\infty x^\alpha \ln x \left[a - \frac{\alpha}{x} - \frac{b}{x^2}\right] \exp\left(-ax - \frac{b}{x}\right) dx = 2\left(\frac{b}{a}\right)^{\alpha/2} K_\alpha\left(2\sqrt{ab}\right)$$
$[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty]$

4.357

1. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{1 + ax^2 - x^4}{x^2}\, dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}$$
$[a > 0]$ BI (357)(8)

2. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4 + ax^2 - 1}{x^4}\, dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[a]{e}}$$ $[a > 0]$ BI (357)(9)

3. $$\int_0^\infty \exp\left(-\frac{1+x^4}{2ax^2}\right) \ln x \frac{x^4 + 3ax - 1}{x^6}\, dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[a]{e}}$$
$[a > 0]$ BI (357)(10)

4.358

1.[6] $$\int_1^\infty x^{\nu-1} e^{-\mu x} (\ln x)^m\, dx = \frac{\partial^m}{\partial \nu^m} \left\{\mu^{-\nu}\, \Gamma(\nu, \mu)\right\}$$ $[m = 0, 1, \ldots, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$
MI 26

2. $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^2\, dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{\left[\psi(\nu) - \ln\mu\right]^2 + \zeta(2, \nu)\right\}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ MI 26

3.[9] $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^3\, dx = \frac{\Gamma(\nu)}{\mu^\nu} \left\{\left[\psi(\nu) - \ln\mu\right]^3 + 3\,\zeta(2, \nu)\left[\psi(\nu) - \ln\mu\right] - 2\,\zeta(3, \nu)\right\}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ MI 26

4.[7] $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^4 \, dx = \frac{\Gamma(\nu)}{\nu} \left\{ [\psi(\nu) - \ln\mu]^4 + 6\,\zeta(2,\nu)\,[\psi(\nu)-\ln\mu]^2 - 8\ \zeta(3,\nu)\,[\psi(\nu)-\ln\mu] + 3\,[\zeta(2,\nu)]^2 + 6\,\zeta(4,\nu) \right\}$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$

5.[3] $$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^n \, dx = \frac{\partial^n}{\partial\nu^n} \left\{ \mu^{-\nu}\,\Gamma(\nu) \right\}$$ $[n = 0, 1, 2, \ldots]$

4.359

1. $$\int_0^\infty e^{-\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} \, dx = \frac{1}{\mu} [\lambda(\mu, p-1) - \lambda(\mu, q-1)]$$
$[\operatorname{Re}\mu > 0, \quad p > 0, \quad q > 0]$ MI 27

2.[11] $$\int_0^1 e^{\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} \, dx = \sum_{k=0}^\infty \frac{\mu^k}{k!} \ln \frac{p+k}{q+k}$$ $[p > 0, \quad q > 0]$ BI (352)(9)

4.361

1. $$\int_0^\infty \frac{(x+1)e^{-\mu x}}{\pi^2 + (\ln x)^2} \, dx = \nu'(\mu) - \nu''(\mu)$$ $[\operatorname{Re}\mu > 0]$ MI 27

2. $$\int_0^\infty \frac{e^{-\mu x}\, dx}{x\left[\pi^2 + (\ln x)^2\right]} = e^\mu - \nu(\mu)$$ $[\operatorname{Re}\mu > 0]$ MI 27

4.362

1. $$\int_0^1 x e^x \ln(1-x) \, dx = 1 - e$$ BI (352)(5)a

2. $$\int_1^\infty e^{-\mu x} \ln(2x-1) \frac{dx}{x} = \frac{1}{2} \left[\operatorname{Ei}\left(-\frac{\mu}{2}\right)\right]^2$$ $[\operatorname{Re}\mu > 0]$ ET I 148(8)

4.363

1. $$\int_0^\infty e^{-\mu x} \ln(a+x) \frac{\mu(x+a)\ln(x+a) - 2}{x+a} \, dx = \frac{1}{4} \int_0^\infty e^{-\mu x} \ln^2(a-x) \frac{\mu(x-a)\ln^2(x-a) - 4}{x-a} \, dx = (\ln a)^2$$
$[\operatorname{Re}\mu > 0, \quad a > 0]$ BI (354)(4, 5)

2. $$\int_0^1 x(1-x)(2-x) e^{-(1-x)^2} \ln(1-x) \, dx = \frac{1-e}{4e}$$ BI (352)(4)

4.364

1. $$\int_0^\infty e^{-\mu x} \ln[(x+a)(x+b)] \frac{dx}{x+a+b} = e^{(a+b)\mu} \left\{ \operatorname{Ei}(-a\mu)\operatorname{Ei}(-b\mu) - \ln(ab)\operatorname{Ei}[-(a+b)\mu] \right\}$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\mu > 0]$ BI (354)(11)

2. $$\int_0^\infty e^{-\mu x}\ln(x+a+b)\left(\frac{1}{x+a}+\frac{1}{x+b}\right)dx$$
$$= (1+\ln a\ln b)\ln(a+b) + e^{-(a+b)\mu}\left\{\operatorname{Ei}(-\alpha\mu)\operatorname{Ei}(-b\mu)\right\}$$
$$+ (1-\ln(ab))\operatorname{Ei}[-(a+b)\mu]$$
$[a>0,\quad b>0,\quad \operatorname{Re}\mu>0]$ BI (354)(12)

4.365 $$\int_0^\infty \left[e^{-x} - \frac{x}{(1+x)^{p+1}\ln(1+x)}\right]\frac{dx}{x} = \ln p$$ $[p>0]$ BI (354)(15)

4.366

1. $$\int_0^\infty e^{-\mu x}\ln\left(1+\frac{x^2}{a^2}\right)\frac{dx}{x} = [\operatorname{ci}(a\mu)]^2 + [\operatorname{si}(a\mu)]^2$$ $[\operatorname{Re}\mu>0]$ NT 32(11)a

2. $$\int_0^\infty e^{-\mu x}\ln\left|1-\frac{x^2}{a^2}\right|\frac{dx}{x} = \operatorname{Ei}(a\mu)\operatorname{Ei}(-a\mu)$$ $[\operatorname{Re}\mu>0]$ ME 18

3. $$\int_0^\infty xe^{-\mu x^2}\ln\left|\frac{1+x^2}{1-x^2}\right|dx = \frac{1}{\mu}\left[\cosh\mu\sinh(i\mu) - \sinh\mu\cosh(i\mu)\right]$$
$[\operatorname{Re}\mu>0]$; (cf. **4.339**) MI 27

4.367 $$\int_0^\infty xe^{-\mu x^2}\ln\frac{x+\sqrt{x^2+2\beta}}{\sqrt{2\beta}}\,dx = \frac{e^{\beta\mu}}{4\mu}K_0(\beta\mu)$$ $[|\arg\beta|<\pi,\quad \operatorname{Re}\mu>0]$ ET I 149(19)

4.368 $$\int_0^{2u} e^{-\mu x^2}\ln\frac{x^2\left(4u^2-x^2\right)}{u^4}\frac{dx}{\sqrt{4u^2-x^2}} = \frac{\pi}{2}e^{-2u^2\mu}\left[\frac{\pi}{2}Y_0\left(2iu^2\mu\right) - (\boldsymbol{C}-\ln 2)J_0\left(2iu^2\mu\right)\right]$$
$[\operatorname{Re}\mu>0]$ ET I 149(21)a

4.369

1. $$\int_0^\infty x^{\nu-1}e^{-\mu x}\left[\psi(\nu)-\ln x\right]dx = \frac{\Gamma(\nu)\ln\mu}{\mu^\nu}$$ $[\operatorname{Re}\nu>0]$ ET I 149(12)

2. $$\int_0^\infty x^n e^{-\mu x}\left\{\left[\ln x - \tfrac{1}{2}\psi(n+1)\right]^2 - \tfrac{1}{2}\psi'(n+1)\right\}dx$$
$$= \frac{n!}{\mu^{n+1}}\left\{\left[\ln\mu - \frac{1}{2}\psi(n+1)\right]^2 + \frac{1}{2}\psi'(n+1)\right\}$$
$[\operatorname{Re}\mu>0]$ MI 26

4.37 Combinations of logarithms and hyperbolic functions

4.371

1. $$\int_0^\infty \frac{\ln x}{\cosh x}\,dx = \pi\ln\left[\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$ LI (260)(1)a

2. $$\int_0^\infty \frac{\ln x\,dx}{\cosh x+\cos t} = \frac{\pi}{\sin t}\ln\frac{(2\pi)^{t/\pi}\,\Gamma\left(\dfrac{\pi+t}{2\pi}\right)}{\Gamma\left(\dfrac{\pi-t}{2\pi}\right)}$$ $\left[t^2<\pi^2\right]$ BI (257)(7)a

3. $$\int_0^\infty \frac{\ln x\,dx}{\cosh^2 x} = \psi\left(\frac{1}{2}\right) + \ln\pi = \ln\pi - 2\ln 2 - \boldsymbol{C}$$ BI (257)(4)a

4.372

1. $$\int_1^\infty \ln x \frac{\sinh mx}{\sinh nx}\,dx = \frac{\pi}{2n}\tan\frac{m\pi}{2n}\ln 2\pi + \frac{\pi}{n}\sum_{k=1}^{n-1}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \qquad [m+n \text{ is odd}]$$
$$= \frac{\pi}{2n}\tan\frac{m\pi}{2n}\ln\pi + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\sin\frac{km\pi}{n}\ln\frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \qquad [m+n \text{ is even}]$$
BI (148)(3)a

2. $$\int_1^\infty \ln x \frac{\cosh mx}{\cosh nx}\,dx$$
$$= \frac{\pi}{2n}\frac{\ln 2\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{n}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \qquad [m+n \text{ is odd}]$$
$$= \frac{\pi}{2n}\frac{\ln\pi}{\cos\frac{m\pi}{2n}} + \frac{\pi}{n}\sum_{k=1}^{\frac{n-1}{2}}(-1)^{k-1}\cos\frac{(2k-1)m\pi}{2n}\ln\frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \qquad [m+n \text{ is even}]$$
BI (148)(6)a

4.373

1. $$\int_0^\infty \frac{\ln\left(a^2+x^2\right)}{\cosh bx}\,dx = \frac{\pi}{b}\left[2\ln\frac{2\,\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln\frac{2b}{\pi}\right] \qquad \left[b>0, \quad a>-\frac{\pi}{2b}\right].$$ BI (258)(11)a

2. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{\cosh\frac{\pi x}{2}} = 2\ln\frac{4}{\pi}$$ BI (258)(1)a

3. $$\int_0^\infty \ln\left(a^2+x^2\right)\frac{\sinh\left(\frac{2}{3}\pi x\right)}{\sinh\pi x}\,dx = 2\sin\frac{\pi}{3}\ln\frac{6\,\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)}$$
$[a>-1]$. BI (258)(12)

4. $$\int_0^\infty \ln\left(1+x^2\right)\frac{dx}{\sinh^2 ax} = \frac{2}{a}\left[\ln\frac{a}{\pi} + \frac{\pi}{2a} - \psi\left(\frac{\pi+a}{\pi}\right)\right]$$
$[a>0]$ BI (258)(5)

5. $$\int_0^\infty \ln\left(1+x^2\right)\frac{\cosh\left(\frac{\pi}{2}x\right)}{\sinh^2\left(\frac{\pi}{2}x\right)}\,dx = \frac{2\pi-4}{\pi}$$ BI (258)(3)

6. $$\int_0^\infty \ln\left(1+x^2\right)\frac{\cosh\left(\frac{\pi}{4}x\right)}{\sinh^2\left(\frac{\pi}{4}x\right)}\,dx = 4\sqrt{2} - \frac{16}{\pi} + \frac{8\sqrt{2}}{\pi}\ln\left(\sqrt{2}+1\right)$$ BI (258)(2)

4.374

1. $$\int_0^\infty \ln\left(\cos^2 t + e^{-2x}\sin^2 t\right)\frac{dx}{\sinh x} = -2t^2$$ BI (259)(10)a

2. $$\int_0^\infty \ln\left(a+be^{-2x}\right)\frac{dx}{\cosh^2 x} = \frac{2}{(b-a)}\left[\frac{a+b}{2}\ln(a+b) - a\ln a - b\ln 2\right]$$

$[a>0, \quad a+b>0]$ LI (259)(14)

4.375

1.[11] $$\int_0^\infty \ln\cosh\frac{x}{2}\frac{dx}{\cosh x} = \boldsymbol{G} - \frac{\pi}{4}\ln 2$$ BI (259)(11)

2. $$\int_0^\infty \ln\coth x\frac{dx}{\cosh x} = \frac{\pi}{2}\ln 2$$ BI (259)(16)

4.376

1. $$\int_0^\infty \frac{\ln x}{\sqrt{x}\cosh x}\,dx = 2\sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}}\left\{\ln(2k+1) + 2\ln 2 + \boldsymbol{C}\right\}$$ BI (147)(4)

2. $$\int_0^\infty \ln x\frac{(\mu+1)\cosh x - x\sinh x}{\cosh^2 x}x^\mu\,dx = 2\,\Gamma(\mu+1)\sum_{k=0}^\infty \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}}$$

$[\operatorname{Re}\mu > -1]$ BI (356)(10)

3. $$\int_0^\infty \ln x\frac{(n+1)\cosh x - x\sinh x}{\cosh^2 x}x^n\,dx = \frac{(-1)^n}{2^n}\beta^{(n)}\left(\frac{1}{2}\right)$$

4. $$\int_0^\infty \ln 2x\frac{n\sinh 2ax - ax}{\sinh^2 ax}x^{2n-1}\,dx = -\frac{1}{n}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$

$[n=1,2,\ldots]$ BI (356)(9)a

5. $$\int_0^\infty \ln x\frac{ax\cosh ax - (2n+1)\sinh ax}{\sinh^2 ax}x^{2n}\,dx = 2\frac{2^{2n+1}-1}{(2a)^{2n+1}}(2n)!\,\zeta(2n+1)$$ BI (356)(14)

6. $$\int_0^\infty \ln x\frac{ax\cosh ax - 2n\sinh ax}{\sinh^2 ax}x^{2n-1}\,dx = \frac{2^{2n-1}-1}{2n}|B_{2n}|\left(\frac{\pi}{a}\right)^{2n}$$

$[n=1,2,\ldots, a>0]$ BI (356)(15)

7. $$\int_0^\infty \ln\frac{(2n+1)\cosh ax - ax\sinh ax}{\cosh^2 ax}x^{2n}\,dx = -\left(\frac{\pi}{2a}\right)^{2n+1}|E_{2n}|$$

$[a>0]$ BI (356)(11)

8.[6] $$\int_0^\infty \ln x\frac{2ax\sinh ax - (2n+1)\cosh ax}{\cosh^3 ax}x^{2n}\,dx = \begin{cases} \frac{2}{a}\left(2^{2n-1}-1\right)\left(\frac{\pi}{2a}\right)^{2n}|B_{2n}| & n=1,2,\ldots \\ \frac{1}{a} & n=0 \end{cases}$$

$[a>0]$ BI (356)(2)

9.[6] $$\int_0^\infty \ln x\frac{2ax\cosh ax - (2n+1)\sinh ax}{\sinh^3 ax}x^{2n}\,dx = \frac{1}{a}\left(\frac{\pi}{a}\right)^{2n}|B_{2n}|$$

$[a>0, \quad n=1,2,\ldots]$ BI (356)(6)a

10. $$\int_0^\infty \ln x \frac{x\sinh x - 6\sinh^2\left(\frac{x}{2}\right) - 6\cos^2\frac{t}{2}}{(\cosh x + \cos t)^2} x^2\,dx = \frac{\left(\pi - t^2\right)t}{3\sin t}$$
$[0 < t < \pi]$ BI (356)(16)a

11. $$\int_0^\infty \ln\left(1 + x^2\right) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4 - \pi$$ BI (356)(12)

12. $$\int_0^\infty \ln\left(1 + 4x^2\right) \frac{\cosh \pi x + \pi x \sinh \pi x}{\cosh^2 \pi x} \frac{dx}{x^2} = 4\ln 2$$ BI (356)(13)

4.377 $$\int_0^\infty \ln 2x \frac{ax - n\left(1 - e^{-2ax}\right)}{\sinh^2 ax} x^{2n-1}\,dx = \frac{1}{2n}\left(\frac{\pi}{a}\right)^{2n} |B_{2n}|$$
$[n = 1, 2, \ldots]$ LI (356)(8)a

4.38–4.41 Logarithms and trigonometric functions

4.381

1. $$\int_0^1 \ln x \sin ax\,dx = -\frac{1}{a}\left[\boldsymbol{C} + \ln a - \operatorname{ci}(a)\right]$$ $[a > 0]$ GW (338)(2a)

2. $$\int_0^1 \ln x \cos ax\,dx = -\frac{1}{a}\left[\operatorname{si}(a) + \frac{\pi}{2}\right]$$ $[a > 0]$ BI (284)(2)

3. $$\int_0^{2\pi} \ln x \sin nx\,dx = -\frac{1}{n}\left[\boldsymbol{C} + \ln(2n\pi) - \operatorname{ci}(2n\pi)\right]$$ GW (338)(1a)

4. $$\int_0^{2\pi} \ln x \cos nx\,dx = -\frac{1}{n}\left[\operatorname{si}(2n\pi) + \frac{\pi}{2}\right]$$ GW (338)(1b)

4.382

1. $$\int_0^\infty \ln\left|\frac{x+a}{x-a}\right| \sin bx\,dx = \frac{\pi}{b}\sin ab$$ $[a < 0, \quad b > 0]$ ET I 77(11)

2.[10] $$\int_0^\infty \ln\left|\frac{x+a}{x-a}\right| \cos bx\,dx = \frac{2}{b}\left[\cos(ab)\left\{\operatorname{si}(ab) + \frac{\pi}{2}\right\} - \sin(ab)\operatorname{ci}(ab)\right]$$
$[a > 0, \quad b > 0]$ ET I 18(9)

3. $$\int_0^\infty \ln\frac{a^2 + x^2}{b^2 + x^2} \cos cx\,dx = \frac{\pi}{c}\left(e^{-bc} - e^{-ac}\right)$$ $[a > 0, \quad b > 0, \quad c > 0]$
FI III 648a, BI (337)(5)

4. $$\int_0^\infty \ln\frac{x^2 + x + a^2}{x^2 - x + a^2} \sin bx\,dx = \frac{2\pi}{b}\exp\left(-b\sqrt{a^2 - \frac{1}{4}}\right)\sin\frac{b}{2}$$
$[b > 0]$ ET I 77(12)

5. $$\int_0^\infty \ln\frac{(x+\beta)^2 + \gamma^2}{(x-\beta)^2 + \gamma^2} \sin bx\,dx = \frac{2\pi}{b} e^{-\gamma b} \sin \beta b$$ $[\operatorname{Re}\gamma > 0, \quad |\operatorname{Im}\beta| \le \operatorname{Re}\gamma, \quad b > 0]$
ET I 77(13)

4.383

1. $$\int_0^\infty \ln\left(1+e^{-\beta x}\right)\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b\sinh\left(\dfrac{\pi b}{\beta}\right)} \qquad [\operatorname{Re}\beta>0, \quad b>0]$$ ET I 18(13)

2. $$\int_0^\infty \ln\left(1-e^{-\beta x}\right)\cos bx\,dx = \frac{\beta}{2b^2} - \frac{\pi}{2b}\coth\left(\frac{\pi b}{\beta}\right) \qquad [\operatorname{Re}\beta>0, \quad b>0]$$ ET I 18(14)

4.384

1. $$\int_0^1 \ln(\sin\pi x)\sin 2n\pi x\,dx = 0$$ GW (338)(3a)

2.[7] $$\int_0^1 \ln(\sin\pi x)\sin(2n+1)\pi x\,dx = 2\int_0^{1/2}\ln(\sin\pi x)\sin(2n+1)\pi x\,dx = \frac{2}{(2n+1)\pi}\left[\ln 2 - \frac{1}{2n+1} - 2\sum_{k=1}^{n}\frac{1}{2k-1}\right]$$ GW (338)(3b)

3.[6] $$\int_0^1 \ln(\sin\pi x)\cos 2n\pi x\,dx = 2\int_0^{1/2}\ln(\sin\pi x)\cos 2n\pi x\,dx$$
$$= -\ln 2 \qquad [n=0]$$
$$= -\frac{1}{2n} \qquad [n>0]$$ GW (338)(3c)

4. $$\int_0^1 \ln(\sin\pi x)\cos(2n+1)\pi x\,dx = 0$$ GW (338)(3d)

5. $$\int_0^{\pi/2} \ln\sin x\sin x\,dx = \ln 2 - 1$$ BI (305)(4)

6. $$\int_0^{\pi/2} \ln\sin x\cos x\,dx = -1$$ BI (305)(5)

7. $$\int_0^{\pi/2} \ln\sin x\cos 2nx\,dx = \begin{cases} -\dfrac{\pi}{4n}, & \text{for } n>0 \\ -\dfrac{\pi}{2}\ln 2, & \text{for } n=0 \end{cases}$$ LI (305)(6)

8. $$\int_0^{\pi} \ln\sin x\cos[2m(x-n)]\,dx = -\frac{\pi\cos 2mn}{2m}$$ LI (330)(8)

9. $$\int_0^{\pi/2} \ln\sin x\sin^2 x\,dx = \frac{\pi}{8}(1-\ln 4)$$ BI (305)(7)

10. $$\int_0^{\pi/2} \ln\sin x\cos^2 x\,dx = -\frac{\pi}{8}(1+\ln 4)$$ BI (305)(8)

11. $$\int_0^{\pi/2} \ln\sin x\sin x\cos^2 x\,dx = \frac{1}{9}(\ln 8 - 4)$$ BI (305)(9)

12. $$\int_0^{\pi/2} \ln\sin x\tan x\,dx = -\frac{\pi^2}{24}$$ BI (305)(11)

13. $$\int_0^{\pi/2} \ln \sin 2x \sin x \, dx = \int_0^{\pi/2} \ln \sin 2x \cos x \, dx = 2\left(\ln 2 - 1\right)$$ BI (305)(16, 17)

14. $$\int_0^{\pi} \frac{\ln\left(1 + p \cos x\right)}{\cos x} \, dx = \pi \arcsin p \qquad \left[p^2 < 1\right]$$ FI II 484

15. $$\int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^2}{2} \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi}{a^2 - 1} \ln \frac{a^2 - 1}{2a^2} \qquad \left[a^2 > 1\right]$$
BI (331)(8)

16. $$\int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^{2b}}{2} \qquad \left[a^2 < 1\right]$$ BI (331)(10)

17. $$\int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^{2b}}{2} \qquad \left[a^2 < 1\right]$$ BI (331)(11)

18. $$\int_0^{\pi/2} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{1}{2} \int_0^{\pi} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2}$$
$$= \frac{\pi}{2\left(1 - a^2\right)} \ln \frac{1 - a}{2} \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi}{2\left(a^2 - 1\right)} \ln \frac{a - 1}{2a} \qquad \left[a^2 > 1\right]$$
BI (321)(1), BI (331)(13)

19. $$\int_0^{\pi} \ln \sin bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{2} \qquad \left[a^2 < 1\right]$$ BI (331)(18)

20. $$\int_0^{\pi} \ln \cos bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^b}{2} \qquad \left[a^2 < 1\right]$$ BI (331)(21)

21. $$\int_0^{\pi/2} \frac{\ln \cos x \, dx}{1 - 2p \cos 2x + p^2} = \frac{\pi}{2\left(1 - p^2\right)} \ln \frac{1 + p}{2} \qquad \left[p^2 < 1\right]$$
$$= \frac{\pi}{2\left(p^2 - 1\right)} \ln \frac{p + 1}{2p} \qquad \left[p^2 > 1\right]$$
BI (321)(8)

22. $$\int_0^{\pi} \ln \sin x \frac{\cos x \, dx}{1 - 2a \cos x + a^2} = \frac{\pi}{2a} \frac{1 + a^2}{1 - a^2} \ln\left(1 - a^2\right) - \frac{a\pi \ln 2}{1 - a^2} \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi}{2a} \frac{a^2 + 1}{a^2 - 1} \ln \frac{a^2 - 1}{a^2} - \frac{\pi \ln 2}{a\left(a^2 - 1\right)} \qquad \left[a^2 > 1\right]$$
LI (331)(9)

23. $$\int_0^{\pi} \ln \sin bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^{\pi} \ln \cos bx \frac{\cos x \, dx}{1 - 2a \cos 2x + a^2} = 0$$
$$[0 < a < 1]$$ BI (331)(19, 22)

24. $$\int_0^{\pi} \ln \sin x \frac{\cos^2 x \, dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{4a} \frac{1 + a}{1 - a} \ln(1 - a) - \frac{\pi \ln 2}{2(1 - a)} \qquad [0 < a < 1]$$
$$= \frac{\pi}{4a} \frac{a + 1}{a - 1} \ln \frac{a - 1}{a} - \frac{\pi \ln 2}{2a(a - 1)} \qquad [a > 1]$$
BI (331)(16)

25. $$\int_0^{\pi/2} \ln\sin x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2} = \frac{1}{2}\int_0^{\pi} \ln\sin x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2}$$
$$= \frac{\pi}{2a\left(1-a^2\right)}\left\{\frac{1+a^2}{2}\ln(1-a)-a^2\ln 2\right\} \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{2a\left(a^2-1\right)}\left\{\frac{1+a^2}{2}\ln\frac{a-1}{a}-\ln 2\right\} \qquad \left[a^2>1\right]$$
BI (321)(2), BI (331)(15), LI (321))(2)

26. $$\int_0^{\pi/2} \ln\cos x \frac{\cos 2x\,dx}{1-2a\cos 2x+a^2} = \frac{\pi}{2a\left(1-a^2\right)}\left\{\frac{1+a^2}{2}\ln(1+a)-a^2\ln 2\right\} \qquad \left[a^2<1\right]$$
$$= \frac{\pi}{2a\left(a^2-1\right)}\left\{\frac{1+a^2}{2}\ln\frac{1+a}{a}-\ln 2\right\} \qquad \left[a^2>1\right]$$
BI (321)(9)

4.385

1. $$\int_0^{\pi} \ln\sin x \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \qquad [a>0, \quad a>b]$$ BI (331)(6)

2. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{(a\sin x \pm b\cos x)^2} = \int_0^{\pi/2} \ln\cos x \frac{dx}{(a\cos x \pm b\sin x)^2}$$
$$= \frac{1}{b\left(a^2+b^2\right)}\left(\mp a\ln\frac{a}{b}-\frac{b\pi}{2}\right)$$
$$[a>0, \quad b>0]$$ BI (319)(1,6)a

3. $$\int_0^{\pi/2} \frac{\ln\sin x\,dx}{a^2\sin^2 x+b^2\cos^2 x} = \int_0^{\pi/2} \frac{\ln\cos x\,dx}{b^2\sin^2 x+a^2\cos^2 x} = \frac{\pi}{2ab}\ln\frac{b}{a+b}$$
$$[a>0, \quad b>0]$$ BI (317)(4, 10)

4. $$\int_0^{\pi/2} \ln\sin x \frac{\sin 2x\,dx}{\left(a\sin^2 x+b\cos^2 x\right)^2} = \int_0^{\pi/2} \ln\cos x \frac{\sin 2x\,dx}{\left(b\sin^2 x+a\cos^2 x\right)^2}$$
$$= \frac{1}{2b(b-a)}\ln\frac{a}{b}$$
$$[a>0, \quad b>0]$$ BI (319)(3, 7), LI (319)(3)

5. $$\int_0^{\pi/2} \ln\sin x \frac{a^2\sin^2 x-b^2\cos^2 x}{\left(a^2\sin^2 x+b^2\cos^2 x\right)^2}\,dx = \int_0^{\pi/2} \ln\cos x \frac{a^2\cos^2 x-b^2\sin^2 x}{\left(a^2\cos^2 x+b^2\sin^2 x\right)^2}\,dx$$
$$= \frac{\pi}{2b(a+b)}$$
$$[a>0, \quad b>0]$$ LI (319)(2, 8)

4.386

1. $$\int_0^{\pi/2} \ln\sin x \frac{\sin x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^{\pi/2} \frac{\cos x\ln\cos x}{\sqrt{1+\cos^2 x}}\,dx = -\frac{\pi}{8}\ln 2$$ BI (322)(1, 6)

2. $$\int_0^{\pi/2} \frac{\sin^3 x\ln\sin x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^{\pi/2} \frac{\cos^3 x\ln\cos x}{\sqrt{1+\cos^2 x}}\,dx = \frac{\ln 2-1}{4}$$ BI (322)(2, 7)

3. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{2}\boldsymbol{K}(k)\ln k - \frac{\pi}{4}\boldsymbol{K}(k')$$ BI (322)(3)

4. $$\int_0^{\pi/2} \frac{\ln\cos x\, dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{2}\boldsymbol{K}(k)\ln\frac{k'}{k} - \frac{\pi}{4}\boldsymbol{K}(k')$$ BI (322)(9)

4.387

1. $$\int_0^{\pi/2} \ln\sin x \sin^\mu x \cos^\nu x\, dx = \int_0^{\pi/2} \ln\cos x \cos^\mu x \sin^\nu x\, dx = \frac{1}{4}\mathrm{B}\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right)\left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right)\right]$$
$[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ GW (338)(6c)

2. $$\int_0^{\pi/2} \ln\sin x \sin^{\mu-1} x\, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\mu}{2}\right)}{4\,\Gamma\left(\frac{\mu+1}{2}\right)}\left[\psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right)\right]$$
$[\operatorname{Re}\mu > 0]$ GW (338)(6a)

3. $$\int_0^{\pi/2} \ln\sin x \cos^{\nu-1} x\, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}{4\,\Gamma\left(\frac{\nu+1}{2}\right)}\left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right]$$
$[\operatorname{Re}\nu > 0]$ GW (338)(6b)

4. $$\int_0^{\pi/2} \ln\sin x \sin^{2n} x\, dx = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}\left\{\sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k} - \ln 2\right\}$$ FI II 811

5. $$\int_0^{\pi/2} \ln\sin x \sin^{2n+1} x\, dx = \frac{(2n)!!}{(2n+1)!!}\left\{\sum_{k=1}^{2n+1}\frac{(-1)^k}{k} + \ln 2\right\}$$ BI (305)(13)

6. $$\int_0^{\pi/2} \ln\sin x \cos^{2n} x\, dx = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}\left[\sum_{k=1}^{n}\frac{1}{k} + \ln 4\right] = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4}\left[\boldsymbol{C} + \psi(n+1) + \ln 4\right]$$
BI (305)(14)

7. $$\int_0^{\pi/2} \ln\sin x \cos^{2n+1} x\, dx = -\frac{(2n)!!}{(2n+1)!!}\sum_{k=0}^{n}\frac{1}{2k+1} = -\frac{(2n)!!}{2(2n+1)!!}\left[\psi\left(n+\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right)\right]$$
GW (338)(7b)

8. $$\int_0^{\pi/2} \ln\cos x \sin^{2n} x\, dx = -\frac{(2n-1)!!}{2^{n+1}\cdot n!}\frac{\pi}{2}\left\{\boldsymbol{C} + 2\ln 2 + \psi(n+1)\right\}$$
BI (306)(8)

9. $$\int_0^{\pi/2} \ln\cos x \cos^{2n} x\, dx = -\frac{(2n-1)!!}{2^n n!}\frac{\pi}{2}\left(\ln 2 + \sum_{k=1}^{2n}\frac{(-1)^k}{k}\right)$$ BI (306)(10)

10. $$\int_0^{\pi/2} \ln\cos x \cos^{2n} x\, dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!}\left[\ln 2 + \sum_{k=1}^{2n-1}\frac{(-1)^k}{k}\right]$$ BI (306)(9)

4.388

1. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n} x}{\cos^{2n+2} x}\, dx = \frac{1}{2n+1}\left[\frac{1}{2}\ln 2 + (-1)^n\frac{\pi}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\right]$$ BI (288)(1)

2. $$\int_0^{\pi/4} \ln\sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\, dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=1}^{n-1}\frac{(-1)^k}{n-k}\right]$$ LI (288)(2)

3. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n} x}{\cos^{2n+2} x}\, dx = \frac{1}{2n+1}\left[-\frac{1}{2}\ln 2 + (-1)^{n+1}\frac{\pi}{4} + \sum_{k=0}^{n}\frac{(-1)^{k-1}}{2n-2k+1}\right]$$ BI (288)(10)

4. $$\int_0^{\pi/4} \ln\cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x}\, dx = \frac{1}{4n}\left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=0}^{n-1}\frac{(-1)^k}{n-k}\right]$$ BI (288)(11)

5. $$\int_0^{\pi/2} \ln\sin x \frac{\sin^{p-1} x}{\cos^{p+1} x}\, dx = -\frac{\pi}{2p}\operatorname{cosec}\frac{p\pi}{2} \qquad [0 < p < 2]$$ BI (310)(4)

6. $$\int_0^{\pi/2} \ln\sin x \frac{dx}{\tan^{p-1} x \sin 2x} = \frac{1}{4}\frac{\pi}{p-1}\sec\frac{p\pi}{2} \qquad [p^2 < 1]$$ BI (310)(3)

4.389

1. $$\int_0^{\pi} \ln\sin x \sin^{2n} 2x \cos 2x\, dx = -\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{4n+2}$$ BI (330)(9)

2. $$\int_0^{\pi/4} \ln\sin x \cos^n 2x \sin 2x\, dx = -\frac{1}{4(n+1)}\left\{\boldsymbol{C} + \psi(n+2) + \ln 2\right\}$$ BI (285)(2)

3. $$\int_0^{\pi/4} \ln\cos x \cos^{\mu-1} 2x \tan 2x\, dx = \frac{1}{4(1-\mu)}\beta(\mu) \qquad [\operatorname{Re}\mu > 0]$$ BI (286)(2)

4. $$\int_0^{\pi/2} \ln\sin x \sin^{\mu-1} x \cos x\, dx = \int_0^{\pi/2} \ln\cos x \cos^{\mu-1} x \sin x\, dx = -\frac{1}{\mu^2} \qquad [\operatorname{Re}\mu > 0]$$ BI (306)(11)

5.[3] $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\cos x \cos^p x \cos px\, dx = \frac{\pi}{2^{p+1}}\left[\boldsymbol{C} + \psi(p+1) - 2\ln 2\right] \qquad [p > -1]$$

6. $$\int_0^{\pi/2} \ln\cos x \cos^{p-1} x \sin px \sin x\, dx = \frac{\pi}{2^{p+2}}\left[\boldsymbol{C} + \psi(p) - \frac{1}{p} - 2\ln 2\right] \qquad [p > 0]$$ BI (306)(12)

4.391

1. $$\int_0^{\pi/4} (\ln\cos 2x)^n \cos^{p-1} 2x \tan x \, dx = \int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x \tan\left(\frac{\pi}{4} - x\right) dx = \frac{1}{2}\beta^{(n)}(p)$$
$[p > 0]$ BI (286)(10), BI (285)(18)

2. $$\int_0^{\pi/4} (\ln\sin 2x)^n \sin^{p-1} 2x \tan\left(\frac{\pi}{4} + x\right) dx = \frac{(-1)^n n!}{2} \zeta(n+1, p)$$
BI (285)(17)

3. $$\int_0^{\pi/4} (\ln\cos 2x)^{2n-1} \tan x \, dx = \frac{1 - 2^{2n-1}}{4n} \pi^{2n} |B_{2n}| \qquad [n = 1, 2, \ldots]$$
BI (286)(7)

4. $$\int_0^{\pi/4} (\ln\cos 2x)^{2n} \tan x \, dx = \frac{2^{2n}1}{2^{2n+1}} (2n)! \, \zeta(2n+1)$$
BI (286)(8)

4.392

1. $$\int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[(-1)^{n+1} \frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{2n-2k-1}\right]$$
BI (294)(8)

2. $$\int_0^{\pi/4} \ln(\sin x \cos x) \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{2n} \left[(-1)^n \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n \sum_{k=1}^{n-1} \frac{(-1)^k}{k}\right]$$
BI (294)(9)

4.393

1. $$\int_0^{\pi/2} \ln\tan x \sin x \, dx = \ln 2$$
BI (307)(3)

2. $$\int_0^{\pi/2} \ln\tan x \cos x \, dx = -\ln 2$$
BI (307)(4)

3. $$\int_0^{\pi/2} \ln\tan x \sin^2 x \, dx = -\int_0^{\pi/2} \ln\tan x \cos^2 x \, dx = \frac{\pi}{4}$$
BI (307)(5, 6)

4. $$\int_0^{\pi/4} \frac{\ln\tan x}{\cos 2x} dx = -\frac{\pi^2}{8}$$
GW (338)(10b)a

5. $$\int_0^{\pi/2} \sin x \ln\cot \frac{x}{2} \, dx = \ln 2$$
LO III 290

4.394

1. $$\int_0^{\pi/2} \frac{\ln\tan x \, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{2(1-a^2)} \ln\frac{1-a}{1+a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{2(a^2-1)} \ln\frac{a-1}{a+1} \qquad [a^2 > 1]$$
BI (321)(15)

2. $$\int_0^{\pi/2} \frac{\ln\tan x \cos 2x \, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{4a} \frac{1+a^2}{1-a^2} \ln\frac{1-a}{1+a} \qquad [a^2 < 1]$$
$$= \frac{\pi}{4a} \frac{a^2+1}{a^2-1} \ln\frac{a-1}{a+1} \qquad [a^2 > 1]$$
BI (321)(16)

3. $$\int_0^{\pi} \frac{\ln \tan bx \, dx}{1 - 2a\cos 2x + a^2} = \frac{\pi}{1-a^2} \ln \frac{1-a^b}{1+a^b}$$ $[0 < a < 1, \quad b > 0]$ BI (331)(24)

4. $$\int_0^{\pi} \frac{\ln \tan bx \cos x \, dx}{1 - 2a\cos 2x + a^2} = 0$$ $[0 < a < 1]$ BI (331)(25)

5. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 - a\sin 2x} = -\frac{\arcsin a}{4a} (\pi + \arcsin a)$$ $[a^2 \le 1]$ BI (291)(2,3)

6. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 - a^2\sin^2 2x} = -\frac{\pi}{4a} \arcsin a$$ $[a^2 < 1]$ BI (291)(9)

7. $$\int_0^{\pi/4} \ln \tan x \frac{\cos 2x \, dx}{1 + a^2\sin^2 2x} = -\frac{\pi}{4a} \operatorname{arcsinh} a = -\frac{\pi}{4a} \ln\left(a + \sqrt{1+a^2}\right)$$ $[a^2 < 1]$ BI (291)(10)

8. $$\int_0^{u} \frac{\sin x \ln \cot \frac{x}{2}}{1 - \cos^2\alpha \sin^2 x} \, dx = \operatorname{cosec} 2\alpha \left\{ \frac{\pi}{2} \ln 2 + L(\varphi - \alpha) - L(\varphi + \alpha) - L\left(\frac{\pi}{2} - 2\alpha\right) \right\}$$ $[\tan\varphi = \cot\alpha \cos u; \quad 0 < u < \pi]$ LO III 290

9. $$\int_0^{\pi/4} \frac{\ln \tan x \sin 2x \, dx}{1 - \cos^2 t \sin^2 2x} = \operatorname{cosec} 2t \left[L\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right) \ln 2 \right]$$ LO III 290a

4.395

1. $$\int_0^{\pi/2} \frac{\ln \tan x \, dx}{\sqrt{1 - k^2\sin^2 x}} = -\ln k' \, \boldsymbol{K}(k)$$ BI (322)(11)

2. $$\int_u^{\pi/4} \frac{\ln \tan x \sin 4x \, dx}{\left(\sin^2 u + \tan^2 v \sin^2 2x\right) \sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2} \frac{\cos^2 v}{\sin u \sin v} \ln \frac{\sin v + \sqrt{1 - \cos^2 u \cos^2 v}}{\sin u \,(1 + \sin v)}$$ $\left[0 < u < \frac{\pi}{2}, \quad 0 < v < \frac{\pi}{2}\right]$ LO III 285a

4.396

1. $$\int_0^{\pi/2} \ln(a \tan x) \sin^{\mu-1} 2x \, dx = 2^{\mu-2} \ln a \frac{\left\{\Gamma\left(\frac{a}{2}\right)\right\}^2}{\Gamma(a)}$$ $[a > 0, \quad \operatorname{Re}\mu > 0]$ LI (307)(8)

2. $$\int_0^{\pi/2} \ln \tan x \cos^{2(\mu-1)x} \, dx = -\frac{\sqrt{\pi}}{4} \frac{\Gamma\left(u - \frac{1}{2}\right)}{\Gamma(\mu)} \left[\boldsymbol{C} + \psi\left(\frac{2\mu - 1}{2}\right) + \ln 4 \right]$$ $\left[\operatorname{Re}\mu > \frac{1}{2}\right]$ BI (307)(9)

3. $$\int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cot x \sin[(q+1)x] \, dx = -\frac{\pi}{2} \left[\boldsymbol{C} + \psi(q+1) \right]$$ $[q > -1]$ BI (307)(11)

4. $$\int_0^{\pi/2} \ln \tan x \cos^{q-1} x \cos[(q+1)x] \, dx = -\frac{\pi}{2q}$$ $[q > 0]$ BI (307)(10)

5. $$\int_0^{\pi/4} (\ln \tan x)^n \tan^p x \, dx = \frac{1}{2^{n+1}} B^{(n)}\left(\frac{p+1}{2}\right)$$ $[p > -1]$ LI (286)(22)

6. $$\int_0^{\pi/2} (\ln\tan x)^{2n-1} \frac{dx}{\cos 2x} = \frac{1-2^{2n}}{2n}\pi^{2n}|B_{2n}| \qquad [n = 1, 2, \ldots]$$ BI (312)(6)

7. $$\int_0^{\pi/4} \ln\tan x \tan^{2n+1} x\, dx = \frac{(-1)^{n+1}}{4}\left[\frac{\pi^2}{12} + \sum_{k=1}^{n} \frac{(-1)^k}{k^2}\right]$$ GW (338)(8a)

4.397

1. $$\int_0^{\pi/2} \ln(1 + p\sin x)\frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2}\left(\arccos p^2\right) \qquad \left[p^2 < 1\right]$$ BI (313)(1)

2. $$\int_0^{\pi/2} \ln(1 + p\cos x)\frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2}(\arccos p)^2 \qquad \left[p^2 < 1\right]$$ BI (313)(8)

3. $$\int_0^{\pi} \ln(1 + p\cos x)\frac{dx}{\cos x} = \pi\arcsin p \qquad \left[p^2 < 1\right]$$ BI (331)(1)

4. $$\int_0^{\pi/2} \frac{\cos x\ln(1+\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\, dx = \frac{L\left(\frac{\pi}{2}-\alpha\right) - \alpha\ln\sin\alpha}{\sin\alpha\cos\alpha} \qquad \left[0 < \alpha < \frac{\pi}{2}\right]$$ LO III 291

5. $$\int_0^{\pi/2} \frac{\cos x\ln(1-\cos\alpha\cos x)}{1-\cos^2\alpha\cos^2 x}\, dx = \frac{L\left(\frac{\pi}{2}-\alpha\right) + (\pi-\alpha)\ln\sin\alpha}{\sin\alpha\cos\alpha} \qquad \left[0 < \alpha < \frac{\pi}{2}\right]$$ LO III 291

6. $$\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right)\cos nx\, dx$$
$$= \frac{1}{2}\int_0^{2\pi} \ln\left(1 - 2a\cos x + a^2\right)\cos nx\, dx$$
$$= -\frac{\pi}{n}a^n \qquad \left[a^2 < 1\right]$$ BI (330)(11), BI (332)(5)
$$= -\frac{\pi}{na^n} \qquad \left[a^2 > 1\right]$$ GW (338)(13a)

7. $$\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right)\sin nx\sin x\, dx = \frac{1}{2}\int_0^{2\pi} \ln\left(1 - 2a\cos x + a^2\right)\sin nx\sin x\, dx$$
$$= \frac{\pi}{2}\left(\frac{a^{n+1}}{n+1} - \frac{a^{n-1}}{n-1}\right) \qquad \left[a^2 > 1\right]$$ BI (330)(10), BI (332)(4)

8. $$\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right)\sin nx\sin x\, dx = \frac{1}{2}\int_0^{2\pi} \ln\left(1 - 2a\cos x + a^2\right)\cos nx\cos x\, dx$$
$$= -\frac{\pi}{2}\left(\frac{a^{n+1}}{n+1} + \frac{a^{n-1}}{n-1}\right)$$

BI (330)(12), BI (332)(6)

9. $$\int_0^{\pi} \ln\left(1 - 2a\cos 2x + a^2\right)\cos(2n-1)x\,dx = 0 \qquad \left[a^2 < 1\right]$$ BI (330)(15)

10. $$\int_0^{\pi} \ln\left(1 - 2a\cos 2x + a^2\right)\sin 2nx \sin x\,dx = 0 \qquad \left[a^2 < 1\right]$$ BI (330)(13)

11. $$\int_0^{\pi} \ln\left(1 - 2a\cos 2x + a^2\right)\sin(2n-1)x \sin x\,dx = \frac{\pi}{2}\left(\frac{a^n}{n} - \frac{a^{n-1}}{n-1}\right) \qquad \left[a^2 < 1\right]$$ BI (330)(14)

12. $$\int_0^{\pi} \ln\left(1 - 2a\cos 2x + a^2\right)\cos 2nx \cos x\,dx = 0 \qquad \left[a^2 < 1\right]$$ BI (330)(16)

13. $$\int_0^{\pi} \ln\left(1 - 2a\cos 2x + a^2\right)\cos(2n-1)x \cos x\,dx = -\frac{\pi}{2}\left(\frac{a^n}{n} + \frac{a^{n-1}}{n-1}\right) \qquad \left[a^2 < 1\right]$$ BI (330)(17)

14. $$\int_0^{\pi/2} \ln\left(1 + 2a\cos 2x + a^2\right)\sin^2 x\,dx = -\frac{a\pi}{4} \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi \ln a^2}{4} - \frac{\pi}{4a} \qquad \left[a^2 > 1\right]$$
BI (309)(22), LI (309)(22)

15. $$\int_0^{\pi/2} \ln\left(1 + 2a\cos 2x + a^2\right)\cos^2 x\,dx = \frac{a\pi}{4} \qquad \left[a^2 < 1\right]$$
$$= \frac{\pi \ln a^2}{4} + \frac{\pi}{4a} \qquad \left[a^2 > 1\right]$$
BI (309)(23), LI (309)(23)

16. $$\int_0^{\pi} \frac{\ln\left(1 - 2a\cos x + a^2\right)}{1 - 2b\cos x + b^2}\,dx = \frac{2\pi\ln(1-ab)}{1-b^2} \qquad \left[a^2 \le 1, \quad b^2 < 1\right]$$ BI (331)(26)

4.398

1. $$\int_0^{\pi} \ln\frac{1 + 2a\cos x + a^2}{1 - 2a\cos x + a^2}\sin(2n+1)x\,dx = (-1)^n\frac{2\pi a^{2n+1}}{2n+1} \qquad \left[a^2 < 1\right]$$ BI (330)(18)

2. $$\int_0^{2\pi} \ln\frac{1 - 2a\cos x + a^2}{1 - 2a\cos nx + a^2}\cos mx\,dx = 2\pi\left(\frac{n}{m}a^{m/n} - \frac{a^m}{m}\right) \qquad \left[a^2 \le 1\right]$$
$$= 2\pi\left(\frac{n}{m}a^{-m/n} - \frac{a^{-m}}{m}\right) \qquad \left[a^2 \ge 1\right]$$
BI (332)(9)

3. $$\int_0^{\pi} \ln\frac{1 + 2a\cos 2x + a^2}{1 + 2a\cos 2nx + a^2}\cot x\,dx = 0$$ BI (331)(5), LI(331)(5)

4.399

1. $$\int_0^{\pi/2} \ln\left(1 + a\sin^2 x\right)\sin^2 x\,dx = \frac{\pi}{2}\left(\ln\frac{1+\sqrt{1+a}}{2} - \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right) \qquad [a > -1]$$ BI (309)(14)

2. $$\int_0^{\pi/2} \ln\left(1 + a\sin^2 x\right) \cos^2 x \, dx = \frac{\pi}{2}\left(\ln \frac{1+\sqrt{1+a}}{2} + \frac{1}{2}\frac{1-\sqrt{1+a}}{1+\sqrt{1+a}}\right)$$
$[a > -1]$ BI (309)(15)

3. $$\int_0^{\pi/2} \frac{\ln\left(1 - \cos^2\beta\cos^2 x\right)}{1 - \cos^2\alpha\cos^2 x} \, dx = -\frac{\pi}{\sin\alpha} \ln \frac{1+\sin\alpha}{\sin\alpha + \sin\beta}$$
$\left[0 < \beta < \frac{\pi}{2}, \quad 0 < \alpha < \frac{\pi}{2}\right]$ LO III 285

4.411

1. $$\int_0^{\pi} \ln \frac{1+\sin x}{1+\cos\lambda\sin x} \frac{dx}{\sin x} = \lambda^2$$
$\left[\lambda^2 < \pi^2\right]$ BI (331)(2)

2. $$\int_0^{\pi/2} \ln \frac{p + q\sin ax}{p - q\sin ax} \frac{dx}{\sin ax} = \int_0^{\pi/2} \ln \frac{p + q\cos ax}{p - q\cos ax} \frac{dx}{\cos ax} = \int_0^{\pi/2} \ln \frac{p + q\tan ax}{p - q\tan ax} \frac{dx}{\tan ax} = \pi \arcsin \frac{q}{p}$$
$[p > q > 0]$
FI II 695a, BI (315)(5, 13,17)a

3. $$\int_0^{\pi/2} \frac{\cos x}{1 - \cos^2\alpha\cos^2 x} \ln \frac{1 + \cos\beta\cos x}{1 - \cos\beta\cos x} \, dx = \frac{2\pi}{\sin 2\alpha} \ln \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$
$\left[0 < \alpha \le \beta < \frac{\pi}{2}\right]$ LO III 284

4.412

1. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4} \pm x\right) \frac{dx}{\sin 2x} = \pm\frac{\pi^2}{8}$$ BI (293)(1)

2. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4} \pm x\right) \frac{dx}{\tan 2x} = \pm\frac{\pi^2}{16}$$ BI (293)(2)

3. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4} \pm x\right) (\ln\tan x)^{2n} \frac{dx}{\sin 2x} = \pm\frac{2^{2n+2}-1}{4(n+1)(2n+1)}\pi^{2n+2}\left|B_{2n+2}\right|$$ BI (294)(24)

4. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4} \pm x\right) (\ln\tan x)^{2n-1} \frac{dx}{\sin 2x} = \pm\frac{1-2^{2n+1}}{2^{2n+2}n}(2n)!\,\zeta(2n+1)$$ BI (294)(25)

5. $$\int_0^{\pi/4} \ln\tan\left(\frac{\pi}{4} \pm x\right) (\ln\sin 2x)^{n-1} \frac{dx}{\tan 2x} = \frac{(-1)^{n-1}}{2}(n-1)!\,\zeta(n+1)$$ LI (294)(20)

4.413

1. $$\int_0^{\pi/2} \ln\left(p^2 + q^2\tan^2 x\right) \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{ab} \ln \frac{ap + bq}{a}$$
$[a > 0, \quad b > 0, \quad p > 0, \quad q > 0]$
BI (318)(1–4)a

2. $$\int_0^{\pi/2} \ln\left(1 + q^2\tan^2 x\right) \frac{1}{p^2\sin^2 x + r^2\cos^2 x} \frac{dx}{s^2\sin^2 x + t^2\cos^2 x}$$
$$= \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{p^2 - r^2}{pr} \ln\left(1 + \frac{qr}{p}\right) + \frac{t^2 - s^2}{st} \ln\left(1 + \frac{qt}{s}\right)\right\}$$
$[q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0]$ BI (320)(18)

3. $$\int_0^{\pi/2} \ln\left(1+q^2\tan^2 x\right) \frac{\sin^2 x}{p^2\sin^2 x + r^2\cos^2 x} \frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{t}{s}\ln\left(1+\frac{qr}{p}\right) - \frac{r}{p}\ln\left(1+\frac{qt}{s}\right)\right\}$$
$[q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0]$ BI (320)(20)

4. $$\int_0^{\pi/2} \ln\left(1+q^2\tan^2 x\right) \frac{\cos^2 x}{p^2\sin^2 x + r^2\cos^2 x} \frac{dx}{s^2\sin^2 x + t^2\cos^2 x} = \frac{\pi}{p^2t^2 - s^2r^2}\left\{\frac{p}{r}\ln\left(1+\frac{qr}{p}\right) - \frac{s}{t}\ln\left(1+\frac{qt}{s}\right)\right\}$$
$[q>0, \quad p>0, \quad r>0, \quad s>0, \quad t>0]$ BI (320)(21)

5. $$\int_0^{\pi} \frac{\ln\tan rx\,dx}{1-2p\cos x+p^2} = \frac{\pi}{1-p^2}\ln\frac{1-p^{2r}}{1+p^{2r}}$$ $\left[p^2<1\right]$ BI (331)(12)

4.414

1. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{dx}{\sqrt{1-k^2\sin^2 x}} = \ln k' \boldsymbol{K}(k)$$ BI (323)(1)

2. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{1}{k^2}\left\{\left(k^2-2+\ln k'\right)\boldsymbol{K}(k) + \left(2-\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (323)(3)

3. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\cos^2 x}{dx}\sqrt{1-k^2\sin^2 x} = \frac{1}{k^2}\left[\left(1+k'^2-k'^2\ln k'\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(6)

4. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{dx}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} = \frac{1}{k'^2}\left[\left(k^2-2\right)\boldsymbol{K}(k) + \left(2+\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(9)

5. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin^2 x}{dx}\sqrt{\left(1-k^2\sin^2 x\right)^3} = \frac{1}{k^2k'^2}\left[\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(1+k'^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right]$$
BI (323)(10)

6. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\cos^2 x\,dx}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} = \frac{1}{k^2}\left[\left(1+k'^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right]$$
BI (323)(16)

7. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,dx = \left(1+k'^2\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$ BI (324)(18)

8. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sin^2 x\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{9k^2}\left\{\left(-2+11k^2-6k^4+3k'^2\ln k'\right)\boldsymbol{K}(k) + \left[2-10k^2-3\left(1-2k^2\right)\ln k'\right]\boldsymbol{E}(k)\right\}$$

BI (324)(20)

9. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\cos^2 x\sqrt{1-k^2\sin^2 x}\,dx = \frac{1}{9k^2}\left\{\left(2+7k^2-3k^4-3k'^2\ln k'\right)\boldsymbol{K}(k) - \left[2+8k^2-3\left(1+k^2\right)\ln k'\right]\boldsymbol{E}(k)\right\}$$

BI (324)(21), LI (324)(21)

10. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\frac{\sin x\cos x\,dx}{\sqrt{\left(1-k^2\sin^2 x\right)^{2n+1}}} = \frac{2}{(2n-1)^2k^2}\left\{\left[1+(2n-1)\ln k'\right]k'^{1-2n}-1\right\}$$

BI (324)(17)

4.415

1. $$\int_0^{\infty} \ln x\sin ax^2\,dx = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left(\ln 4a+\boldsymbol{C}-\frac{\pi}{2}\right) \qquad [a>0]$$ GW (338)(19)

2. $$\int_0^{\infty} \ln x\cos ax^2\,dx = -\frac{1}{4}\sqrt{\frac{\pi}{2a}}\left(\ln 4a+\boldsymbol{C}-\frac{\pi}{2}\right) \qquad [a>0]$$ GW (338)(19)

4.416

1. $$\int_0^{\pi/2} \frac{\cos x\ln\left(1+\sqrt{\sin^2\beta-\cos^2\beta\tan^2\alpha\sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx = \operatorname{cosec}2\alpha\left\{(2\alpha+2\gamma-\pi)\ln\cos\beta+2\,L(\alpha)-2\,L(\gamma)+L(\alpha+\gamma)-L(\alpha-\gamma)\right\}$$
$$\left[\cos\gamma=\frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right]$$ LO III 291

2. $$\int_0^{\pi/2} \frac{\cos x\ln\left(1-\sqrt{\sin^2\beta-\cos^2\beta\tan^2\alpha\sin^2 x}\right)}{1-\sin^2\alpha\cos^2 x}\,dx = \operatorname{cosec}2\alpha\left\{(\pi+2\alpha-2\gamma)\ln\cos\beta+2\,L(\alpha)+2\,L(\gamma)-L(\alpha+\gamma)+L(\alpha-\gamma)\right\}$$
$$\left[\cos\gamma=\frac{\sin\alpha}{\sin\beta};\quad 0<\alpha<\beta<\frac{\pi}{2}\right]$$ LO III 291

3. $$\int_\beta^{\pi/2} \frac{\ln\left(\sin x+\sqrt{\sin^2 x-\sin^2\beta}\right)}{1-\cos^2\alpha\cos^2 x}\,dx = -\operatorname{cosec}\alpha\left\{\arctan\left(\frac{\tan\beta}{\sin\alpha}\right)\ln\sin\beta+\frac{\pi}{2}\ln\frac{1+\sin\alpha}{\sin\alpha+\sqrt{1-\cos^2\alpha\cos^2\beta}}\right\}$$
$$\left[0<\alpha<\pi,\quad 0<\beta<\frac{\pi}{2}\right]$$ LO III 285

4.[7] $$\int_0^{\pi/4} \ln\tan x\left(\ln\cos 2x\right)^{n-1}\tan 2x\,dx = \tfrac{1}{2}(-1)^n(n-1)!\left(1-2^{-(n+1)}\right)\zeta(n+1)$$

BI (287)(20)

4.42–4.43 Combinations of logarithms, trigonometric functions, and powers

4.421

1. $$\int_0^\infty \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2}\left(\boldsymbol{C} + \ln a\right) \qquad [a > 0]$$ FI II 810a

2. $$\int_0^\infty \ln ax \sin bx \frac{x\,dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln\left(a\beta'\right) - \frac{\pi}{4}\left[e^{b\beta'}\operatorname{Ei}\left(-b\beta'\right) + e^{-b\beta'}\operatorname{Ei}\left(b\beta'\right)\right]$$ $$[\beta' = \beta \operatorname{sign}\beta; \quad a > 0, \quad b > 0]$$ ET I 76(5), NT 27(10)a

3. $$\int_0^\infty \ln ax \cos bx \frac{\beta'\,dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln\left(a\beta'\right) + \frac{\pi}{4}\left[e^{b\beta'}\operatorname{Ei}\left(-b\beta'\right) - e^{-b\beta'}\operatorname{Ei}\left(b\beta'\right)\right]$$ $$[\beta' = \beta \operatorname{sign}\beta; \quad a > 0, \quad b > 0]$$ ET I 17(3), NT 27(11)a

4. $$\int_0^\infty \ln ax \sin bx \frac{x\,dx}{x^2 - c^2} = \frac{\pi}{2}\left\{-\operatorname{si}(bc)\sin bc + \cos bc\left[\ln ac - ci(bc)\right]\right\}$$ $$[a > 0, \quad b > 0, \quad c > 0]$$ BI (422)(5)

5. $$\int_0^\infty \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c}\left\{\sin bc\left[\operatorname{ci}(bc) - \ln ac\right] - \cos bc \operatorname{si}(bc)\right\}$$ $$[a > 0, \quad b > 0, \quad c > 0]$$ BI (422)(6)

4.422

1. $$\int_0^\infty \ln x \sin ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a + \frac{\pi}{2}\cot\frac{\mu\pi}{2}\right]$$ $$[a > 0, \quad |\operatorname{Re}\mu| < 1]$$ BI (411)(5)

2. $$\int_0^\infty \ln x \cos ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\cos\frac{\mu\pi}{2}\left[\psi(\mu) - \ln a - \frac{\pi}{2}\tan\frac{\mu\pi}{2}\right]$$ $$[a > 0, \quad 0 < \operatorname{Re}\mu < 1]$$ BI (411)(6)

4.423

1. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x}\,dx = \ln\frac{a}{b}\left(\boldsymbol{C} + \frac{1}{2}\ln ab\right) \qquad [a > 0, \quad b > 0]$$ GW (338)(21a)

2. $$\int_0^\infty \ln x \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}\left[(a - b)\left(\boldsymbol{C} - 1\right) + a\ln a - b\ln b\right]$$ $$[a > 0, \quad b > 0]$$ GW (338)(21b)

3. $$\int_0^\infty \ln x \frac{\sin^2 ax}{x^2}\,dx = -\frac{a\pi}{2}\left(\boldsymbol{C} + \ln 2a - 1\right) \qquad [a > 0]$$ GW (338)(20b)

4.424

1. $$\int_0^\infty (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2}\boldsymbol{C}^2 + \frac{\pi^3}{24} + \pi\boldsymbol{C}\ln a + \frac{\pi}{2}(\ln a)^2$$ $$[a > 0]$$ ET I 77(9), FI II 810a

2.[6] $$\int_0^\infty (\ln x)^2 \sin ax x^{\mu-1}\,dx = \frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}\left[\psi'(\mu)+\psi^2(\mu)+\pi\,\psi(\mu)\cot\frac{\mu\pi}{2}-2\,\psi(\mu)\ln a - \pi\ln a\cot\frac{\mu\pi}{2}+(\ln a)^2-\frac{1}{4}\pi^2\right]$$ $[a>0,\quad 0<\operatorname{Re}\mu<1]$ ET I 77(10)

4.425

1. $$\int_0^\infty \ln(1+x)\cos ax\frac{dx}{x} = \frac{1}{2}\left\{[\operatorname{si}(a)]^2+[\operatorname{ci}(a)]^2\right\}$$ $[a>0]$ ET I 18(8)

2. $$\int_0^\infty \ln^2\left(\frac{b+x}{b-x}\right)\cos ax\frac{dx}{x} = -2\pi\operatorname{si}(ab)$$ $[a\ge 0,\quad b>0]$ ET I 18(11)

3. $$\int_0^\infty \ln\left(1+b^2x^2\right)\sin ax\frac{dx}{x} = -\pi\operatorname{Ei}\left(-\frac{a}{b}\right)$$ $[a>0,\quad b>0]$ GW (338)(24), ET I 77(14)

4. $$\int_0^1 \ln\left(1-x^2\right)\cos\left(p\ln x\right)\frac{dx}{x} = \frac{1}{2p^2}+\frac{\pi}{2p}\coth\frac{p\pi}{2}$$ LI (309)(1)a

4.426

1.[11] $$\int_0^\infty x\ln\frac{b^2+x^2}{c^2+x^2}\sin ax\,dx = \frac{\pi}{a^2}\left[(1+ac)e^{-ac}-(1+ab)e^{-ab}\right]$$ $[b\ge 0,\quad c\ge 0,\quad a>0]$ GW (338)(23)

2. $$\int_0^\infty \ln\frac{b^2x^2+p^2}{c^2x^2+p^2}\sin ax\frac{dx}{x} = \pi\left[\operatorname{Ei}\left(-\frac{ap}{c}\right)-\operatorname{Ei}\left(-\frac{ap}{b}\right)\right]$$ $[b>0,\quad c>0,\quad p>0,\quad a>0]$ ET I 77(15)

4.427 $$\int_0^\infty \ln\left(x+\sqrt{\beta^2+x^2}\right)\frac{\sin ax}{\sqrt{\beta^2+x^2}}\,dx = \frac{\pi}{2}K_0(a\beta)+\frac{\pi}{2}\ln(\beta)\left[I_0(a\beta)-\mathbf{L}(a\beta)\right]$$ $[\operatorname{Re}\beta>0,\quad a>0]$ ET I 77(16)

4.428

1. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2}\,dx = \pi b\ln 2 - a\pi$$ $[a>0,\quad b>0]$ ET I 22(29)

2. $$\int_0^\infty \ln\left(4\cos^2 ax\right)\frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c}\cosh(bc)\ln\left(1+e^{-2ac}\right)$$ $\left[a<b<2a<\frac{\pi}{c}\right]$ ET I 22(30)

3. $$\int_0^\infty \ln\cos^2 ax\frac{\sin bx}{x\left(1+x^2\right)}\,dx = \pi\ln\left(1+e^{-2a}\right)\sinh b-\pi\ln 2\left(1-e^{-b}\right)$$ $[a>0,\quad b>0]$ ET I 82(36)

4. $$\int_0^\infty \ln\cos^2 ax\frac{\cos bx}{x^2\left(1+x^2\right)}\,dx = -\pi\ln\left(1+e^{-2a}\right)\cosh b+\left(b+e^{-b}\right)\pi\ln 2-a\pi$$ $[a>0,\quad b>0]$ ET I 22(31)

4.429 $$\int_0^1 \frac{(1+x)x}{\ln x}\sin\left(\ln x\right)\,dx = \frac{\pi}{4}$$ BI (326)(2)a

4.431

1. $$\int_0^\infty \ln(2 \pm 2\cos x) \frac{\sin bx}{x^2+c^2} x\,dx = -\,\pi \sinh(bc) \ln\left(1 \pm e^{-c}\right)$$

$[b > 0, \quad c > 0]$ ET I 22(32)

2. $$\int_0^\infty \ln(2 \pm 2\cos x) \frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c} \cosh(bc) \ln\left(1 \pm e^{-c}\right)$$

$[b > 0, \quad c > 0]$ ET I 22(32)

3. $$\int_0^\infty \ln\left(1 + 2a\cos x + a^2\right) \frac{\sin bx}{x}\,dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{(-a)^k}{k} \left[1 + \operatorname{sign}(b-k)\right]$$

$[0 < a < 1, \quad b > 0]$ ET I 82(25)

4. $$\int_0^\infty \ln\left(1 - 2a\cos x + a^2\right) \frac{\cos bx}{x^2+c^2}\,dx = \frac{\pi}{c} \ln\left(1 - ae^{-c}\right) \cosh(bc) + \frac{\pi}{c} \sum_{k=1}^{\lfloor b \rfloor} \frac{a^k}{k} \sinh[c(b-k)]$$

$[|a| < 1, \quad b > 0, \quad c > 0]$ ET I 22(33)

4.432

1. $$\int_0^\infty \ln\left(1 - k^2\sin^2 x\right) \frac{\sin x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln\left(1 - k^2\cos^2 x\right) \frac{\sin x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \ln k' \, \boldsymbol{K}(k)$$

BI ((412, 414))(4)

2. $$\int_0^{\pi/2} \ln\left(1 - k^2\sin^2 x\right) \frac{\sin x \cos x}{\sqrt{1-k^2\sin^2 x}} x\,dx = \frac{1}{k^2} \left\{\pi k' \left(1 - \ln k'\right) + \left(2 - k^2\right) \boldsymbol{K}(k) - \left(4 - \ln k'\right) \boldsymbol{E}(k)\right\}$$

BI (426)(3)

3. $$\int_0^{\pi/2} \ln\left(1 - k^2\cos^2 x\right) \frac{\sin x \cos x}{\sqrt{1-k^2\cos^2 x}} x\,dx = \frac{1}{k^2} \left\{-\pi - \left(2 - k^2\right) \boldsymbol{K}(k) + \left(4 - \ln k'\right) \boldsymbol{E}(k)\right\}$$

BI (426)(6)

4. $$\int_0^\infty \ln\left(1 - k^2\sin^2 x\right) \frac{\sin x \cos x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{\left(2 - k^2 - k'^2 \ln k'\right) \boldsymbol{K}(k) - \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$

BI (412)(5)

5. $$\int_0^\infty \ln\left(1 - k^2\cos^2 x\right) \frac{\sin x \cos x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \left\{\left(k^2 - 2 + \ln k'\right) \boldsymbol{K}(k) + \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$

BI (414)(5)

6. $$\int_0^\infty \ln\left(1 \pm k \sin^2 x\right) \frac{\sin x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln\left(1 \pm k \cos^2 x\right) \frac{\sin x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k \sin^2 x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k \cos^2 x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k \sin^2 2x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 2x}} \frac{dx}{x}$$
$$= \int_0^\infty \ln\left(1 \pm k^2 \cos^2 2x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 2x}} \frac{dx}{x}$$
$$= \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} \boldsymbol{K}(k) - \frac{\pi}{8} \boldsymbol{K}(k')$$
BI (413)(1–6), BI (415)(1–6)

7. $$\int_0^\infty \ln\left(1 - k^2 \sin^2 x\right) \frac{\sin^3 x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2 - 2 + \ln k'\right) \boldsymbol{K}(k) + \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (412)(6)

8. $$\int_0^\infty \ln\left(1 - k^2 \cos^2 x\right) \frac{\sin^3 x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2 - k^2 - {k'}^2 \ln k'\right) \boldsymbol{K}(k) - \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (414)(6)a

9. $$\int_0^\infty \ln\left(1 - k^2 \sin^2 x\right) \frac{\sin x \cos^2 x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2 - k^2 - {k'}^2 \ln k'\right) \boldsymbol{K}(k) - \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (412)(7)

10. $$\int_0^\infty \ln\left(1 - k^2 \cos^2 x\right) \frac{\sin x \cos^2 x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2 - 2 + \ln k'\right) \boldsymbol{K}(k) + \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (414)(7)

11. $$\int_0^\infty \ln\left(1 - k^2 \sin^2 x\right) \frac{\tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln\left(1 - k^2 \cos^2 x\right) \frac{\tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \ln k' \, \boldsymbol{K}(k)$$
BI ((412, 414))(9)

12. $$\int_0^\infty \ln\left(1 - k^2 \sin^2 x\right) \frac{\sin^2 x \tan x}{\sqrt{1-k^2\sin^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(k^2 - 2 + \ln k'\right) \boldsymbol{K}(k) + \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (412)(8)

13. $$\int_0^\infty \ln\left(1 - k^2 \cos^2 x\right) \frac{\sin^2 x \tan x}{\sqrt{1-k^2\cos^2 x}} \frac{dx}{x} = \frac{1}{k^2}\left\{\left(2 - k^2 - {k'}^2 \ln k'\right) \boldsymbol{K}(k) - \left(2 - \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI (414)(8)

14. $$\int_0^\infty \ln\left(1 - k^2 \sin^2 x\right) \frac{\sin^2 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^\infty \ln\left(1 - k^2 \cos^2 x\right) \frac{\sin x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{{k'}^2}\left\{\left(k^2 - 2\right) \boldsymbol{K}(k) + \left(2 + \ln k'\right) \boldsymbol{E}(k)\right\}$$
BI ((412, 414))(13)

15. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x \cos x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} x\,dx = \frac{1}{k^2}\left\{\left(1+\ln k'\right)\frac{\pi}{k'} - \left(2+\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (426)(9)

16. $$\int_0^{\pi/2} \ln\left(1-k^2\cos^2 x\right) \frac{\sin x \cos x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} x\,dx = \frac{1}{k^2}\left\{-\pi + \left(2+\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (426)(15)

17. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x \cos x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin^3 x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2}\left\{\left(2-k^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(14), BI(414)(15)

18. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin^3 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(1-k^2 cos^2 x\right) \frac{\sin x \cos x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2 k'^2}\left\{\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(2-k^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (412)(15), BI(414)(14)

19. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin x \cos^2 x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin^2 x \tan x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2}\left\{\left(2-k^2+\ln k'\right)\boldsymbol{K}(k) - \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI (412)(16), BI(414)(17)

20. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\sin^2 x \tan x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\sin x \cos^2 x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k^2 k'^2}\left\{\left(2+\ln k'\right)\boldsymbol{E}(k) - \left(2-k^2+k'^2\ln k'\right)\boldsymbol{K}(k)\right\}$$

BI (412)(17), BI(414)(16)

21. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right) \frac{\tan x}{\sqrt{\left(1-k^2\sin^2 x\right)^3}} \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right) \frac{\tan x}{\sqrt{\left(1-k^2\cos^2 x\right)^3}} \frac{dx}{x}$$
$$= \frac{1}{k'^2}\left\{\left(k^2-2\right)\boldsymbol{K}(k) + \left(2+\ln k'\right)\boldsymbol{E}(k)\right\}$$

BI ((412, 414))(18)

22. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,\sin x \frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\,\sin x \frac{dx}{x}$$
$$= \left(2-k^2\right)\boldsymbol{K}(k) - \left(2-\ln k'\right)\boldsymbol{E}(k)$$

BI ((412, 414))(1)

23. $$\int_0^{\pi/2} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,\sin x\cos x\cdot x\,dx$$
$$= \frac{1}{27k^2}\left\{3\pi k'^3\left(1-3\ln k'\right)+\left(22k'^2+6k^4-3k'^2\ln k'\right)\boldsymbol{K}(k)\right\} \quad -\left(2-k^2\right)\left(14-6\ln k'\right)\boldsymbol{E}(k)$$
BI (426)(1)

24. $$\int_0^{\pi/2} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\,\sin x\cos x\cdot x\,dx$$
$$= \frac{1}{27k^2}\left\{-3\pi-\left(22k'^2+6k^4-3k'^2\ln k'\right)\boldsymbol{K}(k)+\left(2-k^2\right)\left(14-6\ln k'\right)\boldsymbol{E}(k)\right\}$$
BI (426)(2)

25. $$\int_0^{\infty} \ln\left(1-k^2\sin^2 x\right)\sqrt{1-k^2\sin^2 x}\,\tan x\frac{dx}{x} = \int_0^{\infty} \ln\left(1-k^2\cos^2 x\right)\sqrt{1-k^2\cos^2 x}\,\tan x\frac{dx}{x}$$
$$= \left(2-k^2\right)\boldsymbol{K}(k)-\left(2-\ln k'\right)\boldsymbol{E}(k)$$
((412,414))(2)

26. $$\int_0^{\infty} \ln\left(\sin^2 x+k'\cos^2 x\right)\frac{\sin x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x} = \int_0^{\infty} \ln\left(\sin^2 x+k'\cos^2 x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 x}}\frac{dx}{x}$$
$$= \int_0^{\infty} \ln\left(\sin^2 2x+k'\cos^2 2x\right)\frac{\tan x}{\sqrt{1-k^2\cos^2 2x}}\frac{dx}{x}$$
$$= \frac{1}{2}\ln\left[\frac{2\left(\sqrt{k'}\right)^3}{1+k'}\right]\boldsymbol{K}(k)$$
BI (415)(19–21)

4.44 Combinations of logarithms, trigonometric functions, and exponentials

4.441

1.[7] $$\int_0^{\infty} e^{-qx}\sin px\ln x\,dx = \frac{1}{p^2+q^2}\left[q\arctan\frac{p}{q}-p\boldsymbol{C}-\frac{p}{c}\ln\left(p^2-q^2\right)\right]$$
$[q>0, \quad p>0]$ BI (467)(1)

2. $$\int_0^{\infty} e^{-qx}\cos px\ln x\,dx = -\frac{1}{p^2+q^2}\left[\frac{q}{2}\ln\left(p^2+q^2\right)+p\arctan\frac{p}{q}+q\boldsymbol{C}\right]$$
$[q>0]$ BI (467)(2)

4.442 $$\int_0^{\pi/2} \frac{e^{-p\tan x}\ln\cos x\,dx}{\sin x\cos x} = -\frac{1}{2}\left[\operatorname{ci}(p)\right]^2+\frac{1}{2}\left[\operatorname{si}(p)\right]^2$$
$[\operatorname{Re} p>0]$ NT 32(11)

4.5 Inverse Trigonometric Functions

4.51 Inverse trigonometric functions

4.511 $$\int_0^{\infty} \operatorname{arccot} px\operatorname{arccot} qx\,dx = \frac{\pi}{2}\left\{\frac{1}{p}\ln\left(1+\frac{p}{q}\right)+\frac{1}{q}\ln\left(1+\frac{q}{p}\right)\right\}$$
$[p>0, \quad q>0]$ BI (77)(8)

4.512 $$\int_0^{\pi} \arctan(\cos x)\, dx = 0$$ BI (345)(1)

4.52 Combinations of arcsines, arccosines, and powers

4.521

1. $$\int_0^1 \frac{\arcsin x}{x}\, dx = \frac{\pi}{2} \ln 2$$ FI II 614, 623

2. $$\int_0^1 \frac{\arccos x}{1 \pm x}\, dx = \mp \frac{\pi}{2} \ln 2 + 2\boldsymbol{G}$$ BI (231)(7, 8)

3. $$\int_0^1 \arcsin x \frac{x}{1 + qx^2}\, dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \qquad [q > -1]$$ BI (231)(1)

4. $$\int_0^1 \arcsin x \frac{x}{1 - p^2x^2}\, dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \qquad \left[p^2 < 1\right]$$ LI (231)(3)

5. $$\int_0^1 \arccos x \frac{dx}{\sin^2 \lambda - x^2} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$$ BI (231)(10)

6. $$\int_0^1 \arcsin x \frac{dx}{x\left(1 + qx^2\right)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \qquad [q > -1]$$ BI (235)(10)

7. $$\int_0^1 \arcsin x \frac{x}{\left(1 + qx^2\right)^2}\, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \qquad [q > -1]$$ BI (234)(2)

8. $$\int_0^1 \arccos x \frac{x}{\left(1 + qx^2\right)^2}\, dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \qquad [q > -1]$$ BI (234)(4)

4.522

1. $$\int_0^1 x\sqrt{1-k^2x^2} \arccos x\, dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\, \boldsymbol{K}(k) - 2\left(1 + k'^2\right) \boldsymbol{E}(k)\right]$$ BI (236)(9)

2. $$\int_0^1 x\sqrt{1-k^2x^2} \arcsin x\, dx = \frac{1}{9k^2}\left[-\frac{3}{2}\pi k'^3 - k'^2\, \boldsymbol{K}(k) + 2\left(1 + k'^2\right) \boldsymbol{E}(k)\right]$$ BI (236)(1)

3. $$\int_0^1 x\sqrt{k'^2+k^2x^2} \arcsin x\, dx = \frac{1}{9k^2}\left[\frac{3}{2}\pi + k'^2\, \boldsymbol{K}(k) - 2\left(1 + k'^2\right) \boldsymbol{E}(k)\right]$$ BI(236)(5)

4. $$\int_0^1 \frac{x \arcsin x}{\sqrt{1-k^2x^2}}\, dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right]$$ BI (237)(1)

5. $$\int_0^1 \frac{x \arccos x}{\sqrt{1-k^2x^2}}\, dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right]$$ BI (240)(1)

6. $$\int_0^1 \frac{x \arcsin x}{\sqrt{k'^2+k^2x^2}}\, dx = \frac{1}{k^2}\left[\frac{\pi}{2} - \boldsymbol{E}(k)\right]$$ BI (238)(1)

7. $$\int_0^1 \frac{x \arccos x}{\sqrt{k'^2+k^2x^2}}\, dx = \frac{1}{k^2}\left[-\frac{\pi}{2}k' + \boldsymbol{E}(k)\right]$$ BI (241)(1)

8. $$\int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda)\sqrt{1-x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}$$ BI (243)(11)

9. $$\int_0^1 \frac{x \arcsin kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = -\frac{\pi}{2k} \ln k'$$ BI (239)(1)

10. $$\int_0^1 \frac{x \arccos kx}{\sqrt{(1-x^2)(1-k^2x^2)}} \, dx = \frac{\pi}{2k} \ln(1+k)$$ BI (242)(1)

4.523

1. $$\int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1} \left[\frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!} \right]$$ BI (229)(1)

2. $$\int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n} \left[1 - \frac{(2n-1)!!}{2^n n!} \right]$$ BI (229)(2)

3. $$\int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!}$$ BI (229)(4)

4. $$\int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n} \frac{(2n-1)!!}{2^n n!}$$ BI (229)(5)

5. $$\int_{-1}^1 \left(1-x^2\right)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!}$$ BI (254)(2)

6. $$\int_{-1}^1 \left(1-x^2\right)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2} \frac{(2n-1)!!}{2^n n!}$$ BI (254)(3)

4.524

1. $$\int_0^1 (\arcsin x)^2 \frac{dx}{x^2\sqrt{1-x^2}} = \pi \ln 2$$ BI (243)(13)

2. $$\int_0^1 (\arccos x)^2 \frac{dx}{\left(\sqrt{1-x^2}\right)^3} = \pi \ln 2$$ BI (244)(9)

4.53–4.54 Combinations of arctangents, arccotangents, and powers

4.531

1. $$\int_0^1 \frac{\arctan x}{x} \, dx = \int_1^\infty \frac{\operatorname{arccot} x}{x} \, dx = \boldsymbol{G}$$ FI II 482, BI (253)(8)

2. $$\int_0^\infty \frac{\operatorname{arccot} x}{1 \pm x} \, dx = \pm \frac{\pi}{4} \ln 2 + \boldsymbol{G}$$ BI (248)(6, 7)

3. $$\int_0^1 \frac{\operatorname{arccot} x}{x(1+x)} \, dx = -\frac{\pi}{8} \ln 2 + \boldsymbol{G}$$ BI (235)(11)

4. $$\int_0^\infty \frac{\arctan x}{1-x^2} \, dx = -\boldsymbol{G}.$$ BI (248)(2)

5. $$\int_0^1 \arctan qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)\left(p^2+q^2\right)}\arctan q$$
$[p > -1]$ BI (243)(7)

6. $$\int_0^1 \operatorname{arccot} qx \frac{dx}{(1+px)^2} = \frac{1}{2}\frac{q}{p^2+q^2}\ln\frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2}\arctan q + \frac{1}{1+p}\operatorname{arccot} q$$
$[p > -1]$ BI (234)(10)

7. $$\int_0^1 \frac{\arctan x}{x\left(1+x^2\right)}\,dx = \frac{\pi}{8}\ln 2 + \frac{1}{2}\boldsymbol{G}$$ BI (235)(12)

8. $$\int_0^\infty \frac{x\arctan x}{1+x^4}\,dx = \frac{\pi^2}{16}$$ BI (248)(3)

9. $$\int_0^\infty \frac{x\arctan x}{1-x^4}\,dx = -\frac{\pi}{8}\ln 2$$ BI (248)(4)

10.[11] $$\int_0^\infty \frac{x\operatorname{arccot} x}{1-x^4}\,dx = \frac{\pi}{8}\ln 2$$ BI (248)(12)

11. $$\int_0^\infty \frac{\operatorname{arccot} x}{x\sqrt{1+x^2}}\,dx = \int_0^\infty \frac{\operatorname{arccot} x}{\sqrt{1+x^2}}\,dx = 2\boldsymbol{G}$$ BI (251)(3, 10)

12. $$\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$$ FI II 694

13. $$\int_0^1 \frac{x\arctan x\,dx}{\sqrt{\left(1+x^2\right)\left(1+k'^2x^2\right)}} = \frac{1}{k^2}\left[F\left(\frac{\pi}{4},k\right) - \frac{\pi}{2\sqrt{2\left(1+k'^2\right)}}\right]$$ BI (294)(14)

4.532

1. $$\int_0^1 x^p\arctan x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} - \beta\left(\frac{p}{2}+1\right)\right]$$ $[p > -2]$ BI (229)(7)

2. $$\int_0^\infty x^p\arctan x\,dx = \frac{\pi}{2(p+1)}\operatorname{cosec}\frac{p\pi}{2}$$ $[-1 > p > -2]$ BI (246)(1)

3. $$\int_0^1 x^p\operatorname{arccot} x\,dx = \frac{1}{2(p+1)}\left[\frac{\pi}{2} + \beta\left(\frac{p}{2}+1\right)\right]$$ $[p > -1]$ BI (229)(8)

4. $$\int_0^\infty x^p\operatorname{arccot} x\,dx = -\frac{\pi}{2(p+1)}\operatorname{cosec}\frac{p\pi}{2}$$ $[-1 < p < 0]$ BI (246)(2)

5. $$\int_0^\infty \left(\frac{x^p}{1+x^{2p}}\right)^{2q}\arctan x\frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2}p}\frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)}$$ $[q > 0]$ BI (250)(10)

4.533

1. $$\int_0^\infty \left(1 - x\operatorname{arccot} x\right)\,dx = \frac{\pi}{4}$$ BI (246)(3)

2. $$\int_0^1 \left(\frac{\pi}{4} - \arctan x\right)\frac{dx}{1-x} = -\frac{\pi}{8}\ln 2 + \boldsymbol{G}$$ BI (232)(2)

3. $$\int_0^1 \left(\frac{\pi}{4} - \arctan x\right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} \boldsymbol{G}$$ BI (235)(25)

4. $$\int_0^1 \left(x \operatorname{arccot} x - \frac{1}{x} \arctan x\right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2$$ BI (232)(1)

4.534 $$\int_0^\infty (\arctan x)^2 \frac{dx}{x^2\sqrt{1+x^2}} = \int_0^\infty (\operatorname{arccot} x)^2 \frac{x\,dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4\boldsymbol{G}$$ BI (251)(9, 17)

4.535

1. $$\int_0^1 \frac{\arctan px}{1+p^2x}\,dx = \frac{1}{2p^2} \arctan p \ln\left(1+p^2\right)$$ BI (231)(19)

2. $$\int_0^1 \frac{\operatorname{arccot} px}{1+p^2x}\,dx = \frac{1}{p^2}\left\{\frac{\pi}{4} + \frac{1}{2}\operatorname{arccot} p\right\} \ln\left(1+p^2\right) \qquad [p>0]$$ BI (231)(24)

3. $$\int_0^\infty \frac{\arctan qx}{(p+x)^2}\,dx = -\frac{q}{1+p^2q^2}\left(\ln pq - \frac{\pi}{2} pq\right) \qquad [p>0, \quad q>0]$$ BI (249)(1)

4. $$\int_0^\infty \frac{\operatorname{arccot} qx}{(p+x)^2}\,dx = \frac{q}{1+p^2q^2}\left(\ln pq + \frac{\pi}{2pq}\right) \qquad [p>0, \quad q>0]$$ BI (249)(8)

5. $$\int_0^\infty \frac{x \operatorname{arccot} px}{q^2+x^2}\,dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \qquad [p>0, \quad q>0]$$ BI (248)(9)

6. $$\int_0^\infty \frac{x \operatorname{arccot} px\,dx}{x^2-q^2} = \frac{\pi}{4} \ln \frac{1+p^2q^2}{p^2q^2} \qquad [p>0, \quad q>0]$$ BI (248)(10)

7. $$\int_0^\infty \frac{\arctan px}{x\left(1+x^2\right)}\,dx = \frac{\pi}{2} \ln(1+p) \qquad [p \ge 0]$$ FI II 745

8. $$\int_0^\infty \frac{\arctan px}{x\left(1-x^2\right)}\,dx = \frac{\pi}{4} \ln\left(1+p^2\right) \qquad [p \ge 0]$$ BI (250)(6)

9. $$\int_0^\infty \arctan qx \frac{dx}{x\left(p^2+x^2\right)} = \frac{\pi}{2p^2} \ln(1+pq) \qquad [p>0, \quad q \ge 0]$$ BI (250)(3)

10. $$\int_0^\infty \arctan qx \frac{dx}{x\left(1-p^2x^2\right)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \qquad [p \ge 0]$$ BI (250)(6)

11. $$\int_0^\infty \frac{x \arctan qx}{\left(p^2+x^2\right)^2}\,dx = \frac{\pi q}{4p(1+pq)} \qquad [p>0, \quad q \ge 0]$$ BI (252)(12)a

12. $$\int_0^\infty \frac{x \operatorname{arccot} qx}{\left(p^2+x^2\right)^2}\,dx = \frac{\pi}{4p^2(1+pq)} \qquad [p>0, \quad q \ge 0]$$ BI (252)(20)a

13. $$\int_0^1 \frac{\arctan qx}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2} \ln\left(q + \sqrt{1+q^2}\right)$$ BI (244)(11)

14.[9] $$\int_{-\infty}^\infty \frac{x \arctan(\alpha x)\,dx}{\left(x^2+\beta^2\right)\left(x^2+\gamma^2\right)} = \begin{cases} \dfrac{\pi}{\beta^2-\gamma^2} \ln\left(\dfrac{1+|\alpha\beta|}{1+|\alpha\gamma|}\right) \operatorname{sign}(\alpha) & \text{for } \beta \neq \gamma \\ \dfrac{\pi\alpha}{2|\beta|\left(1+|\alpha\beta|\right)} & \text{for } \beta = \gamma \end{cases}$$

for α, β, γ real

15.[9] $$\int_{-\infty}^{\infty} \frac{x \arctan(\alpha/x)\, dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \dfrac{\pi}{\beta^2-\gamma^2} \ln\left(\dfrac{1+|\alpha/\gamma|}{1+|\alpha/\beta|}\right) \operatorname{sign}(\alpha) & (\alpha,\beta,\gamma \text{ real}; \quad \beta \neq \gamma) \\ \dfrac{\pi\alpha}{2\beta^2(|\beta|+|\alpha|)} & (\beta = \gamma) \end{cases}$$

4.536

1. $$\int_0^\infty \arctan qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q\pi \ln \frac{1+\sqrt{1+q^2}}{\sqrt{1+q^2}} + \frac{\pi}{2} \ln\left(q+\sqrt{1+q^2}\right) - \frac{\pi}{2} - \arctan q$$ BI (230)(7)

2. $$\int_0^\infty \frac{\arctan px - \arctan qx}{x}\, dx = \frac{\pi}{2} \ln \frac{p}{q} \qquad [p>0, \quad q>0]$$ FI II 635

3. $$\int_0^\infty \frac{\arctan px \arctan qx}{x^2}\, dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p q^q} \qquad [p>0, \quad q>0]$$ FI II 745

4.537

1.[8] $$\int_0^1 \arctan\left(\sqrt{1-x^2}\right) \frac{dx}{1-x^2\cos^2\lambda} = \frac{\pi}{2\cos\lambda} \ln\left[\cos\left(\frac{\pi-4\lambda}{8}\right) \operatorname{cosec}\left(\frac{\pi+4\lambda}{8}\right)\right]$$ BI (245)(9)

2. $$\int_0^1 \arctan\left(p\sqrt{1-x^2}\right) \frac{dx}{1-x^2} = \frac{1}{2}\pi \ln\left(p+\sqrt{1+p^2}\right) \qquad [p>0]$$ BI (245)(10)

3. $$\int_0^1 \arctan\left(\tan\lambda\sqrt{1-k^2x^2}\right) \sqrt{\frac{1-x^2}{1-k^2x^2}}\, dx = \frac{\pi}{2k^2}\left[E(\lambda,k) - k'^2 F(\gamma,k)\right] - \frac{\pi}{2k^2} \cot\gamma \left(1-\sqrt{1-k^2\sin^2\gamma}\right)$$ BI (245)(12)

4. $$\int_0^1 \arctan\left(\tan\lambda\sqrt{1-k^2x^2}\right) \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx = \frac{\pi}{2} E(\lambda,k) - \frac{\pi}{2} \cot\lambda \left(1-\sqrt{1-k^2\sin^2\lambda}\right)$$ BI (245)(11)

5. $$\int_0^1 \frac{\arctan\left(\tan\lambda\sqrt{1-k^2x^2}\right)}{\sqrt{(1-x^2)(1-k^2x^2)}}\, dx = \frac{\pi}{2} F(\lambda,k)$$ BI (245)(13)

4.538

1. $$\int_0^\infty \arctan x^2 \frac{dx}{1+x^2} = \int_0^\infty \arctan x^3 \frac{dx}{1+x^2}$$ BI (252)(10, 11)
$$= \int_0^\infty \operatorname{arccot} x^2 \frac{dx}{1+x^2} = \int_0^\infty \operatorname{arccot} x^3 \frac{dx}{1+x^2} = \frac{\pi^2}{8}$$ BI (252)(18, 19)

2. $$\int_0^\infty \frac{1-x^2}{x^2} \arctan x^2\, dx = \frac{\pi}{2}\left(\sqrt{2}-1\right)$$ BI (244)(10)a

4.539 $$\int_0^\infty x^{s-1} \arctan\left(ae^{-x}\right)\, dx = 2^{-s-1}\,\Gamma(s) a\, \Phi\left(-a^2, s+1, \tfrac{1}{2}\right)$$ ET I 222(47)

4.541 $$\int_0^\infty \arctan\left(\frac{p\sin qx}{1+p\cos qx}\right) \frac{x\, dx}{1+x^2} = \frac{\pi}{2} \ln\left(1+pe^{-q}\right) \qquad [p>-e^q]$$ BI (341)(14)a

4.55 Combinations of inverse trigonometric functions and exponentials

4.551

1.[9] $$\int_0^1 (\arcsin x)\, e^{-bx}\, dx = \frac{\pi}{2b}\left[I_0(b) - \mathbf{L}_0(b)\right] - \frac{\pi e^{-b}}{2b}$$ ET I 160(1)

2. $$\int_0^1 x\,(\arcsin x)\, e^{-bx}\, dx = \frac{\pi}{2b^2}\left[\mathbf{L}_0(b) - I_0(b) + b\,\mathbf{L}_1(b) - b\, I_1(b)\right] + \frac{1}{b}$$ ET I 161(2)

3.[9] $$\int_0^\infty \left(\arctan\frac{x}{a}\right) e^{-bx}\, dx = \frac{1}{b}\left[\operatorname{ci}(ab)\sin(ab) - \operatorname{si}(ab)\cos(ab)\right]$$
$[\operatorname{Re} b > 0]$ ET I 161(3)

4.[9] $$\int_0^\infty \left(\operatorname{arccot}\frac{x}{a}\right) e^{-bx}\, dx = \frac{1}{b}\left[\frac{\pi}{2} - \operatorname{ci}(ab)\sin(ab) + \operatorname{si}(ab)\cos(ab)\right]$$
$[\operatorname{Re} b > 0]$ ET I 161(4)

4.552 $$\int_0^\infty \frac{\arctan\frac{x}{q}}{e^{2\pi x} - 1}\, dx = \frac{1}{2}\left[\ln\Gamma(q) - \left(q - \frac{1}{2}\right)\ln q + q - \frac{1}{2}\ln 2\pi\right]$$
$[q > 0]$ WH

4.553 $$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arccot} x - e^{-px}\right)\frac{dx}{x} = \boldsymbol{C} + \ln p$$ $[p > 0]$ NT 66(12)

4.56 A combination of the arctangent and a hyperbolic function

4.561 $$\int_{-\infty}^\infty \frac{\arctan e^{-x}}{\cosh^{2q} px}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\Pi(x)}{\cosh^{2q} px}\, dx = \frac{\sqrt{\pi^3}}{4p}\frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)}$$
$[q > 0]$ LI (282)(10)

4.57 Combinations of inverse and direct trigonometric functions

4.571 $$\int_0^{\pi/2} \arcsin(k\sin x)\,\frac{\sin x\, dx}{\sqrt{1 - k^2\sin^2 x}} = -\frac{\pi}{2k}\ln k'$$ BI (344)(2)

4.572 $$\int_0^\infty \left(\frac{2}{\pi}\operatorname{arccot} x - \cos px\right) dx = \boldsymbol{C} + \ln p$$ $[p > 0]$ NT 66(12)

4.573

1. $$\int_0^\infty \operatorname{arccot} qx \sin px\, dx = \frac{\pi}{2p}\left(1 - e^{-\frac{p}{q}}\right)$$ $[p > 0, \quad q > 0]$ BI (347)(1)a

2. $$\int_0^\infty \operatorname{arccot} qx \cos px\, dx = \frac{1}{2p}\left[e^{-\frac{p}{q}}\operatorname{Ei}\left(\frac{p}{q}\right) - e^{\frac{p}{q}}\operatorname{Ei}\left(-\frac{p}{q}\right)\right]$$
$[p > 0, \quad q > 0]$ BI (347)(2)a

3. $$\int_0^\infty \operatorname{arccot} rx \frac{\sin px\, dx}{1 \pm 2q\cos px + q^2} = \pm\frac{\pi}{2pq}\ln\frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}}$$ $[p^2 < 1, \quad r > 0, \quad p > 0]$
$$= \pm\frac{\pi}{2pq}\ln\frac{q \pm 1}{q \pm e^{-\frac{p}{r}}}$$ $[q^2 > 1, \quad r > 0, \quad p > 0]$
BI (347)(10)

4. $$\int_0^\infty \operatorname{arccot} px \frac{\tan x\, dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln\left(1 + \frac{r}{q} \tanh \frac{1}{p}\right)$$
$[p > 0, \quad q > 0, \quad r > 0]$ BI (347)(9)

4.574

1. $$\int_0^\infty \arctan\left(\frac{2a}{x}\right) \sin(bx)\, dx = \frac{\pi}{b} e^{-ab} \sinh(ab)$$ $[\operatorname{Re} a > 0, \quad b > 0]$ ET I 87(8)

2.[7] $$\int_0^\infty \arctan \frac{a}{x} \cos(bx)\, dx = \frac{1}{2b} \left[e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab)\right]$$
$[a > 0, \quad b > 0]$ ET I 29(7)

3. $$\int_0^\infty \arctan\left[\frac{2ax}{x^2 + c^2}\right] \sin(bx)\, dx = \frac{\pi}{b} e^{-b\sqrt{a^2+c^2}} \sinh(ab)$$
$[b > 0]$ ET I 87(9)

4. $$\int_0^\infty \arctan\left(\frac{2}{x^2}\right) \cos(bx)\, dx = \frac{\pi}{b} e^{-b} \sin b$$ $[b > 0]$ ET I 29(8)

4.575

1. $$\int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx\, dx = \frac{\pi}{2n} p^n$$ $\left[p^2 < 1\right]$ BI (345)(4)

2. $$\int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \sin nx \cos x\, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} + \frac{p^{n-1}}{n-1}\right)$$
$\left[p^2 < 1\right]$ BI (345)(5)

3. $$\int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \cos nx \sin x\, dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} - \frac{p^{n-1}}{n-1}\right)$$
$\left[p^2 < 1\right]$ BI (345)(6)

4.576

1. $$\int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p}$$ $\left[p^2 < 1\right]$ BI(346)(1)

2. $$\int_0^\pi \arctan \frac{p \sin x}{1 - p \cos x} \frac{dx}{\tan x} = -\frac{\pi}{2} \ln\left(1 - p^2\right)$$ $\left[p^2 < 1\right]$ BI(346)(3)

4.577

1. $$\int_0^{\pi/2} \arctan\left(\tan \lambda \sqrt{1 - k^2 \sin^2 x}\right) \frac{\sin^2 x\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2k^2} \left[F(\lambda, k) - E(\lambda, k) + \cot \lambda \left(1 - \sqrt{1 - k^2 \sin^2 \lambda}\right)\right]$$
BI (344)(4)

2. $$\int_0^{\pi/2} \arctan\left(\tan \lambda \sqrt{1 - k^2 \sin^2 x}\right) \frac{\cos^2 x\, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2k^2} \left[E(\lambda, k) - k'^2 F(\lambda, k) + \cot \lambda \left(\sqrt{1 - k^2 \sin^2 \lambda} - 1\right)\right]$$
BI (344)(5)

4.58 A combination involving an inverse and a direct trigonometric function and a power

4.581[10] $$\int_0^\infty \arctan x \cos px \frac{dx}{x} = \int_0^\infty \arctan \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2} \operatorname{Ei}(-p)$$

$[\operatorname{Re}(p) > 0]$ ET I 29(3), NT 25(13)

4.59 Combinations of inverse trigonometric functions and logarithms

4.591

1. $$\int_0^1 \arcsin x \ln x \, dx = 2 - \ln 2 - \frac{1}{2}\pi$$ BI (339)(1)

2. $$\int_0^1 \arccos x \ln x \, dx = \ln 2 - 2$$ BI (339)(2)

4.592 $$\int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{\ln(2k+2)}{2k+1}$$ BI (339)(8)

4.593

1. $$\int_0^1 \arctan x \ln x \, dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48}\pi^2$$ BI (339)(3)

2. $$\int_0^1 \operatorname{arccot} x \ln x \, dx = -\frac{1}{48}\pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2$$ BI (339)(4)

4.594 $$\int_0^1 \arctan x \, (\ln x)^{n-1} (\ln x + n) \, dx = \frac{n!}{(-2)^{n+1}} \left(2^{-n} - 1\right) \zeta(n+1)$$ BI (339)(7)

4.6 Multiple Integrals

4.60 Change of variables in multiple integrals

4.601

1. $$\iint\limits_{(\sigma)} f(x,y) \, dx \, dy = \iint\limits_{(\sigma')} f\left[\varphi(u,v), \psi(u,v)\right] |\Delta| \, du \, dv$$

where $x = \varphi(u,v), y = \psi(u,v)$, and $\Delta = \frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v} - \frac{\partial\psi}{\partial u}\frac{\partial\varphi}{\partial v} \equiv \frac{D(\varphi,\psi)}{D(u,v)}$ is the Jacobian determinant of the functions φ and ψ.

2. $$\iiint\limits_{(V)} f(x,y,z) \, dx \, dy \, dz = \iiint\limits_{(V')} f\left[\varphi(u,v,w), \psi(u,v,w), \chi(u,v,w) |\Delta| \, du \, dv \, dw\right]$$

where $x = \varphi(u,v,w)$, $y = \psi(u,v,w)$, and $z = \chi(u,v,w)$ and where

$$\Delta = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} & \frac{\partial\varphi}{\partial w} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\chi}{\partial u} & \frac{\partial\chi}{\partial v} & \frac{\partial\chi}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi,\psi,\chi)}{D(u,v,w)}$$

is the Jacobian determinant of the functions φ, ψ, and χ.

Here, we assume, both in (**4.601** 1) and in (**4.601** 2) that

(a) the functions φ, ψ, and χ and also their first partial derivatives are continuous in the region of integration;

(b) the Jacobian does not change sign in this region;

(c) there exists a one-to-one correspondence between the old variables x, y,z and the new ones u, v,w in the region of integration;

(d) when we change from the variables x, y,z to the variables u, v,w, the region V (resp. σ) is mapped into the region V' (resp. σ').

4.602 Transformation to polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi; \quad \frac{D(x,y)}{D(r,\varphi)} = r$$

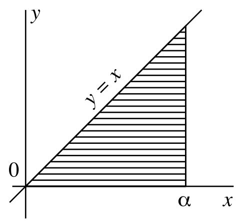
4.603 Transformation to spherical coordinates:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \quad \frac{D(x,y,z)}{D(r,\theta,\varphi)} = r^2\sin\theta$$

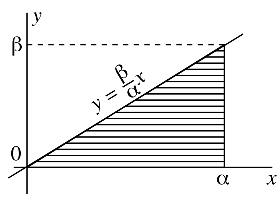
4.61 Change of the order of integration and change of variables

4.611

1. $$\int_0^\alpha dx \int_0^x f(x,y)\,dy = \int_0^\alpha dy \int_y^\alpha f(x,y)\,dx$$

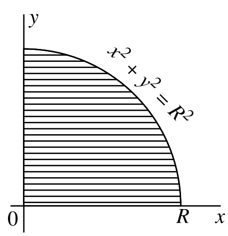


2. $$\int_0^\alpha dx \int_0^{\frac{\beta}{\alpha}x} f(x,y)\,dy = \int_0^\beta dy \int_{\frac{\alpha}{\beta}y}^\alpha f(x,y)\,dx$$

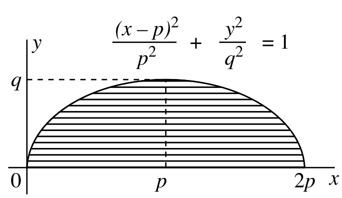


4.612

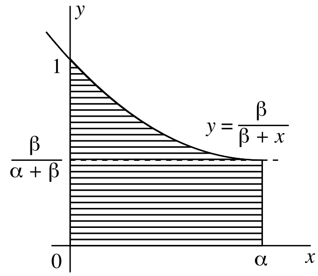
1. $$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f\,(x,y)\;dx$$



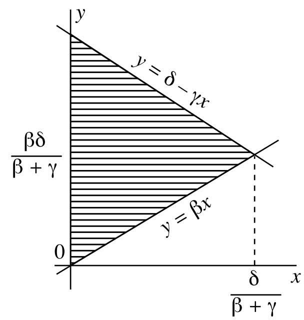
2. $$\int_0^{2p} dx \int_0^{q/p\sqrt{2px-x^2}} f(x,y)\,dy = \int_0^q dy \int_{p\left[1-\sqrt{1-(y/q)^2}\right]}^{p\left[1+\sqrt{1-(y/q)^2}\right]} f(x,y)\,dx$$



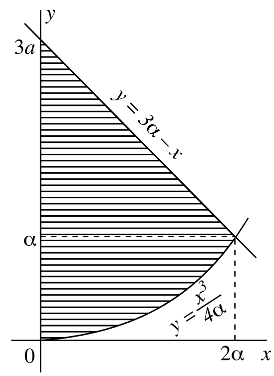
4.613



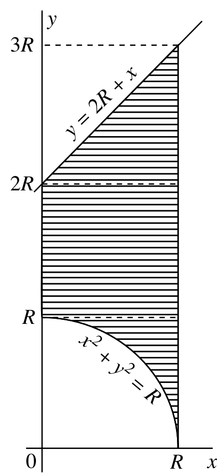
1. $$\int_0^\alpha dx \int_0^{\beta/(\beta+x)} f(x,y)\,dy = \int_0^{\beta/(\beta+\alpha)} dy \int_0^\alpha f(x,y)\,dx + \int_{\beta/(\beta+\alpha)}^1 dy \int_0^{\beta(1-y)/y} f(x,y)\,dx$$



2. $$\int_0^\alpha dx \int_{\beta x}^{\delta-\nu x} f(x,y)\,dy = \int_0^{\alpha\beta} dy \int_0^{y/\beta} f(x,y)\,dx + \int_{\alpha\beta}^{\delta} dy \int_0^{(\delta-y)/\gamma} f(x,y)\,dx$$
$$\left[\alpha = \frac{\delta}{\beta+\gamma}, \quad a > 0, \quad \beta > 0, \quad \gamma > 0\right]$$

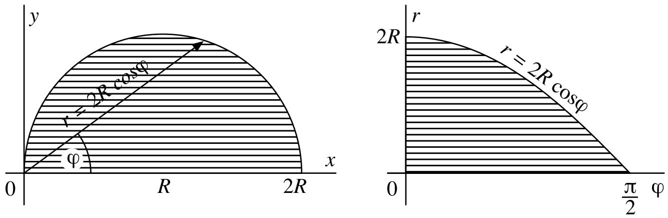


3. $$\int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha-x} f(x,y)\,dy = \int_0^\alpha dy \int_0^{2\sqrt{\alpha y}} f(x,y)\,dx + \int_\alpha^{3\alpha} dy \int_0^{3\alpha-y} f(x,y)\,dx$$

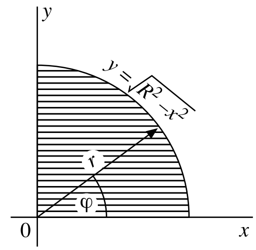


4. $$\int_0^R dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x,y)\,dy = \int_0^R dy \int_{\sqrt{R^2-y^2}}^R f(x,y)\,dx + \int_R^{2R} dy \int_0^R f(x,y)\,dx + \int_{2R}^{3R} dy \int_{y-2R}^R f(x,y)\,dx$$

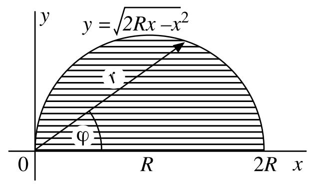
4.614 $$\int_0^{\pi/2} d\varphi \int_0^{2R\cos\varphi} f(r,\varphi)\,dr = \int_0^{2R} dr \int_0^{\arccos\frac{r}{2R}} f(r,\varphi)\,d\varphi$$



4.615 $$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^{\pi/2} d\varphi \int_0^R f\left(r\cos\varphi, r\sin\varphi\right) r\,dr$$



4.616 $$\int_0^{2R} dx \int_0^{\sqrt{2R-x^2}} f(x,y)\,dy = \int_0^{\pi/2} d\varphi \int_0^{2R\cos\varphi} f\left(r\cos\varphi, r\sin\varphi\right) r\,dr$$



4.617 $$\int_\alpha^\beta dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy = \int_0^\beta dx \int_0^{\varphi_2(x)} f(x,y)dy - \int_0^\beta dx \int_0^{\varphi_1(x)} f(x,y)dy - \int_0^\alpha dx \int_0^{\varphi_2(x)} f(x,y)dy + \int_0^\alpha dx \int_0^{\varphi_1(x)} f(x,y)\,dy$$

$[\varphi_1(x) \le \varphi_2(x) \text{ for } \alpha \le x \le \beta]$

4.618 $$\int_0^\gamma dx \int_0^{\varphi(x)} f(x,y)\,dy = \int_0^\gamma dx \int_0^1 f\left[x, z\varphi(x)\right] \varphi(x)dz \qquad [y = z\varphi(x)]$$

$$= \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y)\,dy \qquad [x = \gamma z]$$

4.619 $$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x,y)\,dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0)\, f\left[x, y_0 + (y_1 - y_0)\,t\right]\,dt$$

$[y = y_0 + (y_1 - y_0)\,t]$

4.62 Double and triple integrals with constant limits

4.620 General formulas

1. $$\int_0^\pi d\omega \int_0^\infty f'\left(p\cosh x + q\cos\omega\sinh x\right)\sinh x\,dx = -\frac{\pi \operatorname{sign} p}{\sqrt{p^2-q^2}} f\left(\operatorname{sign} p\sqrt{p^2-q^2}\right)$$

$\left[p^2 > q^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$ LO III 389

2. $$\int_0^{2\pi} d\omega \int_0^\infty f'\left[p\cosh x + (q\cos\omega + r\sin\omega)\sinh x\right]\sinh x\,dx$$
$$= -\frac{2\pi\operatorname{sign} p}{\sqrt{p^2-q^2-r^2}} f\left(\operatorname{sign} p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

3. $$\int_0^\pi \int_0^\pi \frac{dx\,dy}{\sin x \sin^2 y} f'\left[\frac{p - q\cos x}{\sin x \sin y} + r\cot y\right] = -\frac{2\pi\operatorname{sign} p}{\sqrt{p^2-q^2-r^2}} f\left(\operatorname{sign} p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$
LO III 280

4. $$\int_{-\infty}^\infty dx \int_{-\infty}^\infty f'\left(p\cosh x\cosh y + q\sinh x\cosh y + r\sinh y\right)\cosh y\,dy$$
$$= -\frac{2\pi\operatorname{sign} p}{\sqrt{p^2-q^2-r^2}} f\left(\operatorname{sign} p\sqrt{p^2-q^2-r^2}\right)$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

5. $$\int_0^\infty dx \int_0^\pi f\left(p\cosh x + q\cos\omega\sinh x\right)\sinh^2 x\sin\omega\,d\omega = 2\int_0^\infty f\left(\operatorname{sign} p\sqrt{p^2-q^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[\lim_{x\to+\infty} f(x) = 0\right]$$ LO III 391

6. $$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f\left[p\cosh x + (q\cos\omega + r\sin\omega)\sin\theta\sinh x\right]\sinh^2 x\sin\theta\,d\theta$$
$$= 4\int_0^\infty f\left(\operatorname{sign} p\sqrt{p^2-q^2-r^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[p^2 > q^2+r^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 390

7. $$\int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f\left\{p\cosh x + \left[(q\cos\omega + r\sin\omega)\sin\theta + s\cosh\theta\right]\sinh x\right\}\sinh^2 x\sin\theta\,d\theta$$
$$= 4\pi\int_0^\infty f\left(\operatorname{sign} p\sqrt{p^2-q^2-r^2-s^2}\cosh x\right)\sinh^2 x\,dx$$
$$\left[p^2 > q^2+r^2+s^2, \quad \lim_{x\to+\infty} f(x) = 0\right]$$ LO III 391

4.621

1. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\,dx\,dy = \frac{\pi}{2\sqrt{1-k^2}}$$ LO I 252(90)

2. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\cos y\sqrt{1-k^2\sin^2 x\sin^2 y}}{1-k^2\sin^2 y}\,dx\,dy = \boldsymbol{K}(k)$$ LO I 252(91)

3. $$\int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin\alpha\sin y\,dx\,dy}{\sqrt{1-\sin^2\alpha\sin^2 x\sin^2 y}} = \frac{\pi\alpha}{2}$$ LO I 253

4.622

1. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\,dy\,dz}{1-\cos x \cos y \cos z} = 4\pi\,\boldsymbol{K}^2\left(\frac{\sqrt{2}}{2}\right)$$ MO 137

2. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\,dy\,dz}{3-\cos y \cos z - \cos x \cos z - \cos x \cos y} = \sqrt{3}\pi\,\boldsymbol{K}^2\left(\sin\frac{\pi}{12}\right)$$ MO 137

3. $$\int_0^\pi \int_0^\pi \int_0^\pi \frac{dx\,dy\,dz}{3-\cos x - \cos y - \cos z} = 4\pi\left[18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right]\boldsymbol{K}^2\left[\left(2-\sqrt{3}\right)\left(\sqrt{3}-\sqrt{2}\right)\right]$$ MO 137

4.623[3] $$\int_0^\infty \int_0^\infty \varphi\left(a^2x^2+b^2y^2\right)\,dx\,dy = \frac{\pi}{2ab}\int_0^\infty \varphi\left(x^2\right)x\,dx$$

4.624 $$\int_0^\pi \int_0^{2\pi} f\left(\alpha\cos\theta+\beta\sin\theta\cos\psi+\gamma\sin\theta\sin\psi\right)\sin\theta\,d\theta\,d\psi$$
$$= 2\pi\int_0^\pi f\left(R\cos p\right)\sin p\,dp = 2\pi\int_{-1}^1 f(Rt)\,dt$$
$$\left[R=\sqrt{\alpha^2+\beta^2+\gamma^2}\right]$$

4.625[8] $$p_l(a,b) = \int_0^a dx\int_0^b dy\left(x^2+y^2+1\right)^{-3/2} P_l\left(1/\sqrt{x^2+y^2+1}\right)$$

Then, for even and odd subscripts:

- $$p_{2l}(a,b) = \frac{1}{l(2l+1)2^{2l}}\frac{ab}{\sqrt{a^2+b^2+1}}\sum_{k=0}^{l-1}\frac{(-1)^{l-k-1}2^{2k}\binom{2l+2k}{l+k}\binom{l+k}{l-k-1}}{\binom{2k}{k}(2k+1)}$$
$$\times(2l+2k+1)\sum_{j=0}^{k}\frac{\binom{2j}{j}}{2^{2j}}\frac{1}{\left(a^2+b^2+1\right)^j}\left(\frac{1}{\left(a^2+1\right)^{k-j+1}}+\frac{1}{\left(b^2+1\right)^{k-j+1}}\right)$$

- $$p_{2l+1}(a,b) = \frac{1}{2^{2l+1}(2l+1)}\sum_{k=0}^{l}\frac{(-1)^{l+k}}{2^{2k}}\binom{l}{k}\binom{l+k+1}{k}\binom{2l+2k+1}{l+k}$$
$$\times\left\{\frac{1}{\left(b^2+1\right)^k}\frac{b}{\sqrt{b^2+1}}\arctan^{-1}\frac{a}{\sqrt{b^2+1}}+\frac{1}{\left(a^2+1\right)^k}\frac{a}{\sqrt{a^2+1}}\arctan^{-1}\frac{b}{\sqrt{a^2+1}}\right.$$
$$\left.+ab\sum_{j=1}^{k}\frac{2^{2j-1}}{j\binom{2j}{j}}\cdot\frac{1}{\left(a^2+b^2+1\right)^j}\left(\frac{1}{\left(a^2+1\right)^{k-j+1}}+\frac{1}{\left(b^2+1\right)^{k-j+1}}\right)\right\}$$

4.63–4.64 Multiple integrals

4.631 $$\int_p^x dt_{n-1}\int_p^{t_{n-1}} dt_{n-2}\ldots\int_p^{t_1} f(t)\,dt = \frac{1}{(n-1)!}\int_p^x (x-t)^{n-1}f(t)\,dt,$$

where $f(t)$ is continuous on the interval $[p,q]$ and $p \le x \le q$. FI II 692

4.632

1. $$\int\int\cdots\int\limits_{\substack{x_1\ge 0, x_2\ge 0,\ldots,x_n\ge 0\\ x_1+x_2+\cdots+x_n\le h}} dx_1\,dx_2\cdots dx_n = \frac{h^n}{n!}$$

[the volume of an n-dimensional simplex] FI III 472

2. $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le R^2} dx_1\,dx_2\cdots dx_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}+1\right)}R^n$$ [the volume of an n-dimensional sphere]

FI III 473

4.633 $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1} \frac{dx_1\,dx_2\cdots dx_n}{\sqrt{1-x_1^2-x_2^2-\cdots-x_n^2}} = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \qquad [n>1]$$

[half-area of the surface of an $(n+1)$-dimensional sphere $x_1^2+x_2^2+\cdots+x_{n+1}^2=1$] FI III 474

4.634[8] $$\int\int\cdots\int\limits_{\substack{x_1\ge 0,x_2\ge 0,\ldots,x_n\ge 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\le 1}} x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\ldots dx_n$$

$$= \frac{q_1^{p_1}q_2^{p_2}\ldots q_n^{p_n}}{\alpha_1\alpha_2\ldots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}+1\right)}$$

$[\alpha_i>0, \quad p_i>0, \quad q_i>0, \quad i=1,2,\ldots,n]$ FI III 477

4.635

1.[8] $$\int\int\cdots\int\limits_{\substack{x_1\ge 0,x_2\ge 0,\ldots,x_n\ge 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\ge 1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$

$$\times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\cdots dx_n$$

$$= \frac{q_1^{p_1}q_2^{p_2}\ldots q_n^{p_n}}{\alpha_1\alpha_2\cdots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_1^\infty f(x)x^{\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}-1}\,dx$$

under the assumption that the integral on the right converges absolutely. FI III 487

2.[8]
$$\underset{\substack{x_1\geq 0, x_2\geq 0,\cdots,x_n\geq 0\\ \left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\leq 1}}{\iint\cdots\int} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1}+\left(\frac{x_2}{q_2}\right)^{\alpha_2}+\cdots+\left(\frac{x_n}{q_n}\right)^{\alpha_n}\right]$$
$$\times x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}\,dx_1\,dx_2\cdots dx_n$$
$$= \frac{q_1^{p_1}q_2^{p_2}\dots q_n^{p_n}}{\alpha_1\alpha_2\dots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\cdots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}\int_0^1 f(x)x^{\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}-1}\,dx$$

under the assumptions that the one-dimensional integral on the right converges absolutely and that the numbers q_i, α_i, and p_i are positive. FI III 479

In particular,

3.
$$\underset{\substack{x_1\geq 0, x_2\geq 0,\dots,x_n\geq 0\\ x_1+x_2+\cdots+x_n\leq 1}}{\iint\cdots\int} x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}e^{-q(x_1+x_2+\cdots+x_n)}\,dx_1\,dx_2\dots dx_n$$
$$= \frac{\Gamma(p_1)\,\Gamma(p_2)\dots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 x^{p_1+p_2+\cdots+p_n-1}e^{-qx}\,dx$$
$$[n>0, \quad p_1>0, \quad p_2>0,\dots,p_n>0]$$

4.[8]
$$\underset{\substack{x_1\geq 0, x_2\geq 0,\cdots,x_n\geq 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\leq 1}}{\iint\cdots\int} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(1-x_1^{\alpha_1}-x_2^{\alpha_2}-\cdots-x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n$$
$$= \frac{\Gamma(1-\mu)}{\alpha_1\alpha_2\dots\alpha_n}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1-\mu+\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$[p_1>0, \quad p_2>0,\dots,p_n>0, \quad \mu<1]$$ FI III 480

4.636

1.[8]
$$\underset{\substack{x_1\geq 0, x_2\geq 0,\dots,x_n\geq 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\geq 1}}{\iint\cdots\int} \frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{(x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\dots dx_n$$
$$= \frac{1}{\alpha_1\alpha_2\dots\alpha_n\left(\mu-\frac{p_1}{\alpha_1}-\frac{p_2}{\alpha_2}-\cdots-\frac{p_n}{\alpha_n}\right)}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\dots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$\left[p_1>0, \quad p_2>0,\dots,p_n>0; \quad \mu>\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right]$$ FI III 488

2.[8] $$\underset{\substack{x_1\ge 0, x_2\ge 0,\cdots,x_n\ge 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\le 1}}{\int\int\cdots\int} \frac{x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}}{(x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n})^{\mu}}\,dx_1\,dx_2\ldots\,dx_n$$
$$= \frac{1}{\alpha_1\alpha_2\ldots\alpha_n\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}-\mu\right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_2}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right)}$$
$$\left[\mu<\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}\right]$$ FI III 480

3.[8] $$\underset{\substack{x_1\ge 0, x_2\ge 0,\ldots,x_n\ge 0\\ x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}\le 1}}{\int\int\cdots\int} x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}\sqrt{\frac{1-x_1^{\alpha_1}-x_2^{\alpha_2}-\cdots-x_n^{\alpha_n}}{1+x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}}}\,dx_1\,dx_2\ldots\,dx_n$$
$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p_1}{\alpha_1}\right)\Gamma\left(\frac{p_1}{\alpha_2}\right)\ldots\Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1\alpha_2\ldots\alpha_n}\frac{1}{\Gamma(m)}\left\{\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)}-\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\right\},$$
where $m=\frac{p_1}{\alpha_1}+\frac{p_2}{\alpha_2}+\cdots+\frac{p_n}{\alpha_n}$. FI III 480

4.637[8] $$\underset{\substack{x_1\ge 0,\quad x_2\ge 0,\ldots,x_n\ge 0\\ x_1+x_2+\cdots+x_n\le 1}}{\int\int\cdots\int} f\left(x_1+x_2+\cdots+x_n\right)\frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}\,dx_1\,dx_2\ldots\,dx_n}{(q_1x_1+q_2x_2+\cdots+q_nx_n+r)^{p_1+p_2+\cdots+p_n}}$$
$$= \frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1p_2+\ldots p_n)}\int_0^1 f(x)\frac{x^{p_1p_2+\cdots p_n-1}}{(q_1x+r)^{p_1}(q_2x+r)^{p_2}\ldots(q_nx+r)^{p_n}}\,dx,$$
$$[q_1\ge 0,\quad q_2\ge 0,\ldots,q_n\ge 0;\quad r>0]$$
where $f(x)$ is continuous on the interval $(0,1)$.

4.638

1. $$\int_0^\infty\int_0^\infty\ldots\int_0^\infty \frac{x_1^{p_1-1}x_2^{p_2-1}\cdots x_n^{p_n-1}e^{-(q_1x_1+q_2x_2+\cdots+q_nx_n)}}{(r_0+r_1x_1+r_2x_2+\cdots+r_nx_n)^s}\,dx_1\,dx_2\ldots\,dx_n$$
$$= \frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(s)}\int_0^\infty\frac{e^{r_0x}x^{s-1}\,dx}{(q_1r_1x)^{p_1}(q_1r_2x)^{p_2}\ldots(q_nr_nx)^{p_n}}$$
where p_i, q_i, r_i, and s are positive. This result is also valid for $r_0=0$, provided $p_1+p_2+\cdots+p_n>s$.

2. $$\int_0^\infty\int_0^\infty\ldots\int_0^\infty \frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{(r_0+r_1x_1+r_2x_2+\cdots+r_nx_n)^s}\,dx_1\,dx_2\ldots\,dx_n$$
$$= \frac{\Gamma(p_1)\Gamma(p_2)\ldots\Gamma(p_n)\Gamma(sp_1p_2-\cdots-p_n)}{r_1^{p_1}r_2^{p_2}\cdots r_n^{p_n}r_0^{s-p_1-p_2-\cdots-p_n}\Gamma(s)}$$
$$[p_i>0,\quad r_i>0,\quad s>0]$$

3.[8] $$\int_0^\infty\int_0^\infty\ldots\int_0^\infty \frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{[1+(r_1x_1)^{q_1}+(r_2x_2)^{q_2}+\cdots+(r_nx_n)^{q_n}]^s}\,dx_1\,dx_2\ldots\,dx_n$$
$$= \frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\ldots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1q_2\ldots q_nr_1^{p_1q_1}r_2^{p_2q_2}\ldots r_n^{p_nq_n}}\frac{\Gamma\left(s-\frac{p_1}{q_1}-\frac{p_2}{q_2}-\cdots-\frac{p_n}{q_n}\right)}{\Gamma(s)}$$
$$[p_i>0,\quad q_i>0,\quad r_i>0,\quad s>0]$$

4.639

1. $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1} (p_1x_1+p_2x_2+\cdots+p_nx_n)^{2m}\,dx_1\,dx_2\ldots dx_n = \frac{(2m-1)!!}{2^m}\frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}+m+1\right)}\left(p_1^2+p_2^2+\cdots+p_n^2\right)^m$$

FI III 482

2. $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1} (p_1x_1+p_2x_2+\cdots+p_nx_n)^{2m+1}\,dx_1\,dx_2\ldots dx_n = 0$$ FI III 483

4.641

1.[11] $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le 1} e^{p_1x_1+p_2x_2+\cdots+p_nx_n}\,dx_1\,dx_2\ldots dx_n = \sqrt{\pi^n}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma\left(\frac{n}{2}+k+1\right)}\left(\frac{p_1^2+p_2^2+\cdots+p_n^2}{4}\right)^k$$

FI III 483

2. $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_{2n}^2\le 1} e^{p_1x_1p_2x_2+\cdots p_{2n}x_{2n}}\,dx_1\,dx_2\ldots dx_{2n} = \frac{(2\pi)^n\,I_n\left(\sqrt{p_1^2+p_2^2+\cdots+p_{2n}^2}\right)}{\left(p_1^2+p_2^2+\cdots+p_{2n}^2\right)^{n/2}}$$

FI III 483a

4.642 $$\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_n^2\le R^2} f\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right)dx_1\,dx_2\ldots dx_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)}\int_0^R x^{n-1}f(x)\,dx,$$

where $f(x)$ is a function that is continuous on the interval $(0,R)$. FI III 485

4.643 $$\int_0^1\int_0^1\ldots\int_0^1 f\left(x_1x_2\cdots x_n\right)(1-x_1)^{p_1-1}(1-x_2)^{p_2-1}\ldots(1-x_n)^{p_n-1}$$
$$\times x_2^{p_1}x_3^{p_1+p_2}\cdots x_n^{p_1+p_2+\cdots+p_{n-1}}\,dx_1\,dx_2\ldots dx_n$$
$$=\frac{\Gamma(p_1)\,\Gamma(p_2)\ldots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(x)(1-x)^{p_1+p_2+\cdots+p_n-1}\,dx$$

under the assumption that the integral on the right converges absolutely. FI III 488

4.644 $$\overbrace{\int\int\cdots\int}^{n-1}\limits_{x_1^2+x_2^2+\cdots+x_n^2=1} f\left(p_1x_1+p_2x_2+\cdots+p_nx_n\right)\frac{dx_1\,dx_2\ldots dx_{n-1}}{|x_n|}$$
$$=2\int\int\cdots\int\limits_{x_1^2+x_2^2+\cdots+x_{n-1}^2\le 1} f\left(p_1x_1+p_2x_2+\cdots+p_nx_n\right)\frac{dx_1\,dx_2\cdots dx_{n-1}}{\sqrt{1-x_1^2-x_2^2-\cdots-x_{n-1}^2}}$$
$$=\frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi f\left(\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\cos x\right)\sin^{n-2}x\,dx \qquad [n\ge 3]$$

where $f(x)$ is continuous on the interval $\left\{-\sqrt{p_1^2+p_2^2+\cdots+p_n^2},\sqrt{p_1^2+p_2^2+\cdots+p_n^2}\right\}$. FI III 489

4.645 Suppose that two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are continuous in a closed, bounded region D and that the smallest and greatest values of the function g in D are m and M, respectively. Let $\varphi(u)$ denote a function that is continuous for $m \le u \le$M. We denote by $\psi(u)$ the integral

1. $$\psi(u) = \iint\limits_{m \le g(x_1, x_2, \ldots, x_n) \le u} \cdots \int f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n,$$

over that portion of the region D on which the inequality $m \le g(x_1, x_2, \ldots, x_n) \le u$ is satisfied. Then

2. $$\iint\limits_{m \le g(x_1, x_2, \ldots, x_n) \le M} \cdots \int f(x_1, x_2, \ldots, x_n)\, \varphi\left[g(x_1, x_2, \ldots, x_n)\right] dx_1\, dx_2 \ldots dx_n$$
$$= (S)\int_m^M \varphi(u) d\, \psi(u) = (R)\int_m^M \varphi(u) \frac{d\, \psi(u)}{du}\, du$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative $\frac{d\psi}{du}$ exists and is continuous, the Riemann integral on the right exists.

M may be $+\infty$ in formulas **4.645** 2, in which case $\int_m^{+\infty}$ should be understood to mean $\lim\limits_{M \to +\infty} \int_m^M$.

4.646[8] $$\iint\limits_{\substack{x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \\ x_1 + x_2 + \cdots + x_n \le 1}} \cdots \int \frac{x_1^{p_1 - 1} x_2^{p_2 - 1} \ldots x_n^{p_n - 1}}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)^r}\, dx_1\, dx_2 \ldots dx_n$$
$$= \frac{\Gamma(p_1)\,\Gamma(p_2) \ldots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1)\,\Gamma(r)} \int_0^\infty \frac{x^{r-1}\, dx}{(1 + q_1 x)^{p_1}\,(1 + q_2 x)^{p_2} \cdots (1 + q_n x)^{p_n}}$$
$$= [p_1 > 0, \quad p_2 > 0, \ldots, p_n > 0, \quad q_1 > 0, \quad q_2 > 0, \ldots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0]$$

FI III 493

4.647 $$\iint\limits_{0 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le 1} \cdots \int \exp\left\{\frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}\right\} dx_1\, dx_2 \ldots dx_n$$
$$= \frac{2\sqrt{\pi^n}}{n\left(p_1^2 + p_2^2 + \cdots + p_n^2\right)^{\frac{n}{4} - \frac{1}{2}}}\, I_{\frac{n}{2} - 1}\left(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}\right)$$

FI III 495

4.648[8] $$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp\left[-\left(x_1 + x_2 + \cdots + x_n + \frac{\lambda^{n+1}}{x_1 x_2 \ldots x_n}\right)\right]$$
$$\times x c_1^{\frac{1}{n+1} - 1} x_2^{\frac{2}{n+1} - 1} \ldots x_n^{\frac{n}{n+1} - 1}\, dx_1\, dx_2 \cdots dx_n$$
$$= \frac{1}{\sqrt{n+1}} (2\pi)^{\frac{n}{2}} e^{-(n+1)\lambda}$$

FI III 496

5 Indefinite Integrals of Special Functions

5.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

5.11 Complete elliptic integrals

5.111

1. $$\int \boldsymbol{K}(k) k^{2p+3}\, dk = \frac{1}{(2p+3)^2}\left\{4(p+1)^2 \int \boldsymbol{K}(k) k^{2p+1}\, dk + k^{2p+2}\left[\boldsymbol{E}(k) - (2p+3)\,\boldsymbol{K}(k) {k'}^2\right]\right\}$$ BY (610.04)

2. $$\int \boldsymbol{E}(k) k^{2p+3}\, dk = \frac{1}{4p^2+16p+15}\left\{4(p+1)^2 \int \boldsymbol{E}(k) k^{2p+1}\, dk - \boldsymbol{E}(k) k^{2p+2}\left[(2p+3){k'}^2 - 2\right] - k^{2p+2}{k'}^2\,\boldsymbol{K}(k)\right\}$$ BY (611.04)

5.112

1. $$\int \boldsymbol{K}(k)\, dk = \frac{\pi k}{2}\left[1 + \sum_{j=1}^{\infty} \frac{\left[(2j)!\right]^2 k^{2j}}{(2j+1) 2^{4j} \left(j!\right)^4}\right]$$ BY (610.00)

2.[6] $$\int \boldsymbol{E}(k)\, dk = \frac{\pi k}{2}\left[1 - \sum_{j=1}^{\infty} \frac{\left[(2j)!\right]^2 k^{2j}}{(4j^2-1)\, 2^{4j} \left(j!\right)^4}\right]$$ BY (611.00)

3. $$\int \boldsymbol{K}(k) k\, dk = \boldsymbol{E}(k) - {k'}^2\,\boldsymbol{K}(k)$$ BY (610.01)

4. $$\int \boldsymbol{E}(k) k\, dk = \frac{1}{3}\left[\left(1+k^2\right)\boldsymbol{E}(k) - {k'}^2\,\boldsymbol{K}(k)\right]$$ BY (611.01)

5. $$\int \boldsymbol{K}(k) k^3\, dk = \frac{1}{9}\left[\left(4+k^2\right)\boldsymbol{E}(k) - {k'}^2\left(4+3k^2\right)\boldsymbol{K}(k)\right]$$ BY (610.02)

6. $$\int \boldsymbol{E}(k) k^3\, dk = \frac{1}{45}\left[\left(4 + k^2 + 9k^4\right)\boldsymbol{E}(k) - k'^2\left(4 + 3k^2\right)\boldsymbol{K}(k)\right]$$ BY 611.02)

7. $$\int \boldsymbol{K}(k) k^5\, dk = \frac{1}{225}\left[\left(64 + 16k^2 + 9k^4\right)\boldsymbol{E}(k) - k'^2\left(64 + 48k^2 + 45k^4\right)\boldsymbol{K}(k)\right]$$ BY (610.03)

8. $$\int \boldsymbol{E}(k) k^5\, dk = \frac{1}{1575}\left[\left(64 + 16k^2 + 9k^4 + 225k^6\right)\boldsymbol{E}(k) - k'^2\left(64 + 48k^2 + 45k^4\right)\boldsymbol{K}(k)\right]$$ BY (611.03)

9. $$\int \frac{\boldsymbol{K}(k)}{k^2}\, dk = -\frac{\boldsymbol{E}(k)}{k}$$ BY (612.05)

10. $$\int \frac{\boldsymbol{E}(k)}{k^2}\, dk = \frac{1}{k}\left[k'^2\,\boldsymbol{K}(k) - 2\,\boldsymbol{E}(k)\right]$$ BY (612.02)

11. $$\int \frac{\boldsymbol{E}(k)}{k'^2}\, dk = k\,\boldsymbol{K}(k)$$ BY (612.01)

12. $$\int \frac{\boldsymbol{E}(k)}{k^4}\, dk = \frac{1}{9k^3}\left[2\left(k^2 - 2\right)\boldsymbol{E}(k) + k'^2\,\boldsymbol{K}(k)\right]$$ BY (612.03)

13. $$\int \frac{k\,\boldsymbol{E}(k)}{k'^2}\, dk = \boldsymbol{K}(k) - \boldsymbol{E}(k)$$ BY (612.04)

5.113

1. $$\int [\boldsymbol{K}(k) - \boldsymbol{E}(k)]\frac{dk}{k} = -\boldsymbol{E}(k)$$ BY (612.06)

2. $$\int \left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{k} = 2\,\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)$$ BY (612.09)

3. $$\int \left[\left(1 + k^2\right)\boldsymbol{K}(k) - \boldsymbol{E}(k)\right]\frac{dk}{k} = -k'^2\,\boldsymbol{K}(k)$$ BY (612.12)

4. $$\int [\boldsymbol{K}(k) - \boldsymbol{E}(k)]\frac{dk}{k^2} = \frac{1}{k}\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]$$ BY (612.07)

5. $$\int \left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{k^2 k'^2} = \frac{1}{k}[\boldsymbol{K}(k) - \boldsymbol{E}(k)]$$

6. $$\int \left[\left(1 + k^2\right)\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]\frac{dk}{k k'^4} = \frac{\boldsymbol{E}(k)}{k'^2}$$ BY (612.13)

5.114 $$\int \frac{k\,\boldsymbol{K}(k)\, dk}{\left[\boldsymbol{E}(k) - k'^2\,\boldsymbol{K}(k)\right]^2} = \frac{1}{k'^2\,\boldsymbol{K}(k) - \boldsymbol{E}(k)}$$ BY (612.11)

5.115

1. $$\int \Pi\left(\frac{\pi}{2}, r^2, k\right) k\, dk = \left(k^2 - r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right) - \boldsymbol{K}(k) + \boldsymbol{E}(k)$$ BY (612.14)

2. $$\int \left[\boldsymbol{K}(k) - \Pi\left(\frac{\pi}{2}, r^2, k\right)\right] k\, dk = k^2\,\boldsymbol{K}(k) - \left(k^2 - r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right)$$ BY (612.15)

3. $$\int \left[\frac{\boldsymbol{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2}, r^2, k\right)\right] k\, dk = \left(k^2 - r^2\right)\Pi\left(\frac{\pi}{2}, r^2, k\right)$$ BY (612.16)

5.12 Elliptic integrals

5.121 $$\int_0^x \frac{F(x,k)\,dx}{\sqrt{1-k^2\sin^2 x}} = \frac{[F(x,k)]^2}{2} \qquad \left[0 < x \le \frac{\pi}{2}\right]$$ BY (630.01)

5.122[11] $$\int_0^x E(x,k)\sqrt{1-k^2\sin^2 x}\,dx = \frac{[E(x,k)]^2}{2}$$ BY (630.32)

5.123

1. $$\int_0^x F(x,k)\sin x\,dx = -\cos x\,F(x,k) + \frac{1}{k}\arcsin(k\sin x)$$ BY (630.11)

2. $$\int_0^x F(x,k)\cos x\,dx = \sin x\,F(x,k) + \frac{1}{k}\operatorname{arccosh}\sqrt{\frac{1-k^2\sin^2 x}{k'^2}} - \frac{1}{k}\operatorname{arccosh}\left(\frac{1}{k'}\right)$$ BY (630.21)

5.124

1. $$\int_0^x E(x,k)\sin x\,dx = -\cos x\,E(x,k) + \frac{1}{2k}\left[k\sin x\sqrt{1-k^2\sin^2 x} + \arcsin(k\sin x)\right]$$ BY (630.12)

2. $$\int_0^x E(x,k)\cos x\,dx = \sin x\,E(x,k) + \frac{1}{2k}\left[k\cos x\sqrt{1-k^2\sin^2 x} - k'^2\operatorname{arccosh}\sqrt{\frac{1-k^2\sin^2 x}{k'^2}} - k + k'^2\operatorname{arccosh}\left(\frac{1}{k'}\right)\right]$$ BY (630.22)

3.* $$\int_0^a \frac{x\,\boldsymbol{E}(x)\,dx}{\left(k'^2+k^2x^2\right)^2\sqrt{a^2-x^2}} = \frac{\pi}{4}\left(\frac{a\sqrt{1-a^2}}{\left(k'^2+k^2a^2\right)^2} + \frac{a^2\,E(\lambda,k)}{k'^2\left(k'^2+k^2a^2\right)^{3/2}} + \frac{\left(1-a^2\right)F(\lambda,k)}{\left(k'^2+k^2a^2\right)^{3/2}}\right)$$

$$\lambda = \arcsin\left(\frac{a}{\sqrt{k'^2+k^2a^2}}\right) \qquad k' = \sqrt{1-k^2} \qquad [0 < a < 1, \quad 0 < k < 1]$$

4.* $$\int_0^a \frac{x\,\boldsymbol{E}(x)\,dx}{\left(k^2-x^2\right)^2\sqrt{a^2-x^2}} = \frac{\pi}{4}\left(\frac{a\sqrt{1-a^2}}{k^2\left(k^2-a^2\right)} + \frac{F(\phi,k)}{k^2\sqrt{k^2-a^2}} + \frac{a^2\,E(\phi,k)}{k^2\left(k^2-a^2\right)^{3/2}}\right)$$

$$\phi = \arcsin\left(\frac{a}{k}\right) \qquad [0 < a < k < 1]$$

5.* $$\int_0^{\pi/2} \frac{E(x,k')\sin x\cos x\,dx}{\left(1-k'^2\cosh^2 v\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}} = \frac{1}{k'^2\sinh v\cosh v}\left\{\boldsymbol{E}(k')\operatorname{arctanh}\left(\frac{\tanh v}{k}\right) - \frac{\pi\tanh v}{2} - \frac{\pi}{2}\left[F(\phi,k) - E(\phi,k)\right]\right\}$$

$$\phi = \arcsin\left(\frac{\tanh v}{k}\right) \qquad k' = \sqrt{1-k^2} \qquad [0 < \tanh v < k < 1]$$

6.* $$\int_0^{\pi/2} \frac{E(x,k)\sin x\cos x\,dx}{\left(1-k^2\cos^2\psi\sin^2 x\right)\sqrt{1-k^2\sin^2 x}}$$
$$= \frac{1}{k^2\sin\psi\cos\psi}\left\{\boldsymbol{E}(k)\arctan\left(\frac{\tan\psi}{k'}\right) - \frac{\pi}{2}E(\beta,k) + \frac{\pi}{2}\frac{\tan\psi}{\sqrt{1-k^2\cos^2\psi}}\left(1-\sqrt{1-k^2\cos^2\psi}\right)\right\}$$
$$\beta = \arctan\left(\frac{\tan\psi}{k}\right) \quad k' = \sqrt{1-k^2} \qquad \left[0<k<1, \quad 0<\psi<\frac{\pi}{2}\right]$$

7.* $$\int_0^{\pi/2} \frac{E(x,k')\sin x\cos x\,dx}{\left(1+k'^2\sinh^2\mu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{-1}{k'^2\sinh\mu\cosh\mu}\left\{\boldsymbol{E}(k')\operatorname{arctanh}(k\tanh\mu) - \frac{\pi}{2}\left[F(\phi,k)-E(\phi,k)+\tanh\mu\sqrt{1+k'^2\sinh^2\mu}\right]\right.$$
$$\left. -\frac{\pi}{2}\coth\mu\left(1-\sqrt{1+k'^2\sinh^2\mu}\right)\right\}$$
$$\phi = \arcsin(\tanh\mu) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\mu<1]$$

8.* $$\int_0^{\pi/2} \frac{F(x,k')\sin x\cos x\,dx}{\left(1+k'^2\sinh^2\mu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{-1}{k'^2\sinh\mu\cosh\mu}\left[\boldsymbol{K}(k')\operatorname{arctanh}(k\tanh\mu) - \frac{\pi}{2}F(\phi,k)\right]$$
$$\phi = \arcsin(\tanh\mu) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\mu<1]$$

9.* $$\int_0^{\pi/2} \frac{F(x,k')\sin x\cos x\,dx}{\left(1-k'^2\cosh^2\nu\sin^2 x\right)\sqrt{1-k'^2\sin^2 x}}$$
$$= \frac{1}{k'^2\sinh\nu\cosh\nu}\left[\boldsymbol{K}(k')\operatorname{arctanh}\left(\frac{\tanh\nu}{k}\right) - \frac{\pi}{2}F(\phi,k)\right]$$
$$\phi = \arcsin\left(\frac{\tanh\nu}{k}\right) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\tanh\nu<1]$$

10.* $$\int_0^{\pi/2} \frac{F(x,k)\sin x\cos x\,dx}{\left(1-k^2\cos^2\psi\sin^2 x\right)\sqrt{1-k^2\sin^2 x}}$$
$$= \frac{1}{k^2\sin\psi\cos\psi}\left[\boldsymbol{K}(k')\operatorname{arctanh}\left(\frac{\tan\psi}{k'}\right) - \frac{\pi}{2}F(\beta,k)\right]$$
$$\beta = \arctan\left(\frac{\tan\psi}{k'}\right) \quad k' = \sqrt{1-k^2} \qquad [0<k<1, \quad 0<\psi<1]$$

11.* $$\int_a^b \ln\left(\frac{\epsilon+x}{\epsilon-x}\right)\frac{x^2\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \frac{\pi}{\epsilon}\left(\epsilon^2-\sqrt{(\epsilon^2-a^2)(\epsilon^2-b^2)}\right) + \pi\beta\left[F(\phi,k)-E(\phi,k)\right]$$
$$\phi = \arcsin\left(\frac{\beta}{\epsilon}\right) \quad k = \frac{a}{b} \qquad [0<a<b<\epsilon]$$

5.125

1. $$\int_0^x \Pi\left(x,\alpha^2,k\right)\sin x\,dx$$
$$= -\cos x\,\Pi\left(x,\alpha^2,k\right) + \frac{1}{\sqrt{k^2-\alpha^2}}\arctan\left[\sqrt{\frac{k^2-\alpha^2}{1-k^2\sin^2 x}}\sin x\right] \qquad \left[\alpha^2 < k^2\right]$$
$$= -\cos x\,\Pi\left(x,\alpha^2,k\right) + \frac{1}{\sqrt{\alpha^2-k^2}}\operatorname{arctanh}\left[\sqrt{\frac{\alpha^2-k^2}{1-k^2\sin^2 x}}\sin x\right] \qquad \left[\alpha^2 > k^2\right]$$
BY (630.13)

2. $$\int_0^x \Pi\left(x,\alpha^2,k\right)\cos x\,dx = \sin x\,\Pi\left(x,\alpha^2,k\right) - f - f_0$$

where
$$f = \frac{1}{2\sqrt{\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)}}\arctan\left[\frac{2\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)+\left(1-\alpha^2\sin^2 x\right)\left(2k^2-\alpha^2-\alpha^2k^2\right)}{2\alpha^2\sqrt{\left(1-\alpha^2\right)\left(\alpha^2-k^2\right)}\cos x\sqrt{1-k^2\sin^2 x}}\right]$$
for $\left(1-\alpha^2\right)\left(\alpha^2-k^2\right) > 0$;
$$= \frac{1}{2\sqrt{\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)}}\ln\left[\frac{2\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)+\left(1-\alpha^2\sin^2 x\right)\left(\alpha^2+\alpha^2k^2-2k^2\right)}{1-\alpha^2\sin^2 x} + \frac{2\alpha^2\sqrt{\left(\alpha^2-1\right)\left(\alpha^2-k^2\right)}\cos x\sqrt{1-k^2\sin^2 x}}{1-\alpha^2\sin^2 x}\right]$$
for $\left(1-\alpha^2\right)\left(\alpha^2-k^2\right) < 0$,
f_0 is the value of f at $x=0$ BY (630.23)

Integration with respect to the modulus

5.126 $$\int F(x,k)k\,dk = E(x,k) - k'^2\,F(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x$$ BY (613.01)

5.127 $$\int E(x,k)k\,dk = \frac{1}{3}\left[\left(1+k^2\right)E(x,k) - k'^2\,F(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x\right]$$ BY (613.02)

5.128 $$\int \Pi\left(x,r^2,k\right)k\,dk = \left(k^2-r^2\right)\Pi\left(x,r^2,k\right) - F(x,k) + E(x,k) + \left(\sqrt{1-k^2\sin^2 x}-1\right)\cot x$$
BY (613.03)

5.13 Jacobian elliptic functions

5.131

1. $$\int \operatorname{sn}^m u\,du = \frac{1}{m+1}\left[\operatorname{sn}^{m+1}u\operatorname{cn}u\operatorname{dn}u + (m+2)\left(1+k^2\right)\int \operatorname{sn}^{m+2}u\,du - (m+3)k^2\int \operatorname{sn}^{m+4}u\,du\right]$$
SI 259, PE(567)

2. $$\int \operatorname{cn}^m u\,du = \frac{1}{(m+1)k'^2}\left[-\operatorname{cn}^{m+1}u \operatorname{sn} u \operatorname{dn} u + (m+2)\left(1-2k^2\right)\int \operatorname{cn}^{m+2}u\,du + (m+3)k^2\int \operatorname{cn}^{m+4}u\,du\right]$$ PE (568)

3. $$\int \operatorname{dn}^m u\,du = \frac{1}{(m+1)k'^2}\left[k^2 \operatorname{dn}^{m+1}u \operatorname{sn} u \operatorname{cn} u + (m+2)\left(2-k^2\right)\int \operatorname{dn}^{m+2}u\,du - (m+3)\int \operatorname{dn}^{m+4}u\,du\right]$$ PE (569)

By using formulas **5.131**, we can reduce the integrals (for $m \neq 1$) $\int \operatorname{sn}^m u\,du$, $\int \operatorname{cn}^m u\,du$, and $\int \operatorname{dn}^m u\,du$ to the integrals **5.132**, **5.133** and **5.134**.

5.132

1. $$\int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u}$$ H 87(164)

$$= \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u}$$ SI 266(4)

2. $$\int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u}$$ SI 266(5)

3. $$\int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \arctan \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u}$$ H 88(166)

$$= \frac{1}{k'} \arccos \frac{\operatorname{cn} u}{\operatorname{dn} u}$$ JA

$$= \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u}$$ SI 266(6)

$$= \frac{1}{k'} \arcsin \frac{k' \operatorname{sn} u}{\operatorname{dn} u}$$ JA

5.133

1. $$\int \operatorname{sn} u\,du = \frac{1}{k} \ln\left(\operatorname{dn} u - k \operatorname{cn} u\right)$$ H 87(161)

$$= \frac{1}{k} \operatorname{arccosh} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1-k^2}$$ JA

$$= \frac{1}{k} \operatorname{arcsinh} \left(k \frac{\operatorname{dn} u - \operatorname{cn} u}{1-k^2}\right);$$ JA

$$= -\frac{1}{k} \ln\left(\operatorname{dn} u + k \operatorname{cn} u\right)$$ SI 365(1)

2. $$\int \operatorname{cn} u\,du = \frac{1}{k} \arccos\left(\operatorname{dn} u\right);$$ H 87(162)

$$= \frac{i}{k} \ln\left(\operatorname{dn} u - ik \operatorname{sn} u\right);$$ SI 265(2)a, ZH 87(162)

$$= \frac{1}{k} \arcsin\left(k \operatorname{sn} u\right)$$ JA

3. $$\int \operatorname{dn} u \, du = \arcsin(\operatorname{sn} u);$$ H 87(163)

$$= \operatorname{am} u = i \ln(\operatorname{cn} u - i \operatorname{sn} u)$$ SI 266(3), ZH 87(163)

5.134

1. $$\int \operatorname{sn}^2 u \, du = \frac{1}{k^2} [u - E(\operatorname{am} u, k)]$$ PE (564)

2. $$\int \operatorname{cn}^2 u \, du = \frac{1}{k^2} \left[E(\operatorname{am} u, k) - k'^2 u\right]$$ PE (565)

3. $$\int \operatorname{dn}^2 u \, du = E(\operatorname{am} u, k)$$ PE (566)

5.135

1. $$\int \frac{\operatorname{sn} u}{\operatorname{cn} u} \, du = \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u}$$ SI 266(7)

$$= \frac{1}{2k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}$$ H 88(167)

2. $$\int \frac{\operatorname{sn} u}{\operatorname{dn} u} \, du = \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u}$$ SI 266(8)

$$= \frac{1}{kk'} \operatorname{arccot} \frac{k \operatorname{cn} u}{k'}$$

3. $$\int \frac{\operatorname{cn} u}{\operatorname{sn} u} \, du = \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u}$$ SI 266(10)

$$= \frac{1}{2} \ln \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}$$ H 88(168)

4. $$\int \frac{\operatorname{cn} u}{\operatorname{dn} u} \, du = -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u}$$ SI 266(9)

$$= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u}$$

5. $$\int \frac{\operatorname{dn} u}{\operatorname{cn} u} \, du = \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u}$$ H 88(172)

$$= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}$$ JA

6. $$\int \frac{\operatorname{dn} u}{\operatorname{sn} u} \, du = \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}$$ H 87(170)

5.136

1. $$\int \operatorname{sn} u \operatorname{cn} u \, du = -\frac{1}{k^2} \operatorname{dn} u$$

2. $$\int \operatorname{sn} u \operatorname{dn} u \, du = -\operatorname{cn} u$$

3. $$\int \operatorname{cn} u \operatorname{dn} u \, du = \operatorname{sn} u$$

5.137

1. $$\int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u} \, du = \frac{1}{k'^2} \frac{\operatorname{dn} u}{\operatorname{cn} u}$$ H 88(173)

2. $$\int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u}\, du = -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u}$$ H 88(175)

3. $$\int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u}\, du = -\frac{\operatorname{dn} u}{\operatorname{sn} u}$$ H 88(174)

4. $$\int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u}\, du = \frac{\operatorname{sn} u}{\operatorname{dn} u}$$ H 88(177)

5. $$\int \frac{\operatorname{dn} u}{\operatorname{sn}^2 u}\, du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}$$ H 88(176)

6. $$\int \frac{\operatorname{dn} u}{\operatorname{cn}^2 u}\, du = \frac{\operatorname{sn} u}{\operatorname{cn} u}$$ H 88(178)

5.138

1. $$\int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u}\, du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u}$$ H 88(183)

2. $$\int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u}\, du = \frac{1}{k'^2} \ln \frac{\operatorname{dn} u}{\operatorname{cn} u}$$ H 88(182)

3. $$\int \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u}\, du = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u}$$ H 88(184)

5.139

1.[11] $$\int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}\, du = \ln \operatorname{sn} u$$ H 88(179)

2. $$\int \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\, du = \ln \frac{1}{\operatorname{cn} u}$$ H 88(180)

3. $$\int \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\, du = -\frac{1}{k^2} \ln \operatorname{dn} u$$ H 88(181)

5.14 Weierstrass elliptic functions

The invariants g_1 and g_2 used below are defined in 8.161.

5.141

1. $$\int \wp(u)\, du = -\zeta(u)$$

2. $$\int \wp^2(u)\, du = \frac{1}{6} \wp'(u) + \frac{1}{12} g_2 u$$ H 120(192)

3. $$\int \wp^3(u)\, du = \frac{1}{120} \wp'''(u) - \frac{3}{20} g_2 \zeta(u) + \frac{1}{10} g_3 u$$ H 120(193)

4.[8] $$\int \frac{du}{\wp(u) - \wp(v)} = \frac{1}{\wp'(v)} \left[2u\,\zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right] \qquad [\wp(v) \neq e_1, e_2, e_3]$$ (see **8.162**)

H 120(194)

5. $$\int \frac{\alpha\wp(u) + \beta}{\gamma\wp(u) + \delta}\, du = \frac{au}{\gamma} + \frac{\alpha\delta - \beta\gamma}{\gamma^2 \wp'(v)} \left[\ln \frac{\sigma(u+v)}{\sigma(u-v)} - 2u\,\zeta(v) \right]$$

where $v = \wp^{-1}\left(\frac{-\delta}{\gamma}\right)$ H 120(195)

5.2 The Exponential Integral Function

5.21 The exponential integral function

5.211 $$\int_x^\infty \operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x)\,dx = \left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\operatorname{Ei}[-(\beta+\gamma)x] - x\operatorname{Ei}(-\beta x)\operatorname{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta}\operatorname{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma}\operatorname{Ei}(-\beta x)$$
$[\operatorname{Re}(\beta+\gamma)>0]$ NT 53(2)

5.22 Combinations of the exponential integral function and powers

5.221

1. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^{n+1}}\,dx = \left[\frac{1}{x^n} - \frac{(-1)^n}{b^n}\right]\frac{\operatorname{Ei}[-a(x+b)]}{n} + \frac{e^{-ab}}{n}\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}}{b^{n-k}}\int_x^\infty \frac{e^{-ax}}{x^{k+1}}\,dx$$
$[a>0, \quad b>0]$ NT 52(3)

2. $$\int_x^\infty \frac{\operatorname{Ei}[-a(x+b)]}{x^2}\,dx = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{Ei}[-a(x+b)] - \frac{e^{-ab}\operatorname{Ei}(-ax)}{b}$$
$[a>0, b>0]$ NT 52(4)

3.* $$\int x\operatorname{Ei}(-ax)\,dx = \frac{x^2}{2}\operatorname{Ei}(-ax) + \frac{1}{2a^2}e^{-ax} + \frac{xe^{-ax}}{2a}$$ $[a>0]$

4.* $$\int x^n\operatorname{Ei}(-ax)\,dx = \frac{x^{n+1}}{n+1}\operatorname{Ei}(-ax) + \frac{n!e^{-ax}}{(n+1)a^{n+1}}\sum_{k=0}^{\infty}\frac{(ax)^k}{k!}$$
$[a>0]$

5.* $$\int x\operatorname{Ei}(-ax)e^{-bx}\,dx = \frac{1}{b^2}\operatorname{Ei}\left[-(a+b)x\right] - \frac{1}{b^2}\operatorname{Ei}(-ax)e^{-bx} - \frac{x}{b}\operatorname{Ei}(-ax)e^{-bx} - \frac{1}{b(a+b)}e^{-(a+b)x}$$
$[a>0, \quad b>0]$

6.* $$\int \operatorname{Ei}^2(-ax)\,dx = x\operatorname{Ei}^2(-ax) + \frac{2}{a}\left[\operatorname{Ei}(-ax)e^{-ax} - \operatorname{Ei}(-2ax)\right]$$
$[a>0]$

7.* $$\int x\operatorname{Ei}^2(-ax)\,dx = \frac{x^2}{2}\operatorname{Ei}^2(-ax) + \left(\frac{1}{a^2}+\frac{x}{a}\right)\operatorname{Ei}(-ax)e^{-ax} - \frac{1}{a^2}\operatorname{Ei}(-2ax) + \frac{1}{a^2}e^{-2ax}$$
$[a>0]$

8.* $$\int_0^u \operatorname{Ei}(-ax)\,dx = u\operatorname{Ei}(-au) + \frac{e^{-au}-1}{a}$$ $[a>0]$

9.* $$\int_0^\infty x\operatorname{Ei}\left(-\frac{x}{a}\right)\operatorname{Ei}\left(-\frac{x}{b}\right)\,dx = \left(\frac{a^2+b^2}{2}\right)\ln(a+b) - \frac{a^2}{2}\ln a - \frac{b^2}{2}\ln b - \frac{ab}{2}$$
$[a>0, \quad b>0]$

10.* $$\int_0^\infty x^2 \operatorname{Ei}\left(-\frac{x}{a}\right) \operatorname{Ei}\left(-\frac{x}{b}\right) dx = \frac{2}{3}\left[\left(a^3+b^3\right)\ln(a+b) - a^3\ln a - b^3\ln b - \frac{ab}{a+b}\left(a^2-ab+b^2\right)\right]$$

$$[a>0, \quad b>0]$$

5.23 Combinations of the exponential integral and the exponential

5.231

1. $$\int_0^x e^x \operatorname{Ei}(-x)\,dx = -\ln x - \boldsymbol{C} + e^x \operatorname{Ei}(-x)$$ ET II 308(11)

1. $$\int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x)\,dx = -\frac{1}{\beta}\left\{e^{-\beta x}\operatorname{Ei}(-\alpha x) + \ln\left(1+\frac{\beta}{\alpha}\right) - \operatorname{Ei}[-(\alpha+\beta)x]\right\}$$ ET II 308(12)

5.3 The Sine Integral and the Cosine Integral

5.31

1. $$\int \cos\alpha x \operatorname{ci}(\beta x)\,dx = \frac{\sin\alpha x \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)}{2\alpha}$$ NT 49(1)

2. $$\int \sin\alpha x \operatorname{ci}(\beta x)\,dx = -\frac{\cos\alpha x \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)+\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$$ NT 49(2)

5.32

1. $$\int \cos\alpha x \operatorname{si}(\beta x)\,dx = \frac{\sin\alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x+\beta x)-\operatorname{ci}(\alpha x-\beta x)}{2\alpha}$$ NT 49(3)

2. $$\int \sin\alpha x \operatorname{si}(\beta x)\,dx = -\frac{\cos\alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x+\beta x)-\operatorname{si}(\alpha x-\beta x)}{2\alpha}$$ NT 49(4)

5.33

1. $$\int \operatorname{ci}(\alpha x)\operatorname{ci}(\beta x)\,dx = x\operatorname{ci}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{2\alpha}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)\right) + \frac{1}{2\beta}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x-\alpha x)\right) - \frac{1}{\alpha}\sin\alpha x\operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x\operatorname{ci}(\alpha x)$$ NT 53(5)

2. $$\int \operatorname{si}(\alpha x)\operatorname{si}(\beta x)\,dx = x\operatorname{si}(\alpha x)\operatorname{si}(\beta x) - \frac{1}{2\beta}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\alpha x-\beta x)\right) - \frac{1}{2\alpha}\left(\operatorname{si}(\alpha x+\beta x)+\operatorname{si}(\beta x+\alpha x)\right) + \frac{1}{\alpha}\cos\alpha x\operatorname{si}(\beta x) + \frac{1}{\beta}\cos\beta x\operatorname{si}(\alpha x)$$ NT 54(6)

3. $$\int \operatorname{si}(\alpha x)\operatorname{ci}(\beta x)\,dx = x\operatorname{si}(\alpha x)\operatorname{ci}(\beta x) + \frac{1}{\alpha}\cos\alpha x\operatorname{ci}(\beta x) - \frac{1}{\beta}\sin\beta x\operatorname{si}(\alpha x) - \left(\frac{1}{2\alpha}+\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x+\beta x) - \left(\frac{1}{2\alpha}-\frac{1}{2\beta}\right)\operatorname{ci}(\alpha x-\beta x)$$ NT 54(10)

5.34

1. $$\int_x^\infty \operatorname{si}[a(x+b)]\frac{dx}{x^2} = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{si}[a(x+b)] - \frac{\cos ab\operatorname{si}(ax)+\sin ab\operatorname{ci}(ax)}{b}$$
$[a>0, \quad b>0]$ NT 52(6)

2. $$\int_x^\infty \operatorname{ci}[a(x+b)]\frac{dx}{x^2} = \left(\frac{1}{x}+\frac{1}{b}\right)\operatorname{ci}[a(x+b)] + \frac{\sin ab\operatorname{si}(ax)-\cos ab\operatorname{ci}(ax)}{b}$$
$[a>0, \quad b>0]$ NT 52(5)

5.4 The Probability Integral and Fresnel Integrals

5.41[11] $$\int \Phi(\alpha x)\,dx = x\,\Phi(\alpha x) + \frac{e^{-\alpha^2x^2}}{\alpha\sqrt{\pi}}$$ NT 12(20)a

5.42 $$\int S(\alpha x)\,dx = x\,S(\alpha x) + \frac{\cos^2\alpha x^2}{\alpha\sqrt{2\pi}}$$ NT 12(22)a

5.43 $$\int C(\alpha x)\,dx = x\,C(\alpha x) - \frac{\sin^2\alpha x^2}{\alpha\sqrt{2\pi}}$$ NT 12(21)a

5.5 Bessel Functions

Notation: Z and $\mathfrak{Z}$ denote any of J, N, $H^{(1)}$, $H^{(2)}$. In formulae 5.52–5.56, $Z_p(x)$ and $\mathfrak{Z}_p(x)$ are arbitrary Bessel functions of the first, second, or third kinds.

5.51 $$\int J_p(x)\,dx = 2\sum_{k=0}^{\infty} J_{p+2k+1}(x)$$ JA, MO 30

5.52

1. $$\int x^{p+1}\,Z_p(x)\,dx = x^{p+1}\,Z_{p+1}(x)$$ WA 132(1)

2.[11] $$\int x^{-p}\,Z_{p+1}(x)\,dx = -x^{-p}\,Z_p(x)$$ WA 132(2)

5.53[10] $$\int \left[\left(\alpha^2-\beta^2\right)x - \frac{p^2-q^2}{x}\right] Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)\,dx$$
$$= \alpha x\,Z_{p+1}(\alpha x)\,\mathfrak{Z}_q(\beta x) - \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{q+1}(\beta x) - (p-q)\,Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)$$
$$= \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{q-1}(\beta x) - \alpha x\,Z_{p-1}(\alpha x)\,\mathfrak{Z}_q(\beta x) + (p-q)\,Z_p(\alpha x)\,\mathfrak{Z}_q(\beta x)$$
JA, MO 30, WA 134(7)

5.54

1.[10] $$\int x\,Z_p(\alpha x)\,\mathfrak{Z}_p(\beta x)\,dx = \frac{\alpha x\,Z_{p+1}(\alpha x)\,\mathfrak{Z}_p(\beta x) - \beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p+1}(\beta x)}{\alpha^2-\beta^2}$$
$$= \frac{\beta x\,Z_p(\alpha x)\,\mathfrak{Z}_{p-1}(\beta x) - \alpha x\,Z_{p-1}(\alpha x)\,\mathfrak{Z}_p(\beta x)}{\alpha^2-\beta^2}$$
WA 134(8)

2. $$\int x\left[Z_p(\alpha x)\right]^2\,dx = \frac{x^2}{2}\left\{\left[Z_p(\alpha x)\right]^2 - Z_{p-1}(\alpha x)\,Z_{p+1}(\alpha x)\right\}$$ WA 135(11)

3.* $$\int x\, Z_p(ax)\, \mathfrak{Z}_p(ax)\, dx = \frac{x^4}{4}\left[2\, Z_p(ax)\, \mathfrak{Z}_p(ax) - Z_{p-1}(ax)\, \mathfrak{Z}_{p+1}(ax) - Z_{p+1}(ax)\, \mathfrak{Z}_{p-1}(ax)\right]$$

5.55[10] $$\int \frac{1}{x}\, Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)\, dx = \alpha x \frac{Z_p(\alpha x)\, \mathfrak{Z}_{q+1}(\alpha x) - Z_{p+1}(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p^2 - q^2} + \frac{Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p+q}$$
$$= \alpha x \frac{Z_{p-1}(\alpha x)\, \mathfrak{Z}_q(\alpha x) - Z_p(\alpha x)\, \mathfrak{Z}_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x)\, \mathfrak{Z}_q(\alpha x)}{p+q}$$

WA 135(13)

5.56

1. $$\int Z_1(x)\, dx = -Z_0(x)$$ JA

2. $$\int x\, Z_0(x)\, dx = x\, Z_1(x)$$ JA

6–7 Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1-k^2}$ (cf. 8.1).

6.11 Forms containing $F(x,k)$

6.111 $$\int_0^{\pi/2} F(x,k)\cot x\,dx = \frac{\pi}{4}\boldsymbol{K}(k') + \frac{1}{2}\ln k\,\boldsymbol{K}(k)$$ BI (350)(1)

6.112

1. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1+k\sin^2 x}\,dx = \frac{1}{4k}\boldsymbol{K}(k)\ln\frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k}\boldsymbol{K}(k')$$ BI (350)(6)

2. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k\sin^2 x}\,dx = \frac{1}{4k}\boldsymbol{K}(k)\ln\frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k}\boldsymbol{K}(k')$$ BI (350)(7)

3. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 x}\,dx = -\frac{1}{2k^2}\ln k'\,\boldsymbol{K}(k)$$ BI (350)(2)a, BY(802.12)a

6.113

1. $$\int_0^{\pi/2} F\left(x,k'\right)\frac{\sin x\cos x\,dx}{\cos^2 x + k\sin^2 x} = \frac{1}{4(1-k)}\ln\frac{2}{(1+k)\sqrt{k}}\boldsymbol{K}(k')$$ BI (350)(5)

2. $$\int_0^{\pi/2} F(x,k)\frac{\sin x\cos x}{1-k^2\sin^2 t\sin^2 x}\cdot\frac{dx}{\sqrt{1-k^2\sin^2 x}} = -\frac{1}{k^2\sin t\cos t}\left[\boldsymbol{K}(k)\arctan\left(k'\tan t\right) - \frac{\pi}{2}F(t,k)\right]$$ BI (350)(12)

6.114 $$\int_u^v F(x,k)\frac{dx}{\sqrt{\left(\sin^2 x - \sin^2 u\right)\left(\sin^2 v - \sin^2 x\right)}} = \frac{1}{2\cos u\sin v}\boldsymbol{K}(k)\,\boldsymbol{K}\left(\sqrt{1-\tan^2 u\cot^2 v}\right)$$
$\left[k^2 = 1-\cot^2 u\cdot\cot^2 v\right]$ BI (351)(9)

6.115 $$\int_0^1 F\left(\arcsin x, k\right)\frac{x\,dx}{1+kx^2} = \frac{1}{4k}\boldsymbol{K}(k)\ln\frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k}\boldsymbol{K}(k')$$
(cf. **6.112** 2) BI (466)(1)

This and similar formulas can be obtained from formulas **6.111–6.113** by means of the substitution $x = \arcsin t$.

6.12 Forms containing $\boldsymbol{E(x,k)}$

6.121 $$\int_0^{\pi/2} E(x,k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x}\, dx = \frac{1}{2k^2} \left\{ \left(1 + k'^2\right) \boldsymbol{K}(k) - (2 + \ln k')\, \boldsymbol{E}(k) \right\}$$ BI (350)(4)

6.122 $$\int_0^{\pi/2} E(x,k) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \left\{ \boldsymbol{E}(k)\, \boldsymbol{K}(k) - \ln k' \right\}$$ BI (350)(10), BY (630.02)

6.123 $$\int_0^{\pi/2} E(x,k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{k^2 \sin t \cos t} \left[\boldsymbol{E}(k) \arctan\left(k' \tan t\right) - \frac{\pi}{2}\, E(t,k) + \frac{\pi}{2} \cot t \left(1 - \sqrt{1 - k^2 \sin^2 t}\right) \right]$$ BI (350)(13)

6.124 $$\int_u^v E(x,k) \frac{dx}{\sqrt{\left(\sin^2 x - \sin^2 u\right)\left(\sin^2 v - \sin^2 x\right)}} = \frac{1}{2 \cos u \sin v}\, \boldsymbol{E}(k)\, \boldsymbol{K}\left(\sqrt{1 - \frac{tg^2 u}{tg^2 v}}\right) + \frac{k^2 \sin v}{2 \cos u}\, \boldsymbol{K}\left(\sqrt{1 - \frac{\sin^2 2u}{\sin^2 2v}}\right)$$ $\left[k^2 = 1 - \cot^2 u \cot^2 v\right]$ BI (351)(10)

6.13 Integration of elliptic integrals with respect to the modulus

6.131 $$\int_0^1 F(x,k) k\, dk = \frac{1 - \cos x}{\sin x} = \tan \frac{x}{2}$$ BY (616.03)

6.132 $$\int_0^1 E(x,k) k\, dk = \frac{\sin^2 x + 1 - \cos x}{3 \sin x}$$ BY (616.04)

6.133 $$\int_0^1 \Pi\left(x, r^2, k\right) k\, dk = \tan \frac{x}{2} - r \ln \sqrt{\frac{1 + r \sin x}{1 - r \sin x}} - r^2\, \Pi\left(x, r^2, 0\right)$$ BY (616.05)

6.14–6.15 Complete elliptic integrals

6.141

1. $$\int_0^1 \boldsymbol{K}(k)\, dk = 2\boldsymbol{G}$$ FI II 755

2. $$\int_0^1 \boldsymbol{K}(k')\, dk = \frac{\pi^2}{4}$$ BY (615.03)

6.142 $$\int_0^1 \left(\boldsymbol{K}(k) - \frac{\pi}{2}\right) \frac{dk}{k} = \pi \ln 2 - 2\boldsymbol{G}$$ BY (615.05)

6.143[7] $$\int_0^1 \boldsymbol{K}(k) \frac{dk}{k'} = \boldsymbol{K}^2 \left(\frac{\sqrt{2}}{2}\right) = \frac{1}{16\pi} \Gamma^4 \left(\frac{1}{4}\right)$$ BY (615.08)

6.144 $$\int_0^1 \boldsymbol{K}(k) \frac{dk}{1 + k} = \frac{\pi^2}{8}$$ BY (615.09)

6.145 $\int_0^1 \left(\boldsymbol{K}(k') - \ln \frac{4}{k} \right) \frac{dk}{k} = \frac{1}{12} \left[24 (\ln 2)^2 - \pi^2 \right]$ BY (615.13)

6.146 $n^2 \int_0^1 k^n \, \boldsymbol{K}(k) \, dk = (n-1)^2 \int_0^1 k^{n-2} \, \boldsymbol{K}(k) \, dk + 1$ BY (615.12)

6.147 $n \int_0^1 k^n \, \boldsymbol{K}(k') \, dk = (n-1) \int_0^1 k^{n-2} \, \boldsymbol{E}(k) \, dk$ $[n > 1]$ (see **6.152**) BY (615.11)

6.148

1. $\int_0^1 \boldsymbol{E}(k) \, dk = \frac{1}{2} + \boldsymbol{G}$ BY (615.02)

2. $\int_0^1 \boldsymbol{E}(k') \, dk = \frac{\pi^2}{8}$ BY (615.04)

3.* $\int_0^1 \frac{\boldsymbol{E}(k)}{1+k} \, dk = 1$

6.149

1. $\int_0^1 \left(\boldsymbol{E}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2\boldsymbol{G} + 1 - \frac{\pi}{2}$ BY (615.06)

2. $\int_0^1 \left(\boldsymbol{E}(k') - 1 \right) \frac{dk}{k} = 2 \ln 2 - 1$ BY (615.07)

3.* $\int_0^1 \frac{\boldsymbol{E}(k)}{1+k} \, dk = 1$

4.* $\int_0^1 \frac{dx}{x^3} \left(\sqrt{a - x^2} \, \boldsymbol{K}(x) - \frac{\boldsymbol{E}(x)}{\sqrt{1-x^2}} + \frac{\pi}{4} x^2 \right) = -\frac{\pi}{4} \ln \left(\frac{4}{\sqrt{e}} \right)$

6.151 $\int_0^1 \boldsymbol{E}(k) \frac{dk}{k'} = \frac{1}{8} \left[4 \, \boldsymbol{K}^2 \left(\frac{\sqrt{2}}{2} \right) + \frac{\pi^2}{\boldsymbol{K}^2 \left(\frac{\sqrt{2}}{2} \right)} \right]$ BY (615.10)

6.152 $(n+2) \int_0^1 k^n \, \boldsymbol{E}(k') \, dk = (n+1) \int_0^1 k^n \, \boldsymbol{K}(k') \, dk$ $[n > 1]$ (see **6.147**) BY (615.14)

6.153[6] $\int_0^a \frac{\boldsymbol{K}(k) k \, dk}{k'^2 \sqrt{a^2 - k^2}} = \frac{\pi}{4} \frac{1}{\sqrt{1-a^2}} \ln \left(\frac{1+a}{1-a} \right)$ $[0 < a < 1]$ LO I 252

6.154 $\int_0^{\pi/2} \frac{\boldsymbol{E}(p \sin x)}{1 - p^2 \sin^2 x} \sin x \, dx = \frac{\pi}{2\sqrt{1-p^2}}$ $\left[p^2 > 1 \right]$ FI II 489

6.16 The theta function

6.161

1. $\int_0^\infty x^{s-1} \, \vartheta_2 \left(0 \mid ix^2 \right) \, dx = 2^s \left(1 - 2^{-s} \right) \pi^{-\frac{s}{2}} \, \Gamma \left(\tfrac{1}{2} s \right) \zeta(s)$

   $[\operatorname{Re} s > 2]$ ET I 339(20)

2. $\int_0^\infty x^{s-1} \left[\vartheta_3 \left(0 \mid ix^2 \right) - 1 \right] \, dx = \pi^{-\frac{s}{2}} \, \Gamma \left(\tfrac{1}{2} s \right) \zeta(s)$ $[\operatorname{Re} s > 2]$ ET I 339(21)

3. $$\int_0^\infty x^{s-1}\left[1-\vartheta_4\left(0 \mid ix^2\right)\right]dx = \left(1-2^{1-s}\right)\pi^{-\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$$

$[\operatorname{Re} s > 2]$ ET I 339(22)

4. $$\int_0^\infty x^{s-1}\left[\vartheta_4\left(0 \mid ix^2\right)+\vartheta_2\left(0 \mid ix^2\right)-\vartheta_3\left(0 \mid ix^2\right)\right]dx = -\left(2^s-1\right)\left(2^{1-s}-1\right)\pi^{-\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{2}s\right)\zeta(s)$$

ET I 339(24)

6.162

1.[11] $$\int_0^\infty e^{-ax}\,\vartheta_4\left(\frac{b\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)dx = \frac{l}{\sqrt{a}}\cosh\left(b\sqrt{a}\right)\operatorname{cosech}\left(l\sqrt{a}\right)$$

$[\operatorname{Re} a > 0, \quad |b| \le l]$ ET I 224(1)a

2. $$\int_0^\infty e^{-ax}\,\vartheta_1\left(\frac{b\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)dx = -\frac{l}{\sqrt{a}}\sinh\left(b\sqrt{a}\right)\operatorname{sech}\left(l\sqrt{a}\right)$$

$[\operatorname{Re} a > 0, \quad |b| \le l]$ ET I 224(2)a

3.[11] $$\int_0^\infty e^{-ax}\,\vartheta_2\left(\frac{(l+b)\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)dx = -\frac{l}{\sqrt{a}}\sinh\left(b\sqrt{a}\right)\operatorname{sech}\left(l\sqrt{a}\right)$$

$[\operatorname{Re} a > 0, \quad |b| \le l]$ ET I 224(3)a

4.[11] $$\int_0^\infty e^{-ax}\,\vartheta_3\left(\frac{(l+b)\pi}{2l}\,\middle|\,\frac{i\pi x}{l^2}\right)dx = \frac{l}{\sqrt{a}}\cosh\left(b\sqrt{a}\right)\operatorname{cosech}\left(l\sqrt{a}\right)$$

$[\operatorname{Re} a > 0, \quad |b| \le l]$ ET I 224(4)a

6.163[10]

1. $$\int_0^\infty e^{-(a-\mu)x}\,\vartheta_3\left(\pi\sqrt{\mu}x\,|i\pi x\right)dx = \frac{1}{2\sqrt{a}}\left[\coth\left(\sqrt{a}+\sqrt{\mu}\right)+\coth\left(\sqrt{a}-\sqrt{\mu}\right)\right]$$

$[\operatorname{Re} a > 0]$ ET I 224(7)a

2.[10] $$\int_0^\infty \vartheta_3\left(i\pi kx \mid i\pi x\right)e^{-\left(k^2+l^2\right)x}\,dx = \frac{\sinh 2l}{l\left(\cosh 2l-\cos 2k\right)}$$

6.164[11] $$\int_0^\infty \left[\vartheta_4\left(0 \mid ie^{2x}\right)+\vartheta_2\left(0 \mid ie^{2x}\right)-\vartheta_3\left(0 \mid ie^{2x}\right)\right]e^{\frac{1}{2}x}\cos(ax)\,dx$$
$$= \frac{1}{2}\left(2^{\frac{1}{2}+ia}-1\right)\left(1-2^{\frac{1}{2}-ia}\right)\pi^{-\frac{1}{4}-\frac{1}{2}ia}\,\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}ia\right)\zeta\left(\tfrac{1}{2}+ia\right)$$

$[a > 0]$ ET I 61(11)

6.165 $$\int_0^\infty e^{\frac{1}{2}x}\left[\vartheta_3\left(0 \mid ie^{2x}\right)-1\right]\cos(ax)\,dx$$
$$= \frac{2}{1+4a^2}\left\{1+\left[\left(a^2+\tfrac{1}{4}\right)\pi^{-\frac{1}{2}ia-\frac{1}{4}}\,\Gamma\left(\tfrac{1}{2}ia+\tfrac{1}{4}\right)\zeta\left(ia+\tfrac{1}{2}\right)\right]\right\}$$

$[a > 0]$ ET I 61(12)

6.17[10] Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi \left[1 - k^2 \cos\phi\right]^{-\left(j+\frac{1}{2}\right)} \, d\phi,$$

$$\alpha_m(j) = \frac{\pi}{(64)^m} \frac{j!}{(2j)!} \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m!}\right)^2, \qquad \lambda = \frac{\pi}{2}\sqrt{\frac{(2j+1)k^2}{1-k^2}},$$

then

$$\Omega_j(k) = \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} = \sqrt{\frac{\pi}{(2j+1)k^2}} \left(1-k^2\right)^{-j} \left[\operatorname{erf}\lambda + \frac{1}{2}(2j+1)^{-1}\left(1+\frac{1}{2k^2}\right)\right.$$
$$\times \left\{\operatorname{erf}\lambda - \left(\frac{2}{\sqrt{\pi}}\right)\left(\lambda e^{-\lambda^2}\right)\left(1+\frac{2}{3}\lambda^2\right)\right\} - \frac{1}{12}(2j+1)^{-2}\left(16+\frac{13}{k^2}+\frac{1}{k^4}\right)$$
$$\left.\times \left\{\operatorname{erf}\lambda - \left(\frac{2}{\sqrt{\pi}}\right)\left(\lambda e^{-\lambda^2}\right)\left(1+\frac{2}{3}\lambda^2+\frac{4}{15}\lambda^4\right)\right\} + \dots\right]$$

while for large λ

$$\lim_{j\to\infty} \Omega_j(k) = \sqrt{\frac{\pi}{(2j+1)}k^2}\left(1-k^2\right)^{-j}$$
$$\times \left[1 + \frac{1}{2}(2j+1)^{-1}\left\{1+\frac{1}{2k^2}\right\} - \frac{4}{3}(2j+1)^{-2}\left\{1+\frac{13}{16k^2}+\frac{1}{16k^4}\right\} + \dots\right]$$

2. Set

$$R_\mu(k,\alpha,\delta) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\delta-2\alpha-1}(\theta/2)\,d\theta}{\left[1-k^2\cos\theta\right]^{\mu+\frac{1}{2}}},$$
$$0<k<1, \quad \operatorname{Re}\delta > \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -1/2,$$

$$M_\nu(\mu,\alpha,\delta) = \frac{(-1)^\nu 2^\nu \left(\mu+\frac{1}{2}\right)_\nu}{\nu!} \frac{\Gamma(\alpha)\,\Gamma(\delta-\alpha+\nu)}{\Gamma(\delta+\nu)},$$

with $(\lambda)_\nu = \Gamma(\lambda+\nu)/\Gamma(\lambda)$, and

$$W_\nu(\mu,\alpha,\delta) = \frac{2^\nu\left(\mu+\frac{1}{2}\right)_\nu}{\nu!}\frac{\Gamma(\alpha+\nu)\,\Gamma(\delta-\alpha)}{\Gamma(\delta+\nu)},$$

then:

- for small k:

$$R_\mu(k,\alpha,\delta) = \left(1-k^2\right)^{-\left(\mu+\frac{1}{2}\right)} \sum_{\nu=0}^{\infty}\left[k^2/\left(1-k^2\right)\right]^\nu M_\nu(\mu,\alpha,\delta)$$
$$= \left(1+k^2\right)^{-\left(\mu+\frac{1}{2}\right)} \sum_{\nu=0}^{\infty}\left[k^2/\left(1+k^2\right)\right]^\nu W_\nu(\mu,\alpha,\delta),$$

- for k^2 close to 1:

$$R_\mu(k,\alpha,\delta)$$

$$= \left[\Gamma(\delta-\alpha)\,\Gamma\left(\mu+\alpha-\delta+\tfrac{1}{2}\right)\Gamma\left(\mu+\tfrac{1}{2}\right)\right]\left(2k^2\right)^{\alpha-\delta}\left(1-k^2\right)^{\delta-\alpha-\mu-\frac{1}{2}}$$

$$\times\left\{\Gamma\left(\delta-\alpha-\mu-\tfrac{1}{2}\right)\Gamma(\alpha)\left[\Gamma\left(\delta-\mu-\tfrac{1}{2}\right)\left(2k^2\right)^{\mu+\frac{1}{2}}\right]\right\}$$

$$\left[\operatorname{Re}\left(\mu+\alpha-\delta+\tfrac{1}{2}\right)\ \text{not an integer}\right]$$

$$= \left[2^{\mu+\frac{1}{2}}k^{2\mu+1}\,\Gamma\left(\mu+\tfrac{1}{2}\right)\Gamma(1-\alpha)\right]$$

$$\times\sum_{n=0}^{\infty}\left[\Gamma\left(\delta-\alpha+n\right)\Gamma(1-\alpha+n)\,\Gamma\left(\alpha-\delta+\mu-n+\tfrac{1}{2}\right)n!\right]\left[2k^2/\left(1-k^2\right)\right]^{\alpha-\delta+\mu-n+\frac{1}{2}}$$

$$\left[\alpha-\delta+\mu+\tfrac{1}{2}=m,\ \text{with } m \text{ a non-negative integer}\right]$$

6.2–6.3 The Exponential Integral Function and Functions Generated by It

6.21 The logarithm integral

6.211 $\displaystyle\int_0^1 \operatorname{li}(x)\,dx = -\ln 2$ BI (79)(5)

6.212

1. $\displaystyle\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)x\,dx = 0$ BI (255)(1)

2. $\displaystyle\int_0^1 \operatorname{li}(x)x^{p-1}\,dx = -\frac{1}{p}\ln(p+1)$ $[p>-1]$ BI (255)(2)

3. $\displaystyle\int_0^1 \operatorname{li}(x)\frac{dx}{x^{q+1}} = \frac{1}{q}\ln(1-q)$ $[q<1]$ BI (255)(3)

4. $\displaystyle\int_1^\infty \operatorname{li}(x)\frac{dx}{x^{q+1}} = -\frac{1}{q}\ln(q-1)$ $[q>1]$ BI (255)(4)

6.213

1. $\displaystyle\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)\sin\left(a\ln x\right)\,dx = \frac{1}{1+a^2}\left(a\ln a-\frac{\pi}{2}\right)$ $[a>0]$ BI (475)(1)

2. $\displaystyle\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right)\sin\left(a\ln x\right)\,dx = -\frac{1}{1+a^2}\left(\frac{\pi}{2}+a\ln a\right)$ $[a>0]$ BI (475)(9)

3. $\displaystyle\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)\cos\left(a\ln x\right)\,dx = -\frac{1}{1+a^2}\left(\ln a+\frac{\pi}{2}a\right)$ $[a>0]$ BI (475)(2)

4. $\displaystyle\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right)\cos\left(a\ln x\right)\,dx = \frac{1}{1+a^2}\left(\ln a-\frac{\pi}{2}a\right)$ $[a>0]$ BI (475)(10)

5. $\displaystyle\int_0^1 \operatorname{li}(x)\sin\left(a\ln x\right)\frac{dx}{x} = \frac{\ln\left(1+a^2\right)}{2a}$ $[a>0]$ BI(479)(1), ET I 98(20)a

6. $$\int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x} = -\frac{\arctan a}{a}$$ BI (479)(2)

7. $$\int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2}\left(a \ln a + \frac{\pi}{2}\right) \qquad [a > 0]$$ BI (479)(3)

8. $$\int_1^\infty \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2}\left(\frac{\pi}{2} - a \ln a\right) \qquad [a > 0]$$ BI (479)(13)

9. $$\int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2}\left(\ln a - \frac{\pi}{2} a\right) \qquad [a > 0]$$ BI (479)(4)

10. $$\int_1^\infty \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = -\frac{1}{1+a^2}\left(\ln a + \frac{\pi}{2} a\right) \qquad [a > 0]$$ BI (479)(14)

11. $$\int_0^1 \operatorname{li}(x) \sin(a \ln x)\, x^{p-1}\, dx = \frac{1}{a^2+p^2}\left\{\frac{a}{2} \ln\left[(1+p)^2 + a^2\right] - p \arctan\frac{a}{1+p}\right\} \qquad [p > 0]$$ BI (477)(1)

12. $$\int_0^1 \operatorname{li}(x) \cos(a \ln x)\, x^{p-1}\, dx = -\frac{1}{a^2+p^2}\left\{a \arctan\frac{a}{1+p} + \frac{p}{2} \ln\left[(1+p)^2 + a^2\right]\right\} \qquad [p > 0]$$ BI (477)(2)

6.214

1. $$\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{p-1} dx = -\pi \cot p\pi \cdot \Gamma(p) \qquad [0 < p < 1]$$ BI (340)(1)

2. $$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right)(\ln x)^{p-1}\, dx = -\frac{\pi}{\sin p\pi}\,\Gamma(p) \qquad [0 < p < 1]$$ BI (340)(9)

6.215

1. $$\int_0^1 \operatorname{li}(x) \frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}}\, dx = -2\sqrt{\frac{\pi}{p}} \operatorname{arcsinh}\sqrt{p} = -2\sqrt{\frac{\pi}{p}} \ln\left(\sqrt{p} + \sqrt{p+1}\right) \qquad [p > 0]$$ BI (444)(3)

2. $$\int_0^1 \operatorname{li}(x) \frac{dx}{x^{p+1}\sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}} \arcsin\sqrt{p} \qquad [1 > p > 0]$$ BI (444)(4)

6.216

1. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1} \frac{ax}{x} = -\frac{1}{p}\,\Gamma(p) \qquad [0 < p \le 1]$$ BI (444)(1)

2. $$\int_0^1 \operatorname{li}(x)\left[\ln\left(\frac{1}{x}\right)\right]^{p-1} \frac{dx}{x^2} = -\frac{\pi\,\Gamma(p)}{\sin p\pi} \qquad [0 < p < 1]$$ BI (444)(2)

6.22–6.23 The exponential integral function

6.221 $$\int_0^p \operatorname{Ei}(\alpha x)\,dx = p\operatorname{Ei}(\alpha p) + \frac{1-e^{\alpha p}}{\alpha}$$ NT 11(7)

6.222 $$\int_0^\infty \operatorname{Ei}(-px)\operatorname{Ei}(-qx)\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q}$$ $[p > 0, \quad q > 0]$ FI II 653, NT 53(3)

6.223 $$\int_0^\infty \operatorname{Ei}(-\beta x)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu\beta^\mu}$$ $[\operatorname{Re}\beta \geq 0, \quad \operatorname{Re}\mu > 0]$ NT 55(7), ET I 325(10)

6.224

1. $$\int_0^\infty \operatorname{Ei}(-\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(1+\frac{\mu}{\beta}\right)$$ $[\operatorname{Re}(\beta+\mu) \geq 0, \quad \mu > 0]$
$$= -1/\beta$$ $[\mu = 0]$ FI II 652, NT 48(8)

2. $$\int_0^\infty \operatorname{Ei}(ax)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\left(\frac{\mu}{a}-1\right)$$ $[a > 0, \quad \operatorname{Re}\mu > 0, \quad \mu > a]$ ET I 178(23)a, BI (283)(3)

6.225

1. $$\int_0^\infty \operatorname{Ei}\left(-x^2\right)e^{-\mu x^2}\,dx = -\sqrt{\frac{\pi}{\mu}}\operatorname{arcsinh}\sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}}\ln\left(\sqrt{\mu}+\sqrt{1+\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ BI (283)(5), ET I 178(25)a

2. $$\int_0^\infty \operatorname{Ei}\left(-x^2\right)e^{px^2}\,dx = -\sqrt{\frac{\pi}{p}}\arcsin\sqrt{p}$$ $[1 > p > 0]$ NT 59(9)a

6.226

1. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x}\right)e^{-\mu x}\,dx = -\frac{2}{\mu}K_0\left(\sqrt{\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 34

2. $$\int_0^\infty \operatorname{Ei}\left(\frac{a^2}{4x}\right)e^{-\mu x}\,dx = -\frac{2}{\mu}K_0\left(a\sqrt{\mu}\right)$$ $[a > 0, \quad \operatorname{Re}\mu > 0]$ MI 34

3. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right)e^{-\mu x^2}\,dx = \sqrt{\frac{\pi}{\mu}}\operatorname{Ei}\left(-\sqrt{\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 34

4. $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right)e^{-\mu x^2+\frac{1}{4x^2}}\,dx = \sqrt{\frac{\pi}{\mu}}\left[\cos\sqrt{\mu}\operatorname{ci}\sqrt{\mu} - \sin\sqrt{\mu}\operatorname{si}\sqrt{\mu}\right]$$ $[\operatorname{Re}\mu > 0]$ MI 34

6.227

1. $$\int_0^\infty \operatorname{Ei}(-x)e^{-\mu x}x\,dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2}\ln(1+\mu)$$ $[\operatorname{Re}\mu > 0]$ MI 34

2. $$\int_0^\infty \left[\frac{e^{-ax}\operatorname{Ei}(ax)}{x-b} - \frac{e^{ax}\operatorname{Ei}(-ax)}{x+b}\right] dx = 0 \qquad [a>0, \quad b<0]$$
$$= \pi^2 e^{-ab} \qquad [a>0, \quad b>0]$$
ET II 253(1)a

6.228

1. $$\int_0^\infty \operatorname{Ei}(-x) e^x x^{\nu-1}\, dx = -\frac{\pi\,\Gamma(\nu)}{\sin \nu\pi} \qquad [0 < \operatorname{Re}\nu < 1]$$ ET II 308(13)

2. $$\int_0^\infty \operatorname{Ei}(-\beta x) e^{-\mu x} x^{\nu-1}\, dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu}\, {}_2F_1\left(1, \nu; \nu+1; \frac{\mu}{\beta+\mu}\right)$$
$$[|\arg\beta| < \pi, \quad \operatorname{Re}(\beta+\mu) > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 308(14)

6.229 $$\int_0^\infty \operatorname{Ei}\left(-\frac{1}{4x^2}\right) \exp\left(-\mu x^2 + \frac{1}{4x^2}\right) \frac{dx}{x^2} = 2\sqrt{\pi}\,(\cos\sqrt{\mu}\operatorname{si}\sqrt{\mu} - \sin\sqrt{\mu}\operatorname{ci}\sqrt{\mu})$$
$$[\operatorname{Re}\mu > 0]$$ MI 34

6.231 $$\int_{-\ln a}^\infty \left[\operatorname{Ei}(-a) - \operatorname{Ei}\left(-e^{-x}\right)\right] e^{-\mu x}\, dx = \frac{1}{\mu}\gamma(\mu, a) \qquad [a<1, \quad \operatorname{Re}\mu > 0]$$ MI 34

6.232

1. $$\int_0^\infty \operatorname{Ei}(-ax)\sin bx\, dx = -\frac{\ln\left(1 + \frac{b^2}{a^2}\right)}{2b} \qquad [a>0, \quad b>0]$$ BI (473)(1)a

2. $$\int_0^\infty \operatorname{Ei}(-ax)\cos bx\, dx = -\frac{1}{b}\arctan\frac{b}{a} \qquad [a>0, \quad b>0]$$ BI (473)(2)a

6.233

1. $$\int_0^\infty \operatorname{Ei}(-x) e^{-\mu x}\sin\beta x\, dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\beta}{2}\ln\left[(1+\mu)^2+\beta^2\right] - \mu\arctan\frac{\beta}{1+\mu}\right\}$$
$$[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$$ BI (473)(7)a

2. $$\int_0^\infty \operatorname{Ei}(-x) e^{-\mu x}\cos\beta x\, dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\mu}{2}\ln\left[(1+\mu)^2+\beta^2\right] + \beta\arctan\frac{\beta}{1+\mu}\right\}$$
$$[\operatorname{Re}\mu > |\operatorname{Im}\beta|]$$ BI (473)(8)a

6.234 $$\int_0^\infty \operatorname{Ei}(-x)\ln x\, dx = \boldsymbol{C} + 1$$ NT 56(10)

6.24–6.26 The sine integral and cosine integral functions

6.241

1. $$\int_0^\infty \operatorname{si}(px)\operatorname{si}(qx)\, dx = \frac{\pi}{2p} \qquad [p \ge q]$$ BI II 653, NT 54(8)

2. $$\int_0^\infty \operatorname{ci}(px)\operatorname{ci}(qx)\, dx = \frac{\pi}{2p} \qquad [p \ge q]$$ FI II 653, NT 54(7)

3. $$\int_0^\infty \operatorname{si}(px)\operatorname{ci}(qx)\,dx = \frac{1}{4q}\ln\left(\frac{p+q}{p-q}\right)^2 + \frac{1}{4p}\ln\frac{\left(p^2-q^2\right)^2}{q^4} \qquad [p \neq q]$$
$$= \frac{1}{q}\ln 2 \qquad [p = q]$$

FI II 653, NT 54(10, 12)

6.242 $$\int_0^\infty \frac{\operatorname{ci}(ax)}{\beta+x}\,dx = -\frac{1}{2}\left\{[\operatorname{si}(a\beta)]^2 + [\operatorname{ci}(a\beta)]^2\right\} \qquad [a>0, \quad |\arg\beta| < \pi]$$ ET II 224(1)

6.243

1. $$\int_{-\infty}^\infty \frac{\operatorname{si}(a|x|)}{x-b}\operatorname{sign} x\,dx = \pi\operatorname{ci}(a|b|) \qquad [a>0, \quad b>0]$$ ET II 253(3)

2. $$\int_{-\infty}^\infty \frac{\operatorname{ci}(a|x|)}{x-b}\,dx = -\pi\operatorname{sign} b\cdot\operatorname{si}(a|b|) \qquad [a>0]$$ ET II 253(2)

6.244

1.[8] $$\int_0^\infty \operatorname{si}(px)\frac{x\,dx}{q^2+x^2} = \frac{\pi}{2}\operatorname{Ei}(-pq) \qquad [p>0, \quad q>0]$$ BI (255)(6)

2.[8] $$\int_0^\infty \operatorname{si}(px)\frac{x\,dx}{q^2-x^2} = -\frac{\pi}{2}\operatorname{ci}(pq) \qquad [p>0, \quad q>0]$$ BI (255)(6)

6.245

1. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2+x^2} = \frac{\pi}{2q}\operatorname{Ei}(-pq) \qquad [p>0, \quad q>0]$$ BI (255)(7)

2. $$\int_0^\infty \operatorname{ci}(px)\frac{dx}{q^2-x^2} = \frac{\pi}{2q}\operatorname{si}(pq) \qquad [p>0, \quad q>0]$$ BI (255)(8)

6.246

1. $$\int_0^\infty \operatorname{si}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu}\sin\frac{\mu\pi}{2} \qquad [a>0, \quad 0<\operatorname{Re}\mu<1]$$

NT 56(9), ET I 325(12)a

2. $$\int_0^\infty \operatorname{ci}(ax)x^{\mu-1}\,dx = -\frac{\Gamma(\mu)}{\mu a^\mu}\cos\frac{\mu\pi}{2} \qquad [a>0, \quad 0<\operatorname{Re}\mu<1]$$

NT 56(8), ET I 325(13)a

6.247

1. $$\int_0^\infty \operatorname{si}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\arctan\frac{\mu}{\beta} \qquad [\operatorname{Re}\mu>0]$$ NT 49(12), ET I 177(18)

2. $$\int_0^\infty \operatorname{ci}(\beta x)e^{-\mu x}\,dx = -\frac{1}{\mu}\ln\sqrt{1+\frac{\mu^2}{\beta^2}} \qquad [\operatorname{Re}\mu>0]$$ NT 49(11), ET I 178(19)a

6.248

1.[8] $$\int_0^\infty \operatorname{si}(x)e^{-\mu x^2}x\,dx = \frac{\pi}{4\mu}\left[\Phi\left(\frac{1}{2\sqrt{\mu}}\right)-1\right] \qquad [\operatorname{Re}\mu>0]$$ MI 34

2. $$\int_0^\infty \operatorname{ci}(x) e^{-\mu x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}}\, \operatorname{Ei}\left(-\frac{1}{4\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

6.249 $$\int_0^\infty \left[\operatorname{si}\left(x^2\right) + \frac{\pi}{2}\right] e^{-\mu x}\, dx = \frac{\pi}{\mu}\left\{\left[S\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2 + \left[C\left(\frac{\mu^2}{4}\right) - \frac{1}{2}\right]^2\right\} \qquad [\operatorname{Re}\mu > 0]$$ ME 26

6.251

1. $$\int_0^\infty \operatorname{si}\left(\frac{1}{x}\right) e^{-\mu x}\, dx = \frac{2}{\mu}\operatorname{kei}\left(2\sqrt{\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

2. $$\int_0^\infty \operatorname{ci}\left(\frac{1}{x}\right) e^{-\mu x}\, dx = -\frac{2}{\mu}\ker\left(2\sqrt{\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ MI 34

6.252

1. $$\int_0^\infty \sin px \operatorname{si}(qx)\, dx = -\frac{\pi}{2p} \qquad \left[p^2 > q^2\right]$$
$$= -\frac{\pi}{4p} \qquad \left[p^2 = q^2\right]$$
$$= 0 \qquad \left[p^2 < q^2\right]$$
FI II 652, NT 50(8)

2.[6] $$\int_0^\infty \cos px \operatorname{si}(qx)\, dx = -\frac{1}{4p}\ln\left(\frac{p+q}{p-q}\right)^2 \qquad \left[p \neq 0, \quad p^2 \neq q^2\right]$$
$$= \frac{1}{q} \qquad [p = 0]$$
FI II 652, NT 50(10)

3. $$\int_0^\infty \sin px \operatorname{ci}(qx)\, dx = -\frac{1}{4p}\ln\left(\frac{p^2}{q^2} - 1\right)^2 \qquad \left[p \neq 0, \quad p^2 \neq q^2\right]$$
$$= 0 \qquad [p = 0]$$
FI II 652, NT 50(9)

4. $$\int_0^\infty \cos px \operatorname{ci}(qx)\, dx = -\frac{\pi}{2p} \qquad \left[p^2 > q^2\right]$$
$$= -\frac{\pi}{4p} \qquad \left[p^2 = q^2\right]$$
$$= 0 \qquad \left[p^2 < q^2\right]$$
FI II 654, NT 50(7)

6.253 $$\int_0^\infty \frac{\operatorname{si}(ax)\sin bx}{1 - 2r\cos x + r^2}\, dx = -\frac{\pi\left(r^m + r^{m+1}\right)}{4b(1-r)\left(1-r^2\right)} \qquad [b = a - m]$$
$$= -\frac{\pi\left(2 + 2r - r^m - r^{m+1}\right)}{4b(1-r)\left(1-r^2\right)} \qquad [b = a + m]$$
$$= -\frac{\pi r^{m+1}}{2b(1-r)\left(1-r^2\right)} \qquad [a - m - 1 < b < a - m]$$
$$= -\frac{\pi\left(1 + r - r^{m+1}\right)}{2b(1-r)\left(1-r^2\right)} \qquad [a + m < b < a + m + 1]$$
ET I 97(10)

6.254

1.* $$\int_0^\infty \operatorname{ci}(x)\sin^2 x\frac{dx}{x} = \frac{1}{2}\left[L_2\left(\frac{1}{2}\right) - L_2\left(-\frac{1}{2}\right)\right]$$

where $L_2(x)$ is the Euler dilogarithm defined as $L_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$ and this in turn can be expressed as $L_2(z) = \Phi(z,2,1)$ in terms of the Lerch function defined in 9.550, with z real.

2.[11] $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\cos bx \cdot \frac{dx}{x} = \frac{\pi}{2}\ln\frac{a}{b}\,\mathrm{H}(a-b)$$

$[a > 0, \quad b > 0, \quad \mathrm{H}(x)$ is the Heaviside step function$]$ ET I 41(11)

6.255

1. $$\int_{-\infty}^\infty \left[\cos ax \operatorname{ci}(a|x|) + \sin(a|x|)\operatorname{si}(a|x|)\right]\frac{dx}{x-b} = -\pi\left[\operatorname{sign} b\cos ab\operatorname{si}(a|b|) - \sin ab\operatorname{ci}(a|b|)\right]$$

$[a > 0]$ ET II 253(4)

2. $$\int_{-\infty}^\infty \left[\sin ax \operatorname{ci}(a|x|) - \operatorname{sign} x\cos ax\operatorname{si}(a|x|)\right]\frac{dx}{x-b} = -\pi\left[\sin(a|b|)\operatorname{si}(a|b|) + \cos ab\operatorname{ci}(a|b|)\right]$$

$[a > 0]$ ET II 253(5)

6.256

1. $$\int_0^\infty \left[\operatorname{si}^2(x) + \operatorname{ci}^2(x)\right]\cos ax\,dx = \frac{\pi}{a}\ln(1+a) \qquad [a > 0]$$

2.* $$\int_0^\infty \left[\operatorname{si}(x)\cos x - \operatorname{ci}(x)\sin x\right]^2\,dx = \frac{\pi}{2}$$

3.* $$\int_0^\infty \operatorname{si}^2(x)\cos(ax)\,dx = \frac{\pi}{2a}\log(1+a) \qquad [0 \le a \le 2]$$

4.* $$\int_0^\infty \operatorname{ci}^2(x)\cos(ax)\,dx = \frac{\pi}{2a}\log(1+a) \qquad [0 \le a \le 2]$$

6.257 $$\int_0^\infty \operatorname{si}\left(\frac{a}{x}\right)\sin bx\,dx = -\frac{\pi}{2b}J_0\left(2\sqrt{ab}\right) \qquad [b > 0]$$ ET I 42(18)

6.258

1. $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\sin bx\frac{dx}{x^2+c^2}$$
$$= \frac{\pi}{4c}\left\{e^{-bc}\left[\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)\right] + e^{bc}\left[\operatorname{Ei}(-ac) - \operatorname{Ei}(-bc)\right]\right\} \qquad [0 < b \le a, \quad c > 0]$$
$$= \frac{\pi}{4c}e^{-bc}\left[\operatorname{Ei}(ac) - \operatorname{Ei}(-ac)\right] \qquad [0 < a \le b, \quad c > 0]$$

BI (460)(1)

2. $$\int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2}\right]\cos bx\frac{x\,dx}{x^2+c^2}$$
$$= -\frac{\pi}{4}\left\{e^{-bc}\left[\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)\right] + e^{bc}\left[\operatorname{Ei}(-bc) - \operatorname{Ei}(-ac)\right]\right\} \qquad [0 < b \le a, \quad c > 0]$$
$$= \frac{\pi}{4}e^{-bc}\left[\operatorname{Ei}(-ac) - \operatorname{Ei}(ac)\right] \qquad [0 < a \le b, \quad c > 0]$$

BI (460)(2, 5)

6.259

1. $$\int_0^\infty \operatorname{si}(ax)\sin bx\frac{dx}{x^2+c^2} = \frac{\pi}{2c}\operatorname{Ei}(-ac)\sinh(bc) \qquad [0<b\le a, \quad c>0]$$
$$= \frac{\pi}{4c}e^{-cb}\left[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc)-\operatorname{Ei}(-ac)-\operatorname{Ei}(ac)\right] + \frac{\pi}{2c}\operatorname{Ei}(-bc)\sinh(bc) \qquad [0<a\le b, \quad c>0]$$
ET I 96(8)

2. $$\int_0^\infty \operatorname{ci}(ax)\sin bx\frac{x\,dx}{x^2+c^2} = -\frac{\pi}{2}\sinh(bc)\operatorname{Ei}(-ac) \qquad [0<b\le a, \quad c>0]$$
$$= -\frac{\pi}{2}\sinh(bc)\operatorname{Ei}(-bc)+\frac{\pi}{4}e^{-bc}[\operatorname{Ei}(-bc)+\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)-\operatorname{Ei}(ac)] \qquad [0<a\le b, \quad c>0]$$
BI (460)(3)a, ET I 97(15)a

3. $$\int_0^\infty \operatorname{ci}(ax)\cos bx\frac{dx}{x^2+c^2}$$
$$= \frac{\pi}{2c}\cosh bc\operatorname{Ei}(-ac) \qquad [0<b\le a, \quad c>0]$$
$$= \frac{\pi}{4c}\left\{e^{-bc}\left[\operatorname{Ei}(ac)+\operatorname{Ei}(-ac)-\operatorname{Ei}(bc)\right]+e^{bc}\operatorname{Ei}(-bc)\right\} \qquad [0<a\le b, \quad c>0]$$
BI (460)(4), ET I 41(15)

4.* $$\int_0^\infty [\operatorname{ci}(x)\sin x-\operatorname{Si}(x)\cos x]\sin x\frac{x\,dx}{a^2+x^2} = \frac{1}{8}\left[\operatorname{Ei}(a)e^{-a}-\operatorname{Ei}(-a)e^{a}\right]^2$$
$[a \text{ real}]$

5.* $$\int_0^\infty [\operatorname{ci}(x)\sin x-\operatorname{Si}(x)\cos x]^2\frac{x\,dx}{a^2+x^2} = \frac{\pi^3e^{-|a|}}{8a}\sinh(a)-\frac{\pi}{8|a|}\left[\operatorname{Ei}(a)e^{-a}-\operatorname{Ei}(-a)e^{a}\right]^2$$
$[a \text{ real}]$

6.261

1. $$\int_0^\infty \operatorname{si}(bx)\cos axe^{-px}\,dx = -\frac{1}{2\left(a^2+p^2\right)}\left[\frac{a}{2}\ln\frac{p^2+(a+b)^2}{p^2+(a-b)^2}+p\arctan\frac{2bp}{b^2-a^2-p^2}\right]$$
$[a>0, \quad b>0, \quad p>0]$ ET I 40(8)

2. $$\int_0^\infty \operatorname{si}(\beta x)\cos axe^{-\mu x}\,dx = -\frac{\arctan\dfrac{\mu+ai}{\beta}}{2(\mu+ai)}-\frac{\arctan\dfrac{\mu-ai}{\beta}}{2(\mu-ai)}$$
$[a>0, \quad \operatorname{Re}\mu>|\operatorname{Im}\beta|]$ ET I 40(9)

6.262

1. $$\int_0^\infty \operatorname{ci}(bx)\sin axe^{-\mu x}\,dx = \frac{1}{2\left(a^2+\mu^2\right)}\left\{\mu\arctan\frac{2a\mu}{\mu^2+b^2-a^2}-\frac{a}{2}\ln\frac{\left(\mu^2+b^2-a^2\right)^2+4a^2\mu^2}{b^4}\right\}$$
$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0]$
ET I 98(16)a

2. $$\int_0^\infty \operatorname{ci}(bx)\cos ax e^{-px}\,dx = \frac{-1}{2\left(a^2+p^2\right)}\left\{\frac{p}{2}\ln\frac{\left[\left(b^2+p^2-a^2\right)^2+4a^2p^2\right]}{b^4} + a\arctan\frac{2ap}{b^2+p^2-a^2}\right\}$$
$[a>0, \quad b>0, \quad \operatorname{Re} p>0]$ ET I 41(16)

3. $$\int_0^\infty \operatorname{ci}(\beta x)\cos ax e^{-\mu x}\,dx = \frac{-\ln\left[1+\frac{(\mu+ai)^2}{\beta^2}\right]}{4(\mu+ai)} - \frac{\ln\left[1+\frac{(\mu-ai)^2}{\beta^2}\right]}{4(\mu-ai)}$$
$[a>0, \quad \operatorname{Re}\mu>|\operatorname{Im}\beta|]$ ET I 41(17)

6.263

1. $$\int_0^\infty [\operatorname{ci}(x)\cos x+\operatorname{si}(x)\sin x]\,e^{-\mu x}\,dx = \frac{-\frac{\pi}{2}-\mu\ln\mu}{1+\mu^2}$$ $[\operatorname{Re}\mu>0]$ ME 26a, ET I 178(21)a

2. $$\int_0^\infty [\operatorname{si}(x)\cos x-\operatorname{ci}(x)\sin x]\,e^{-\mu x}\,dx = \frac{-\frac{\pi}{2}\mu+\ln\mu}{1+\mu^2}$$ $[\operatorname{Re}\mu>0]$ ME 26a, ET I 178(20)a

3. $$\int_0^\infty [\sin x - x\operatorname{ci}(x)]\,e^{-\mu x}\,dx = \frac{\ln\left(1+\mu^2\right)}{2\mu^2}$$ $[\operatorname{Re}\mu>0]$ ME 26

6.264

1. $$\int_0^\infty \operatorname{si}(x)\ln x\,dx = \boldsymbol{C}+1$$ NT 46(10)

2. $$\int_0^\infty \operatorname{ci}(x)\ln x\,dx = \frac{\pi}{2}$$ NT 56(11)

6.27 The hyperbolic sine integral and hyperbolic cosine integral functions

6.271

1. $$\int_0^\infty \operatorname{shi}(x)e^{-\mu x}\,dx = \frac{1}{2\mu}\ln\frac{\mu+1}{\mu-1} = \frac{1}{\mu}\operatorname{arccoth}\mu$$ $[\operatorname{Re}\mu>1]$ MI 34

2.[11] $$\int_0^\infty \operatorname{chi}(x)e^{-\mu x}\,dx = -\frac{1}{2\mu}\ln\left(\mu^2-1\right)$$ $[\operatorname{Re}\mu>1]$ MI 34

6.272[11] $$\int_0^\infty \operatorname{chi}(x)e^{-px^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{p}}\operatorname{Ei}\left(\frac{1}{4p}\right)$$ $[p>0]$ MI 35

6.273

1.[11] $$\int_0^\infty [\cosh x\operatorname{shi}(x)-\sinh x\operatorname{chi}(x)]\,e^{-\mu x}\,dx = \frac{\ln\mu}{\mu^2-1}$$ $[\operatorname{Re}\mu>0]$ MI 35

2.[11] $$\int_0^\infty [\cosh x\operatorname{chi}(x)+\sinh x\operatorname{shi}(x)]\,e^{-\mu x}\,dx = \frac{\mu\ln\mu}{1-\mu^2}$$ $[\operatorname{Re}\mu>2]$ MI 35

6.274[11] $$\int_0^\infty [\cosh x \operatorname{shi}(x) - \sinh x \operatorname{chi}(x)]\, e^{-\mu x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\mu}} e^{\frac{1}{4\mu}} \operatorname{Ei}\left(-\frac{1}{4\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 35

6.275 $$\int_0^\infty [x \operatorname{chi}(x) - \sinh x]\, e^{-\mu x}\, dx = -\frac{\ln(\mu^2 - 1)}{2\mu^2}$$ $[\operatorname{Re}\mu > 1]$ MI 35

6.276 $$\int_0^\infty [\cosh x \operatorname{chi}(x) + \sinh x \operatorname{shi}(x)]\, e^{-\mu x^2} x\, dx = \frac{1}{8}\sqrt{\frac{\pi}{\mu^3}} \exp\left(\frac{1}{4\mu}\right) \operatorname{Ei}\left(-\frac{1}{4\mu}\right)$$ $[\operatorname{Re}\mu > 0]$ MI 35

6.277

1. $$\int_0^\infty [\operatorname{chi}(x) + \operatorname{ci}(x)]\, e^{-\mu x}\, dx = -\frac{\ln(\mu^4 - 1)}{2\mu}$$ $[\operatorname{Re}\mu > 1]$ MI 34

2. $$\int_0^\infty [\operatorname{chi}(x) - \operatorname{ci}(x)]\, e^{-\mu x}\, dx = \frac{1}{2\mu} \ln \frac{\mu^2 + 1}{\mu^2 - 1}$$ $[\operatorname{Re}\mu > 1]$ MI 35

6.28–6.31 The probability integral

6.281

1.[6] $$\int_0^\infty [1 - \Phi(px)]\, x^{2q-1}\, dx = \frac{\Gamma\left(q + \frac{1}{2}\right)}{2\sqrt{\pi} q p^{2q}}$$ $[\operatorname{Re} q > 0, \quad \operatorname{Re} p > 0]$

NT 56(12), ET II 306(1)a

2.[6] $$\int_0^\infty \left[1 - \Phi\left(a t^\alpha \pm \frac{b}{t^\alpha}\right)\right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a}\right)^{\frac{1-\alpha}{2\alpha}} \left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab)\right] e^{\pm 2ab}$$

$[a > 0, \quad b > 0, \quad \alpha \neq 0]$

6.282

1. $$\int_0^\infty \Phi(qt) e^{-pt}\, dt = \frac{1}{p}\left[1 - \Phi\left(\frac{p}{2q}\right)\right] \exp\left(\frac{p^2}{4q^2}\right)$$ $\left[\operatorname{Re} p > 0, \quad |\arg q| < \frac{\pi}{4}\right]$

MO 175, EH II 148(11)

2. $$\int_0^\infty \left[\Phi\left(x + \frac{1}{2}\right) - \Phi\left(\frac{1}{2}\right)\right] e^{-\mu x + \frac{1}{4}}\, dx = \frac{1}{(\mu + 1)(\mu + 2)} \exp\frac{(\mu + 1)^2}{4} \left[1 - \Phi\left(\frac{\mu + 1}{2}\right)\right]$$

ME 27

6.283

1. $$\int_0^\infty e^{\beta x} \left[1 - \Phi\left(\sqrt{\alpha x}\right)\right] dx = \frac{1}{\beta}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1\right]$$ $[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta < \operatorname{Re}\alpha]$ ET II 307(5)

2. $$\int_0^\infty \Phi\left(\sqrt{qt}\right) e^{-pt}\, dt = \frac{\sqrt{q}}{p} \frac{1}{\sqrt{p + q}}$$ $[\operatorname{Re} p > 0, \quad \operatorname{Re}(q + p) > 0]$

EH II 148(12)

6.284 $$\int_0^\infty \left[1 - \Phi\left(\frac{q}{2\sqrt{x}}\right)\right] e^{-px}\, dx = \frac{1}{p} e^{-q\sqrt{p}}$$ $\left[\operatorname{Re} p > 0, \quad |\arg q| < \frac{\pi}{4}\right]$

EF 147(235), EH II 148(13)

6.285

1. $$\int_0^\infty [1-\Phi(x)]\, e^{-\mu^2x^2}\, dx = \frac{\arctan\mu}{\sqrt{\pi}\mu} \qquad [\operatorname{Re}\mu > 0]$$ MI 37

2. $$\int_0^\infty \Phi(iat) e^{-a^2t^2-st}\, dt = \frac{-1}{2ai\sqrt{\pi}} \exp\left(\frac{s^2}{4a^2}\right) \operatorname{Ei}\left(-\frac{s^2}{4a^2}\right)$$ $$\left[\operatorname{Re}s > 0, \quad |\arg a| < \frac{\pi}{4}\right]$$ EH II 148(14)a

6.286

1. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{\mu^2x^2} x^{\nu-1}\, dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\nu\beta^\nu}\, {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; \frac{\mu^2}{\beta^2}\right)$$ $$\left[\operatorname{Re}^2\beta > \operatorname{Re}\mu^2, \quad \operatorname{Re}\nu > 0\right]$$ ET II 306(2)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1}\, dx = 2^{\frac{\nu}{2}-1} \sec\frac{\nu\pi}{2}\, \Gamma\left(\frac{\nu}{2}\right)$$ $$[0 < \operatorname{Re}\nu < 1]$$ ET I 325(9)

6.287

1. $$\int_0^\infty \Phi(\beta x) e^{-\mu x^2} x\, dx = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \qquad \left[\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0\right]$$ ME 27a, ET I 176(4)

2. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{-\mu x^2} x\, dx = \frac{1}{2\mu}\left(1 - \frac{\beta}{\sqrt{\mu+\beta^2}}\right) \qquad \left[\operatorname{Re}\mu > -\operatorname{Re}\beta^2, \quad \operatorname{Re}\mu > 0\right]$$ NT 49(14), ET I 177(9)

3.* $$I = \int_{-\infty}^\infty \frac{r}{\sigma^2} \exp\left(\frac{r}{\sigma^2}\right) Q(rA)\, Q(rB)\, dr = \frac{1}{4} - \frac{1}{2\pi}\left[\alpha \arctan\left(\frac{A}{\alpha B}\right) + \beta\arctan\left(\frac{B}{\beta A}\right)\right] \qquad B \neq A$$ $$= \frac{1}{4} - \frac{1}{\pi}\alpha\arctan\frac{1}{\alpha} \qquad B = A$$ $$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}\, dt = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad \alpha = \sqrt{\frac{\sigma^2A^2}{1+\sigma^2A^2}}, \quad \beta = \sqrt{\frac{\sigma^2B^2}{1+\sigma^2B^2}},$$

6.288 $$\int_0^\infty \Phi(iax) e^{-\mu x^2} x\, dx = \frac{ai}{2\mu\sqrt{\mu-a^2}} \qquad \left[a > 0, \quad \operatorname{Re}\mu > \operatorname{Re}a^2\right]$$ MI 37a

6.289

1. $$\int_0^\infty \Phi(\beta x) e^{\left(\beta^2-\mu^2\right)x^2} x\, dx = \frac{\beta}{2\mu\left(\mu^2-\beta^2\right)} \qquad \left[\operatorname{Re}^2\mu > \operatorname{Re}\beta^2, \quad |\arg\mu| < \frac{\pi}{4}\right]$$ ET I 176(5)

2. $$\int_0^\infty [1-\Phi(\beta x)]\, e^{\left(\beta^2-\mu^2\right)x^2} x\, dx = \frac{1}{2\mu(\mu+\beta)} \qquad \left[\operatorname{Re}^2\mu > \operatorname{Re}\beta^2, \quad \arg\mu < \frac{\pi}{4}\right]$$ ET I 177(10)

3. $$\int_0^\infty \Phi\left(\sqrt{b-a}x\right) e^{-(a+\mu)x^2} x\,dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \qquad [\operatorname{Re}\mu > -a > 0, \quad b > a]$$ ME 27

6.291 $$\int_0^\infty \Phi(ix) e^{-\left(\mu x + x^2\right)} x\,dx = \frac{i}{\sqrt{\pi}} \left[\frac{1}{\mu} + \frac{\mu}{4} \operatorname{Ei}\left(-\frac{\mu^2}{4}\right)\right] \qquad [\operatorname{Re}\mu > 0]$$ MI 37

6.292 $$\int_0^\infty [1-\Phi(x)]\, e^{-\mu^2 x^2} x^2\,dx = \frac{1}{2\sqrt{\pi}} \left\{\frac{\arctan\mu}{\mu^3} - \frac{1}{\mu^2\left(\mu^2+1\right)}\right\} \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.293 $$\int_0^\infty \Phi(x) e^{-\mu x^2} \frac{dx}{x} = \frac{1}{2} \ln \frac{\sqrt{\mu+1}+1}{\sqrt{\mu+1}-1} = \operatorname{arccoth}\sqrt{\mu+1} \qquad [\operatorname{Re}\mu > 0]$$ MI 37a

6.294

1. $$\int_0^\infty \left[1-\Phi\left(\frac{\beta}{x}\right)\right] e^{-\mu^2 x^2} x\,dx = \frac{1}{2\mu^2} \exp(-2\beta\mu) \qquad \left[|\arg\beta| < \frac{\pi}{4}, \quad |\arg\mu| < \frac{\pi}{4}\right]$$ ET I 177(11)

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] e^{-\mu^2 x^2} \frac{dx}{x} = -\operatorname{Ei}(-2\mu) \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.295

1. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) dx = \frac{1}{\sqrt{\pi}\mu} [\sin 2\mu \operatorname{ci}(2\mu) - \cos 2\mu \operatorname{si}(2\mu)] \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

2. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) x\,dx = \frac{\pi}{2\mu} [\mathbf{H}_1(2\mu) - Y_1(2\mu)] - \frac{1}{\mu^2} \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

3. $$\int_0^\infty \left[1-\Phi\left(\frac{1}{x}\right)\right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) \frac{dx}{x} = \frac{\pi}{2} [\mathbf{H}_0(2\mu) - Y_0(2\mu)] \qquad \left[|\arg\mu| < \frac{\pi}{4}\right]$$ MI 37

6.296 $$\int_0^\infty \left\{\left(x^2+a^2\right)\left[1-\Phi\left(\frac{a}{\sqrt{2}x}\right)\right] - \sqrt{\frac{2}{\pi}} ax \cdot e^{-\frac{a^2}{2x^2}}\right\} e^{-\mu^2 x^2} x\,dx = \frac{1}{2\mu^4} e^{-a\mu\sqrt{2}} \qquad \left[|\arg\mu| < \frac{\pi}{4}, \quad a > 0\right]$$ MI 38a

6.297

1. $$\int_0^\infty \left[1-\Phi\left(\gamma x + \frac{\beta}{x}\right)\right] e^{\left(\gamma^2-\mu\right)x^2} x\,dx = \frac{1}{2\sqrt{\mu}\left(\sqrt{\mu}+\gamma\right)} \exp\left[-2\left(\beta\gamma + \beta\sqrt{\mu}\right)\right] \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$$ ET I 177(12)a

2. $$\int_0^\infty \left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right] \exp\left[-\left(\mu^2-a^2\right)x^2 + ab\right] x\,dx = \frac{e^{-b\mu}}{2\mu(\mu+a)} \qquad [a > 0, \quad b > 0, \quad \operatorname{Re}\mu > 0]$$ MI 38

3. $$\int_0^\infty \left\{\left[1-\Phi\left(\frac{b-2ax^2}{2x}\right)\right]e^{-ab}+\left[1-\Phi\left(\frac{b+2ax^2}{2x}\right)\right]e^{ab}\right\}e^{-\mu x^2}x\,dx=\frac{1}{\mu}\exp\left(-b\sqrt{a^2+\mu}\right)$$

$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0]$ MI 38

6.298 $$\int_0^\infty \left\{2\cosh ab-e^{-ab}\,\Phi\left(\frac{b-2ax^2}{2x}\right)-e^{ab}\,\Phi\left(\frac{b+2ax^2}{2x}\right)\right\}e^{-(\mu-a^2)x^2}x\,dx=\frac{1}{\mu-a^2}\exp\left(-b\sqrt{\mu}\right)$$

$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0]$ MI 38

6.299 $$\int_0^\infty \cosh(2\nu t)\exp\left[(a\cosh t)^2\right]\left[1-\Phi\left(a\cosh t\right)\right]\,dt=\frac{1}{2\cos(\nu\pi)}\exp\left(\tfrac{1}{2}a^2\right)K_\nu\left(a^2\right)$$

$\left[\operatorname{Re}a>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}\right]$

ET II 308(10)

6.311 $$\int_0^\infty [1-\Phi(ax)]\sin bx\,dx=\frac{1}{b}\left(1-e^{-\frac{b^2}{4a^2}}\right)$$

$[a>0, \quad b>0]$ ET I 96(4)

6.312 $$\int_0^\infty \Phi(ax)\sin bx^2\,dx=\frac{1}{4\sqrt{2\pi b}}\left(\ln\frac{b+a^2+a\sqrt{2b}}{b+a^2-a\sqrt{2b}}+2\arctan\frac{a\sqrt{2b}}{b-a^2}\right)$$

$[a>0, \quad b>0]$ ET I 96(3)

6.313

1. $$\int_0^\infty \sin(\beta x)\left[1-\Phi\left(\sqrt{\alpha x}\right)\right]\,dx=\frac{1}{\beta}-\left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}-\alpha\right]^{-\frac{1}{2}}$$

$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET II 307(6)

2. $$\int_0^\infty \cos(\beta x)\left[1-\Phi\left(\sqrt{\alpha x}\right)\right]\,dx=\left(\frac{\frac{\alpha}{2}}{\alpha^2+\beta^2}\right)^{\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{-\frac{1}{2}}$$

$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ ET II 307(7)

6.314

1. $$\int_0^\infty \sin(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]\,dx=b^{-1}\exp\left[-(2ab)^{\frac{1}{2}}\right]\cos\left[(2ab)^{\frac{1}{2}}\right]$$

$[\operatorname{Re}a>0, \quad b>0]$ ET II 307(8)

2. $$\int_0^\infty \cos(bx)\left[1-\Phi\left(\sqrt{\frac{a}{x}}\right)\right]\,dx=-b^{-1}\exp\left[-(2ab)^{\frac{1}{2}}\right]\sin\left[(2ab)^{\frac{1}{2}}\right]$$

$[\operatorname{Re}a>0, \quad b>0]$ ET II 307(9)

6.315

1. $$\int_0^\infty x^{\nu-1}\sin(\beta x)\left[1-\Phi(\alpha x)\right]\,dx=\frac{\Gamma\left(1+\frac{1}{2}\nu\right)\beta}{\sqrt{\pi}(\nu+1)\alpha^{\nu+1}}\;{}_2F_2\left(\frac{\nu+1}{2},\frac{\nu}{2}+1;\frac{3}{2},\frac{\nu+3}{2};-\frac{\beta^2}{4\alpha^2}\right)$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\nu>-1]$ ET II 307(3)

2. $$\int_0^\infty x^{\nu-1}\cos(\beta x)\left[1-\Phi(\alpha x)\right]\,dx=\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)}{\sqrt{\pi}\nu\alpha^\nu}\;{}_2F_2\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{1}{2},\frac{\nu}{2}+1;-\frac{\beta^2}{4\alpha^2}\right)$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\nu>0]$ ET II 307(4)

3. $$\int_0^\infty [1-\Phi(ax)]\cos bx \cdot x\,dx = \frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right) - \frac{1}{b^2}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right]$$
$[a>0, \quad b>0]$ ET I 40(5)

4. $$\int_0^\infty [\Phi(ax)-\Phi(bx)]\cos px\frac{dx}{x} = \frac{1}{2}\left[\operatorname{Ei}\left(-\frac{p^2}{4b^2}\right) - \operatorname{Ei}\left(\frac{p^2}{4a^2}\right)\right]$$
$[a>0, \quad b>0, \quad p>0]$ ET I 40(6)

5. $$\int_0^\infty x^{-\frac{1}{2}}\,\Phi\left(a\sqrt{x}\right)\sin bx\,dx = \frac{1}{2\sqrt{2\pi b}}\left\{\ln\left[\frac{b+a\sqrt{2b}+a^2}{b-a\sqrt{2b}+a^2}\right] + 2\arctan\left[\frac{a\sqrt{2b}}{b-a^2}\right]\right\}$$
$[a>0, \quad b>0]$ ET I 96(3)

6.316 $$\int_0^\infty e^{\frac{1}{2}x^2}\left[1-\Phi\left(\frac{x}{\sqrt{2}}\right)\right]\sin bx\,dx = \sqrt{\frac{\pi}{2}}e^{\frac{b^2}{2}}\left[1-\Phi\left(\frac{b}{\sqrt{2}}\right)\right]$$
$[b>0]$ ET I 96(5)

6.317[6] $$\int_0^\infty e^{-a^2x^2}\,\Phi(iax)\sin bx\,dx = \frac{i}{a}\frac{\sqrt{\pi}}{2}e^{-\frac{b^2}{4a^2}}$$ $[b>0]$ ET I 96(2)

6.318 $$\int_0^\infty [1-\Phi(x)]\operatorname{si}(2px)\,dx = \frac{2}{\pi p}\left(1-e^{-p^2}\right) - \frac{2}{\sqrt{\pi}}\left(1-\Phi(p)\right)$$
$[p>0]$ NT 61(13)a

6.32 Fresnel integrals

6.321

1. $$\int_0^\infty \left[\frac{1}{2}-S(px)\right]x^{2q-1}\,dx = \frac{\sqrt{2}\,\Gamma\left(q+\frac{1}{2}\right)\sin\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}}$$
$\left[0<\operatorname{Re}q<\frac{3}{2}, \quad p>0\right]$ NT 56(14)a

2. $$\int_0^\infty \left[\frac{1}{2}-C(px)\right]x^{2q-1}\,dx = \frac{\sqrt{2}\,\Gamma\left(q+\frac{1}{2}\right)\cos\frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}}$$
$\left[0<\operatorname{Re}q<\frac{3}{2}, \quad p>0\right]$ NT 56(13)a

6.322

1. $$\int_0^\infty S(t)e^{-pt}\,dt = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right] + \sin\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right]\right\}$$ MO 173a

2. $$\int_0^\infty C(t)e^{-pt}\,dt = \frac{1}{p}\left\{\cos\frac{p^2}{4}\left[\frac{1}{2}-S\left(\frac{p}{2}\right)\right] - \sin\frac{p^2}{4}\left[\frac{1}{2}-C\left(\frac{p}{2}\right)\right]\right\}$$ MO 172a

6.323

1. $$\int_0^\infty S\left(\sqrt{t}\right)e^{-pt}\,dx = \frac{\left(\sqrt{p^2+1}-p\right)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}$$ EF 122(58)a

2. $$\int_0^\infty C\left(\sqrt{t}\right) e^{-pt}\,dt = \frac{\left(\sqrt{p^2+1}+p\right)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}$$ EF 122(58)a

6.324

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \sin 2px\,dx = \frac{1+\sin p^2 - \cos p^2}{4p} \qquad [p>0]$$ NT 61(12)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \sin 2px\,dx = \frac{1-\sin p^2 - \cos p^2}{4p} \qquad [p>0]$$ NT 61(11)a

6.325

1. $$\int_0^\infty S(x) \sin b^2x^2\,dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \qquad \left[0<b^2<1\right]$$
$$= 0 \qquad \left[b^2>1\right]$$ ET I 98(21)a

2. $$\int_0^\infty C(x) \cos b^2x^2\,dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \qquad \left[0<b^2<1\right]$$
$$= 0 \qquad \left[b^2>1\right]$$ ET I 42(22)

6.326

1. $$\int_0^\infty \left[\frac{1}{2} - S(x)\right] \operatorname{si}(2px)\,dx = \left(\frac{\pi}{8}\right)^{1/2} (S(p)+C(p)-1) - \frac{1+\sin p^2-\cos p^2}{4p}$$
$$[p>0]$$ NT 61(15)a

2. $$\int_0^\infty \left[\frac{1}{2} - C(x)\right] \operatorname{si}(2px)\,dx = \left(\frac{\pi}{8}\right)^{1/2} (S(p)-C(p)) - \frac{1-\sin p^2-\cos p^2}{4p}$$
$$[p>0]$$ NT 61(14)a

6.4 The Gamma Function and Functions Generated by It

6.41 The gamma function

6.411[11] $$\int_{-\infty}^\infty \Gamma(\alpha+x)\,\Gamma(\beta-x)\,dx = -i\pi 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$$
$$[\operatorname{Re}(\alpha+\beta)<1 \text{ and either } \operatorname{Im}\alpha<0<\operatorname{Im}\beta \text{ or } \operatorname{Im}\beta<0<\operatorname{Im}\alpha\ ]$$ ET II 297(1)

$$= i\pi 2^{1-\alpha-\beta}\,\Gamma(\alpha+\beta)$$
$$[\operatorname{Re}(\alpha+\beta)<1, \quad \operatorname{Im}\alpha<0, \quad \operatorname{Im}\beta<0]$$ ET II 297(2)

$$= 0$$
$$[\operatorname{Re}(\alpha+\beta)<1, \quad \operatorname{Im}\alpha>0, \quad \operatorname{Im}\beta>0]$$ ET II 297(3)

6.412 $$\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds = 2\pi i \frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$

$[\operatorname{Re}\alpha, \quad \operatorname{Re}\beta, \quad \operatorname{Re}\gamma, \quad \operatorname{Re}\delta > 0]$ ET II 302(32)

6.413

1. $$\int_0^\infty |\Gamma(a+ix)\,\Gamma(b+ix)|^2\,dx = \frac{\sqrt{\pi}\,\Gamma(a)\,\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\,\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{2\,\Gamma\left(a+b+\frac{1}{2}\right)}$$

$[a > 0, \quad b > 0]$ ET II 302(27)

2. $$\int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2 dx = \frac{\sqrt{\pi}\,\Gamma(a)\,\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-a-\frac{1}{2}\right)}{2\,\Gamma(b)\,\Gamma\left(b-\frac{1}{2}\right)\Gamma(b-a)}$$

$\left[0 < a < b-\frac{1}{2}\right]$ ET II 302(28)

6.414

1. $$\int_{-\infty}^\infty \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\,dx = 0$$ $[\operatorname{Im}\alpha \neq 0, \quad \operatorname{Re}(\alpha-\beta) < -1]$

ET II 297(4)

2. $$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}$$ $[\operatorname{Re}(\alpha+\beta) > 1]$ ET II 297(5)

3. $$\int_{-\infty}^\infty \frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(\alpha+x)\,\Gamma(\beta+x)}\,dx = 0$$

$[\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im}\gamma, \quad \operatorname{Im}\delta > 0]$ ET II 299(18)

4. $$\int_{-\infty}^\infty \frac{\Gamma(\gamma+x)\,\Gamma(\delta+x)}{\Gamma(\alpha+x)\,\Gamma(\beta+x)}\,dx = \frac{\pm 2\pi^2 i\,\Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)]\,\Gamma(\alpha-\gamma)\,\Gamma(\alpha-\delta)\,\Gamma(\beta-\gamma)\,\Gamma(\beta-\delta)}$$

$[\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im}\gamma < 0, \quad \operatorname{Im}\delta < 0.$ In the numerator, we take the plus sign if $\operatorname{Im}\gamma > \operatorname{Im}\delta$ and the minus sign if $\operatorname{Im}\gamma < \operatorname{Im}\delta.]$ ET II 300(19)

5. $$\int_{-\infty}^\infty \frac{\Gamma(\alpha-\beta-\gamma+x+1)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)} = \frac{\pi\exp\left(\pm\frac{1}{2}\pi(\delta-\gamma)i\right)}{\Gamma(\beta+\gamma-1)\,\Gamma\left(\frac{1}{2}(\alpha+\beta)\right)\Gamma\left(\frac{1}{2}(\gamma-\delta+1)\right)}$$

$[\operatorname{Re}(\beta+\gamma) > 1, \quad \delta = \alpha-\beta-\gamma+1, \quad \operatorname{Im}\delta \neq 0.$ The sign is plus in the argument if the exponential for $\operatorname{Im}\delta > 0$ and minus for $\operatorname{Im}\delta < 0.]$ ET II 300(20)

6. $$\int_{-\infty}^\infty \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)}$$

$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3]$ ET II 300(21)

6.415

1. $$\int_{-\infty}^{-\infty} \frac{R(x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}$$
$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)}\int_0^1 R(t)\,dt$$

$[\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3, \quad R(x+1) = R(x)]$ ET II 301(24)

2. $$\int_{-\infty}^{\infty} \frac{R(x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\displaystyle\int_0^1 R(t)\cos\left[\tfrac{1}{2}\pi(2t+\alpha-\beta)\right]\,dt}{\Gamma\left(\dfrac{\alpha+\beta}{2}\right)\Gamma\left(\dfrac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$

$[\alpha+\delta=\beta+\gamma,\quad \operatorname{Re}(\alpha+\beta+\gamma+\delta)>2,\quad R(x+1)=-R(x)]$ ET II 301(25)

6.42 Combinations of the gamma function, the exponential, and powers

6.421

1. $$\int_{-\infty}^{\infty} \Gamma(\alpha+x)\,\Gamma(\beta-x)\exp\left[2(\pi n+\theta)xi\right]\,dx = 2\pi i\,\Gamma(\alpha+\beta)(2\cos\theta)^{-\alpha-\beta}\exp[(\beta-\alpha)i\theta] \times\left[\eta_n(\beta)\exp(2n\pi\beta i)-\eta_n(-\alpha)\exp(-2n\pi\alpha i)\right]$$

$$\left[\operatorname{Re}(\alpha+\beta)<1,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n \text{ an integer},\quad \eta_n(\xi)=\begin{cases}0 & \text{if } \left(\frac{1}{2}-n\right)\operatorname{Im}\xi>0\\ \operatorname{sign}\left(\frac{1}{2}-n\right) & \text{if } \left(\frac{1}{2}-n\right)\operatorname{Im}\xi<0\end{cases}\right]$$

ET II 298(7)

2. $$\int_{-\infty}^{\infty} \frac{e^{\pi icx}\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+kx)\,\Gamma(\delta-kx)} = 0$$

$[\operatorname{Re}(\alpha+\beta+\gamma+\delta)>2,\quad c \text{ and } k \text{ are real},\quad |c|>|k|+1]$ ET II 301(26)

3. $$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx = 2\pi i\operatorname{sign}\left(n+\tfrac{1}{2}\right)\frac{(2\cos\theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\exp[-(2\pi n+\pi-\theta)\alpha i+\theta i(\beta-1)]$$

$\left[\operatorname{Re}(\beta-\alpha)>0,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n \text{ is an integer},\quad \left(n+\tfrac{1}{2}\right)\operatorname{Im}\alpha<0\right]$ ET II 298(8)

4. $$\int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}\exp[(2\pi n+\pi-2\theta)xi]\,dx = 0$$

$\left[\operatorname{Re}(\beta-\alpha)>0,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2},\quad n \text{ is an integer},\quad \left(n+\tfrac{1}{2}\right)\operatorname{Im}\alpha>0\right]$ ET II 297(6)

6.422

1. $$\int_{-i\infty}^{i\infty} \Gamma(s-k-\lambda)\,\Gamma\left(\lambda+\mu-s+\tfrac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\tfrac{1}{2}\right)z^s\,ds = 2\pi i\,\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-k+\mu\right)z^\lambda e^{\frac{z}{2}}\,W_{k,\mu}(z)$$

$\left[\operatorname{Re}(k+\lambda)<0,\quad \operatorname{Re}\lambda>|\operatorname{Re}\mu|-\tfrac{1}{2},\quad |\arg z|<\tfrac{3}{2}\pi\right]$ ET II 302(29)

2. $$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+s)\,\Gamma(-s)\,\Gamma(1-c-s)x^s\,ds = 2\pi i\,\Gamma(\alpha)\,\Gamma(\alpha-c+1)\Psi(\alpha,c;x)$$

$\left[-\operatorname{Re}\alpha<\gamma<\min\left(0,1-\operatorname{Re}c\right),\quad -\tfrac{3}{2}\pi<\arg x<\tfrac{3}{2}\pi\right]$ EH I 256(5)

3. $$\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s)\,\Gamma(\beta+s)t^s\,ds = 2\pi i\,\Gamma(\beta)(1+t)^{-\beta} \qquad [0>\gamma>\operatorname{Re}(1-\beta), \quad |\arg t|<\pi]$$
EH I 256, BU 75

4. $$\int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right)\Gamma(-t)\left(\sqrt{2}\right)^{t-p-2} z^t\,dt = 2\pi i e^{\frac{1}{4}z^2}\,\Gamma(-p)\,D_p(z)$$
$$\left[|\arg z|<\tfrac{3}{4}\pi, \quad p \text{ is not a positive integer}\right]$$ WH

5. $$\int_{-i\infty}^{i\infty} \Gamma(s)\,\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{4}-s\right)\Gamma\left(\tfrac{1}{2}\nu-\tfrac{1}{4}-s\right)\left(\frac{z^2}{2}\right)^s ds$$
$$= 2\pi i\cdot 2^{\frac{1}{4}-\frac{1}{2}\nu}z^{-\frac{1}{2}}e^{\frac{3}{4}z^2}\,\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{4}\right)\Gamma\left(\tfrac{1}{2}\nu-\tfrac{1}{4}\right)D_\nu(z)$$
$$\left[|\arg z|<\tfrac{3}{4}\pi, \quad \nu\neq\tfrac{1}{2}, \quad -\tfrac{1}{2}, \quad -\tfrac{3}{2},\ldots\right]$$ EH II 120

6.[3] $$\int_{c-i\infty}^{c+i\infty} \left(\tfrac{1}{2}x\right)^{-s}\Gamma\left(\tfrac{1}{2}\nu+\tfrac{1}{2}s\right)\left[\Gamma\left(1+\tfrac{1}{2}\nu-\tfrac{1}{2}s\right)\right]^{-1} ds = 4\pi i\,J_\nu(x)$$
$$[x>0, -\operatorname{Re}\nu<c<1]$$ EH II 21(34)

7. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\,\Gamma(-s)\left(-\tfrac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi}\,H_\nu^{(1)}(z)$$
$$\left[|\arg(-iz)|<\frac{\pi}{2}, \quad 0<\operatorname{Re}\nu<c\right]$$
EH II 83(34)

8. $$\int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\,\Gamma(-s)\left(\tfrac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi}\,H_\nu^{(2)}(z)$$
$$\left[|\arg(iz)|<\frac{\pi}{2}, \quad 0<\operatorname{Re}\nu<c\right]$$
EH II 83(35)

9. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)}\,ds = 2\pi i\,J_\nu(x) \qquad [x>0, \quad \operatorname{Re}\nu>0]$$ EH II 83(36)

10. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\,\Gamma(-2\nu-s)\,\Gamma\left(\nu+s+\tfrac{1}{2}\right)(-2iz)^s\,ds = -\pi^{\frac{5}{2}}e^{-i(z-\nu\pi)}\sec(\nu\pi)(2z)^{-\nu}\,H_\nu^{(1)}(z)$$
$$\left[|\arg(-iz)|<\tfrac{3}{2}\pi, \quad 2\nu\neq\pm1, \quad \pm3\ldots\right]$$
EH II 83(37)

11. $$\int_{-i\infty}^{i\infty} \Gamma(-s)\,\Gamma(-2\nu-s)\,\Gamma\left(\nu+s+\tfrac{1}{2}\right)(2iz)^s\,ds = \pi^{\frac{5}{2}}e^{i(z-\nu\pi)}\sec(\nu\pi)(2z)^{-\nu}\,H_\nu^{(2)}(z)$$
$$\left[|\arg(iz)|<\tfrac{3}{2}\pi, \quad 2\nu\neq\pm1, \quad \pm3\ldots\right]$$
EH II 84(38)

12. $$\int_{-i\infty}^{i\infty} \Gamma(s)\,\Gamma\left(\tfrac{1}{2}-s-\nu\right)\Gamma\left(\tfrac{1}{2}-s+\nu\right)(2z)^s\,ds = 2^{\frac{3}{2}}\pi^{\frac{3}{2}}iz^{\frac{1}{2}}e^z\sec(\nu\pi)\,K_\nu(z)$$
$$\left[|\arg z|<\tfrac{3}{2}\pi, \quad 2\nu\neq\pm1, \quad \pm3,\ldots\right]$$
EH II 84(39)

13. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s\,\Gamma(1+s)}x^{2s}\,ds = 4\pi\int_{2x}^{\infty}\frac{J_0(t)}{t}\,dt \qquad [x>0]$$ MO 41

14. $$\int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s\,ds = 2\pi i\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,F(\alpha,\beta;\gamma;z)$$

[For $\arg(-z) < \pi$, the path of integration must separate the poles of the integrand at the points $s = 0, 1, 2, 3, \ldots$ from the poles $s = -\alpha - n$ and $s = -\beta - n$ (for $n = 0, 1, 2, \ldots$).]

15. $$\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\,\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s\,ds = \frac{2\pi i\,\Gamma(\alpha)}{\Gamma(\gamma)}\;{}_1F_1(\alpha;\gamma;z)$$

$$\left[-\frac{\pi}{2} < \arg(-z) < \frac{\pi}{2}, \quad 0 > \delta > -\operatorname{Re}\alpha, \quad \gamma \neq 0, 1, 2, \ldots\right]$$ EH I 62(15), EH I 256(4)

16. $$\int_{-i\infty}^{i\infty} \left[\frac{\Gamma\left(\frac{1}{2}-s\right)}{\Gamma(s)}\right]^2 z^s\,ds = 2\pi i z^{\frac{1}{2}}\left[2\pi^{-1}\,K_0\left(4z^{\frac{1}{4}}\right) - Y_0\left(4z^{\frac{1}{4}}\right)\right]$$

$[z > 0]$ ET II 303(33)

17. $$\int_{-i\infty}^{i\infty} \frac{\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\frac{1}{2}\right)}{\Gamma(\lambda-k-s+1)} z^s\,ds = 2\pi i z^{\lambda} e^{-\frac{z}{2}}\,W_{k,\mu}(z)$$

$$\left[\operatorname{Re}\lambda > |\operatorname{Re}\mu| - \tfrac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right]$$ ET II 302(30)

18. $$\int_{-i\infty}^{i\infty} \frac{\Gamma(k-\lambda+s)\,\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)}{\Gamma\left(\mu-\lambda+s+\frac{1}{2}\right)} z^s\,ds = 2\pi i\frac{\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma(2\mu+1)} z^{\lambda} e^{-\frac{z}{2}}\,M_{k,\mu}(z)$$

$$\left[\operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(\lambda+\mu) > -\tfrac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right]$$ ET II 302(31)

19. $$\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m}\Gamma\left(b_j-s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-s\right)} z^s\,ds = 2\pi i\,G^{pq}_{mn}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

$$\left[p+q < 2(m+n); \quad |\arg z| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi;\right.$$

$$\left.\operatorname{Re}a_k < 1, \quad k = 1,\ldots,n; \quad \operatorname{Re}b_j > 0, \quad j = 1,\ldots,m\right]$$

ET II 303(34)

6.423

1. $$\int_0^{\infty} e^{-\alpha x}\frac{dx}{\Gamma(1+x)} = \nu\left(e^{-\alpha}\right)$$ MI 39, EH III 222(16)

2. $$\int_0^{\infty} e^{-\alpha x}\frac{dx}{\Gamma(x+\beta+1)} = e^{\beta\alpha}\nu\left(e^{-\alpha},\beta\right)$$ MI 39, EH III 222(16)

3. $$\int_0^{\infty} e^{-\alpha x}\frac{x^m}{\Gamma(x+1)}\,dx = \mu\left(e^{-\alpha},m\right)\Gamma(m+1)$$ $[\operatorname{Re}m > -1]$ MI 39, EH III 222(17)

4. $$\int_0^\infty e^{-\alpha x} \frac{x^m}{\Gamma(x+n+1)}\,dx = e^{n\alpha}\mu\left(e^{-\alpha}, m, n\right)\Gamma(m+1)$$ MI 39, EH III 222(17)

6.424 $$\int_{-\infty}^\infty \frac{R(x)\exp[(2\pi n+\theta)xi]\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{\left[2\cos\left(\frac{\theta}{2}\right)\right]^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\exp\left[\frac{1}{2}\theta(\beta-\alpha)i\right]\int_0^1 R(t)\exp(2\pi nti)\,dt$$
$[\operatorname{Re}(\alpha+\beta)>1, \quad -\pi<\theta<\pi, \quad n \text{ is an integer}, \quad R(x+1)=R(x)]$ ET II 299(16)

6.43 Combinations of the gamma function and trigonometric functions

6.431

1. $$\int_{-\infty}^{-\infty} \frac{\sin rx\,dx}{\Gamma(p+x)\,\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\sin\frac{r(q-p)}{2}}{\Gamma(p+q-1)} \qquad [|r|<\pi]$$
$$= 0 \qquad [|r|>\pi]$$
$[r \text{ is real}; \quad \operatorname{Re}(p+q)>1]$ MO 10a, ET II 298(9, 10)

2. $$\int_{-\infty}^{\infty} \frac{\cos rx\,dx}{\Gamma(p+x)\,\Gamma(q-x)} = \frac{\left(2\cos\frac{r}{2}\right)^{p+q-2}\cos\frac{r(q-p)}{2}}{\Gamma(p+q-1)} \qquad [|r|<\pi]$$
$$= 0 \qquad [|r|>\pi]$$
$[r \text{ is real}; \quad \operatorname{Re}(p+q)>1]$ MO 10a, ET II 299(13, 14)

6.432 $$\int_{-\infty}^{\infty} \frac{\sin(m\pi x)}{\sin(\pi x)}\frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = 0 \qquad [m \text{ is an even integer}]$$
$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \qquad [m \text{ is an odd integer}]$$
$[\operatorname{Re}(\alpha+\beta)>1]$ ET II 298(11, 12)

6.433

1. $$\int_{-\infty}^{\infty} \frac{\sin\pi x\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\sin\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\,\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$[\alpha+\delta=\beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta)>2]$ ET II 300(22)

2. $$\int_{-\infty}^{\infty} \frac{\cos\pi x\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\cos\left[\frac{\pi}{2}(\beta-\alpha)\right]}{2\,\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)}$$
$[\alpha+\delta=\beta+\gamma, \quad \operatorname{Re}(\alpha+\beta+\gamma+\delta)>2]$ ET II 301(23)

6.44 The logarithm of the gamma function*

6.441

1. $$\int_p^{p+1} \ln\Gamma(x)\,dx = \frac{1}{2}\ln 2\pi + p\ln p - p$$ FI II 784

2. $$\int_0^1 \ln\Gamma(x)\,dx = \int_0^1 \ln\Gamma(1-x)\,dx = \frac{1}{2}\ln 2\pi$$ FI II 783

3. $$\int_0^1 \ln\Gamma(x+q)\,dx = \frac{1}{2}\ln 2\pi + q\ln q - q \qquad [q \ge 0]$$ NH 89(17), ET II 304(40)

4. $$\int_0^z \ln\Gamma(x+1)\,dx = \frac{z}{2}\ln 2\pi - \frac{z(z+1)}{2} + z\ln\Gamma(z+1) - \ln G(z+1),$$
$$\text{where } G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right) \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}$$ WH

5. $$\int_0^n \ln\Gamma(\alpha+x)\,dx = \sum_{k=0}^{n-1}(a+k)\ln(a+k) - na + \frac{1}{2}n\ln(2\pi) - \frac{1}{2}n(n-1)$$
$[a \ge 0; \quad n = 1, 2, \dots]$ ET II 304(41)

6.442 $$\int_0^1 \exp(2\pi nxi)\ln\Gamma(a+x)\,dx = (2\pi ni)^{-1}\left[\ln a - \exp(-2\pi nai)\,\mathrm{Ei}(2\pi nai)\right]$$
$[a > 0; \quad n = \pm 1, \pm 2, \dots]$ ET II 304(38)

6.443

1. $$\int_0^1 \ln\Gamma(x)\sin 2\pi nx\,dx = \frac{1}{2\pi n}\left[\ln(2\pi n) + \boldsymbol{C}\right]$$ NH 203(5), ET II 304(42)

2. $$\int_0^1 \ln\Gamma(x)\sin(2n+1)\pi x\,dx = \frac{1}{(2n+1)\pi}\left[\ln\left(\frac{\pi}{2}\right) + 2\left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) + \frac{1}{2n+1}\right]$$ ET II 305(43)

3. $$\int_0^1 \ln\Gamma(x)\cos 2\pi nx\,dx = \frac{1}{4n}$$ NH 203(6), ET II 305(44)

4.[8] $$\int_0^1 \ln\Gamma(x)\cos(2n+1)\pi x\,dx = \frac{2}{\pi^2}\left[\frac{1}{(2n+1)^2}\left(\boldsymbol{C} + \ln 2\pi\right) + 2\sum_{k=2}^{\infty}\frac{\ln k}{4k^2-(2n+1)^2}\right]$$ NH 203(6)

5. $$\int_0^1 \sin(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}\left[\ln a + \cos(2\pi na)\,\mathrm{ci}(2\pi na) - \sin(2\pi na)\,\mathrm{si}(2\pi na)\right]$$
$[a > 0; \quad n = 1, 2, \dots]$ ET II 304(36)

6. $$\int_0^1 \cos(2\pi nx)\ln\Gamma(a+x)\,dx = -(2\pi n)^{-1}\left[\sin(2\pi na)\,\mathrm{ci}(2\pi na) + \cos(2\pi na)\,\mathrm{si}(2\pi na)\right]$$
$[a > 0; \quad n = 1, 2, \dots]$ ET II 304(37)

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*Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.

6.45 The incomplete gamma function

6.451

1. $$\int_0^\infty e^{-\alpha x}\,\gamma(\beta,x)\,dx = \frac{1}{\alpha}\,\Gamma(\beta)(1+\alpha)^{-\beta} \qquad [\beta > 0]$$ MI 39

2. $$\int_0^\infty e^{-\alpha x}\,\Gamma(\beta,x)\,dx = \frac{1}{\alpha}\,\Gamma(\beta)\left[1-\frac{1}{(\alpha+1)^\beta}\right] \qquad [\beta > 0]$$ MI 39

6.452

1. $$\int_0^\infty e^{-\mu x}\gamma\left(\nu, \frac{x^2}{8a^2}\right)\,dx = \frac{1}{\mu}2^{-\nu-1}\,\Gamma(2\nu)e^{(a\mu)^2}\,D_{-2\nu}(2a\mu)$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\mu > 0\right]$$ ET I 179(36)

2. $$\int_0^\infty e^{-\mu x}\gamma\left(\frac{1}{4}, \frac{x^2}{8a^2}\right)\,dx = \frac{2^{\frac{3}{4}}\sqrt{a}}{\sqrt{\mu}}e^{(a\mu)^2}\,K_{\frac{1}{4}}\left(a^2\mu^2\right) \qquad \left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}\mu > 0\right]$$ ET I 179(35)

6.453 $$\int_0^\infty e^{-\mu x}\,\Gamma\left(\nu, \frac{a}{x}\right)\,dx = 2a^{\frac{1}{2}\nu}\mu^{\frac{1}{2}\nu-1}\,K_\nu\left(2\sqrt{\mu a}\right) \qquad \left[|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re}\mu > 0\right]$$ ET I 179(32)

6.454 $$\int_0^\infty e^{-\beta x}\gamma\left(\nu, \alpha\sqrt{x}\right)\,dx = 2^{-\frac{1}{2}\nu}\alpha^\nu\beta^{-\frac{1}{2}\nu-1}\,\Gamma(\nu)\exp\left(\frac{\alpha^2}{8\beta}\right)D_{-\nu}\left(\frac{\alpha}{\sqrt{2\beta}}\right)$$
$$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > 0]$$ ET II 309(19), MI 39a

6.455

1. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\,\Gamma(\nu,\alpha x)\,dx = \frac{\alpha^\nu\,\Gamma(\mu+\nu)}{\mu(\alpha+\beta)^{\mu+\nu}}\;{}_2F_1\left(1, \mu+\nu; \mu+1; \frac{\beta}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta) > 0, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}(\mu+\nu) > 0]$$ ET II 309(16)

2. $$\int_0^\infty x^{\mu-1}e^{-\beta x}\,\gamma(\nu,\alpha x)\,dx = \frac{\alpha^\nu\,\Gamma(\mu+\nu)}{\nu(\alpha+\beta)^{\mu+\nu}}\;{}_2F_1\left(1, \mu+\nu; \nu+1; \frac{\alpha}{\alpha+\beta}\right)$$
$$[\operatorname{Re}(\alpha+\beta) > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\mu+\nu) > 0]$$ ET II 308(15)

6.456

1. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\gamma\left(\nu, \frac{1}{4x}\right)\,dx = \sqrt{\pi}\frac{\gamma\left(2\nu, \sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39a

2. $$\int_0^\infty e^{-\alpha x}(4x)^{\nu-\frac{1}{2}}\,\Gamma\left(\nu, \frac{1}{4x}\right)\,dx = \frac{\sqrt{\pi}\,\Gamma\left(2\nu, \sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39a

6.457

1. $$\int_0^\infty e^{-\alpha x}\frac{(4x)^\nu}{\sqrt{x}}\gamma\left(\nu+1, \frac{1}{4x}\right)\,dx = \sqrt{\pi}\frac{\gamma\left(2\nu+1, \sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39

2. $$\int_0^\infty e^{-\alpha x}\frac{(4x)^\nu}{\sqrt{x}}\,\Gamma\left(\nu+1, \frac{1}{4x}\right)\,dx = \sqrt{\pi}\frac{\Gamma\left(2\nu+1, \sqrt{\alpha}\right)}{\alpha^{\nu+\frac{1}{2}}}$$ MI 39

6.458 $$\int_0^\infty x^{1-2\nu} \exp\left(\alpha x^2\right) \sin(bx)\, \Gamma\left(\nu, \alpha x^2\right)\, dx = \pi^{\frac{1}{2}} 2^{-\nu} \alpha^{\nu-1}\, \Gamma\left(\tfrac{3}{2}-\nu\right) \exp\left(\frac{b^2}{8\alpha}\right) D_{2\nu-2}\left[\frac{b}{(2\alpha)^{\frac{1}{2}}}\right]$$
$$\left[|\arg\alpha| < \frac{3\pi}{2}, \quad 0 < \operatorname{Re}\nu < 1\right]$$
ET II 309(18)

6.46–6.47 The function $\psi(x)$

6.461 $$\int_1^x \psi(t)\, dt = \ln\Gamma(x)$$

6.462 $$\int_0^1 \psi(\alpha + x)\, dx = \ln\alpha \qquad [\alpha > 0]$$ ET II 305(1)

6.463 $$\int_0^\infty x^{-\alpha}\left[\boldsymbol{C} + \psi(1+x)\right] = -\pi \operatorname{cosec}(\pi\alpha)\, \zeta(\alpha) \qquad [1 < \operatorname{Re}\alpha < 2]$$ ET II 305(6)

6.464 $$\int_0^1 e^{2\pi n x i}\, \psi(\alpha + x)\, dx = e^{-2\pi n \alpha i} \operatorname{Ei}(2\pi n \alpha i) \qquad [\alpha > 0; \quad n = \pm i, \pm 2, \ldots]$$ ET II 305(2)

6.465

1.[8] $$\int_0^1 \psi(x) \sin \pi x\, dx = -\frac{2}{\pi}\left[\boldsymbol{C} + \ln 2\pi + 2\sum_{k=2}^\infty \frac{\ln k}{4k^2 - 1}\right]$$
(see **6.443** 4) NH 204

2. $$\int_0^1 \psi(x) \sin(2\pi n x)\, dx = -\frac{1}{2}\pi \qquad [n = 1, 2, \ldots]$$ ET II 305(3)

6.466 $$\int_0^\infty \left[\psi(\alpha + ix) - \psi(\alpha - ix)\right] \sin xy\, dx = i\pi e^{-\alpha y}\left(1 - e^{-y}\right)^{-1}$$
$$[\alpha > 0, \quad y > 0]$$ ET I 96(1)

6.467

1. $$\int_0^1 \sin(2\pi n x)\, \psi(\alpha + x)\, dx = \sin(2\pi n\alpha) \operatorname{ci}(2\pi n\alpha) + \cos(2\pi n\alpha) \operatorname{si}(2\pi n\alpha)$$
$$[\alpha \geq 0; \quad n = 1, 2, \ldots]$$ ET II 305(4)

2. $$\int_0^1 \cos(2\pi n x)\, \psi(\alpha + x)\, dx = \sin(2\pi n\alpha) \operatorname{si}(2\pi n\alpha) - \cos(2\pi n\alpha) \operatorname{ci}(2\pi n\alpha)$$
$$[\alpha > 0; \quad n = 1, 2, \ldots]$$ ET II 305(5)

6.468 $$\int_0^1 \psi(x) \sin^2 \pi x\, dx = -\frac{1}{2}\left[\boldsymbol{C} + \ln(2\pi)\right]$$ NH 204

6.469

1. $$\int_0^1 \psi(x) \sin \pi x \cos \pi x\, dx = -\frac{\pi}{4}$$ NH 204

2.[8] $$\int_0^1 \psi(x) \sin \pi x \sin(n\pi x)\, dx = \frac{n}{1 - n^2} \qquad [n \text{ is even}]$$
$$= \frac{1}{2} \ln \frac{n-1}{n+1} \qquad [n > 1 \text{ is odd}]$$
NH 204(8)a

6.471

1. $$\int_0^\infty x^{-\alpha}\left[\ln x - \psi(1+x)\right]\,dx = \pi\operatorname{cosec}(\pi\alpha)\,\zeta(\alpha) \qquad [0 < \operatorname{Re}\alpha < 1]$$ ET II 306(7)

2. $$\int_0^\infty x^{-\alpha}\left[\ln(1+x) - \psi(1+x)\right]\,dx = \pi\operatorname{cosec}(\pi\alpha)\left[\zeta(\alpha) - (\alpha-1)^{-1}\right]$$
$$[0 < \operatorname{Re}\alpha < 1]$$ ET II 306(8)

3. $$\int_0^\infty \left[\psi(x+1) - \ln x\right]\cos(2\pi xy)\,dx = \frac{1}{2}\left[\psi(y+1) - \ln y\right]$$ ET II 306(12)

6.472

1. $$\int_0^\infty x^{-\alpha}\left[(1+x)^{-1} - \psi'(1+x)\right]\,dx = -\pi\alpha\operatorname{cosec}(\pi\alpha)\left[\zeta(1+\alpha) - \alpha^{-1}\right]$$
$$[|\operatorname{Re}\alpha| < 1]$$ ET II 306(9)

2. $$\int_0^\infty x^{-\alpha}\left[x^{-1} - \psi'(1+x)\right]\,dx = -\pi\alpha\operatorname{cosec}(\pi\alpha)\,\zeta(1+\alpha)$$
$$[-2 < \operatorname{Re}\alpha < 0]$$ ET II 306(10)

6.473 $$\int_0^\infty x^{-\alpha}\psi^{(n)}(1+x)\,dx = (-1)^{n-1}\frac{\pi\,\Gamma(\alpha+n)}{\Gamma(\alpha)\sin\pi\alpha}\,\zeta(\alpha+n)$$
$$[n = 1, 2, \ldots; \quad 0 < \operatorname{Re}\alpha < 1]$$
ET II 306(11)

6.5–6.7 Bessel Functions

6.51 Bessel functions

6.511

1. $$\int_0^\infty J_\nu(bx)\,dx = \frac{1}{b} \qquad [\operatorname{Re}\nu > -1, \quad b > 0]$$ ET II 22(3)

2. $$\int_0^\infty Y_\nu(bx)\,dx = -\frac{1}{b}\tan\left(\frac{\nu\pi}{2}\right) \qquad [|\operatorname{Re}\nu| < 1, \quad b > 0]$$
WA 432(7), ET II 96(1)

3. $$\int_0^a J_\nu(x)\,dx = 2\sum_{k=0}^\infty J_{\nu+2k+1}(a) \qquad [\operatorname{Re}\nu > -1]$$ ET II 333(1)

4. $$\int_0^a J_{\frac{1}{2}}(t)\,dt = 2\,S\left(\sqrt{a}\right)$$ WA 599(4)

5. $$\int_0^a J_{-\frac{1}{2}}(t)\,dt = 2\,C\left(\sqrt{a}\right)$$ WA 599(3)

6. $$\int_0^a J_0(x)\,dx = a\,J_0(a) + \frac{\pi a}{2}\left[J_1(a)\,\mathbf{H}_0(a) - J_0(a)\,\mathbf{H}_1(a)\right]$$
$$[a > 0]$$ ET II 7(2)

7. $$\int_0^a J_1(x)\,dx = 1 - J_0(a) \qquad [a>0]$$ ET II 18(1)

8. $$\int_a^\infty J_0(x)\,dx = 1 - a\,J_0(a) + \frac{\pi a}{2}\left[J_0(a)\,\mathbf{H}_1(a) - J_1(a)\,\mathbf{H}_0(a)\right] \qquad [a>0]$$ ET II 7(3)

9. $$\int_a^\infty J_1(x)\,dx = J_0(a) \qquad [a>0]$$ ET II 18(2)

10. $$\int_a^b Y_\nu(x)\,dx = 2\sum_{n=0}^\infty \left[Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)\right]$$ ET II 339(46)

11. $$\int_0^a I_\nu(x)\,dx = 2\sum_{n=0}^\infty (-1)^n\,I_{\nu+2n+1}(a) \qquad [\operatorname{Re}\nu > -1]$$ ET II 364(1)

12.* $$\int_0^\infty K_0(ax) = \frac{\pi}{2a} \qquad [a>0]$$

13.* $$\int_0^\infty K_0^2(ax) = \frac{\pi^2}{4a} \qquad [a>0]$$

6.512

1.[11] $$\int_0^\infty J_\mu(ax)\,J_\nu(bx)\,dx = b^\nu a^{-\nu-1}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)}\,F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

[$a>0$, $b>0$, $\operatorname{Re}(\mu+\nu)>-1$, $b<a$. For $a>b$, the positions of μ and ν should be reversed.] ET II 48(6)

2.[7] $$\int_0^\infty J_{\nu+n}(\alpha t)\,J_{\nu-n-1}(\beta t)\,dt = \frac{\beta^{\nu-n-1}\,\Gamma(\nu)}{\alpha^{\nu-n} n!\,\Gamma(\nu-n)}\,F\left(\nu, -n; \nu-n; \frac{\beta^2}{\alpha^2}\right) \qquad [0<\beta<\alpha]$$
$$= (-1)^n\frac{1}{2\alpha} \qquad [0<\beta=\alpha]$$
$$= 0 \qquad [0<\alpha<\beta]$$
[$\operatorname{Re}(\nu)>0$] MO 50

3.[8] $$\int_0^\infty J_\nu(\alpha x)\,J_{\nu-1}(\beta x)\,dx = \frac{\beta^{\nu-1}}{\alpha^\nu} \qquad [\beta<\alpha]$$
$$= \frac{1}{2\beta} \qquad [\beta=\alpha]$$
$$= 0 \qquad [\beta>\alpha]$$
[$\operatorname{Re}\nu>0$] WA 444(8), KU (40)a

4. $$\int_0^\infty J_{\nu+2n+1}(ax)\,J_\nu(bx)\,dx = b^\nu a^{-\nu-1}\,P_n^{(\nu,0)}\left(1-\frac{2b^2}{a^2}\right) \qquad [\operatorname{Re}\nu>-1-n,\quad 0<b<a]$$
$$= 0 \qquad [\operatorname{Re}\nu>-1-n,\quad 0<a<b]$$
ET II 47(5)

5. $$\int_0^\infty J_{\nu+n}(ax)\, Y_{\nu-n}(ax)\, dx = (-1)^{n+1} \frac{1}{2a} \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad a > 0, \quad n = 0, 1, 2, \ldots\right]$$ ET II 347(57)

6. $$\int_0^\infty J_1(bx)\, Y_0(ax)\, dx = -\frac{b^{-1}}{\pi} \ln\left(1 - \frac{b^2}{a^2}\right) \qquad [0 < b < a]$$ ET II 21(31)

7. $$\int_0^a J_\nu(x)\, J_{\nu+1}(x)\, dx = \sum_{n=0}^{\infty} \left[J_{\nu+n+1}(a)\right]^2 \qquad [\operatorname{Re}\nu > -1]$$ ET II 338(37)

8.[9] $$\int_0^\infty k\, J_n(ka)\, J_n(kb)\, dk = \frac{1}{a}\, \delta(b-a) \qquad [n = 0, 1, \ldots]$$ JAC 110

9.* $$\int_0^\infty K_0(ax)\, J_1(bx) = \frac{1}{2b} \ln\left(1 + \frac{b^2}{a^2}\right) \qquad [a > 0, \quad b > 0]$$

10.* $$\int_0^\infty K_0(ax)\, I_1(bx) = -\frac{1}{2b} \ln\left(1 - \frac{b^2}{a^2}\right) \qquad [a > 0, \quad b > 0]$$

6.513

1. $$\int_0^\infty \left[J_\mu(ax)\right]^2 J_\nu(bx)\, dx = a^{2\mu} b^{-2\mu-1} \frac{\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{\left[\Gamma(\mu+1)\right]^2 \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \times \left[F\left(\frac{1-\nu+2\mu}{2}, \frac{1+\nu+2\mu}{2}; \mu+1; \frac{1-\sqrt{1-\frac{4a^2}{b^2}}}{2}\right)\right]^2$$
$$[\operatorname{Re}\nu + \operatorname{Re}2\mu > -1, \quad 0 < 2a < b]$$ ET II 52(33)

2. $$\int_0^\infty \left[J_\mu(ax)\right]^2 K_\nu(bx)\, dx = \frac{b^{-1}}{2} \Gamma\left(\frac{2\mu+\nu+1}{2}\right) \Gamma\left(\frac{2\mu-\nu+1}{2}\right) \left[P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2$$
$$[2\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1, \quad \operatorname{Re}b > 2|\operatorname{Im}a|]$$ ET II 138(18)

3. $$\int_0^\infty I_\mu(ax)\, K_\mu(ax)\, J_\nu(bx)\, dx = \frac{e^{\mu\pi i}\, \Gamma\left(\frac{\nu+2\mu+1}{2}\right)}{b\, \Gamma\left(\frac{\nu-2\mu+1}{2}\right)} P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)$$
$$[\operatorname{Re}a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu+2\mu) > -1]$$ ET II 65(20)

4. $$\int_0^\infty J_\mu(ax)\, J_{-\mu}(ax)\, K_\nu(bx)\, dx = \frac{\pi}{2b} \sec\left(\frac{\nu\pi}{2}\right) P^{\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) P^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)$$
$$[|\operatorname{Re}\nu| < 1, \quad \operatorname{Re}b > 2|\operatorname{Im}a|]$$ ET II 138(21)

5. $$\int_0^\infty [K_\mu(ax)]^2 J_\nu(bx)\,dx = \frac{e^{2\mu\pi i}\,\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b\,\Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \left[Q^{-\mu}_{\frac{1}{2}\nu-\frac{1}{2}}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\left(\tfrac{1}{2}\nu \pm \mu\right) > -\tfrac{1}{2}\right]$ ET II 66(28)

6. $$\int_0^z J_\mu(x)\,J_\nu(z-x)\,dx = 2\sum_{k=0}^\infty (-1)^k J_{\mu+\nu+2k+1}(z)$$ $[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1]$ WA 414(2)

7. $$\int_0^z J_\mu(x)\,J_{-\mu}(z-x)\,dx = \sin z$$ $[-1 < \operatorname{Re}\mu < 1]$ WA 415(4)

8. $$\int_0^z J_\mu(x)\,J_{1-\mu}(z-x)\,dx = J_0(z) - \cos(z)$$ $[-1 < \operatorname{Re}\mu < 2]$ WA 415(4)

9.* $$\int_0^\infty J_0^2(ax)\,J_1(bx) = \frac{1}{b}$$ $[b > 2a > 0]$
$$= \frac{2}{\pi b}\arcsin\left(\frac{b}{2a}\right)$$ $[2a > b > 0]$

6.514

1. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu(bx)\,dx = b^{-1} J_{2\nu}\left(2\sqrt{ab}\right)$$ $\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 57(9)

2. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\,dx = b^{-1}\left[Y_{2\nu}\left(2\sqrt{ab}\right) + \frac{2}{\pi} K_{2\nu}\left(\sqrt{2ab}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{3}{2}\right]$ ET II 110(12)

3. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = b^{-1}e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left[2e^{\frac{1}{4}i\pi}\sqrt{ab}\right] + b^{-1}e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left[2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right]$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \tfrac{5}{2}\right]$ ET II 141(31)

4. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu(bx)\,dx = -\frac{2b^{-1}}{\pi}\left[K_{2\nu}\left(2\sqrt{ab}\right) - \frac{\pi}{2} Y_{2\nu}\left(2\sqrt{ab}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 62(37)a

5. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\,dx = -b^{-1} J_{2\nu}\left(2\sqrt{ab}\right)$$ $\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 110(14)

6. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = -b^{-1}e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) - b^{-1}e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right)$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \tfrac{5}{2}\right]$ ET II 143(37)

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu(bx)\,dx = -2b^{-1}\left[\sin\left(\frac{3\nu\pi}{2}\right)\mathrm{ker}_{2\nu}\left(2\sqrt{ab}\right) + \cos\left(\frac{3\nu\pi}{2}\right)\mathrm{kei}_{2\nu}\left(2\sqrt{ab}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 113(28)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \pi b^{-1} K_{2\nu}\left(2\sqrt{ab}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$ ET II 146(54)

6.515

1. $$\int_0^\infty J_\mu\left(\frac{a}{x}\right) Y_\mu\left(\frac{a}{x}\right) K_0(bx)\,dx = -2b^{-1} J_{2\mu}\left(2\sqrt{ab}\right) K_{2\mu}\left(2\sqrt{ab}\right)$$
$[a > 0, \quad \operatorname{Re} b > 0]$ ET II 143(42)

2. $$\int_0^\infty \left[K_\mu\left(\frac{a}{x}\right)\right]^2 K_0(bx)\,dx = 2\pi b^{-1} K_{2\mu}\left(2e^{\frac{1}{4}\pi i}\sqrt{ab}\right) K_{2\mu}\left(2e^{-\frac{1}{4}\pi i}\sqrt{ab}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$ ET II 147(59)

3. $$\int_0^\infty H_\mu^{(1)}\left(\frac{a^2}{x}\right) H_\mu^{(2)}\left(\frac{a^2}{x}\right) J_0(bx)\,dx = 16\pi^{-2}b^{-1}\cos\mu\pi\, K_{2\mu}\left(2e^{\pi i/4}a\sqrt{b}\right) K_{2\mu}\left(2e^{-\pi i/4}a\sqrt{b}\right)$$
$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad |\operatorname{Re}\mu| < \tfrac{1}{4}\right]$ ET II 17(36)

6.516

1. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right)$$
$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 58(16)

2. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx = -b^{-1}\,\mathbf{H}_\nu\left(\frac{a^2}{4b}\right)$$
$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 111(18)

3. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi}{2}b^{-1}\left[I_\nu\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 144(45)

4.[10] $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{1}{b} J_\nu\left(\frac{a^2}{4b}\right)\cot(2\pi\nu) - \frac{1}{2b} J_{-\nu}\left(\frac{a^2}{4b}\right)\operatorname{cosec}(2\pi\nu) - \frac{2^{3\nu-3}a^{2-2\nu}b^{\nu-2}}{\pi^{3/2}}\Gamma\left(\nu - \tfrac{1}{2}\right)\ {}_1F_2\left(1;\ \frac{3}{2}, \frac{3}{2} - \nu;\ \frac{a^4}{64b^2}\right)$$
$[a > 0, \quad b > 0]$ MC

5. $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx = \frac{b^{-1}}{2}\left[\sec(\nu\pi)\, J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}(\nu\pi)\,\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - 2\cot(2\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 111(19)

6. $$\int_0^\infty Y_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[\operatorname{cosec}(2\nu\pi)\,\mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \cot(2\nu\pi)\,\mathbf{L}_\nu\left(\frac{a^2}{4b}\right) - \tan(\nu\pi)\,I_\nu\left(\frac{a^2}{4b}\right) - \frac{\sec(\nu\pi)}{\pi}K_\nu\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 144(46)

7. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{1}{4}\pi b^{-1}\sec(\nu\pi)\left[\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) - Y_{-\nu}\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 70(22)

8. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) Y_\nu(bx)\,dx = -\frac{1}{4}\pi b^{-1}\left[\sec(\nu\pi)\,J_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{cosec}(\nu\pi)\,\mathbf{H}_{-\nu}\left(\frac{a^2}{4b}\right) + 2\operatorname{cosec}(2\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 114(34)

9. $$\int_0^\infty K_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{4\cos(\nu\pi)}\left\{K_\nu\left(\frac{a^2}{4b}\right) + \frac{\pi}{2\sin(\nu\pi)}\left[\mathbf{L}_{-\nu}\left(\frac{a^2}{4b}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]\right\}$$
$\left[\operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 147(63)

10. $$\int_0^\infty I_{2\nu}\left(a\sqrt{x}\right) K_\nu(bx)\,dx = \frac{\pi b^{-1}}{2}\left[I_\nu\left(\frac{a^2}{4b}\right) + \mathbf{L}_\nu\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 147(60)

6.517 $$\int_0^z J_0\left(\sqrt{z^2 - x^2}\right) dx = \sin z$$ MO 48

6.518 $$\int_0^\infty K_{2\nu}\left(2z\sinh x\right)\,dx = \frac{\pi^2}{8\cos\nu\pi}\left(J_\nu^2(z) + N_\nu^2(z)\right)$$ $\left[\operatorname{Re} z > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$ MO 45

6.519

1. $$\int_0^{\pi/2} J_{2\nu}\left(2z\cos x\right)\,dx = \frac{\pi}{2}J_\nu^2(z)$$ $\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ WH

2. $$\int_0^{\pi/2} J_{2\nu}\left(2z\sin x\right)\,dx = \frac{\pi}{2}J_\nu^2(z)$$ $\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ WA 42(1)a

6.52 Bessel functions combined with x and x^2

6.521

1. $$\int_0^1 x\,J_\nu(\alpha x)\,J_\nu(\beta x)\,dx = \frac{\beta\,J_{\nu-1}(\beta)\,J_\nu(\alpha) - \alpha\,J_{\nu-1}(\alpha)\,J_\nu(\beta)}{\alpha^2 - \beta^2}$$ $[\alpha \neq \beta, \quad \nu > -1]$
$$= \frac{\alpha\,J_\nu(\beta)\,J_\nu'(\alpha) - \beta\,J_\nu(\alpha)\,J_\nu'(\beta)}{\beta^2 - \alpha^2}$$ $[\alpha \neq \beta, \quad \nu > -1]$
WH

2.[10] $$\int_0^\infty x\,K_\nu(ax)\,J_\nu(bx)\,dx = \frac{b^\nu}{a^\nu\left(b^2+a^2\right)}$$ $[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 63(2)

3. $$\int_0^\infty x\,K_\nu(ax)\,K_\nu(bx)\,dx = \frac{\pi(ab)^{-\nu}\left(a^{2\nu}-b^{2\nu}\right)}{2\sin(\nu\pi)\left(a^2-b^2\right)}$$ $[|\operatorname{Re}\nu| < 1, \quad \operatorname{Re}(a+b) > 0]$ ET II 145(48)

4. $$\int_0^a x\,J_\nu(\lambda x)\,K_\nu(\mu x)\,dx = \left(\mu^2+\lambda^2\right)^{-1}\left[\left(\frac{\lambda}{\mu}\right)^\nu + \lambda a\,J_{\nu+1}(\lambda a)\,K_\nu(\mu a) - \mu a\,J_\nu(\lambda a)\,K_{\nu+1}(\mu a)\right]$$ $[\operatorname{Re}\nu > -1]$ ET II 367(26)

5.* $$\int_0^\infty x\,K_1(ax) = \frac{\pi}{2a^2}$$ $[a > 0]$

6.* $$\int_0^\infty x\,K_0^2(ax) = \frac{1}{2a^2}$$ $[a > 0]$

7.* $$\int_0^\infty x\,K_1(ax)\,J_1(bx) = \frac{b}{a\left(a^2+b^2\right)}$$ $[a > 0, \quad b > 0]$

8.* $$\int_0^\infty x\,K_0(ax)\,I_0(bx) = \frac{1}{a^2-b^2}$$ $[a > b > 0]$

9.* $$\int_0^\infty x\,K_1(ax)\,I_1(bx) = \frac{b}{a\left(a^2-b^2\right)}$$ $[a > b > 0]$

10.* $$\int_0^\infty x^2\,K_0(ax) = \frac{\pi}{2a^3}$$ $[a > 0]$

11.* $$\int_0^\infty x^2\,K_1(ax) = \frac{2}{a^3}$$ $[a > 0]$

12.* $$\int_0^\infty x^2\,K_0(ax)\,J_1(bx) = \frac{2b}{\left(a^2+b^2\right)^2}$$ $[a > 0, \quad b > 0]$

13.* $$\int_0^\infty x^2\,K_1(ax)\,J_0(bx) = \frac{2a}{\left(a^2+b^2\right)^2}$$ $[a > b > 0]$

14.* $$\int_0^\infty x^2\,K_0(ax)\,I_1(bx) = \frac{2b}{\left(a^2-b^2\right)^2}$$ $[a > b > 0]$

15.* $$\int_0^\infty x^2\,K_1(ax)\,I_0(bx) = \frac{2a}{\left(a^2-b^2\right)^2}$$ $[a > b > 0]$

6.522 **Notation**: $\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right]$, $\ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$

1.[8] $$\int_0^\infty x\left[J_\mu(ax)\right]^2 K_\nu(bx)\,dx = \Gamma\left(\mu+\tfrac{1}{2}\nu+1\right)\Gamma\left(\mu-\tfrac{1}{2}\nu+1\right)b^{-2} \times\left(1+4a^2b^{-2}\right)^{-\frac{1}{2}} P_{\frac{1}{2}\nu}^{-\mu}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right] P_{-\frac{1}{2}\nu}^{-\mu}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]$$ $[\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re}\mu > |\operatorname{Re}\nu| - 2]$ ET II 138(19)

2. $$\int_0^\infty x\left[K_\mu(ax)\right]^2 J_\nu(bx)\,dx = \frac{2e^{2\mu\pi i}\,\Gamma\left(1+\frac{1}{2}\nu+\mu\right)}{b\left(4a^2+b^2\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\nu-\mu\right)} \times Q^{-\mu}_{\frac{1}{2}\nu}\left(\sqrt{(1+4a^2b^{-2})}\right) Q^{-\mu}_{\frac{1}{2}\nu-1}\left(\sqrt{(1+4a^2b^{-2})}\right)$$

$\left[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\left(\frac{1}{2}\nu\pm\mu\right)>-1\right]$ ET II 66(27)a

3.[11] $$\int_0^\infty x\,K_0(ax)\,J_\nu(bx)\,J_\nu(cx)\,dx = r_1^{-1}r_2^{-1}\left(r_2-r_1\right)^\nu\left(r_2-r_1\right)^{-\nu} = \frac{\ell_1^\nu}{\ell_2^\nu\left(\ell_2^2-\ell_1^2\right)},$$

$\left[r_1=\sqrt{a^2+(b-c)^2}, \quad r_2=\sqrt{a^2+(b+c)^2}, \quad c>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re} a>|\operatorname{Im} b|\right]$

ET II 63(6)

4.[10] $$\int_0^\infty x\,I_0(ax)\,K_0(bx)\,J_0(cx)\,dx = \left(a^4+b^4+c^4-2a^2b^2+2a^2c^2+2b^2c^2\right)^{-\frac{1}{2}}$$

$[\operatorname{Re} b>\operatorname{Re} a, \quad c>0]$ ET II 16(27)

alternatively, with a and c interchanged

$$\int_0^\infty x\,I_0(cx)\,K_0(bx)\,J_0(ax)\,dx = \frac{1}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} b>\operatorname{Re} c, \quad a>0]$$

5.[10] $$\int_0^\infty x\,J_0(ax)\,K_0(bx)\,J_0(cx)\,dx = \left(a^4+b^4+c^4-2a^2c^2+2a^2b^2+2b^2c^2\right)^{-\frac{1}{2}}$$

$[\operatorname{Re} b>|\operatorname{Im} a|, \quad c>0]$ ET II 15(25)

alternatively, with a and b interchanged

$$\int_0^\infty x\,J_0(bx)\,K_0(ax)\,J_0(cx)\,dx = \frac{1}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} a>|\operatorname{Im} b|, \quad c>0]$$

6. $$\int_0^\infty x\,J_0(ax)\,Y_0(ax)\,J_0(bx)\,dx = 0 \qquad [0<b<2a]$$

$$= -2\pi^{-1}b^{-1}\left[b^2-4a^2\right]^{-\frac{1}{2}} \qquad [0<2a<b<\infty]$$

ET II 15(21)

7. $$\int_0^\infty x\,J_\mu(ax)\,J_{\mu+1}(ax)\,K_\nu(bx)\,dx = \Gamma\left(\mu+\frac{3+\nu}{2}\right)\Gamma\left(\mu+\frac{3-\nu}{2}\right)b^{-2}\left(1+4a^2b^{-2}\right)^{-\frac{1}{2}} \times P^{\frac{1}{2}\nu-\frac{1}{2}}_{-\mu}\left[\sqrt{1+4a^2b^{-2}}\right]P^{\frac{1}{2}\nu-\frac{1}{2}}_{-\mu-1}\left[\sqrt{1+4a^2b^{-2}}\right]$$

$[\operatorname{Re} b>2|\operatorname{Im} a|, \quad 2\operatorname{Re}\mu>|\operatorname{Re}\nu|-3]$ ET II 138(20)

8. $$\int_0^\infty x\,K_{\mu-\frac{1}{2}}(ax)\,K_{\mu+\frac{1}{2}}(ax)\,J_\nu(bx)\,dx = -\frac{2e^{2\mu\pi i}\,\Gamma\left(\frac{1}{2}\nu+\mu+1\right)}{b\,\Gamma\left(\frac{1}{2}\nu-\mu\right)\left(b^2+4a^2\right)^{\frac{1}{2}}}\,Q^{-\mu+\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]Q^{-\mu-\frac{1}{2}}_{\frac{1}{2}\nu-\frac{1}{2}}\left[\left(1+4a^2b^{-2}\right)^{\frac{1}{2}}\right]$$

$\left[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1, \quad |\operatorname{Re}\mu|<1+\frac{1}{2}\operatorname{Re}\nu\right]$ ET II 67(29)a

9.[8] $$\int_0^\infty x\,I_{\frac{1}{2}\nu}(ax)\,K_{\frac{1}{2}\nu}(ax)\,J_\nu(bx)\,dx = b^{-1}\left(b^2+4a^2\right)^{-\frac{1}{2}}$$

$[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1]$

ET II 65(16)

10. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}(ax)\, Y_{\frac{1}{2}\nu}(ax)\, J_\nu(bx)\, dx$$
$$= 0 \qquad [a > 0, \quad \operatorname{Re}\nu > -1, \quad 0 < b < 2a]$$
$$= -2\pi^{-1}b^{-1}\left(b^2 - 4a^2\right)^{-\frac{1}{2}} \qquad [a > 0, \quad \operatorname{Re}\nu > -1, \quad 2a < b < \infty]$$
ET II 55(48)

11.[8] $$\int_0^\infty x\, J_{\frac{1}{2}(\nu+n)}(ax)\, J_{\frac{1}{2}(\nu-n)}(ax)\, J_\nu(bx)\, dx$$
$$= 2\pi^{-1}b^{-1}\left(4a^2 - b^2\right)^{-\frac{1}{2}}\, T_n\left(\frac{b}{2a}\right) \qquad [a > 0, \quad \operatorname{Re}\nu > -1, \quad 0 < b < 2a]$$
$$= 0 \qquad [a > 0, \quad \operatorname{Re}\nu > -1, \quad 2a < b]$$
ET II 52(32)

12. $$\int_0^\infty x\, I_{\frac{1}{2}(\nu-\mu)}(ax)\, K_{\frac{1}{2}(\nu+\mu)}(ax)\, J_\nu(bx)\, dx = 2^{-\mu}a^{-\mu}b^{-1}\left(b^2 + 4a^2\right)^{-\frac{1}{2}}\left[b + \left(b^2 + 4a^2\right)^{\frac{1}{2}}\right]^\mu$$
$$[b > 0, \quad \operatorname{Re}a > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu - \mu) > -2]$$ ET II 66(23)

13.[8] $$\int_0^\infty x\, J_\mu\left(xa\sin\varphi\right) K_{\nu-\mu}\left(ax\cos\varphi\cos\psi\right) J_\nu\left(xa\sin\psi\right)\, dx = \frac{(\sin\varphi)^\mu\,(\sin\psi)^\nu\,(\cos\varphi)^{\nu-\mu}\,(\cos\psi)^{\mu-\nu}}{a^2\left(1 - \sin^2\varphi\sin^2\psi\right)}$$
$$\left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1\right]$$ ET II 64(10)

14.[8] $$\int_0^\infty x\, J_\mu\left(xa\sin\varphi\cos\psi\right) J_{\nu-\mu}(ax)\, J_\nu\left(xa\cos\varphi\sin\psi\right)\, dx$$
$$= -2\pi^{-1}a^{-2}\sin(\mu\pi)\,(\sin\varphi)^\mu\,(\sin\psi)^\nu\,(\cos\varphi)^{-\nu}\,(\cos\psi)^{-\mu}\left[\cos(\varphi+\psi)\cos(\varphi-\psi)\right]^{-1}$$
$$\left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \tfrac{1}{2}\pi, \quad \operatorname{Re}\nu > -1\right]$$ ET II 54(39)

15.[10] $$\int_0^\infty x^{\nu+1}\, J_\nu(bx)\, K_\nu(ax)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\,\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\left(\ell_2^2 - \ell_1^2\right)^{2\nu+1}}$$
$$[\operatorname{Re}a > |\operatorname{Im}b|, \quad c > 0]$$

16.[10] $$\int_0^\infty x^{\nu+1}\, I_\nu(cx)\, K_\nu(bx)\, J_\nu(ax)\, dx = \frac{2^{3\nu}(abc)^\nu\,\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\left(\ell_2^2 - \ell_1^2\right)^{2\nu+1}}$$
$$[\operatorname{Re}b > |\operatorname{Im}a| + |\operatorname{Im}c|]$$

17.[11] $$\int_0^\infty t^{\nu-\mu-\rho+1}\, J_\mu(ct)\, J_\nu(bt)\, K_\rho(at)\, dt$$
$$= \frac{2^{1+\nu-\mu-\rho}}{c^\mu b^\nu a^\rho\,\Gamma\left(\mu - \nu + \rho\right)}\int_0^{\ell_1}\frac{x^{1+2\nu-2\rho}\left[\left(\ell_1^2 - x^2\right)\left(\ell_2^2 - x^2\right)\right]^{\mu-\nu+\rho-1}}{\left(b^2 - x^2\right)^{\mu-\nu}}\, dx$$
$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2}\right]$$
$$[\operatorname{Re}a > |\operatorname{Im}b|, \quad c > 0]$$

18.[11] $$\int_0^\infty t^{\mu-\nu+\rho+1} J_\mu(ct)\, J_\nu(bt)\, K_\rho(at)\, dt = \frac{2^{1+\mu-\nu+\rho} a^\rho}{c^\mu b^\nu\, \Gamma(\nu-\mu-\rho)} \int_0^{\ell_1} \frac{x^{1+2\mu+2\rho}\left[\left(\ell_1^2-x^2\right)\left(\ell_2^2-x^2\right)\right]^{\nu-\mu-\rho-1}}{\left(c^2-x^2\right)^{\nu-\mu}}\, dx$$

$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$

$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$

6.523 $$\int_0^\infty x\left[2\pi^{-1} K_0(ax) - Y_0(ax)\right] K_0(bx)\, dx = 2\pi^{-1}\left[\left(a^2+b^2\right)^{-1} + \left(b^2-a^2\right)^{-1}\right] \ln\frac{b}{a}$$

$[\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a+b) > 0]$ ET II 145(50)

6.524

1. $$\int_0^\infty x\, J_\nu^2(ax)\, J_\nu(bx)\, Y_\nu(bx)\, dx = 0 \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
$$= -(2\pi ab)^{-1} \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 352(14)

2. $$\int_0^\infty x\left[J_0(ax)\, K_0(bx)\right]^2\, dx = \frac{\pi}{8ab} - \frac{1}{4ab}\arcsin\left(\frac{b^2-a^2}{b^2+a^2}\right)$$
$[a > 0, \quad b > 0]$ ET II 373(9)

6.525 **Notation**: $\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$

1.[10] $$\int_0^\infty x^2 J_1(ax)\, K_0(bx)\, J_0(cx)\, dx = 2a\left(a^2+b^2-c^2\right)\left[\left(a^2+b^2+c^2\right)^2 - 4a^2c^2\right]^{-\frac{3}{2}}$$
$[c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0]$
ET II 15(26)

alternatively, with a and b interchanged
$$\int_0^\infty x^2 J_1(bx)\, K_0(ax)\, J_0(cx)\, dx = \frac{2b\left(a^2+b^2-c^2\right)}{\left(\ell_2^2-\ell_1^2\right)^3} \qquad [\operatorname{Re} a > |\operatorname{Im} b|, \quad \operatorname{Re} b > 0, \quad c > 0]$$

2.[10] $$\int_0^\infty x^2 I_0(ax)\, K_1(bx)\, J_0(cx)\, dx = 2b\left(b^2+c^2-a^2\right)\left[\left(a^2+b^2+c^2\right)^2 - 4a^2b^2\right]^{-\frac{3}{2}}$$
$[\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0]$ ET II 16(28)

3.[10] $$\int_0^\infty x^2 I_0(cx)\, K_0(bx)\, J_0(ax)\, dx = \frac{2b\left(a^2+b^2-c^2\right)}{\left(\ell_2^2-\ell_1^2\right)^3} \qquad [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$$

6.526

1. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = (2a)^{-1} J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 56(1)

2. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right) Y_\nu(bx)\, dx = (4a)^{-1}\left[Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \sec\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 109(9)

3. $$\int_0^\infty x\, J_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{8a\cos\left(\frac{\nu\pi}{2}\right)}\left[\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - Y_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 140(27)

4. $$\int_0^\infty x\, Y_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = -(2a)^{-1}\, \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 61(35)

5. $$\int_0^\infty x\, Y_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{4a\sin(\nu\pi)}\left[\cos\left(\frac{\nu\pi}{2}\right)\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right) J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 141(28)

6. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\, dx = \frac{\pi}{4a}\left[I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 68(9)

7. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right) Y_\nu(bx)\, dx = \frac{\pi}{4a}\left[\operatorname{cosec}(\nu\pi)\, \mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \cot(\nu\pi)\, \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \tan\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \frac{1}{\pi}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 112(25)

8. $$\int_0^\infty x\, K_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\, dx = \frac{\pi}{8a}\left\{\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi\operatorname{cosec}(\nu\pi)\left[\mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)\right]\right\}$$
$[\operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 146(52)

6.527

1. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, J_{\nu-\frac{1}{2}}\left(x^2\right) dx = \frac{1}{2}a\, J_{\nu+\frac{1}{2}}\left(a^2\right)$$ $\left[a > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 355(33)

2. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, J_{\nu+\frac{1}{2}}\left(x^2\right) dx = \frac{1}{2}a\, J_{\nu-\frac{1}{2}}\left(a^2\right)$$ $[a > 0, \quad \operatorname{Re}\nu > -2]$ ET II 355(35)

3. $$\int_0^\infty x^2\, J_{2\nu}(2ax)\, Y_{\nu+\frac{1}{2}}\left(x^2\right) dx = -\frac{1}{2}a\, \mathbf{H}_{\nu-\frac{1}{2}}\left(a^2\right)$$ $[a > 0, \quad \operatorname{Re}\nu > -2]$ ET II 355(36)

6.528 $$\int_0^\infty x\,K_{\frac14\nu}\left(\frac{x^2}{4}\right)I_{\frac14\nu}\left(\frac{x^2}{4}\right)J_\nu(bx)\,dx = K_{\frac14\nu}\left(\frac{x^2}{4}\right)I_{\frac14\nu}\left(\frac{b^2}{4}\right)$$
$[b>0, \quad \nu>-1]$ MO 183a

6.529

1. $$\int_0^\infty x\,J_\nu\left(2\sqrt{ax}\right)K_\nu\left(2\sqrt{ax}\right)J_\nu(bx)\,dx = \frac{1}{2}b^{-2}e^{-\frac{2a}{b}}$$ $[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}\nu>-1]$ ET II 70(23)

2. $$\int_0^a x\,J_\lambda(2a)\,I_\lambda(2x)\,J_\mu\left(2\sqrt{a^2-x^2}\right)I_\mu\left(2\sqrt{a^2-x^2}\right)dx$$
$$= \frac{a^{2\lambda+2\mu+2}}{2\,\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma(\lambda+\mu+2)} \times {}_1F_4\left(\frac{\lambda+\mu+1}{2};\lambda+1,\mu+1,\lambda+\mu+1,\frac{\lambda+\mu+3}{2};-a^4\right)$$
$[\operatorname{Re}\lambda>-1, \quad \operatorname{Re}\mu>-1]$ ET II 376(31)

6.53–6.54 Combinations of Bessel functions and rational functions

6.531

1.[10] $$\int_0^\infty \frac{Y_\nu(bx)}{x+a}\,dx$$
$$= -\pi\,J_\nu(ab)\cot(\pi\nu)\operatorname{cosec}(\pi\nu) - \pi\,J_{-\nu}(ab)\operatorname{cosec}^2(\pi\nu) + \frac{1}{\nu}\cot\frac{\pi\nu}{2}\;{}_1F_2\left(1;\ \frac{2-\nu}{2},\frac{2+\nu}{2};\ -\frac{a^2b^2}{4}\right)$$
$$+\frac{ab}{\nu^2-1}\;{}_1F_2\left(1;\ \frac{3-\nu}{2},\frac{3+\nu}{2};\ -\frac{a^2b^2}{4}\right)\tan\frac{\pi\nu}{2}$$
$[\operatorname{Re}\nu<1, \quad \arg a\neq\pi, \quad b>0]$ MC

2. $$\int_0^\infty \frac{Y_\nu(bx)}{x-a}\,dx = \pi\left\{\cot(\nu\pi)\left[Y_\nu(ab)+\mathbf{E}_\nu(ab)\right]+\mathbf{J}_\nu(ab)+2\left[\cot(\nu\pi)\right]^2\left[\mathbf{J}_\nu(ab)-J_\nu(ab)\right]\right\}$$
$[b>0, \quad a>0, \quad |\operatorname{Re}\nu|<1]$ ET II 98(9)

3. $$\int_0^\infty \frac{K_\nu(bx)}{x+a}\,dx = \frac{\pi^2}{2}\left[\operatorname{cosec}(\nu\pi)\right]^2\left[I_\nu(ab)+I_{-\nu}(ab)-e^{-\frac12 i\nu\pi}\,\mathbf{J}_\nu(iab)-e^{\frac12 i\nu\pi}\,\mathbf{J}_{-\nu}(iab)\right]$$
$[\operatorname{Re} b>0, \quad |\arg a|<\pi, \quad |\operatorname{Re}\nu|<1]$ ET II 128(5)

6.532

1.[11] $$\int_0^\infty \frac{J_\nu(x)}{x^2+a^2}\,dx = \frac{i}{a}\left[S_{0,\nu}(ia)-e^{-i\nu\pi/2}\,K_\nu(a)\right] = \frac{1}{a}\left[i\,s_{0,\nu}(ia)+\frac{\pi}{2}\sec\left(\frac{\nu\pi}{2}\right)I_\nu(a)\right]$$
$[\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1]$

2. $$\int_0^\infty \frac{Y_\nu(x)}{x^2+a^2}\,dx = \frac{1}{\cos\frac{\nu\pi}{2}}\left[-\frac{\pi}{2a}\tan\left(\frac{\nu\pi}{2}\right)I_\nu(ab) - \frac{1}{a}K_\nu(ab) + \frac{b\sin\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\,{}_1F_2\left(1;\frac{3-\nu}{2},\frac{3+\nu}{2};\frac{a^2b^2}{4}\right)\right]$$

$[b>0, \quad \operatorname{Re} a>0, \quad |\operatorname{Re}\nu|<1]$ ET II 99(13)

3. $$\int_0^\infty \frac{Y_\nu(bx)}{x^2-a^2}\,dx = \frac{\pi}{2a}\left\{J_\nu(ab) + \tan\left(\frac{\nu\pi}{2}\right)\left\{\tan\left(\frac{\nu\pi}{2}\right)\left[\mathbf{J}_\nu(ab) - J_\nu(ab)\right] - \mathbf{E}_\nu(ab) - Y_\nu(ab)\right\}\right\}$$

$[b>0, \quad a>0, \quad |\operatorname{Re}\nu|<1]$

ET II 101(21)

4. $$\int_0^\infty \frac{x\,J_0(ax)}{x^2+k^2}\,dx = K_0(ak)$$ $[a>0, \quad \operatorname{Re} k>0]$ WA 466(5)

5. $$\int_0^\infty \frac{Y_0(ax)}{x^2+k^2}\,dx = -\frac{K_0(ak)}{k}$$ $[a>0, \quad \operatorname{Re} k>0]$ WA 466(6)

6. $$\int_0^\infty \frac{J_0(ax)}{x^2+k^2}\,dx = \frac{\pi}{2k}\left[I_0(ak) - \mathbf{L}_0(ak)\right]$$ $[a>0, \quad \operatorname{Re} k>0]$ WA 467(7)

6.533

1. $$\int_0^z J_p(x)\,J_q(z-x)\frac{dx}{x} = \frac{J_{p+q}(z)}{p}$$ $[\operatorname{Re} p>0, \quad \operatorname{Re} q>-1]$ WA 415(3)

2. $$\int_0^z \frac{J_p(x)}{x}\frac{J_q(z-x)}{z-x}\,dx = \left(\frac{1}{p}+\frac{1}{q}\right)\frac{J_{p+q}(z)}{z}$$ $[\operatorname{Re} p>0, \quad \operatorname{Re} q>0]$ WA 415(5)

3.[11] $$\int_0^\infty [J_0(ax)-1]\,J_1(bx)\frac{dx}{x^2} = -\frac{b}{4}\left[1+2\ln\frac{a}{b}\right]$$ $[0<b<a]$

$$= -\frac{a^2}{4b}$$ $[0<a<b]$

ET II 21(28)a

3b $$\int_0^\infty [J_0(ax)-1]\,J_1(bx)\frac{dx}{x} = \begin{cases} \frac{b}{2a}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2,\frac{b^2}{a^2}\right)-1 & [0<b<a] \\ \frac{2}{\pi}\boldsymbol{E}\left(\frac{b^2}{a^2}\right)-1 & [0<a<b] \end{cases}$$

4. $$\int_0^\infty [1-J_0(ax)]\,J_0(bx)\frac{dx}{x} = 0$$ $[0<a<b]$

$$= \ln\frac{a}{b}$$ $[0<b<a]$

ET II 14(16)

6.534 $$\int_0^\infty \frac{x^3\,J_0(x)}{x^4-a^4}\,dx = \frac{1}{2}K_0(a) - \frac{1}{4}\pi\,Y_0(a)$$ $[a>0]$ ET II 340(5)

6.535 $$\int_0^\infty \frac{x}{x^2+a^2}\left[J_\nu(x)\right]^2\,dx = I_\nu(a)\,K_\nu(a)$$ $[\operatorname{Re} a>0, \quad \operatorname{Re}\nu>-1]$ ET II 342(26)

6.536 $$\int_0^\infty \frac{x^3\,J_0(bx)}{x^4+a^4}\,dx = \operatorname{ker}(ab)$$ $\left[b>0, \quad |\arg a|<\tfrac{1}{4}\pi\right]$

ET II 8(9), MO 46a

6.537 $$\int_0^\infty \frac{x^2\, J_0(bx)}{x^4+a^4}\, dx = -\frac{1}{a^2}\,\operatorname{kei}(ab) \qquad \left[b>0, \quad |\arg a| < \frac{\pi}{4}\right]$$ MO 46a

6.538

1. $$\int_0^\infty J_1(ax)\, J_1(bx) \frac{dx}{x^2} = \frac{a+b}{\pi}\left[E\left(\frac{2i\sqrt{ab}}{|b-a|}\right) - K\left(\frac{2i\sqrt{ab}}{|b-a|}\right)\right]$$ $[a>0, \quad b>0]$ ET II 21(30)

2.[8] $$\int_0^\infty x^{-1}\, J_{\nu+2n+1}(x)\, J_{\nu+2m+1}(x)\, dx = 0 \qquad [m \neq n \text{ with } m, n \text{ integers}, \nu > -1]$$
$$= (4n+2\nu+2)^{-1} \qquad [m=n, \quad \nu > -1]$$ EH II 64

6.539

1. $$\int_a^b \frac{dx}{x\,[J_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)}\right] \qquad [J_\nu(x) \neq 0 \quad \text{for } x \in [a,b]]$$ ET II 338(41)

2. $$\int_a^b \frac{dx}{x\,[Y_\nu(x)]^2} = \frac{\pi}{2}\left[\frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)}\right] \qquad [Y_\nu(x) \neq 0 \quad \text{for } x \in [a,b]]$$ ET II 339(49)

3. $$\int_a^b \frac{dx}{x\, J_\nu(x)\, Y_\nu(x)} = \frac{\pi}{2}\ln\left[\frac{J_\nu(a)\, Y_\nu(b)}{J_\nu(b)\, Y_\nu(a)}\right]$$ ET II 339(50)

6.541

1. $$\int_0^\infty x\, J_\nu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2} = I_\nu(bc)\, K_\nu(ac) \qquad [0<b<a, \quad \operatorname{Re} c > 0, \quad \operatorname{Re}\nu > -1]$$
$$= I_\nu(ac)\, K_\nu(bc) \qquad [0<a<b, \quad \operatorname{Re} c > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 49(10)

2.[8] $$\int_0^\infty x^{1-2n}\, J_\nu(ax)\, J_\nu(bx) \frac{dx}{x^2+c^2}$$
$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc)\, K_\nu(ac) - \frac{1}{2}\left(\frac{b}{a}\right)^\nu \frac{\pi}{\sin(\pi\nu)} \sum_{p=0}^{n-1} \frac{\left(a^2c^2/4\right)^p}{p!\,\Gamma(1-\nu+p)} \sum_{k=0}^{n-1-p} \frac{\left(b^2c^2/4\right)^k}{k!\,\Gamma(1-\nu+k)}\right] \qquad [0<b<a]$$
$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc)\, K_\nu(ac) - \frac{1}{2\nu}\left(\frac{b}{a}\right)^\nu \sum_{p=0}^{n-1} \frac{\left(a^2c^2/4\right)^p}{p!(1-\nu)_p} \sum_{k=0}^{n-1-p} \frac{\left(b^2c^2/4\right)^k}{k!(1+\nu)_k}\right]$$
$[n = 1, 2, \ldots, \quad \operatorname{Re}\nu > n-1, \quad \operatorname{Re} c > 0, \quad 0<b<a]$

3.[8]
$$\int_0^\infty \frac{x^{\alpha-1}}{(x^2+z^2)^\rho} J_\mu(cx)\, J_\nu(cx)\, dx = \frac{1}{2}\left(\frac{c}{2}\right)^{2\rho-\alpha} \times \Gamma\left[\begin{matrix}(\mu+\nu+\alpha)/2-\rho, 1+2\rho-\alpha \\ (\mu-\nu-\alpha)/2+\rho+1, (\mu+\nu-\alpha)/2+\rho+1, (\nu-\mu-\alpha)/2+\rho+1\end{matrix}\right]$$
$$\times\, {}_3F_4\left(\frac{1-\alpha}{2}+\rho, 1-\frac{\alpha}{2}+\rho, \rho; \rho+1-\frac{\mu+\nu+\alpha}{2}, \rho+1+\frac{\mu-\nu-\alpha}{2},\right.$$
$$\left.\rho+1+\frac{\mu+\nu-\alpha}{2}, \rho+1+\frac{\nu-\mu-\alpha}{2}; c^2z^2\right) + \frac{z^{\alpha-2\rho}}{2}\left(\frac{cz}{2}\right)^{\mu+\nu},$$
$$\Gamma\left[\begin{matrix}\rho-(\alpha+\mu+\nu)/2, (\alpha+\mu+\nu)/2 \\ \rho, \mu+1, \nu+1\end{matrix}\right] {}_3F_4\left(\frac{1+\mu+\nu}{2}, 1+\frac{\mu+\nu}{2},\right.$$
$$\left.\frac{\alpha+\mu+\nu}{2}; 1-\rho+\frac{\alpha+\mu+\nu}{2}, \mu+1, \nu+1, \mu+\nu+1; c^2z^2\right)$$
$$\left[\Gamma\left[\begin{matrix}a_1,\dots,a_p \\ b_1,\dots,b_q\end{matrix}\right] = \frac{\Gamma(a_1)\dots\Gamma(a_p)}{\Gamma(b_1)\dots\Gamma(b_q)}, \quad c>0, \quad \operatorname{Re} z>0, \quad \operatorname{Re}(\alpha+\mu+\nu)>0; \quad \operatorname{Re}(\alpha-2\rho)>1\right]$$

6.542
$$\int_0^\infty \frac{J_\nu(ax)\, Y_\nu(bx) - J_\nu(bx)\, Y_\nu(ax)}{x\left\{[J_\nu(bx)]^2 + [Y_\nu(bx)]^2\right\}}\, dx = -\frac{\pi}{2}\left(\frac{b}{a}\right)^\nu \qquad [0<b<a]$$
ET II 352(16)

6.543
$$\int_0^\infty J_\mu(bx)\left\{\cos\left[\frac{1}{2}(\nu-\mu)\pi\right] J_\nu(ax) - \sin\left[\frac{1}{2}(\nu-\mu)\pi\right] Y_\nu(ax)\right\}\frac{x\,dx}{x^2+r^2} = I_\mu(br)\, K_\nu(ar)$$
$$[\operatorname{Re} r>0, \quad a\ge b>0, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|-2]$$

6.544

1. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = -\frac{1}{a}\left[\frac{2}{\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - Y_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
EI II 357(47)

2. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{1}{a} J_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \qquad \left[a>0, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 57(10)

3. $$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{1}{a}e^{\frac{1}{2}i\nu\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right) + \frac{1}{a}e^{-\frac{1}{2}i\nu\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)$$
$$\left[\operatorname{Re} b>0, \quad a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 142(32)

4. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{2}{a\pi}\left[K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) + \frac{\pi}{2} Y_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$$\left[a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 62(38)

5. $$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{4}{a}\left[e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{\frac{1}{4}i\pi}\right) + e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}e^{-\frac{1}{4}i\pi}\right)\right]$$
$$\left[\operatorname{Re} b>0, \quad a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 143(38)

6. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{i}{a}\left[e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right) - e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{-\frac{1}{4}\pi i}\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$ ET II 70(19)

7. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{2}{a}\left[\sin\left(\frac{3}{2}\pi\nu\right)\operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right)\operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{5}{2}\right]$ ET II 113(29)

8. $$\int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right)\frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$ ET II 146(55)

6.55 Combinations of Bessel functions and algebraic functions

6.551[10]

1. $$\int_0^1 x^{1/2} J_\nu(xy)\, dx = \sqrt{2} y^{-3/2}\frac{\Gamma\left(\frac{3}{4}+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\nu\right)} + y^{-1/2}\left[\left(\nu-\frac{1}{2}\right) J_\nu(y)\, S_{-1/2,\nu-1}(y) - J_{\nu-1}(y)\, S_{1/2,\nu}(y)\right]$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET II 21(1)

2. $$\int_1^\infty x^{1/2} J_\nu(xy)\, dx = y^{-1/2}\left[J_{\nu-1}(y)\, S_{1/2,\nu}(y) + \left(\frac{1}{2}-\nu\right) J_\nu(y)\, S_{-1/2,\nu-1}(y)\right]$$
$[y > 0]$ ET II 22(2)

6.552

1. $$\int_0^\infty J_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right) K_{\nu/2}\left(\frac{1}{2}ay\right)$$
$[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 23(11), WA 477(3), MO 44

2. $$\int_0^\infty Y_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = -\frac{1}{\pi}\sec\left(\frac{1}{2}\nu\pi\right) K_{\nu/2}\left(\frac{1}{2}ay\right)\left[K_{\nu/2}\left(\frac{1}{2}ay\right) + \pi\sin\left(\frac{1}{2}\nu\pi\right) I_{\nu/2}\left(\frac{1}{2}ay\right)\right]$$
$[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 100(18)

3. $$\int_0^\infty K_\nu(xy)\frac{dx}{\left(x^2+a^2\right)^{1/2}} = \frac{\pi^2}{8}\sec\left(\frac{1}{2}\nu\pi\right)\left\{\left[J_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2 + \left[Y_{\nu/2}\left(\frac{1}{2}ay\right)\right]^2\right\}$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad |\operatorname{Re}\nu| < 1]$ ET II 128(6)

4. $$\int_0^1 J_\nu(xy)\frac{dx}{\left(1-x^2\right)^{1/2}} = \frac{\pi}{2}\left[J_{\nu/2}\left(\frac{1}{2}y\right)\right]^2$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 24(22)a

5. $$\int_0^1 Y_0(xy)\frac{dx}{\left(1-x^2\right)^{1/2}} = \frac{\pi}{2} J_0\left(\frac{1}{2}y\right) Y_0\left(\frac{1}{2}y\right)$$
$[y > 0]$ ET II 102(26)a

6. $$\int_1^\infty J_\nu(xy)\frac{dx}{\left(x^2-1\right)^{1/2}} = -\frac{\pi}{2} J_{\nu/2}\left(\frac{1}{2}y\right) Y_{\nu/2}\left(\frac{1}{2}y\right)$$
$[y > 0]$ ET II 24(23)a

7. $$\int_1^\infty Y_\nu(xy)\frac{dx}{\left(x^2-1\right)^{1/2}} = \frac{\pi}{4}\left\{\left[J_{\nu/2}\left(\tfrac{1}{2}y\right)\right]^2 - \left[Y_{\nu/2}\left(\tfrac{1}{2}y\right)\right]^2\right\}$$
$[y > 0]$ ET II 102(27)

6.553 $$\int_0^\infty x^{-1/2}\, I_\nu(x)\, K_\nu(x)\, K_\mu(2x)\, dx = \frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{4}+\nu-\frac{1}{2}\mu\right)}{4\,\Gamma\left(\frac{3}{4}+\nu+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}+\nu-\frac{1}{2}\mu\right)}$$
$\left[|\operatorname{Re}\mu| < \frac{1}{2}, \quad 2\operatorname{Re}\nu > |\operatorname{Re}\mu| - \frac{1}{2}\right]$ ET II 372(2)

6.554

1. $$\int_0^\infty x\, J_0(xy)\frac{dx}{\left(a^2+x^2\right)^{1/2}} = y^{-1}e^{-ay}$$ $[y > 0, \quad \operatorname{Re} a > 0]$ ET II 7(4)

2. $$\int_0^1 x\, J_0(xy)\frac{dx}{\left(1-x^2\right)^{1/2}} = y^{-1}\sin y$$ $[y > 0]$ ET II 7(5)a

3. $$\int_1^\infty x\, J_0(xy)\frac{dx}{\left(x^2-1\right)^{1/2}} = y^{-1}\cos y$$ $[y > 0]$ ET II 7(6)a

4. $$\int_0^\infty x\, J_0(xy)\frac{dx}{\left(x^2+a^2\right)^{3/2}} = a^{-1}e^{-ay}$$ $[y > 0, \quad \operatorname{Re} a > 0]$ ET II 7(7)a

5.[11] $$\int_0^\infty \frac{x^{\nu+1}\, J_\nu(ax)}{\left(x^4+4k^4\right)^{\nu+1/2}}\, dx = \frac{\left(\frac{1}{2}a\right)^\nu\sqrt{\pi}}{(2k)^{2\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)}\, J_\nu(ak)\, K_\nu(ak)$$
$\left[a > 0, \quad |\arg k| > \frac{\pi}{4}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 473(1)

6.555 $$\int_0^\infty x^{1/2}\, J_{2\nu-1}\left(ax^{1/2}\right)\, Y_\nu(xy)\, dx = -\frac{a}{2y^2}\,\mathbf{H}_{\nu-1}\left(\frac{a^2}{4y}\right)$$
$\left[a > 0, \quad y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 111(17)

6.556 $$\int_0^\infty J_\nu\left[a\left(x^2+1\right)^{1/2}\right]\frac{dx}{\sqrt{x^2+1}} = -\frac{\pi}{2}\, J_{\nu/2}\left(\frac{a}{2}\right)\, Y_{\nu/2}\left(\frac{a}{2}\right)$$ $[\operatorname{Re}\nu > -1, \quad a > 0]$ MO 46

6.56–6.58 Combinations of Bessel functions and powers

6.561

1. $$\int_0^1 x^\nu\, J_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[J_\nu(a)\,\mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\, J_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 333(2)a

2. $$\int_0^1 x^\nu\, Y_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[Y_\nu(a)\,\mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a)\, Y_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 338(43)a

3. $$\int_0^1 x^\nu\, I_\nu(ax)\, dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[I_\nu(a)\,\mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a)\, I_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 364(2)a

4. $$\int_0^1 x^\nu K_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[K_\nu(a)\,\mathbf{L}_{\nu-1}(a)+\mathbf{L}_\nu(a)\,K_{\nu-1}(a)\right]$$
$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 367(21)a

5. $$\int_0^1 x^{\nu+1} J_\nu(ax)\,dx = a^{-1} J_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 333(3)a

6. $$\int_0^1 x^{\nu+1} Y_\nu(ax)\,dx = a^{-1} Y_{\nu+1}(a) + 2^{\nu+1}a^{-\nu-2}\pi^{-1}\,\Gamma(\nu+1)$$
$[\operatorname{Re}\nu > -1]$ ET II 339(44)a

7. $$\int_0^1 x^{\nu+1} I_\nu(ax)\,dx = a^{-1} I_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 365(3)a

8. $$\int_0^1 x^{\nu+1} K_\nu(ax)\,dx = 2^\nu a^{-\nu-2}\,\Gamma(\nu+1) - a^{-1} K_{\nu+1}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 367(22)a

9. $$\int_0^1 x^{1-\nu} J_\nu(ax)\,dx = \frac{a^{\nu-2}}{2^{\nu-1}\,\Gamma(\nu)} - a^{-1} J_{\nu-1}(a)$$
ET II 333(4)a

10. $$\int_0^1 x^{1-\nu} Y_\nu(ax)\,dx = \frac{a^{\nu-2}\cot(\nu\pi)}{2^{\nu-1}\,\Gamma(\nu)} - a^{-1} Y_{\nu-1}(a)$$
$[\operatorname{Re}\nu < 1]$ ET II 339(45)a

11. $$\int_0^1 x^{1-\nu} I_\nu(ax)\,dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1}\,\Gamma(\nu)}$$
ET II 365(4)a

12. $$\int_0^1 x^{1-\nu} K_\nu(ax)\,dx = 2^{-\nu}a^{\nu-2}\,\Gamma(1-\nu) - a^{-1} K_{\nu-1}(a)$$
$[\operatorname{Re}\nu < 1]$ ET II 367(23)a

13.[7] $$\int_0^1 x^\mu J_\nu(ax)\,dx = \frac{2^\mu\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{a^{\mu+1}\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)} + a^{-\mu}\left\{(\mu+\nu-1)\,J_\nu(a)\,S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a)\,S_{\mu,\nu}(a)\right\}$$
$[a > 0, \quad \operatorname{Re}(\mu+\nu) > -1]$ ET II 22(8)a

14. $$\int_0^\infty x^\mu J_\nu(ax)\,dx = 2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$$
$\left[-\operatorname{Re}\nu - 1 < \operatorname{Re}\mu < \tfrac{1}{2}, \quad a > 0\right]$
EH II 49(19)

15. $$\int_0^\infty x^\mu Y_\nu(ax)\,dx = 2^\mu \cot\left[\tfrac{1}{2}(\nu+1-\mu)\pi\right] a^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$$
$\left[|\operatorname{Re}\nu| - 1 < \mu < \tfrac{1}{2}, \quad a > 0\right]$
ET II 97(3)a

16. $$\int_0^\infty x^\mu K_\nu(ax)\,dx = 2^{\mu-1}a^{-\mu-1}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$$
$[\operatorname{Re}(\mu+1\pm\nu) > 0, \quad \operatorname{Re}a > 0]$
EH II 51(27)

17. $$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}}\,dx = \frac{\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{2^{\nu-q}a^{q-\nu+1}\,\Gamma\left(\nu-\frac{1}{2}q+\frac{1}{2}\right)} \qquad \left[-1 < \operatorname{Re} q < \operatorname{Re}\nu - \tfrac{1}{2}\right]$$

WA 428(1), KU 144(5)

18. $$\int_0^\infty \frac{Y_\nu(x)}{x^{\nu-\mu}}\,dx = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\nu\right)\sin\left(\frac{1}{2}\mu-\nu\right)\pi}{2^{\nu-\mu}\pi}$$
$$\left[|\operatorname{Re}\nu| < \operatorname{Re}(1+\mu-\nu) < \tfrac{3}{2}\right]$$

WA 430(5)

19. $$\int_0^1 x^{2m+n+1/2}\,K_{n+1/2}(\alpha x)\,dx = \sqrt{\frac{\pi}{2}}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{\gamma(2m+n-k+1,\alpha)}{\alpha^{2m+n+3/2}2^k}$$ STR

6.562

1. $$\int_0^\infty x^\mu\,Y_\nu(bx)\frac{dx}{x+a} = (2a)^\mu\pi^{-1}\left\{\sin\left[\tfrac{1}{2}\pi(\mu-\nu)\right]\Gamma\left[\tfrac{1}{2}(\mu+\nu+1)\right]\Gamma\left[\tfrac{1}{2}(1+\mu-\nu)\right]S_{-\mu,\nu}(ab)\right.$$
$$\left. -2\cos\left[\tfrac{1}{2}\pi\left(\mu-\nu\right)\right]\Gamma\left(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu\right)\Gamma\left(1+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\right)S_{-\mu-1,\nu}(ab)\right\}$$
$$\left[b>0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\mu\pm\nu) > -1, \quad \operatorname{Re}\mu < \tfrac{3}{2}\right]$$ ET II 98(8)

2. $$\int_0^\infty \frac{x^\nu\,J_\nu(ax)}{x+k}\,dx = \frac{\pi k^\nu}{2\cos\nu\pi}\left[\mathbf{H}_{-\nu}(ak) - Y_{-\nu}(ak)\right] \qquad \left[-\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{3}{2}, \quad a>0, \quad |\arg k| < \pi\right]$$

WA 479(7)

3. $$\int_0^\infty x^\mu\,K_\nu(bx)\frac{dx}{x+a}$$
$$= 2^{\mu-2}\,\Gamma\left[\tfrac{1}{2}(\mu+\nu)\right]\Gamma\left[\tfrac{1}{2}(\mu-\nu)\right]b^{-\mu}\;{}_1F_2\left(1; 1-\frac{\mu+\nu}{2}, 1-\frac{\mu-\nu}{2}; \frac{a^2b^2}{4}\right)$$
$$-2^{\mu-3}\,\Gamma\left[\tfrac{1}{2}\left(\mu-\nu-1\right)\right]\Gamma\left[\tfrac{1}{2}(\mu+\nu-1)\right]ab^{1-\mu}\;{}_1F_2\left(1; \frac{3-\mu-\nu}{2}, \frac{3-\mu+\nu}{2}; \frac{a^2b^2}{4}\right)$$
$$-\pi a^\mu \operatorname{cosec}[\pi(\mu-\nu)]\left\{K_\nu(ab) + \pi\cos(\mu\pi)\operatorname{cosec}[\pi(\nu+\mu)]\,I_\nu(ab)\right\}$$
$$[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu| - 1]$$ ET II 127(4)

6.563 $$\int_0^\infty x^{\varrho-1}\,J_\nu(bx)\frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho+\nu-\mu)\pi]\,\Gamma(\mu+1)}$$
$$\times\left\{\sum_{m=0}^\infty \frac{(-1)^m\left(\frac{1}{2}ab\right)^{\nu+2m}\Gamma(\varrho+\nu+2m)}{m!\,\Gamma(\nu+m+1)\,\Gamma\left(\varrho+\nu-\mu+2m\right)}\right.$$
$$\left. -\sum_{m=0}^\infty \frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m}\Gamma(\mu+m+1)}{m!\,\Gamma\left[\frac{1}{2}(\mu+\nu-\varrho+m+3)\right]}\frac{\sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{\Gamma\left[\frac{1}{2}(\mu-\nu-\varrho+m+3)\right]}\right\}$$
$$\left[b>0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\varrho+\nu) > 0, \quad \operatorname{Re}(\varrho-\mu) < \tfrac{5}{2}\right]$$ ET II 23(10), WA 479

6.564

1. $$\int_0^\infty x^{\nu+1}\,J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{2}{\pi b}}a^{\nu+\frac{1}{2}}\,K_{\nu+\frac{1}{2}}(ab) \qquad \left[\operatorname{Re} a > 0, \quad b>0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$

ET II 23(15)

2. $$\int_0^\infty x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{\pi}{2b}} a^{\frac{1}{2}-\nu} \left[I_{\nu-\frac{1}{2}}(ab) - \mathbf{L}_{\nu-\frac{1}{2}}(ab)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 23(16)

6.565

1. $$\int_0^\infty x^{-\nu} \left(x^2+a^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
WA 477(4), ET II 23(17)

2. $$\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 24(18)

3. $$\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^{-\nu-\frac{3}{2}} J_\nu(bx)\,dx = \frac{b^\nu \sqrt{\pi}}{2^{\nu+1} a e^{ab}\,\Gamma\left(\nu+\frac{3}{2}\right)}$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$
ET II 24(19)

4. $$\int_0^\infty \frac{J_\nu(bx) x^{\nu+1}}{\left(x^2+a^2\right)^{\mu+1}}\,dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu\,\Gamma(\mu+1)} K_{\nu-\mu}(ab)$$
$\left[-1 < \operatorname{Re}\nu < \operatorname{Re}\left(2\mu+\frac{3}{2}\right), \quad a > 0, \quad b > 0\right]$ MO 43

5. $$\begin{aligned}\int_0^\infty x^{\nu+1} \left(x^2+a^2\right)^\mu Y_\nu(bx)\,dx = {} & 2^{\nu-1}\pi^{-1} a^{2\mu+2} (1+\mu)^{-1}\,\Gamma(\nu) b^{-\nu} \\ & \times {}_1F_2\left(1; 1-\nu, 2+\mu; \frac{a^2b^2}{4}\right) - 2^\mu a^{\mu+\nu+1} \left[\sin(\nu\pi)\right]^{-1} \\ & \times \Gamma(\mu+1) b^{-1-\mu} \left[I_{\mu+\nu+1}(ab) - 2\cos(\mu\pi)\,K_{\mu+\nu+1}(ab)\right]\end{aligned}$$
$[b > 0, \quad \operatorname{Re} a > 0, \quad -1 < \operatorname{Re}\nu < -2\operatorname{Re}\mu]$ ET II 100(19)

6.[10] $$\begin{aligned}\int_0^\infty x^{1-\nu} \left(x^2+a^2\right)^\mu Y_\nu(bx)\,dx = {} & \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu}\pi}{\Gamma(-\mu)} I_{-1-\mu+\nu}(ab) \cot[\pi(\mu-\nu)] \operatorname{cosec}(\pi\mu) \\ & - \frac{2^\mu a^{1+\mu-\nu} b^{-1-\mu}\pi}{\Gamma(-\mu)} I_{1+\mu-\nu}(ab) \operatorname{cosec}[\pi(\mu-\nu)] \operatorname{cosec}(\pi\nu) \\ & + \frac{2^{-1-\nu} a^{2+2\mu} b^\nu}{(1+\mu)\pi} \cos(\pi\nu)\,\Gamma(-\mu)\ {}_1F_2\left(1;\ 2+\mu, 1+\nu;\ \frac{a^2b^2}{4}\right)\end{aligned}$$
$\left[\operatorname{Re}\nu < 1, \quad \operatorname{Re}(\nu-2\mu) > -3, \quad \arg a^2 \neq \pi, \quad b > 0\right]$ MC

7. $$\int_0^\infty x^{1+\nu} \left(x^2+a^2\right)^\mu K_\nu(bx)\,dx = 2^\nu\,\Gamma(\nu+1) a^{\nu+\mu+1} b^{-1-\mu}\,S_{\mu-\nu,\mu+\nu+1}(ab)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1]$
ET II 128(8)

8.[11] $$\int_0^\infty \frac{x^{\varrho-1}\,J_\nu(ax)}{\left(x^2+k^2\right)^{\mu+1}}\,dx = \frac{a^\nu k^{\varrho+\nu-2\mu-2}\,\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu\right)\Gamma\left(\mu+1-\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{2^{\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)} \times {}_1F_2\left(\frac{\varrho+\nu}{2};\frac{\varrho+\nu}{2}-\mu,\nu+1;\frac{a^2k^2}{4}\right) + \frac{a^{2\mu+2-\varrho}\,\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\varrho-\mu-1\right)}{2^{2\mu+3-\varrho}\,\Gamma\left(\mu+2+\frac{1}{2}\nu-\frac{1}{2}\varrho\right)} \times {}_1F_2\left(\mu+1;\mu+2+\frac{\nu-\varrho}{2},\mu+2-\frac{\nu+\varrho}{2};\frac{a^2k^2}{4}\right)$$
$\left[a>0, \quad -\operatorname{Re}\nu<\operatorname{Re}\varrho<2\operatorname{Re}\mu+\frac{7}{2}, \quad \operatorname{Re}k>0\right]$ WA 477(1)

6.566

1. $$\int_0^\infty x^\mu\,Y_\nu(bx)\frac{dx}{x^2+a^2} = 2^{\mu-2}\pi^{-1}b^{1-\mu} \times \cos\left[\frac{\pi}{2}(\mu-\nu+1)\right]\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\right) \times {}_1F_2\left(1;2-\frac{\mu+1+\nu}{2},2-\frac{\mu+1-\nu}{2};\frac{a^2b^2}{4}\right) - \frac{1}{2}\pi a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu+\nu+1)\right]\cot\left[\frac{\pi}{2}(\mu-\nu+1)\right]I_\nu(ab) - a^{\mu-1}\operatorname{cosec}\left[\frac{\pi}{2}(\mu-\nu+1)\right]K_\nu(ab)$$
$\left[b>0, \quad \operatorname{Re}a>0, \quad |\operatorname{Re}\nu|-1<\operatorname{Re}\mu<\frac{5}{2}\right]$ ET II 100(17)

2. $$\int_0^\infty x^{\nu+1}\,J_\nu(ax)\frac{dx}{x^2+b^2} = b^\nu\,K_\nu(ab)$$ $\left[a>0, \quad \operatorname{Re}b>0, \quad -1<\operatorname{Re}\nu<\frac{3}{2}\right]$
EH II 96(58)

3. $$\int_0^\infty x^\nu\,K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2 b^{\nu-1}}{4\cos\nu\pi}\left[\mathbf{H}_{-\nu}(ab)-Y_{-\nu}(ab)\right]$$
$\left[a>0, \quad \operatorname{Re}b>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right]$
WA 468(9)

4. $$\int_0^\infty x^{-\nu}\,K_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi^2}{4b^{\nu+1}\cos\nu\pi}\left[\mathbf{H}_\nu(ab)-Y_\nu(ab)\right]$$
$\left[a>0, \quad \operatorname{Re}b>0, \quad \operatorname{Re}\nu<\frac{1}{2}\right]$
WA 468(10)

5. $$\int_0^\infty x^{-\nu}\,J_\nu(ax)\frac{dx}{x^2+b^2} = \frac{\pi}{2b^{\nu+1}}\left[I_\nu(ab)-\mathbf{L}_\nu(ab)\right]$$ $\left[a>0, \quad \operatorname{Re}b>0, \quad \operatorname{Re}\nu>-\frac{5}{2}\right]$
WA 468(11)

6.567

1. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^\mu J_\nu(bx)\,dx = 2^\mu\,\Gamma(\mu+1)b^{-(\mu+1)}\,J_{\nu+\mu+1}(b)$$
$[b>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>-1]$
ET II 26(33)a

2. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^{\mu} Y_\nu(bx)\,dx$$
$$= b^{-(\mu+1)}\left[2^{\mu}\,\Gamma(\mu+1)\,Y_{\mu+\nu+1}(b)+2^{\nu+1}\pi^{-1}\,\Gamma(\nu+1)\,S_{\mu-\nu,\mu+\nu+1}(b)\right]$$
$[b>0, \quad \operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu>-1]$ ET II 103(35)a

3. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} J_\nu(bx)\,dx=\frac{2^{1-\nu}\,S_{\nu+\mu,\mu-\nu+1}(b)}{b^{\mu+1}\,\Gamma(\nu)}$$ $[b>0, \quad \operatorname{Re}\mu>-1]$ ET II 25(31)a

4. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} Y_\nu(bx)\,dx=b^{-(\mu+1)}\left[2^{1-\nu}\pi^{-1}\cos(\nu\pi)\,\Gamma\left(1-\nu\right)\right.$$
$$\left.\times\, S_{\mu+\nu,\mu-\nu+1}(b)-2^{\mu}\operatorname{cosec}(\nu\pi)\,\Gamma(\mu+1)\,J_{\mu-\nu+1}(b)\right]$$
$[b>0, \quad \operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu<1]$ ET II 104(37)a

5. $$\int_0^1 x^{1-\nu}\left(1-x^2\right)^{\mu} K_\nu(bx)\,dx=2^{-\nu-2}b^{\nu}(\mu+1)^{-1}\,\Gamma(-\nu)\;{}_1F_2\left(1;\nu+1,\mu+2;\frac{b^2}{4}\right)$$
$$+\pi 2^{\mu-1}b^{-(\mu+1)}\operatorname{cosec}\left(\nu\pi\right)\Gamma(\mu+1)\,I_{\mu-\nu+1}(b)$$
$[\operatorname{Re}\mu>-1, \quad \operatorname{Re}\nu<1]$ ET II 129(12)a

6. $$\int_0^1 x^{1-\nu}\,J_\nu(bx)\frac{dx}{\sqrt{1-x^2}}=\sqrt{\frac{\pi}{2b}}\,\mathbf{H}_{\nu-\frac{1}{2}}(b)$$ $[b>0]$ ET II 24(24)a

7. $$\int_0^1 x^{1+\nu}\,Y_\nu(bx)\frac{dx}{\sqrt{1-x^2}}=\sqrt{\frac{\pi}{2b}}\operatorname{cosec}(\nu\pi)\left[\cos(\nu\pi)\,J_{\nu+\frac{1}{2}}(b)-\mathbf{H}_{-\nu-\frac{1}{2}}(b)\right]$$
$[b>0, \quad \operatorname{Re}\nu>-1]$ ET II 102(28)a

8. $$\int_0^1 x^{1-\nu}\,Y_\nu(bx)\frac{dx}{\sqrt{1-x^2}}=\sqrt{\frac{\pi}{2b}}\left\{\cot(\nu\pi)\left[\mathbf{H}_{\nu-\frac{1}{2}}(b)-Y_{\nu-\frac{1}{2}}(b)\right]-J_{\nu-\frac{1}{2}}(b)\right\}$$
$[b>0, \quad \operatorname{Re}\nu<1]$ ET II 102(30)a

9. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} J_\nu(bx)\,dx=2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[J_\nu\left(\frac{b}{2}\right)\right]^2$$
$\left[b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 24(25)a

10. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} Y_\nu(bx)\,dx=2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)J_\nu\left(\frac{b}{2}\right)Y_\nu\left(\frac{b}{2}\right)$$
$\left[b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 102(31)a

11. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx=2^{\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)I_\nu\left(\frac{b}{2}\right)K_\nu\left(\frac{b}{2}\right)$$
$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ ET II 129(10)a

12. $$\int_0^1 x^{\nu}\left(1-x^2\right)^{\nu-\frac{1}{2}} I_\nu(bx)\,dx=2^{-\nu-1}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[I_\nu\left(\frac{b}{2}\right)\right]^2$$ ET II 365(5)a

13. $$\int_0^1 x^{\nu+1}\left(1-x^2\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx=2^{-\nu}\frac{b^{\nu-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\sin b$$
$\left[b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$ ET II 25(27)a

14. $$\int_1^\infty x^\nu \left(x^2-1\right)^{\nu-\frac{1}{2}} Y_\nu(bx)\,dx = 2^{\nu-2}\sqrt{\pi}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[J_\nu\left(\frac{b}{2}\right)J_{-\nu}\left(\frac{b}{2}\right) - Y_\nu\left(\frac{b}{2}\right)Y_{-\nu}\left(\frac{b}{2}\right)\right]$$
$\left[|\operatorname{Re}\nu| < \frac{1}{2}, \quad b > 0\right]$ ET II 103(32)a

15. $$\int_1^\infty x^\nu \left(x^2-1\right)^{\nu-\frac{1}{2}} K_\nu(bx)\,dx = \frac{2^{\nu-1}}{\sqrt{\pi}}b^{-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[K_\nu\left(\frac{b}{2}\right)\right]^2$$
$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 129(11)a

16. $$\int_1^\infty x^{-\nu} \left(x^2-1\right)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = -2^{-\nu-1}\sqrt{\pi}b^{\nu}\,\Gamma\left(\frac{1}{2}-\nu\right)J_\nu\left(\frac{b}{2}\right)Y_\nu\left(\frac{b}{2}\right)$$
$\left[b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 25(26)a

17.[8] $$\int_1^\infty x^{-\nu+1} \left(x^2-1\right)^{\nu-\frac{1}{2}} J_\nu(bx)\,dx = \frac{2^{\nu}}{\sqrt{\pi}}b^{-\nu-1}\,\Gamma\left(\frac{1}{2}+\nu\right)\cos b$$
$\left[b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 25(28)

6.568

1. $$\int_0^\infty x^\nu\, Y_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\nu-1}\,J_\nu(ab)$$
$\left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{5}{2}\right]$ ET II 101(22)

2. $$\int_0^\infty x^\mu\, Y_\nu(bx)\frac{dx}{x^2-a^2} = \frac{\pi}{2}a^{\mu-1}\,J_\nu(ab) + 2^\mu\pi^{-1}a^{\mu-1}\cos\left[\frac{\pi}{2}(\mu-\nu+1)\right] \times\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)S_{-\mu,\nu}(ab)$$
$\left[a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| - 1 < \operatorname{Re}\mu < \frac{5}{2}\right]$ ET II (101)(25)

6.569 $$\int_0^1 x^\lambda(1-x)^{\mu-1}\,J_\nu(ax)\,dx = \frac{\Gamma(\mu)\,\Gamma(1+\lambda+\nu)2^{-\nu}a^\nu}{\Gamma(\nu+1)\,\Gamma(1+\lambda+\mu+\nu)} \times {}_2F_3\left(\frac{\lambda+1+\nu}{2},\frac{\lambda+2+\nu}{2};\nu+1,\frac{\lambda+1+\mu+\nu}{2},\frac{\lambda+2+\mu+\nu}{2};-\frac{a^2}{4}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}(\lambda+\nu) > -1]$ ET II 193(56)a

6.571

1. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}} \pm x\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = a^\mu\, I_{\frac{1}{2}(\nu\mp\mu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\nu\pm\mu)}\left(\frac{ab}{2}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu < \frac{3}{2}\right]$ ET II 26(38)

2. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}} - x\right]^\mu Y_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = a^\mu\left[\cot(\nu\pi)\,I_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right) - \operatorname{cosec}(\nu\pi)\,I_{\frac{1}{2}(\mu-\nu)}\left(\frac{ab}{2}\right)K_{\frac{1}{2}(\mu+\nu)}\left(\frac{ab}{2}\right)\right]$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\mu > -\frac{3}{2}, \quad |\operatorname{Re}\nu| < 1\right]$ ET II 104(40)

3. $$\int_0^\infty \left[\left(x^2+a^2\right)^{\frac{1}{2}}+x\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= \frac{\pi^2}{4}a^\mu \operatorname{cosec}(\nu\pi)\left[J_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)Y_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)-Y_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right)J_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right)\right]$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0]$ ET II 130(15)

6.572

1. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}+a\right]^\mu J_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}} = \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab\,\Gamma(\nu+1)}\,W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\,M_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)$$
$[\operatorname{Re} a>0, \quad b>0, \quad \operatorname{Re}(\nu-\mu)>-1]$
ET II 26(40)

2. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}+a\right]^\mu K_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab}\,W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(iab)\,W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(-iab)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} b>0, \quad \operatorname{Re}\mu+|\operatorname{Re}\nu|<1]$ ET II 130(18), BU 87(6a)

3. $$\int_0^\infty x^{-\mu}\left[\left(x^2+a^2\right)^{\frac{1}{2}}-a\right]^\mu Y_\nu(bx)\frac{dx}{\sqrt{x^2+a^2}}$$
$$= -\frac{1}{ab}\,W_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\left\{\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)}\tan\left(\frac{\nu-\mu}{2}\pi\right)M_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\right.$$
$$\left.+\sec\left(\frac{\nu-\mu}{2}\pi\right)W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ab)\right\}$$
$\left[\operatorname{Re} a>0, \quad b>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}+\frac{1}{2}\operatorname{Re}\mu\right]$ ET II 105(42)

6.573

1. $$\int_0^\infty x^{\nu-M+1}J_\nu(bx)\prod_{i=1}^k J_{\mu_i}(a_i x)\,dx = 0 \qquad M=\sum_{i=1}^k \mu_i$$
$\left[a_i>0, \quad \sum_{i=1}^k a_i<b<\infty, \quad -1<\operatorname{Re}\nu<\operatorname{Re} M+\frac{1}{2}k-\frac{1}{2}\right]$ ET II 54(42)

2. $$\int_0^\infty x^{\nu-M-1}J_\nu(bx)\prod_{i=1}^k J_{\mu_i}(a_i x)\,dx = 2^{\nu-M-1}b^{-\nu}\,\Gamma(\nu)\prod_{i=1}^k\frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \qquad M=\sum_{i=1}^k\mu_i$$
$\left[a_i>0, \quad \sum_{i=1}^k a_i<b<\infty, \quad 0<\operatorname{Re}\nu<\operatorname{Re} M+\frac{1}{2}k+\frac{3}{2}\right]$ WA 460(16)a, ET II 54(43)

6.574

1.[8] $$\int_0^\infty J_\nu(\alpha t)\,J_\mu(\beta t)t^{-\lambda}\,dt = \frac{\alpha^\nu\,\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda\beta^{\nu-\lambda+1}\,\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma(\nu+1)} \times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{\alpha^2}{\beta^2}\right)$$

$[\operatorname{Re}(\nu+\mu-\lambda+1)>0, \quad \operatorname{Re}\lambda>-1, \quad 0<\alpha<\beta]$ WA 439(2)a, MO 49

If we reverse the positions of ν and μ and at the same time reverse the positions of α and β, the function on the right-hand side of this equation will change. Thus, the right-hand side represents a function of $\frac{\alpha}{\beta}$ that is not analytic at $\frac{\alpha}{\beta}=1$.

For $\alpha=\beta$, we have the following equation:

2. $$\int_0^\infty J_\nu(\alpha t)\,J_\mu(\alpha t)t^{-\lambda}\,dt = \frac{\alpha^{\lambda-1}\,\Gamma(\lambda)\,\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda\,\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right)\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)}$$

$[\operatorname{Re}(\nu+\mu+1)>\operatorname{Re}\lambda>0, \quad \alpha>0]$
MO 49, WA 441(2)a

If $\mu-\nu+\lambda+1$ (or $\nu-\mu+\lambda+1$) is a negative integer, the right-hand side of equation **6.574** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.574** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

3.* $$\int_0^\infty J_\nu(\alpha t)\,J_\mu(\beta t)t^{-\lambda}\,dt = \frac{\beta^\nu\,\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda\alpha^{\mu-\lambda+1}\,\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)\Gamma(\nu+1)} \times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{-\nu+\mu-\lambda+1}{2}; \mu+1; \frac{\beta^2}{\alpha^2}\right)$$

$[\operatorname{Re}(\nu+\mu-\lambda+1)>0, \quad \operatorname{Re}\lambda>-1, \quad 0<\beta<\alpha]$ MO 50, WA 440(3)a

If $\mu-\nu+\lambda+1$ (or $\nu-\mu+\lambda+1$) is a negative integer, the right-hand side of equation **6.754** 1 (or **6.574** 3) vanishes. The cases in which the hypergeometric function F in **6.754** 3 (or **6.574** 1) can be reduced to an elementary function are then especially important.

6.575

1.[11] $$\int_0^\infty J_{\nu+1}(\alpha t)\,J_\mu(\beta t)t^{\mu-\nu}\,dt = 0 \qquad [\alpha<\beta]$$

$$= \frac{\left(\alpha^2-\beta^2\right)^{\nu-\mu}\beta^\mu}{2^{\nu-\mu}\alpha^{\nu+1}\,\Gamma(\nu-\mu+1)} \qquad [\alpha\ge\beta]$$

$[\operatorname{Re}(\nu+1)>\operatorname{Re}\mu>-1]$ MO 51

2. $$\int_0^\infty \frac{J_\nu(x)\,J_\mu(x)}{x^{\nu+\mu}}\,dx = \frac{\sqrt{\pi}\,\Gamma(\nu+\mu)}{2^{\nu+\mu}\,\Gamma\left(\nu+\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)}$$

$[\operatorname{Re}(\nu+\mu)>0]$ KU 147(17), WA 434(1)

6.576

1. $$\int_0^\infty x^{\mu-\nu+1} J_\mu(x)\, K_\nu(x)\, dx = \tfrac{1}{2}\Gamma(\mu-\nu+1) \qquad [\operatorname{Re}\mu > -1, \quad \operatorname{Re}(\mu-\nu) > -1]$$
ET II 370(47)

2.[11] $$\int_0^\infty x^{-\lambda} J_\nu(ax)\, J_\nu(bx)\, dx = \frac{a^\nu b^\nu\, \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1}\, \Gamma(\nu+1)\, \Gamma\left(\frac{1+\lambda}{2}\right)} \times F\left(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}\right)$$
$[a > 0, \quad b > 0, \quad 2\operatorname{Re}\nu + 1 > \operatorname{Re}\lambda > -1]$ ET II 47(4)

3. $$\int_0^\infty x^{-\lambda} K_\mu(ax)\, J_\nu(bx)\, dx = \frac{b^\nu\, \Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1}\, \Gamma(1+\nu)} \times F\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(a \pm ib) > 0, \quad \operatorname{Re}(\nu-\lambda+1) > |\operatorname{Re}\mu|]$ EH II 52(31), ET II 63(4), WA 449(1)

4. $$\int_0^\infty x^{-\lambda} K_\mu(ax)\, K_\nu(bx)\, dx = \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \times F\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right)$$
$[\operatorname{Re} a+b > 0, \quad \operatorname{Re}\lambda < 1 - |\operatorname{Re}\mu| - |\operatorname{Re}\nu|]$ ET II 145(49), EH II 93(36)

5. $$\int_0^\infty x^{-\lambda} K_\mu(ax)\, I_\nu(bx)\, dx = \frac{b^\nu\, \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{2^{\lambda+1}\, \Gamma(\nu+1) a^{-\lambda+\nu+1}} \times F\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu; \nu+1; \frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(\nu+1-\lambda\pm\mu) > 0, \quad a > b]$ EH II 93(35)

6. $$\int_0^\infty x^{-\lambda} Y_\mu(ax)\, J_\nu(bx)\, dx = \frac{2}{\pi}\sin\frac{\pi(\nu-\mu-\lambda)}{2}\int_0^\infty x^{-\lambda} K_\mu(ax)\, I_\nu(bx)\, dx$$
$[a > b, \quad \operatorname{Re}\lambda > -1, \quad \operatorname{Re}(\nu-\lambda+1\pm\mu) > 0]$ (see **6.576** 5) EH II 93(37)

7.[8] $$\int_0^\infty x^{\mu+\nu+1} J_\mu(ax)\, K_\nu(bx)\, dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}$$
$[\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1, \quad \operatorname{Re} b > |\operatorname{Im} a|]$
ET 137(16), EH II 93(36)

6.577

1.[8] $$\int_0^\infty x^{\nu-\mu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\nu-\mu+2n} I_\mu(ac) K_\nu(bc)$$

$[a>0, \quad b>a, \quad \operatorname{Re} c>0, \quad 2+\operatorname{Re}\mu-2n>\operatorname{Re}\nu>-1-n, \quad n\geq 0$ an integer$]$ ET II 49(13)

2.[8] $$\int_0^\infty x^{\mu-\nu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} = (-1)^n c^{\mu-\nu+2n} I_\nu(bc) K_\mu(ac)$$

$[b>0, \quad a>b, \quad \operatorname{Re}\nu-2n+2>\operatorname{Re}\mu>-n-1, \quad n\geq 0$ an integer$]$ ET II 49(15)

6.578

1. $$\int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) J_\nu(cx)\,dx = \frac{2^{\varrho-1}a^\lambda b^\mu c^{-\lambda-\mu-\varrho}\,\Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma\left(1-\frac{\lambda+\mu-\nu+\varrho}{2}\right)} \times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right)$$

$\left[\operatorname{Re}(\lambda+\mu+\nu+\varrho)>0, \quad \operatorname{Re}\varrho<\frac{5}{2}, \quad a>0, \quad b>0, \quad c>0, \quad c>a+b\right]$ ET II 351(9)

2. $$\int_0^\infty x^{\varrho-1} J_\lambda(ax) J_\mu(bx) K_\nu(cx)\,dx = \frac{2^{\varrho-2}a^\lambda b^\mu c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1)\,\Gamma(\mu+1)}\Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right) \times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right)$$

$[\operatorname{Re}(\varrho+\lambda+\mu)>|\operatorname{Re}\nu|, \quad \operatorname{Re} c>|\operatorname{Im} a|+|\operatorname{Im} b|]$ ET II 373(8)

3. $$\int_0^\infty x^{\lambda-\mu-\nu+1} J_\nu(ax) J_\mu(bx) J_\lambda(cx)\,dx = 0$$

$\left[\operatorname{Re}\lambda>-1, \quad \operatorname{Re}(\lambda-\mu-\nu)<\tfrac{1}{2}, \quad c>b>0, \quad 0<a<c-b\right]$ ET II 53(36)

4. $$\int_0^\infty x^{\lambda-\mu-\nu-1} J_\nu(ax) J_\mu(bx) J_\lambda(cx)\,dx = \frac{2^{\lambda-\mu-\nu-1}a^\nu b^\mu\,\Gamma(\lambda)}{c^\lambda\,\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$\left[\operatorname{Re}\lambda>0, \quad \operatorname{Re}(\lambda-\mu-\nu)<\tfrac{5}{2}, \quad c>b>0, \quad 0<a<c-b\right]$ ET II 53(37)

5. $$\int_0^\infty x^{1+\mu} Y_\mu(ax) J_\nu(bx) J_\nu(cx)\,dx = 0$$

$[0<b<c, \quad 0<a<c-b]$ ET II 352(13)

6.[11] $$\int_0^\infty x^{\mu+1} K_\mu(ax) J_\nu(bx) J_\nu(cx)\,dx = \frac{1}{\sqrt{2\pi}} a^\mu b^{-\mu-1} c^{-\mu-1} e^{-\left(\mu+\frac{1}{2}\right)\pi i}\left(u^2-1\right)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(u)$$

$\left[2bcu=a^2+b^2+c^2, \quad \operatorname{Re} a>|\operatorname{Im} b|+|\operatorname{Im} c|, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1\right]$

WA 452(2), ET II 64(12)

7.[11] $$\int_0^\infty x^{\mu+1} I_\nu(ax) K_\mu(bx) J_\nu(cx)\,dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-\left(\mu-\frac{1}{2}\nu+\frac{1}{4}\right)\pi i}\left(v^2+1\right)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iv),$$

$2acv=b^2-a^2+c^2$ $[\operatorname{Re} b>|\operatorname{Re} a|+|\operatorname{Im} c|; \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\mu+\nu)>-1]$ ET II 66(22)

8.[11] $$\int_0^\infty x^{1-\mu} J_\mu(ax) J_\nu(bx) J_\nu(cx)\,dx$$
$$= \sqrt{\frac{2}{\pi^3}} a^{-\mu}(bc)^{\mu-1} (\sinh u)^{\mu-\frac{1}{2}} \sin[(\mu-\nu)\pi] e^{\left(\mu-\frac{1}{2}\right)\pi i} Q^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\cosh u) \qquad [a > b+c]$$
$$= \frac{1}{\sqrt{2\pi}} a^{-\mu}(bc)^{\mu-1} (\sin v)^{\mu-\frac{1}{2}} P^{\frac{1}{2}-\mu}_{\nu-\frac{1}{2}}(\cos v) \qquad [|b-c| < a < b+c]$$
$$= 0 \qquad [0 < a < |b-c|]$$
$$\left[2bc\cosh u = a^2-b^2-c^2, \quad 2bc\cos v = b^2+c^2-a^2, \quad b>0, \quad c>0; \quad \operatorname{Re}\nu > -1, \operatorname{Re}\mu > -\tfrac{1}{2}\right]$$

9. $$\int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu}\,dx = 0 \qquad [0 < c \le |a-b| \text{ or } c \ge a+b]$$
$$= \frac{2^{\nu-1}\Delta^{2\nu-1}}{(abc)^\nu \Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \qquad [|a-b| < c < a+b]$$
$$\Delta = \tfrac{1}{4}\sqrt{[c^2-(a-b)^2][(a+b)^2-c^2]}, \qquad \left[a>0, \quad b>0, \quad c>0; \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$

($\Delta > 0$ is equal to the area of a triangle whose sides are a, b, and c.)

10.[11] $$\int_0^\infty x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx)\,dx = \frac{\sqrt{\pi} c^\nu \Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}{2^{3/2}(ab)^{\nu+1}(u^2-1)^{\frac{1}{2}\nu+\frac{1}{4}}} P^{-\nu-\frac{1}{2}}_{\mu-\frac{1}{2}}(u)$$
$$\left[2abu = a^2+b^2+c^2, \quad \operatorname{Re}(a+b) > |\operatorname{Im} c|, \quad \operatorname{Re}(\nu\pm\mu) > -1, \quad \operatorname{Re}\nu > -1\right]$$
ET II 67(30)

11.[11] $$\int_0^\infty x^{\nu+1} K_\mu(ax) I_\mu(bx) J_\nu(cx)\,dx = \frac{(ab)^{-\nu-1} c^\nu e^{-\left(\nu+\frac{1}{2}\right)\pi i} Q^{\nu+\frac{1}{2}}_{\mu-\frac{1}{2}}(u)}{\sqrt{2\pi}(u^2-1)^{\frac{1}{2}\nu+\frac{1}{4}}} \qquad 2abu = a^2+b^2+c^2$$
$$[\operatorname{Re} a > |\operatorname{Re} b| + |\operatorname{Im} c|; \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1]$$
ET II 66(24)

12.[8] $$\int_0^\infty x^{\nu+1}[J_\nu(ax)]^2 Y_\nu(bx)\,dx = 0 \qquad \left[0 < b < 2a, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= \frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < 2a < b, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET II 109(3)

13. $$\int_0^\infty x^{\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx)\,dx$$
$$= 0 \qquad \left[a>0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}, \quad 0 < b < 2a\right]$$
$$= -\frac{2^{3\nu+1}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[a>0, \quad 2a < b < \infty, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET II 55(49)

14. $$\int_0^\infty x^{\nu+1} J_\mu(xa\sin\psi) J_\nu(xa\sin\varphi) K_\mu(xa\cos\varphi\cos\psi)\,dx$$
$$= \frac{2^\nu\Gamma(\mu+\nu+1)(\sin\varphi)^\nu\left(\cos\frac{\alpha}{2}\right)^{2\nu+1}}{a^{\nu+2}(\cos\psi)^{2\nu+2}} P^{-\mu}_\nu(\cos\alpha)$$
$$\left[\tan\tfrac{1}{2}\alpha = \tan\psi\cos\varphi, \quad a>0, \quad \frac{\pi}{2} > \varphi > 0, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\mu+\nu) > -1\right]$$
ET II 64(11)

15. $$\int_0^\infty x^{\nu+1}\, J_\nu(ax)\, K_\nu(bx)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(a^2+b^2+c^2\right)^2 - 4a^2c^2\right]^{\nu+\frac{1}{2}}}$$
$\left[\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 63(8)

16.[8] $$\int_0^\infty x^{\nu+1}\, I_\nu(ax)\, K_\nu(bx)\, J_\nu(cx)\, dx = \frac{2^{3\nu}(abc)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(b^2-a^2+c^2\right)^2 + 4a^2c^2\right]^{\nu+\frac{1}{2}}}$$
$\left[\operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|; \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 65(18)

6.579

1. $$\int_0^\infty x^{2\nu+1}\, J_\nu(ax)\, Y_\nu(ax)\, J_\nu(bx)\, Y_\nu(bx)\, dx$$
$$= \frac{a^{2\nu}\, \Gamma(3\nu+1)}{2\pi b^{4\nu+2}\, \Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(2\nu+\frac{3}{2}\right)}\, F\left(\nu+\frac{1}{2}, 3\nu+1; 2\nu+\frac{3}{2}; \frac{a^2}{b^2}\right)$$
$\left[0 < a < b, \quad -\frac{1}{3} < \operatorname{Re}\nu < \frac{1}{2}\right]$ EH II 94(45), ET II 352(15)

2. $$\int_0^\infty x^{2\nu+1}\, J_\nu(ax)\, K_\nu(ax)\, J_\nu(bx)\, K_\nu(bx)\, dx$$
$$= \frac{2^{\nu-3}a^{2\nu}\, \Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{3\nu+1}{2}\right)}{\sqrt{\pi}b^{4\nu+2}\, \Gamma(\nu+1)}\, F\left(\nu+\frac{1}{2}, \frac{3\nu+1}{2}; 2\nu+1; 1-\frac{a^4}{b^4}\right)$$
$\left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{3}\right]$ ET II 373(10)

3. $$\int_0^\infty x^{1-2\nu}\left[J_\nu(ax)\right]^4\, dx = \frac{\Gamma(\nu)\,\Gamma(2\nu)}{2\pi\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2\Gamma(3\nu)}$$ $[\operatorname{Re}\nu > 0]$ ET II 342(25)

4. $$\int_0^\infty x^{1-2\nu}\left[J_\nu(ax)\right]^2\left[J_\nu(bx)\right]^2\, dx = \frac{a^{2\nu-1}\,\Gamma(\nu)}{2\pi b\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)}\, F\left(\nu, \frac{1}{2}-\nu; 2\nu+\frac{1}{2}; \frac{a^2}{b^2}\right)$$
ET II 351(10)

6.581

1. $$\int_0^a x^{\lambda-1}\, J_\mu(x)\, J_\nu(a-x)\, dx = 2^\lambda \sum_{m=0}^\infty \frac{(-1)^m\, \Gamma(\lambda+\mu+m)\,\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)\,\Gamma(\mu+m+1)}\, J_{\lambda+\mu+\nu+2m}(a)$$
$[\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > -1]$
ET II 354(25)

2.[8] $$\int_0^a x^{\lambda-1}(a-x)^{-1}\, J_\mu(x)\, J_\nu(a-x)\, dx$$
$$= \frac{2^\lambda}{a\nu}\sum_{m=0}^\infty \frac{(-1)^m\, \Gamma(\lambda+\mu+m)\,\Gamma(\lambda+m)}{m!\,\Gamma(\lambda)\,\Gamma(\mu+m+1)}(\lambda+\mu+\nu+2m)\, J_{\lambda+\mu+\nu+2m}(a)$$
$[\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re}\nu > 0]$ ET II 354(27)

3. $$\int_0^a x^\mu (a-x)^\nu\, J_\mu(x)\, J_\nu(a-x)\, dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+1)} a^{\mu+\nu+\frac{1}{2}}\, J_{\mu+\nu+\frac{1}{2}}(a)$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 354(28), EH II 46(6)

4. $$\int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x)\,dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{2\pi}\,\Gamma(\mu+\nu+2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$$
$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET II 354(29)

5. $$\int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x)\,dx = \frac{2^\mu\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu)}{\sqrt{\pi}\,\Gamma(\mu+\nu+1)} a^\mu J_\nu(a)$$
$\left[\operatorname{Re}\nu > \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET II 355(30)

6.582 $$\int_0^\infty x^{\mu-1}|x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x)\,dx = \frac{1}{\sqrt{\pi}}(2b)^{-\mu}\,\Gamma\left(\frac{1}{2}-\mu\right)\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)\,K_\nu(b)$$
$\left[b>0, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|\right]$ ET II 374(14)

6.583 $$\int_0^\infty x^{\mu-1}(x+b)^{-\mu} K_\mu(x+b) K_\nu(x)\,dx = \frac{\sqrt{\pi}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)}{2^\mu b^\mu\,\Gamma\left(\mu+\frac{1}{2}\right)} K_\nu(b)$$
$[|\arg b| < \pi, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|]$ ET II 374(15)

6.584

1.[8] $$\int_0^\infty \frac{x^{\varrho-1}\left[H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}\left(axe^{\pi i}\right)\right]}{\left(x^2-r^2\right)^{m+1}}\,dx = \frac{\pi i}{2^m m!}\left(\frac{d}{r\,dr}\right)^m \left[r^{\varrho-2} H_\nu^{(1)}(ar)\right]$$
$\left[m=0,1,2,\ldots, \quad \operatorname{Im} r > 0, \quad a>0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m+\frac{7}{2}\right]$ WA 465

2.[8] $$\int_0^\infty \left[\cos\frac{1}{2}(\varrho-\nu)\pi\, J_\nu(ax) + \sin\frac{1}{2}(\varrho-\nu)\pi\, Y_\nu(ax)\right] \frac{x^{\varrho-1}}{\left(x^2+k^2\right)^{m+1}}\,dx = \frac{(-1)^{m+1}}{2^m\cdot m!}\left(\frac{d}{k\,dk}\right)^m \left[k^{\varrho-2} K_\nu(ak)\right]$$
$\left[m=0,1,2,\ldots, \quad \operatorname{Re}k>0, \quad a>0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2m+\frac{7}{2}\right]$ WA 466(2)

3. $$\int_0^\infty \{\cos\nu\pi\, J_\nu(ax) - \sin\nu\pi\, Y_\nu(ax)\} \frac{x^{1-\nu}\,dx}{\left(x^2+k^2\right)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m\cdot m! k^{\nu+m}}$$
$\left[m=0,1,2,\ldots, \quad \operatorname{Re}k>0, \quad a>0, \quad -2m-\frac{3}{2} < \operatorname{Re}\nu < 1\right]$ WA 466(3)

4. $$\int_0^\infty \left\{\cos\left[\left(\tfrac{1}{2}\varrho-\tfrac{1}{2}\nu-\mu\right)\pi\right] J_\nu(ax) + \sin\left[\left(\tfrac{1}{2}\varrho-\tfrac{1}{2}\nu-\mu\right)\pi\right] Y_\nu(ax)\right\} \frac{x^{\varrho-1}}{\left(x^2+k^2\right)^{\mu+1}}\,dx$$
$$= \frac{\pi k^{\varrho-2\mu-2}}{2\sin\nu\pi\cdot\Gamma(\mu+1)}\left[\frac{\left(\frac{1}{2}ak\right)^\nu\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\nu-\mu\right)}\,{}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2}-\mu, \nu+1; \frac{a^2k^2}{4}\right)\right.$$
$$\left. - \frac{\left(\frac{1}{2}ak\right)^{-\nu}\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\,\Gamma\left(\frac{1}{2}\varrho-\frac{1}{2}\nu-\mu\right)}\,{}_1F_2\left(\frac{\varrho-\nu}{2}; \frac{\varrho-\nu}{2}-\mu, 1-\nu; \frac{a^2k^2}{4}\right)\right]$$
$\left[a>0, \quad \operatorname{Re}k>0, \quad |\operatorname{Re}\nu| < \operatorname{Re}\varrho < 2\operatorname{Re}\mu+\frac{7}{2}\right]$ WA 407(1)

5.[8] $$\int_0^\infty \left[\prod_{j=1}^n J_{\mu_j}(b_n x)\right] \left\{\cos\left[\frac{1}{2}\left(\varrho + \sum_j \mu_j - \nu\right)\pi\right] J_\nu(ax) + \sin\left[\frac{1}{2}\left(\varrho + \sum_j \mu_j - \nu\right)\pi\right] Y_\nu(ax)\right\} \frac{x^{\varrho-1}}{x^2+k^2}\,dx = -\left[\prod_{j=1}^n I_{\mu_j}(b_n k)\right] K_\nu(ak) k^{\varrho-2}$$

$$\left[\operatorname{Re} k > 0, \quad a > \sum_j |\operatorname{Re} b_j|, \quad \operatorname{Re}\left(\varrho + \sum_j \mu_j\right) > |\operatorname{Re}\nu|\right]$$ WA 472(9)

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.591

1. $$\int_0^\infty x^{2\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} J_{1+2\nu}\left(\sqrt{2ab}\right) K_{1+2\nu}\left(\sqrt{2ab}\right)$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 142(35)

2. $$\int_0^\infty x^{2\nu+\frac{1}{2}} Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} Y_{2\nu+1}\left(\sqrt{2ab}\right) K_{2\nu+1}\left(\sqrt{2ab}\right)$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -1]$$ ET II 143(41)

3. $$\int_0^\infty x^{2\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} K_{2\nu+1}\left(e^{\frac{1}{4}i\pi}\sqrt{2ab}\right) K_{2\nu+1}\left(e^{-\frac{1}{4}i\pi}\sqrt{2ab}\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$$ ET II 146(56)

4. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} K_{2\nu-1}\left(\sqrt{2ab}\right) \times \left[\sin(\nu\pi) J_{2\nu-1}\left(\sqrt{2ab}\right) + \cos(\nu\pi) Y_{2\nu-1}\left(\sqrt{2ab}\right)\right]$$
$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu < 1]$$ ET II 142(34)

5. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} Y_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_\nu(bx)\,dx = -\sqrt{\frac{\pi}{2}} b^{\nu-1} a^{\frac{1}{2}-\nu} \sec(\nu\pi) K_{2\nu-1}\left(\sqrt{2ab}\right) \times \left[J_{2\nu-1}\left(\sqrt{2ab}\right) - J_{1-2\nu}\left(\sqrt{2ab}\right)\right]$$
$$[a > 0, \quad \operatorname{Re}\nu < 1]$$ ET II 143(40)

6. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} J_{\frac{1}{2}-\nu}\left(\frac{a}{x}\right) J_\nu(bx)\,dx = -\frac{1}{2} i \operatorname{cosec}(2\nu\pi) b^{\nu-1} a^{\frac{1}{2}-\nu} \left[e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v) - e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)\right]$$
$$\left[u = \left(\tfrac{1}{2}ab\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}, \quad v = \left(\tfrac{1}{2}ab\right)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}, \quad a > 0, \quad b > 0, \quad -\tfrac{1}{2} < \operatorname{Re}\nu < 3\right]$$ ET II 58(12)

7. $$\int_0^\infty x^{-2\nu+\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) Y_\nu(bx)\,dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} Y_{2\nu-1}\left(\sqrt{2ab}\right) K_{2\nu-1}\left(\sqrt{2ab}\right)$$

$\left[b>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\nu>\frac{1}{6}\right]$ ET II 113(30)

8. $$\int_0^\infty x^{\varrho-1} J_\mu(ax)\, J_\nu\left(\frac{b}{x}\right) dx = \frac{a^{\nu-\varrho} b^\nu\, \Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\varrho-\frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1}\,\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\varrho+1\right)} \times {}_0F_3\left(\nu+1, \frac{\nu-\mu-\varrho}{2}+1, \frac{\nu+\mu-\varrho}{2}+1; \frac{a^2b^2}{16}\right) + \frac{a^\mu b^{\mu+\varrho}\,\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1}\,\Gamma(\mu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\varrho+1\right)} \times {}_0F_3\left(\mu+1, \frac{\mu-\nu+\varrho}{2}+1, \frac{\nu+\mu+\varrho}{2}+1; \frac{a^2b^2}{16}\right)$$

$\left[a>0, \quad b>0, \quad -\operatorname{Re}\left(\mu+\frac{3}{2}\right) < \operatorname{Re}\varrho < \operatorname{Re}\left(\nu+\frac{3}{2}\right)\right]$ WA 480(1)

6.592

1. $$\int_0^\infty x^\lambda (1-x)^{\mu-1}\, Y_\nu\left(a\sqrt{x}\right) dx = 2^{-\nu} a^\nu \cot(\nu\pi) \frac{\Gamma(\mu)\,\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)\,\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)} \times {}_1F_2\left(\lambda+1+\frac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; -\frac{a^2}{4}\right) - 2^\nu a^{-\nu} \operatorname{cosec}(\nu\pi) \frac{\Gamma(\mu)\,\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\,\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)} \times {}_1F_2\left(\lambda-\frac{1}{2}\nu+1; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; -\frac{a^2}{4}\right)$$

$\left[\operatorname{Re}\lambda > -1+\frac{1}{2}|\operatorname{Re}\nu|, \quad \operatorname{Re}\mu>0\right]$ ET II 197(76)a

2.[10] $$\int_0^1 x^\lambda (1-x)^{\mu-1} K_\nu\left(a\sqrt{x}\right) dx$$

$$= 2^{-\nu-1} a^{-\nu} \frac{\Gamma(\nu)\,\Gamma(\mu)\,\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)}\ {}_1F_2\left(\lambda+1-\frac{1}{2}\nu; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; \frac{a^2}{4}\right)$$

$$+ 2^{-1-\nu} a^\nu \frac{\Gamma(-\nu)\,\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)\Gamma(\mu)}{\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)}\ {}_1F_2\left(\lambda+1+\frac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; \frac{a^2}{4}\right)$$

$$= \frac{2^{\nu-1}}{a^\nu}\,\Gamma(\mu)\, G_{13}^{21}\left(\frac{a^2}{4}\,\middle|\,\begin{matrix}\frac{\nu}{2}-\lambda \\ \nu, 0, \frac{\nu}{2}-\lambda-\mu\end{matrix}\right)$$

OB 159 (3.16)

$\left[\operatorname{Re}\lambda > -1+\frac{1}{2}|\operatorname{Re}\nu|, \quad \operatorname{Re}\mu>0\right]$ ET II 198(87)a

3.[11] $$\int_1^\infty x^\lambda (x-1)^{\mu-1} J_\nu\left(a\sqrt{x}\right) dx = 2^{2\lambda} a^{-2\lambda} G^{20}_{13}\left(\frac{a^2}{4}\middle|\begin{matrix}0\\ -\mu, \lambda+\frac{1}{2}\nu, \lambda-\frac{1}{2}\nu\end{matrix}\right)\Gamma(\mu)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{3}{4}-\operatorname{Re}\lambda\right]$ ET II 205(36)a

4. $$\int_1^\infty x^\lambda (x-1)^{\mu-1} K_\nu\left(a\sqrt{x}\right) dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G^{30}_{13}\left(\frac{a^2}{4}\middle|\begin{matrix}0\\ -\mu, \frac{1}{2}\nu+\lambda, -\frac{1}{2}\nu+\lambda\end{matrix}\right)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re}\mu>0]$ ET II 209(60)a

5. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} J_\nu\left(a\sqrt{x}\right) dx = \pi\left[J_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2$$ $[\operatorname{Re}\nu>-1]$ ET II 194(59)a

6. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} I_\nu\left(a\sqrt{x}\right) dx = \pi\left[I_{\frac{1}{2}\nu}\left(\frac{1}{2}a\right)\right]^2$$ $[\operatorname{Re}\nu>-1]$ ET II 197(79)

7. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} K_\nu\left(a\sqrt{x}\right) dx = \frac{1}{2}\pi\sec\left(\frac{1}{2}\nu\pi\right)\left[I_{\frac{\nu}{2}}\left(\frac{a}{2}\right)+I_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)$$
$[|\operatorname{Re}\nu|<1]$ ET II 198(85)a

8. $$\int_1^\infty x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}} K_\nu\left(a\sqrt{x}\right) dx = \left[K_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2$$ $[\operatorname{Re} a>0]$ ET II 208(56)a

9. $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} Y_\nu\left(a\sqrt{x}\right) dx = \pi\left\{\cot(\nu\pi)\left[J_{\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2-\operatorname{cosec}(\nu\pi)\left[J_{-\frac{\nu}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$[|\operatorname{Re}\nu|<1]$ ET II 195(68)a

10. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} J_\nu\left(a\sqrt{x}\right) dx = \Gamma(\mu) 2^\mu a^{-\mu} J_{\nu-\mu}(a)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 205(34)a

11. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} J_{-\nu}\left(a\sqrt{x}\right) dx = \Gamma(\mu) 2^\mu a^{-\mu}\left[\cos(\nu\pi) J_{\nu-\mu}(a)-\sin(\nu\pi) Y_{\nu-\mu}(a)\right]$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 205(35)a

12. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} K_\nu\left(a\sqrt{x}\right) dx = \Gamma(\mu) 2^\mu a^{-\mu} K_{\nu-\mu}(a)$$
$[\operatorname{Re} a>0, \quad \operatorname{Re}\mu>0]$ ET II 209(59)a

13. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} Y_\nu\left(a\sqrt{x}\right) dx = 2^\mu a^{-\mu} Y_{\nu-\mu}(a)\,\Gamma(\mu)$$
$\left[a>0, \quad 0<\operatorname{Re}\mu<\frac{1}{2}\operatorname{Re}\nu+\frac{3}{4}\right]$ ET II 206(40)a

14. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} H^{(1)}_\nu\left(a\sqrt{x}\right) dx = 2^\mu a^{-\mu} H^{(1)}_{\nu-\mu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Im} a>0]$ ET II 206(45)a

15. $$\int_1^\infty x^{-\frac{1}{2}\nu}(x-1)^{\mu-1} H^{(2)}_\nu\left(a\sqrt{x}\right) dx = 2^\mu a^{-\mu} H^{(2)}_{\nu-\mu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu>0, \quad \operatorname{Im} a<0]$ ET II 207(48)a

16. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} J_\nu\left(a\sqrt{x}\right) dx = \frac{2^{2-\nu}a^{-\mu}}{\Gamma(\nu)} s_{\mu+\nu-1,\mu-\nu}(a)$$
$[\operatorname{Re}\mu > 0]$ ET II 194(64)a

17. $$\int_0^1 x^{-\frac{1}{2}\nu}(1-x)^{\mu-1} Y_\nu\left(a\sqrt{x}\right) dx = \frac{2^{2-\nu}a^{-\mu}\cot(\nu\pi)}{\Gamma(\nu)} s_{\mu+\nu-1,\mu-\nu}(a) - 2^\mu a^{-\mu}\operatorname{cosec}(\nu\pi) J_{\mu-\nu}(a)\,\Gamma(\mu)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu < 1]$ ET II 196(75)a

6.593

1. $$\int_0^\infty \sqrt{x}\, J_{2\nu-1}\left(a\sqrt{x}\right) J_\nu(bx)\, dx = \frac{1}{2}ab^{-2} J_{\nu-1}\left(\frac{a^2}{4b}\right) \qquad \left[b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$
ET II 58(15)

2. $$\int_0^\infty \sqrt{x}\, J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \frac{\pi a}{4b^2}\left[I_{\nu-1}\left(\frac{a^2}{4b}\right) - \mathbf{L}_{\nu-1}\left(\frac{a^2}{4b}\right)\right]$$
$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 144(44)

6.594

1. $$\int_0^\infty x^\nu I_{2\nu-1}\left(a\sqrt{x}\right) J_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx = \sqrt{\pi}2^{-\nu}a^{2\nu-1}b^{-2\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(65)

2. $$\int_0^\infty x^\nu I_{2\nu-1}\left(a\sqrt{x}\right) Y_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx$$
$$= \sqrt{\pi}2^{-\nu-1}a^{2\nu-1}b^{-2\nu-\frac{1}{2}}\operatorname{cosec}(\nu\pi) \times\left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi) Y_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(66)

3. $$\int_0^\infty x^\nu J_{2\nu-1}\left(a\sqrt{x}\right) K_{2\nu-1}\left(a\sqrt{x}\right) K_\nu(bx)\, dx$$
$$= \pi^2 2^{-\nu-2}a^{2\nu-1}b^{-2\nu-\frac{1}{2}}\operatorname{cosec}(\nu\pi)\left[\mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - Y_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right)\right]$$
$[\operatorname{Re} b > 0, \quad \operatorname{Re}\nu > 0]$ ET II 148(67)

6.595

1. $$\int_0^\infty x^{\nu+1} J_\nu(cx)\prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i)\, dx = 0 \qquad z_i = \sqrt{x^2+b_i^2}$$
$$\left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c; \quad \operatorname{Re}\left(\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2}\right) > \operatorname{Re}\nu > -1\right]$$
EH II 52(33), ET II 60(26)

2. $$\int_0^\infty x^{\nu-1} J_\nu(cx)\prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i)\, dx = 2^{\nu-1}\Gamma(\nu)c^{-\nu}\prod_{i=1}^n \left[b_i^{-\mu_i} J_{\mu_i}(a_i b_i)\right] \qquad z_i = \sqrt{x^2+b_i^2}$$
$$\left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c, \quad \operatorname{Re}\left(\frac{1}{2}n + \sum_{i=1}^n \mu_i + \frac{3}{2}\right) > \operatorname{Re}\nu > 0\right]$$
EH II 52(34), ET II 60(27)

6.596

1. $$\int_0^\infty J_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{\left(x^2+z^2\right)^\nu}}\,dx = \frac{2^\mu\,\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}\,J_{\nu-\mu-1}(\alpha z)$$
$$\left[\alpha>0,\quad \operatorname{Re}\left(\frac{1}{2}\nu-\frac{1}{4}\right)>\operatorname{Re}\mu>-1\right]$$
WA 457(5)

2. $$\int_0^\infty \frac{J_\nu\left(\alpha\sqrt{t^2+1}\right)}{\sqrt{t^2+1}}\,dt = -\frac{\pi}{2}\,J_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right)Y_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right) \qquad [\operatorname{Re}\nu>-1,\quad \alpha>0]$$
MO 46

3. $$\int_0^\infty K_\nu\left(\alpha\sqrt{x^2+z^2}\right)\frac{x^{2\mu+1}}{\sqrt{\left(x^2+z^2\right)^\nu}}\,dx = \frac{2^\mu\,\Gamma(\mu+1)}{\alpha^{\mu+1}z^{\nu-\mu-1}}\,K_{\nu-\mu-1}(\alpha z)$$
$$[\alpha>0,\quad \operatorname{Re}\mu>-1]$$
WA 457(6)

4.[8] $$\int_0^\infty J_\nu(\beta x)\frac{J_{\mu-1}\left\{\alpha\sqrt{x^2+z^2}\right\}}{\left(x^2+z^2\right)^{\frac{1}{2}\mu+\frac{1}{2}}}x^{\nu+1}\,dx = \frac{\alpha^{\mu-1}z^\nu}{2^{\mu-1}\,\Gamma(\mu)}\,K_\nu(\beta z)$$
$$[\alpha<\beta,\quad \operatorname{Re}(\mu+2)>\operatorname{Re}\nu>-1]$$
ET II 59(19)

5.[8] $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left\{\alpha\sqrt{x^2+z^2}\right\}}{\sqrt{\left(x^2+z^2\right)^\mu}}x^{\nu-1}\,dx = \frac{2^{\nu-1}\,\Gamma(\nu)}{\beta^\nu}\frac{J_\mu(\alpha z)}{z^\mu}$$
$$[\operatorname{Re}(\mu+2)>\operatorname{Re}\nu>0,\quad \beta>\alpha>0]$$
WA 459(12)

6.[6] $$\int_0^\infty J_\nu(\beta x)\frac{J_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{\left(x^2+z^2\right)^\mu}}x^{\nu+1}\,dx$$
$$=0 \qquad [0<\alpha<\beta]$$
$$=\frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2-\beta^2}}{z}\right)^{\mu-\nu-1}J_{\mu-\nu-1}\left\{z\sqrt{\alpha^2-\beta^2}\right\} \qquad [\alpha>\beta>0]$$
$$[\operatorname{Re}\mu>\operatorname{Re}\nu>-1]$$
WA 415(1)

7.[8] $$\int_0^\infty J_\nu(\beta x)\frac{K_\mu\left(\alpha\sqrt{x^2+z^2}\right)}{\sqrt{\left(x^2+z^2\right)^\mu}}x^{\nu+1}\,dx = \frac{\beta^\nu}{\alpha^\mu}\left(\frac{\sqrt{\alpha^2+\beta^2}}{z}\right)^{\mu-\nu-1}K_{\mu-\nu-1}\left(z\sqrt{\alpha^2+\beta^2}\right)$$
$$\left[\alpha>0,\quad \beta>0,\quad \operatorname{Re}\nu>-1,\quad |\arg z|<\frac{\pi}{2}\right]$$
KU 151(31), WA 416(2)

8.[8] $$\int_0^\infty J_\nu(ux)\,K_\mu\left(v\sqrt{x^2-y^2}\right)\left(x^2-y^2\right)^{-\frac{\mu}{2}}x^{\nu+1}\,dx = \frac{\pi}{2}\exp\left[-i\pi\left(\mu-\nu-\frac{1}{2}\right)\right]\cdot\frac{u^\nu}{v^\mu}$$
$$\cdot\left[\frac{\sqrt{u^2+v^2}}{y}\right]^{\mu-\nu-1}H^{(2)}_{\mu-\nu-1}\left(y\sqrt{u^2+v^2}\right)$$
$$\left[\operatorname{Re}\mu<1,\quad \operatorname{Re}\nu>-1, u>0,\quad v>0, y>0;\quad \left(x^2-y^2\right)^{\frac{1}{2}\alpha}=e^{\frac{1}{2}\alpha\pi i}\left(y^2-n^2\right)^{\frac{1}{2}\alpha}\text{ if }x<y\right]$$

9.[8] $$\int_0^\infty J_\nu(ux)\, H_\mu^{(2)}\left(v\sqrt{x^2+y^2}\right)\left(x^2+y^2\right)^{-\frac{\mu}{2}} x^{\nu+1}\, dx = \frac{u^\nu}{v^\mu}\left[\frac{\sqrt{v^2-u^2}}{y}\right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)}\left(y\sqrt{v^2-u^2}\right)$$
$[u < v]$

$$\left[\operatorname{Re}\mu > \operatorname{Re}\nu > -1, \quad u > 0, \quad v > 0, \quad y > 0;, \quad \arg\sqrt{v^2-u^2} = 0, \text{ for } v > u\right.$$
$$\left.\arg\left(v^2-u^2\right)^\sigma = -\pi\sigma \text{ for } v < u, \text{ where } \sigma = \frac{1}{2} \text{ or } \sigma = \frac{\mu-\nu-1}{2}\right]$$

MO 43

10.[8] $$\int_0^\infty J_\nu(\beta x)\, J_\mu\left(\alpha\sqrt{x^2+z^2}\right) J_\mu\left(\gamma\sqrt{x^2+z^2}\right)\frac{x^{\nu-1}}{\left(x^2+z^2\right)^\mu}\, dx = \frac{2^{\nu-1}\,\Gamma(\nu)}{\beta^\nu}\frac{J_\mu(\alpha z)}{z^\mu}\frac{J_\mu(\gamma z)}{z^\mu}$$
$\left[\alpha > 0; \quad \beta > \alpha+\gamma; \quad \gamma > 0, \quad \operatorname{Re}\left(2\mu+\frac{5}{2}\right) > \operatorname{Re}\nu > 0\right]$ WA 459(14)

11.[8] $$\int_0^\infty J_\nu(\beta t) t^{\nu-1}\prod_{k=1}^n J_\mu\left(\alpha_k\sqrt{t^2+x^2}\right)\sqrt{\left(t^2+x^2\right)^{-n\mu}}\, dt = 2^{\nu-1}\beta^{-\nu}\,\Gamma(\nu)\prod_{k=1}^n\left[x^{-\mu} J_\mu\left(\alpha_k x\right)\right]$$
$$\left[x > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \ldots, \alpha_n > 0, \quad \beta > \prod_{k=1}^n \alpha_k; \quad \operatorname{Re}\left(n\mu+\frac{1}{2}n+\frac{1}{2}\right) > \operatorname{Re}\nu > 0\right]$$

MO 43

12.[8] $$\int_0^\infty \frac{J_\mu^2\left(\sqrt{a^2+x^2}\right)}{\left(a^2+x^2\right)^\nu} x^{2\nu-2}\, dx = \frac{\Gamma\left(\nu-\frac{1}{2}\right)}{2a^{\nu+1}\sqrt{\pi}}\mathbf{H}_\nu(2a) \qquad \left[\operatorname{Re}\nu > \frac{1}{2}\right]$$ WA 457(8)

6.597 $$\int_0^\infty t^{\nu+1} J_\mu\left[b\left(t^2+y^2\right)^{\frac{1}{2}}\right]\left(t^2+y^2\right)^{-\frac{1}{2}\mu}\left(t^2+\beta^2\right)^{-1} J_\nu(at)\, dt = \beta^\nu J_\mu\left[b\left(y^2-\beta^2\right)^{\frac{1}{2}}\right]\left(y^2-\beta^2\right)^{-\frac{1}{2}\mu} K_\nu(a\beta)$$
$[a \ge b, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < 2+\operatorname{Re}\mu]$ EH II 95(56)

6.598 $$\int_0^1 x^{\frac{\mu}{2}}(1-x)^{\frac{\nu}{2}} J_\mu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{1-x}\right)\, dx = 2a^\mu b^\nu\left(a^2+b^2\right)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}\left(\sqrt{a^2+b^2}\right)$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ EH II 46a

6.61 Combinations of Bessel functions and exponentials

6.611

1. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left[\sqrt{\alpha^2+\beta^2}-\alpha\right]^\nu}{\sqrt{\alpha^2+\beta^2}} \qquad [\operatorname{Re}\nu > -1, \quad \operatorname{Re}(\alpha \pm i\beta) > 0]$$

EH II 49(18), WA 422(8)

2. $$\int_0^\infty e^{-\alpha x}\,Y_\nu(\beta x)\,dx = \left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\operatorname{cosec}(\nu\pi) \times\left\{\beta^\nu\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{-\nu}\cos(\nu\pi)-\beta^{-\nu}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\alpha\right]^{\nu}\right\}$$

$[\operatorname{Re}\alpha>0, \quad \beta>0, \quad |\operatorname{Re}\nu|<1]$ MO 179, ET II 105(1)

3. $$\int_0^\infty e^{-\alpha x}\,K_\nu(\beta x)\,dx = \frac{\pi}{\beta\sin(\nu\pi)}\frac{\sin(\nu\theta)}{\sin\theta}$$

$\left[\cos\theta=\dfrac{\alpha}{\beta}; \quad \theta\to\dfrac{\pi}{2} \quad \text{for } \beta\to\infty\right]$
ET II 131(22)

$$= \frac{\pi\operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2-\beta^2}}\left[\beta^{-\nu}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^{\nu}-\beta^{\nu}\left(\sqrt{\alpha^2-\beta^2}+\alpha\right)^{-\nu}\right]$$

$[|\operatorname{Re}\nu|<1, \quad \operatorname{Re}(\alpha+\beta)>0]$
ET I 197(24), MO 180

4.[8] $$\int_0^\infty e^{-\alpha x}\,I_\nu(\beta x)\,dx = \frac{\beta^{-\nu}\left[\alpha-\sqrt{\alpha^2-\beta^2}\right]^{\nu}}{\sqrt{\alpha^2-\beta^2}}$$ $[\operatorname{Re}\nu>-1, \quad \operatorname{Re}\alpha>|\operatorname{Re}\beta|]$

MO 180, ET I 195(1)

5. $$\int_0^\infty e^{-\alpha x}\,H_\nu^{(1,2)}(\beta x)\,dx = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^{\nu}}{\beta^\nu\sqrt{\alpha^2+\beta^2}}\left\{1\pm\frac{i}{\sin(\nu\pi)}\left[\cos(\nu\pi)-\frac{\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)^{2\nu}}{b^{2\nu}}\right]\right\}$$

[$-1<\operatorname{Re}\nu<1$; a plus sign corresponds to the function $H_\nu^{(1)}$, a minus sign to the function $H_\nu^{(2)}$.]
MO 180, ET I188(54, 55)

6. $$\int_0^\infty e^{-\alpha x}\,H_0^{(1)}(\beta x)\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\left\{1-\frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\}$$

$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ MO 180, ET I 188(53)

7. $$\int_0^\infty e^{-\alpha x}\,H_0^{(2)}(\beta x)\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\left\{1+\frac{2i}{\pi}\ln\left[\frac{\alpha}{\beta}+\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}\right]\right\}$$

$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ MO 180, ET I 188(53)

8. $$\int_0^\infty e^{-\alpha x}\,Y_0(\beta x)\,dx = \frac{-2}{\pi\sqrt{\alpha^2+\beta^2}}\ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}$$

$[\operatorname{Re}\alpha>|\operatorname{Im}\beta|]$ MO 47, ET I 187(44)

9.[11] $$\int_0^\infty e^{-\alpha x}\,K_0(\beta x)\,dx = \frac{\arccos\frac{\alpha}{\beta}}{\sqrt{\beta^2-\alpha^2}}$$ $[\operatorname{Re}(\alpha+\beta)>0]$ WA 424, ET II 131(22)

$$= \frac{1}{\sqrt{\alpha^2-\beta^2}}\ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}-1}\right)$$ $[\operatorname{Re}(\alpha+\beta)>0]$

MO 48

10.[10] $$\int_a^b \alpha\,d\alpha \int_0^\infty dk\, J_1(k\alpha) e^{-k|\beta|} = \int_a^b \left(1 - \frac{|\beta|}{\sqrt{\alpha^2+\beta^2}}\right) d\alpha$$

(see **3.241** 6)

6.612

1. $$\int_0^\infty e^{-2\alpha x} J_0(x)\, Y_0(x)\, dx = \frac{\boldsymbol{K}\left[\alpha\left(\alpha^2+1\right)^{-\frac{1}{2}}\right]}{\pi\left(\alpha^2+1\right)^{\frac{1}{2}}} \qquad [\operatorname{Re}\alpha > 0]$$ ET II 347(58)

2. $$\int_0^\infty e^{-2\alpha x} I_0(x)\, K_0(x)\, dx = \frac{1}{2}\boldsymbol{K}\left[\left(1-\alpha^2\right)^{\frac{1}{2}}\right] \qquad [0 < \alpha < 1]$$
$$= \frac{1}{2\alpha}\boldsymbol{K}\left[\left(1-\frac{1}{\alpha^2}\right)^{\frac{1}{2}}\right] \qquad [1 < \alpha < \infty]$$ ET II 370(48)

3. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{\pi\sqrt{\gamma\beta}}\, Q_{\nu-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}\right)$$
$$\left[\operatorname{Re}\left(\alpha \pm i\beta \pm i\gamma\right) > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ WA 426(2), ET II 50(17)

4. $$\int_0^\infty e^{-\alpha x}\left[J_0(\beta x)\right]^2\, dx = \frac{2}{\pi\sqrt{\alpha^2+4\beta^2}}\boldsymbol{K}\left(\frac{2\beta}{\sqrt{\alpha^2+4\beta^2}}\right)$$ MO 178

5. $$\int_0^\infty e^{-2\alpha x} J_1^2(\beta x)\, dx = \frac{\left(2\alpha^2+\beta^2\right)\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) - 2\left(\alpha^2+\beta^2\right)\boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{\pi\beta^2\sqrt{\alpha^2+\beta^2}}$$ WA 428(3)

6. $$\int_0^\infty e^{-3x} I_l(x)\, I_m(x)\, I_n(x)\, dx = r_1 g + \frac{r_2}{\pi^2 g} + r_3$$
where
$$g = \frac{\sqrt{3}-1}{96\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right)$$
and

(lmn)	r_1	r_2	r_3
000	1	0	0
100	1	0	$-1/3$
110	$5/12$	$-1/2$	0
111	$-1/8$	$3/4$	0
200	$10/3$	2	-2
210	$3/8$	$-9/4$	$1/3$
211	$-2/3$	2	0
220	$73/36$	$-29/6$	0
221	$-15/16$	$21/8$	0
222	$5/8$	$-27/20$	0
300	$35/2$	21	-13
310	$-79/36$	$-85/6$	4
311	$-11/4$	$21/2$	$-2/3$
320	$319/48$	$-119/8$	$-1/3$
321	$-125/36$	$269/30$	0
322	$35/16$	$-213/40$	0
330	$50/3$	$-1046/25$	0
331	$-35/3$	$148/5$	0
332	$35/9$	$-1012/105$	0
333	$-35/16$	$1587/280$	0
400	$994/9$	$542/3$	-92
410	$-515/16$	$-879/8$	$115/3$
411	$-9/2$	$357/5$	-12
420	$12907/120$	$-13903/10$	-6
421	$-229/16$	$1251/40$	1
422	$35/3$	$-1024/35$	0
430	$2641/48$	$-28049/200$	$1/3$
431	$-1505/36$	$118051/1050$	0

(lmn)	r_1	r_2	r_3
432	$525/32$	$-4617/112$	0
433	$-595/72$	$8809/420$	0
440	$6025/36$	$-620161/1470$	0
441	$-29175/224$	$131379/400$	0
442	$2975/48$	$-31231/200$	0
443	$-539/32$	$119271/2800$	0
444	$77/8$	$-186003/7700$	0
500	$9287/12$	$3005/2$	$-2077/3$
510	$-189029/180$	$-138331/50$	348
511	$275/4$	$5751/10$	-150
520	$2897/16$	$-15123/20$	$-229/3$
521	$-937/12$	$27059/30$	24
522	$509/8$	$-4209/28$	0
530	$3589/18$	$-1993883/3075$	0
531	$-1329/8$	$297981/700$	$-4/3$
532	$2555/36$	$-187777/1050$	0
533	$-2233/48$	$164399/1400$	0
540	$18471/32$	$-28493109/19600$	$-1/3$
541	$-1390/3$	$286274/245$	0
542	$7777/32$	$-1715589/2800$	0
543	$-5621/72$	$4550057/23100$	0
544	$1155/32$	$-560001/6160$	0
550	$197045/108$	$-101441689/22050$	0
551	$-12023/8$	$18569853/4900$	0
552	$1683/2$	$-5718309/2695$	0
553	$-5159/16$	$2504541/3080$	0
554	$24563/312$	$-1527851/77000$	0
555	$-9251/208$	$12099711/107800$	0

6.613[11] $$\int_0^\infty e^{-xz} J_{\nu+\frac{1}{2}}\left(\frac{x^2}{2}\right) dx = \frac{\Gamma(\nu+1)}{\sqrt{\pi}} D_{-\nu-1}\left(ze^{\frac{\pi}{4}i}\right) D_{-\nu-1}\left(ze^{-\frac{\pi i}{4}}\right) \qquad [\operatorname{Re}\nu > -1]$$ MO 122

6.614

1. $$\int_0^\infty e^{-\alpha x} J_\nu\left(\beta\sqrt{x}\right) dx = \frac{\beta}{4}\sqrt{\frac{\pi}{\alpha^3}} \exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$$= \frac{1}{\alpha} e^{-\beta^2/4\alpha} \qquad [\nu = 0]$$
MO 178

2. $$\int_0^\infty e^{-\alpha x} Y_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}\left\{\cot(\nu\pi)\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right) - \operatorname{cosec}(\nu\pi)\, W_{\frac{1}{2},nu}\left(\frac{\beta}{\alpha}\right)\right\}$$
$$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$$ ET I 188(50)a

3. $$\int_0^\infty e^{-\alpha x} I_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}}\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right)$$
$$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$$ ET I 197(20)a

4. $$\int_0^\infty e^{-\alpha x} K_{2\nu}\left(2\sqrt{\beta x}\right) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}} \Gamma(\nu+1)\,\Gamma(1-\nu)\; W_{-\frac{1}{2},\nu}\left(\frac{\beta}{\alpha}\right)$$

$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$ ET I 199(37)a

5. $$\int_0^\infty e^{-\alpha x} K_1\left(\beta\sqrt{x}\right) dx = \frac{\beta}{8}\sqrt{\frac{\pi}{\alpha^3}} \exp\left(\frac{\beta^2}{8\alpha}\right)\left[K_1\left(\frac{\beta^2}{8\alpha}\right) - K_0\left(\frac{\beta^2}{8\alpha}\right)\right]$$ MO 181

6.615 $$\int_0^\infty e^{-\alpha x} J_\nu\left(2\beta\sqrt{x}\right) J_\nu\left(2\gamma\sqrt{x}\right) dx = \frac{1}{\alpha} I_\nu\left(\frac{2\beta\gamma}{\alpha}\right) \exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right) \qquad [\operatorname{Re}\nu > -1]$$

MO 178

6.616

1. $$\int_0^\infty e^{-\alpha x} J_0\left(\beta\sqrt{x^2+2\gamma x}\right) dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \exp\left[\gamma\left(\alpha - \sqrt{\alpha^2+\beta^2}\right)\right]$$ MO 179

2. $$\int_1^\infty e^{-\alpha x} J_0\left(\beta\sqrt{x^2-1}\right) dx = \frac{1}{\sqrt{\alpha^2+\beta^2}} \exp\left(-\sqrt{\alpha^2+\beta^2}\right)$$ MO 179

3. $$\int_{-\infty}^\infty e^{itx} H_0^{(1)}\left(r\sqrt{\alpha^2-t^2}\right) dt = -2i\frac{e^{i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$

$\left[0 \le \arg\sqrt{\alpha^2-t^2} < \pi, \quad 0 \le \arg\alpha < \pi; \quad r \text{ and } x \text{ are real}\right]$ MO 49

4. $$\int_{-\infty}^\infty e^{-itx} H_0^{(2)}\left(r\sqrt{\alpha^2-t^2}\right) dt = 2i\frac{e^{-i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$

$\left[-\pi < \arg\sqrt{\alpha^2-t^2} \le 0, \quad -\pi < \arg\alpha \le 0, \quad r \text{ and } x \text{ are real}\right]$ MO 49

5.[3] $$\int_{-1}^1 e^{-ax} I_0\left(b\sqrt{1-x^2}\right) dx = 2\left(a^2+b^2\right)^{-1/2} \sinh\sqrt{a^2+b^2}$$

$[a > 0, \quad b > 0]$

6.[8] $$\int_0^\infty e^{-xy} J_0\left[y\sqrt{1-x^2}\right]/(\alpha+y)\,dy = \sum_{n=0}^\infty n!\frac{P_n(x)}{\alpha^{n+1}}$$

6.617

1. $$\int_0^\infty K_{q-p}\left(2z\sinh x\right) e^{(p+q)x}\,dx = \frac{\pi^2}{4\sin[(p-q)\pi]}\left[J_p(z)\;Y_q(z) - J_q(z)\;Y_p(z)\right]$$

$[\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(p-q) < 1]$

MO 44

2. $$\int_0^\infty K_0\left(2z\sinh x\right) e^{-2px}\,dx = -\frac{\pi}{4}\left\{J_p(z)\frac{\partial\,Y_p(z)}{\partial p} - Y_p(z)\frac{\partial\,J_p(z)}{\partial p}\right\}$$

$[\operatorname{Re} z > 0]$ MO 44

6.618

1. $$\int_0^\infty e^{-\alpha x^2} J_\nu(\beta x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \qquad [\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > -1]$$

WA 432(5), ET II 29(8)

2. $$\int_0^\infty e^{-\alpha x^2}\, Y_\nu(\beta x)\,dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[\tan\frac{\nu\pi}{2}\, I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi}\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)\right]$$
$[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad |\operatorname{Re}\nu| < 1]$
WA 432(6), ET II 106(3)

3. $$\int_0^\infty e^{-\alpha x^2}\, K_\nu(\beta x)\,dx = \frac{1}{4}\sec\left(\frac{\nu\pi}{2}\right)\frac{\sqrt{\pi}}{\sqrt{\alpha}}\exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)$$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\nu| < 1]$
EH II 51(28), ET II 132(24)

4. $$\int_0^\infty e^{-\alpha x^2}\, I_\nu(\beta x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right)$$ $[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\alpha > 0]$ EH II 92(27)

5. $$\int_0^\infty e^{-\alpha x^2}\, J_\mu(\beta x)\, J_\nu(\beta x)\,dx$$
$$= 2^{-\nu-\mu-1}\alpha^{-\frac{\nu+\mu+1}{2}}\beta^{\nu+\mu}\frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\, {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}\alpha > 0]$ EH II 50(21)a

6.62–6.63 Combinations of Bessel functions, exponentials, and powers

6.621 Notation:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}\right], \quad \ell_2 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} + \sqrt{(a-\rho)^2+z^2}\right]$$

1. $$\int_0^\infty e^{-\alpha x}\, J_\nu(\beta x) x^{\mu-1}\,dx$$
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu\,\Gamma(\nu+1)}\, F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$
WA 421(2)
$$= \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu\,\Gamma(\nu+1)}\left(1+\frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2}+1; \nu+1; -\frac{\beta^2}{\alpha^2}\right)$$
WA 421(3)
$$= \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{\left(\alpha^2+\beta^2\right)^{\nu+\mu}}\,\Gamma(\nu+1)}\, F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$[\operatorname{Re}(\nu+\mu) > 0, \quad \operatorname{Re}(\alpha+i\beta) > 0, \quad \operatorname{Re}(\alpha-i\beta) > 0]$
WA 421(3)
$$= \left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\mu}\Gamma(\nu+\mu)\, P_{\mu-1}^{-\nu}\left[\alpha\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\right]$$
$[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}(\nu+\mu) > 0]$
ET II 29(6)

2. $$\int_0^\infty e^{-\alpha x}\, Y_\nu(\beta x) x^{\mu-1}\, dx$$
$$= \cot\nu\pi \frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu+\mu)}{\sqrt{(\alpha^2+\beta^2)^{\nu+\mu}}\,\Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$$- \operatorname{cosec}\nu\pi \frac{\left(\frac{\beta}{2}\right)^{-\nu} \Gamma(\mu-\nu)}{\sqrt{(\alpha^2+\beta^2)^{\mu-\nu}}\,\Gamma(1-\nu)} F\left(\frac{\mu-\nu}{2}, \frac{1-\nu-\mu}{2}; 1-\nu; \frac{\beta^2}{\alpha^2+\beta^2}\right)$$
$[\operatorname{Re}\mu \geq |\operatorname{Re}\nu|, \quad \operatorname{Re}(\alpha \pm i\beta) > 0]$

WA 421(4)

$$= -\frac{2}{\pi}\Gamma(\nu+\mu)\left(\beta^2+\alpha^2\right)^{-\frac{1}{2}\mu} Q_{\mu-1}^{-\nu}\left[\alpha\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\right]$$
$[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\mu > |\operatorname{Re}\nu|]$

ET II 105(2)

3. $$\int_0^\infty x^{\mu-1} e^{-\alpha x}\, K_\nu(\beta x)\, dx = \frac{\sqrt{\pi}(2\beta)^\nu}{(\alpha+\beta)^{\mu+\nu}} \frac{\Gamma(\mu+\nu)\,\Gamma(\mu-\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)} F\left(\mu+\nu, \nu+\frac{1}{2}; \mu+\frac{1}{2}; \frac{\alpha-\beta}{\alpha+\beta}\right)$$
$[\operatorname{Re}\mu > |\operatorname{Re}\nu|, \quad \operatorname{Re}(\alpha+\beta) > 0]$

ET II 131(23)a, EH II 50(26)

4. $$\int_0^\infty x^{m+1} e^{-\alpha x}\, J_\nu(\beta x)\, dx = (-1)^{m+1}\beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}}\left[\frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\sqrt{\alpha^2+\beta^2}}\right]$$
$[\beta > 0, \quad \operatorname{Re}\nu > -m-2]$ ET II 28(3)

5.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{-3/2}\, dx$$
$$= \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\left\{\frac{\ell_1}{2}\sqrt{a^2-\ell_1^2} + \frac{a^2}{2}\arcsin\left(\frac{\ell_1}{2}\right) + z\left[\sqrt{\rho^2-\ell_1^2}-\rho\right]\right\}$$
$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$

6.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{-1/2}\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\left[\rho - \sqrt{\rho^2-\ell_1^2}\right]$$
$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$

7.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{1/2}(\rho x)\, x^{1/2}\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi\rho}}\frac{\ell_1\sqrt{a^2-\ell_1^2}}{\ell_2^2-\ell_1^2}$$
$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$

8.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{3/2}(\rho x)\, x^{1/2}\, dx = \sqrt{\frac{2}{\pi}}\frac{\ell_1^2\sqrt{\rho^2-\ell_1^2}}{\rho^{3/2}a\left(\ell_2^2-\ell_1^2\right)}$$
$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$

9.[10] $$\int_0^\infty e^{-zx}\, J_1(ax)\, J_{3/2}(\rho x)\, x^{-3/2}\, dx = \frac{1}{\sqrt{2\pi}}\frac{1}{\rho^{3/2}a}\left[a^2\arcsin\left(\frac{\ell_1}{a}\right) - \ell_1\sqrt{a^2-\ell_1^2}\right]$$
$[\arg a > 0, \quad \arg\rho > 0, \quad \arg z > 0]$

10.[10] $$\int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-1/2}\,dx = \frac{1}{\sqrt{2\pi}} \frac{z}{\rho^{5/2} a} \left[\ell_1 \sqrt{a^2 - \ell_1^2} + \frac{2a^2 \ell_1}{\sqrt{a^2 - \ell_1^2}} - 3a^2 \arcsin\left(\frac{\ell_1}{a}\right) \right]$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

11.[10] $$\int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-3/2}\,dx$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left[\frac{\ell_1}{\sqrt{a^2 - \ell_1^2}} \left(\frac{7a^2}{8} - a^2 z^2 - \frac{\ell_1^4}{4} - \frac{5a^2 \ell_1^2}{8} \right) - \frac{1}{2}\left(\ell_1^2 + \ell_2^2\right) \ell_1 \sqrt{a^2 - \ell_1^2} + \arcsin\left(\frac{\ell_1}{a}\right) \left(\frac{3}{2} a^2 z^2 + \frac{1}{2} a^2 \rho^2 - \frac{3a^4}{8} \right) \right]$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

12.[10] $$\int_0^\infty e^{-zx} J_1(ax) J_{5/2}(\rho x) x^{-5/2}\,dx$$
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\rho^{5/2} a} \left\{ \frac{2\left[\rho^{5/2} - \left(\rho^2 - \ell_1^2\right)^{5/2}\right]}{15} + z a^2 \arcsin\left(\frac{\ell_1}{a}\right) \left[\frac{3a^2}{8} - \frac{\rho^2}{2} - \frac{z^2}{2} \right] + z \ell_1 \sqrt{a^2 - \ell_1^2} \left[\frac{\rho^2}{2} - \frac{3a^2}{8} + \frac{z^2}{6} - \frac{\ell_1^2}{4} \right] + \frac{z^3 a^2 \ell_1}{3\sqrt{a^2 - \ell_1^2}} \right\}$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

13.[10] $$\int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{1/2}\,dx = \sqrt{\frac{2}{\pi}} a^2 \rho^{3/2} \frac{\sqrt{\ell_2^2 - \rho^2}}{\left(\ell_2^2 - \ell_1^2\right) \ell_2^4}$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

14.[10] $$\int_0^\infty e^{-zx} J_2(ax) J_{3/2}(\rho x) x^{-1/2}\,dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{a^2} \left[\frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{\left(\rho^2 - \ell_1^2\right)^{3/2}}{3\rho^3} \right]$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

15.[10] $$\int_0^\infty e^{-zx} J_3(ax) J_{1/2}(\rho x) x^{-1/2}\,dx$$
$$= \sqrt{\frac{2}{\pi\rho}} \frac{1}{3a^3} \left\{ \rho\left[3a^2 - 4\rho^2 + 12z^2\right] - \sqrt{\rho^2 - \ell_1^2} \left\{12\ell_2^2 - 16\rho^2 + 4\ell_1^2 - 3a^2\right\} \right\}$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

16.[10] $$\int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{1/2}\,dx$$
$$= \sqrt{\frac{2}{\pi}} \rho^{3/2} \left\{ \frac{4}{a^3} \left[\frac{2}{3} - \frac{\sqrt{\rho^2 - \ell_1^2}}{\rho} + \frac{\left(\rho^2 - \ell_1^2\right)^{3/2}}{3\rho^2} \right] - \frac{a\sqrt{\ell_2^2 - a^2}}{\left(\ell_2^2 - \ell_1^2\right) \ell_2^3} \right\}$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

17.[10] $$\int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-1/2}\,dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left[\sqrt{\ell_2^2 - \rho^2} \left(\frac{4\rho^2\left(2\rho^2 - \ell_1^2\right) - \ell_1^4}{\rho^4} \right) - 8z \right]$$
$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

18.[10] $$\int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-3/2}\, dx = \sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ a^2 - \frac{4}{5}\rho^2 + 4z^2 - \sqrt{\rho^2 - \ell_1^2} \left[\frac{4\ell_2^2}{\rho} - \frac{24\rho}{5} + \frac{8\ell_1^2}{5\rho} - \frac{a^2}{\rho} + \frac{\ell_1^4}{5\rho^3} \right] \right\}$$

$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

19.[10] $$\int_0^\infty e^{-zx} J_3(ax) J_{3/2}(\rho x) x^{-5/2}\, dx = -\sqrt{\frac{2}{\pi}} \frac{\rho^{3/2}}{3a^3} \left\{ \left(a^2 - \frac{4}{5}\rho^2 \right) z + \frac{4z^3}{3} + \sqrt{\ell_2^2 - \rho^2} \left[a^2 + \frac{32}{15}\rho^2 - \frac{12}{5}\ell_1^2 - \frac{4}{3}\ell_2^2 + \frac{2\ell_1^4}{5\rho^2} + \frac{a^4\ell_1^2}{16\rho^4} + \frac{a^2\ell_1^2}{24\rho^4} + \frac{\ell_1^6}{30\rho^4} \right] - \frac{a^6}{16\rho^3} \arcsin\left(\frac{\rho}{\ell_2} \right) \right\}$$

$[\arg a > 0, \quad \arg \rho > 0, \quad \arg z > 0]$

6.622

1. $$\int_0^\infty \left(J_0(x) - e^{-\alpha x} \right) \frac{dx}{x} = \ln 2\alpha$$ $[\alpha > 0]$ NT 66(13)

2. $$\int_0^\infty \frac{e^{i(u+x)}}{u+x} J_0(x)\, dx = \frac{\pi}{2} i\, H_0^{(1)}(u)$$ MO 44

3.[8] $$\int_0^\infty e^{-x\cosh\alpha} I_\nu(x) x^{\mu-1}\, dx = \sqrt{\frac{2}{\pi}} e^{-\left(\mu-\frac{1}{2}\right)\pi i} \frac{Q_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(\cosh\alpha)}{\sinh^{\mu-\frac{1}{2}}\alpha}$$ $[\mathrm{Re}(\mu+\nu) > 0, \quad \mathrm{Re}(\cosh\alpha) > 1]$ WA 388(6)a

6.623

1. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu\, dx = \frac{(2\beta)^\nu\, \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}\left(\alpha^2+\beta^2\right)^{\nu+\frac{1}{2}}}$$ $\left[\mathrm{Re}\,\nu > -\frac{1}{2}, \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|\right]$ WA 422(5)

2. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1}\, dx = \frac{2\alpha(2\beta)^\nu\, \Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{\pi}\left(\alpha^2+\beta^2\right)^{\nu+\frac{3}{2}}}$$ $[\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|]$ WA 422(6)

3. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) \frac{dx}{x} = \frac{\left(\sqrt{\alpha^2+\beta^2} - \alpha\right)^\nu}{\nu\beta^\nu}$$ $[\mathrm{Re}\,\nu > 0; \quad \mathrm{Re}\,\alpha > |\mathrm{Im}\,\beta|]$ (cf. **6.611** 1) WA 422(7)

6.624

1. $$\int_0^\infty x e^{-\alpha x} K_0(\beta x)\, dx = \frac{1}{\alpha^2 - \beta^2} \left\{ \frac{\alpha}{\sqrt{\alpha^2-\beta^2}} \ln\left[\frac{\alpha}{\beta} + \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - 1} \right] - 1 \right\}$$ MO 181

2. $$\int_0^\infty \sqrt{x} e^{-\alpha x} K_{\pm\frac{1}{2}}(\beta x)\,dx = \sqrt{\frac{\pi}{2\beta}} \frac{1}{\alpha+\beta}$$ MO 181

3. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-1/2}} K_\mu(t) t^\nu\,dt = \frac{\Gamma(\nu-\mu+1)}{\left(z^2-1\right)^{-\frac{1}{2}(\nu+1)}} e^{i\mu\pi} Q_\nu^\mu(z)$$
$[\operatorname{Re}(\nu\pm\mu) > -1]$ EH II 57(7)

4. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-1/2}} I_{-\mu}(t) t^\nu\,dt = \frac{\Gamma(-\nu-\mu)}{\left(z^2-1\right)^{\frac{1}{2}\nu}} P_\nu^\mu(z)$$ $[\operatorname{Re}(\nu+\mu) < 0]$ EH II 57(8)

5. $$\int_0^\infty e^{-tz\left(z^2-1\right)^{-\frac{1}{2}}} I_\mu(t) t^\nu\,dt = \frac{\Gamma(\nu+\mu+1)}{\left(z^2-1\right)^{-\frac{1}{2}(\nu+1)}} P_\nu^{-\mu}(z)$$
$[\operatorname{Re}(\nu+\mu) > -1]$ EH II 57(9)

6. $$\int_0^\infty e^{-t\cos\theta} J_\mu(t\sin\theta)\, t^\nu\,dt = \Gamma(\nu+\mu+1)\, P_\nu^{-\mu}(\cos\theta)$$
$\left[\operatorname{Re}(\nu+\mu) > -1, \quad 0 \le \theta < \frac{1}{2}\pi\right]$ EH II 57(10)

7. $$\int_0^\infty \frac{J_\nu(bx)x^\nu}{e^{\pi x}-1}\,dx = \frac{(2b)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{n=1}^\infty \frac{1}{\left(n^2\pi^2+b^2\right)^{\nu+\frac{1}{2}}}$$
$[\operatorname{Re}\nu > 0, \quad |\operatorname{Im} b| < \pi]$ WA 423(9)

6.625

1. $$\int_0^1 x^{\lambda-\nu-1}(1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x)\,dx = \frac{2^{-\nu}\alpha^\nu\, \Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)\,\Gamma(\nu+1)}\, {}_2F_2\left(\lambda, \nu+\frac{1}{2}; \lambda+\mu, 2\nu+1; \pm 2i\alpha\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 194(58)a

2. $$\int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x)\,dx = \frac{(2\alpha)^\nu\, \Gamma(\mu)\, \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\mu+2\nu+1)}\, {}_1F_1\left(\nu+\frac{1}{2}; \mu+2\nu+1; \pm 2i\alpha\right)$$
$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 194(57)a

3. $$\int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm\alpha x} J_\nu(\alpha x)\,dx = \frac{(2\alpha)^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(\mu+2\nu+1)}\, {}_1F_1\left(\nu+\frac{1}{2}; \mu+2\nu+1; \pm 2\alpha\right)$$
$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ BU 9(16a), ET II 197(77)a

4. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} e^{\pm\alpha x} I_\nu(\alpha x)\,dx = \frac{\left(\frac{1}{2}\alpha\right)^\nu \Gamma(\lambda+\nu)\,\Gamma(\mu)}{\Gamma(\nu+1)\,\Gamma(\lambda+\mu+\nu)} \times {}_2F_2\left(\nu+\frac{1}{2}, \lambda+\nu; 2\nu+1, \mu+\lambda+\nu; \pm 2\alpha\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}(\lambda+\nu) > 0]$ ET II 197(78)a

5. $$\int_0^1 x^{\mu-\kappa}(1-x)^{2\kappa-1} I_{\mu-\kappa}\left(\frac{1}{2}xz\right) e^{-\frac{1}{2}xz}\,dx = \frac{\Gamma(2\kappa)}{\sqrt{\pi}\,\Gamma(1+2\mu)} e^{\frac{z}{2}} z^{-\kappa-\frac{1}{2}} M_{\kappa,u}(z)$$
$\left[\operatorname{Re}\left(\kappa-\frac{1}{2}-\mu\right) < 0, \quad \operatorname{Re}\kappa > 0\right]$ BU 129(14a)

6. $$\int_1^\infty x^{-\lambda}(x-1)^{\mu-1}e^{-\alpha x}\,I_\nu(\alpha x)\,dx = \frac{(2\alpha)^\lambda\,\Gamma(\mu)}{\sqrt{\pi}}\,G^{21}_{23}\left(2\alpha\,\middle|\,\begin{matrix}\frac{1}{2}-\lambda,0\\ -\mu,\nu-\lambda,-\nu-\lambda\end{matrix}\right)$$
$$\left[0<\operatorname{Re}\mu<\tfrac{1}{2}+\operatorname{Re}\lambda,\quad \operatorname{Re}\alpha>0\right]$$
ET II 207(50)a

7. $$\int_1^\infty x^{-\lambda}(x-1)^{\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx = \Gamma(\mu)\sqrt{\pi}(2\alpha)^\lambda\,G^{30}_{23}\left(2\alpha\,\middle|\,\begin{matrix}0,\frac{1}{2}-\lambda\\ -\mu,\nu-\lambda,-\nu-\lambda\end{matrix}\right)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 208(55)a

8. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}\,I_\nu(\alpha x)\,dx = \frac{(2\alpha)^{\nu-\mu}\,\Gamma\left(\frac{1}{2}-\mu+\nu\right)\Gamma(\mu)}{\sqrt{\pi}\,\Gamma(1-\mu+2\nu)} \times {}_1F_1\left(\frac{1}{2}-\mu+\nu;1-\mu+2\nu;-2\alpha\right)$$
$$\left[0<\operatorname{Re}\mu<\tfrac{1}{2}+\operatorname{Re}\nu,\quad \operatorname{Re}\alpha>0\right]$$
ET II 207(49)a

9. $$\int_1^\infty x^{-\nu}(x-1)^{\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx = \sqrt{\pi}\,\Gamma(\mu)(2\alpha)^{-\frac{1}{2}\mu-\frac{1}{2}}e^{-\alpha}\,W_{-\frac{1}{2}\mu,\nu-\frac{1}{2}\mu}(2\alpha)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 208(53)a

10. $$\int_1^\infty x^{-\mu-\frac{1}{2}(x-1)\mu-1}e^{-\alpha x}\,K_\nu(\alpha x)\,dx = \sqrt{\pi}\,\Gamma(\mu)(2\alpha)^{-\frac{1}{2}}e^{-\alpha}\,W_{-\mu,\nu}(2\alpha)$$
$$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\alpha>0]$$
ET II 207(51)a

11.[3] $$\int_{-1}^1 \left(1-x^2\right)^{-1/2} xe^{-ax}\,I_1\left(b\sqrt{1-x^2}\right)\,dx = \frac{2}{b}\left\{\sinh a - a\left(a^2+b^2\right)^{-1/2}\sinh\sqrt{a^2+b^2}\right\}$$
$$[a>0,\quad b>0]$$

6.626

1.[11] $$\int_0^\infty x^{\lambda-1}e^{-\alpha x}\,J_\mu(\beta x)\,J_\nu(\gamma x)\,dx = \frac{\beta^\mu\gamma^\nu}{\Gamma(\nu+1)}2^{-\nu-\mu}\alpha^{-\lambda-\mu-\nu}\sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)} \times F\left(-m,-\mu-m;\nu+1;\frac{\gamma^2}{\beta^2}\right)\left(-\frac{\beta^2}{4\alpha^2}\right)^m$$
$$[\operatorname{Re}(\lambda+\mu+\nu)>0,\quad \operatorname{Re}(\alpha\pm i\beta\pm i\gamma)>1]$$
EH II 48(15)

2. $$\int_0^\infty e^{-2\alpha x}\,J_\nu(\beta x)\,J_\mu(\beta x)x^{\nu+\mu}\,dx = \frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\beta^{\nu+\mu}}{\sqrt{\pi^3}} \times \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu+\mu}\varphi\cos(\nu-\mu)\varphi}{\left(\alpha^2+\beta^2\cos^2\varphi\right)^{\nu+\mu}\sqrt{\alpha^2+\beta^2\cos^2\varphi}}\,d\varphi$$
$$\left[\operatorname{Re}\alpha>|\operatorname{Im}\beta|,\quad \operatorname{Re}(\nu+\mu)>-\tfrac{1}{2}\right]$$
WA 427(1)

3. $$\int_0^\infty e^{-2\alpha x}\,J_0(\beta x)\,J_1(\beta x)x\,dx = \frac{\boldsymbol{K}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)-\boldsymbol{E}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{2\pi\beta\sqrt{\alpha^2+\beta^2}}$$
WA 427(2)

4. $$\int_0^\infty e^{-2\alpha x}\,I_0(\beta x)\,I_1(\beta x)x\,dx = \frac{1}{2\pi\beta}\left\{\frac{\alpha}{\alpha^2-\beta^2}\,\boldsymbol{E}\left(\frac{\beta}{\alpha}\right)-\frac{1}{\alpha}\,\boldsymbol{K}\left(\frac{\beta}{\alpha}\right)\right\}$$
$$[\operatorname{Re}\alpha>\operatorname{Re}\beta]$$
WA 428(5)

5.[10] $$\int_0^\infty x^{\nu-\mu+2n} e^{-zx}\, J_\mu(\alpha x)\, J_\nu(\rho x)\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{a}{2}\right)^{\mu-\nu-2n-1} \left(\frac{\rho}{a}\right)^{\nu} \times \frac{1}{\Gamma\left(\mu-\nu-n+\frac{1}{2}\right)} \sum_{q=0}^{\infty} \frac{\Gamma\left(\nu+n+q+\frac{1}{2}\right)\left(\nu-\mu+n+\frac{1}{2}\right)_q}{q!\,\Gamma\left(\nu+q+\frac{1}{2}\right)} \times a^{-2q} \int_0^{\ell_1/\rho} \frac{dx}{\sqrt{1-x^2}} x^{2\nu+2q} \left(\rho^2 + \frac{z^2}{1-x^2}\right)^q$$

where $\ell_1 = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}\right]$ $\quad\left[\mu > \nu + 2n, \quad n = 0, 1, \ldots, \quad \nu > -\frac{1}{2}\right]$

6.627 $$\int_0^\infty \frac{x^{-1/2}}{x+a} e^{-x}\, K_\nu(x)\, dx = \frac{\pi e^a\, K_\nu(a)}{\sqrt{a}\cos(\nu\pi)} \qquad \left[|\arg a| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$ ET II 368(29)

6.628

1. $$\int_0^\infty e^{-x\cos\beta}\, J_{-\nu}(x\sin\beta)\, x^\mu\, dx = \Gamma(\mu-\nu+1)\, P_\mu^\nu(\cos\beta)$$
$$\left[0 < \beta < \frac{\pi}{2}, \quad \operatorname{Re}(\mu-\nu) > -1\right]$$
WA 424(3), WH

2. $$\int_0^\infty e^{-x\cos\beta}\, Y_\nu(x\sin\beta)\, x^\mu\, dx = -\frac{\sin\mu\pi}{\sin(\mu+\nu)\pi} \frac{\Gamma(\mu-\nu+1)}{\pi} \times \left[Q_\mu^\nu(\cos\beta + 0\cdot i)\, e^{\frac{1}{2}\nu\pi i} + Q_\mu^\nu(\cos\beta - 0\cdot i)\, e^{-\frac{1}{2}\nu\pi i}\right]$$
$$\left[\operatorname{Re}(\mu+\nu) > -1, \quad 0 < \beta < \frac{\pi}{2}\right]$$ WA 424(4)

3. $$\int_0^1 e^{\frac{xu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu}\, J_{\mu-\nu}\left(\frac{ixu}{2}\right) dx = 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{\mathrm{B}(2\nu, 2\mu-2\nu+1)}{\Gamma(\mu-\nu+1)} \frac{e^{\frac{u}{2}}}{u^{\nu+\frac{1}{2}}} M_{\nu,\mu}(u)$$
MO 118a

4.[8] $$\int_0^\infty e^{-x\cosh\alpha}\, I_\nu(x\sinh\alpha)\, x^\mu\, dx = \Gamma(\nu+\mu+1)\, P_\mu^{-\nu}(\cosh\alpha)$$
$$\left[\operatorname{Re}(\mu+\nu) > -1, \quad |\operatorname{Im}\alpha| < \frac{1}{2}\pi\right]$$
WA 423(1)

5. $$\int_0^\infty e^{-x\cosh\alpha}\, K_\nu(x\sinh\alpha)\, x^\mu\, dx = \frac{\sin\mu\pi}{\sin(\nu+\mu)\pi} \Gamma(\mu-\nu+1)\, Q_\mu^\nu(\cosh\alpha)$$
$$\left[\operatorname{Re}(\mu+1) > |\operatorname{Re}\nu|\right]$$ WA 423(2)

6. $$\int_0^\infty e^{-x\cosh\alpha}\, I_\nu(x) x^{\mu-1}\, dx = \frac{\cos\nu\pi}{\sin(\mu+\nu)\pi} \frac{Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh\alpha)}{\sqrt{\frac{\pi}{2}}\,(\sinh\alpha)^{\mu-\frac{1}{2}}}$$
$$\left[\operatorname{Re}(\mu+\nu) > 0, \quad \operatorname{Re}(\cosh\alpha) > 1\right]$$
WA 424(6)

7. $$\int_0^\infty e^{-x\cosh\alpha}\, K_\nu(x) x^{\mu-1}\, dx = \sqrt{\frac{\pi}{2}}\, \Gamma(\mu-\nu)\, \Gamma(\mu+\nu) \frac{P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh\alpha)}{(\sinh\alpha)^{\mu-\frac{1}{2}}}$$
$$\left[\operatorname{Re}\mu > |\operatorname{Re}\nu|, \quad \operatorname{Re}(\cosh\alpha) > -1\right]$$
WA 424(7)

6.629[8] $$\int_0^\infty x^{-1/2} e^{-x\alpha\cos\varphi\cos\psi}\, J_\mu(\alpha x\sin\varphi)\, J_\nu(\alpha x\sin\psi)\,dx = \Gamma\left(\mu+\nu+\tfrac{1}{2}\right)\alpha^{-\frac{1}{2}}\, P^{-\mu}_{\nu-\frac{1}{2}}(\cos\varphi)\, P^{-\nu}_{\mu-\frac{1}{2}}(\cos\psi)$$

$$\left[\alpha>0,\quad 0<\varphi<\frac{\pi}{2},\quad 0<\psi<\frac{\pi}{2},\quad \operatorname{Re}(\mu+\nu)>-\frac{1}{2}\right]$$ ET II 50(19)

6.631

1. $$\int_0^\infty x^\mu e^{-\alpha x^2}\, J_\nu(\beta x)\,dx = \frac{\beta^\nu\,\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{2^{\nu+1}\alpha^{\frac{1}{2}(\mu+\nu+1)}\,\Gamma(\nu+1)}\ {}_1F_1\left(\frac{\nu+\mu+1}{2};\nu+1;-\frac{\beta^2}{4\alpha}\right)$$ BU 8(15)

$$= \frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\right)}{\beta\alpha^{\frac{1}{2}\mu}\,\Gamma(\nu+1)}\exp\left(-\frac{\beta^2}{8\alpha}\right) M_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}(\mu+\nu)>-1]$$ EH II 50(22), ET II 30(14), BU 14(13b)

2. $$\int_0^\infty x^\mu e^{-\alpha x^2}\, Y_\nu(\beta x)\,dx = -\alpha^{-\frac{1}{2}\mu}\beta^{-1}\sec\left(\frac{\nu-\mu}{2}\pi\right)\exp\left(-\frac{\beta^2}{8\alpha}\right) \times\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)}\sin\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) + W_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)\right\}$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\mu>|\operatorname{Re}\nu|-1,\quad \beta>0]$$ ET II 106(4)

3. $$\int_0^\infty x^\mu e^{-\alpha x^2}\, K_\nu(\beta x)\,dx = \frac{1}{2}\alpha^{-\frac{1}{2}\mu}\beta^{-1}\,\Gamma\left(\frac{1+\nu+\mu}{2}\right)\Gamma\left(\frac{1-\nu+\mu}{2}\right)\exp\left(\frac{\beta^2}{8\alpha}\right) W_{-\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$

$$[\operatorname{Re}\mu>|\operatorname{Re}\nu|-1]$$ ET II 132(25)

4.[11] $$\int_0^\infty x^{\nu+1} e^{-\alpha x^2}\, J_\nu(\beta x)\,dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right) \qquad [\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-1]$$ WA 431(4), ET II 29(10)

5. $$\int_0^\infty x^{\nu-1} e^{-\alpha x^2}\, J_\nu(\beta x)\,dx = 2^{\nu-1}\beta^{-\nu}\left[1-\gamma\left(\nu,\frac{\beta^2}{4\alpha}\right)\right]$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>0]$$ ET II 30(11)

6. $$\int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2}\, J_\nu(\beta x)\,dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left[\pm i\left(\frac{\nu+1}{2}\pi-\frac{\beta^2}{4\alpha}\right)\right]$$

$$\left[\alpha>0,\quad -1<\operatorname{Re}\nu<\tfrac{1}{2},\quad \beta>0\right]$$ ET II 30(12)

7. $$\int_0^\infty x e^{-\alpha x^2}\, J_\nu(\beta x)\,dx = \frac{\sqrt{\pi}\beta}{8\alpha^{\frac{3}{2}}}\exp\left(-\frac{\beta^2}{8\alpha}\right)\left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right)\right]$$

$$[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\nu>-2]$$ ET II 29(9)

8. $$\int_0^1 x^{n+1} e^{-\alpha x^2} I_n(2\alpha x)\,dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=-n}^{n} I_r(2\alpha)\right]$$

$[n = 0, 1, \ldots]$ ET II 365(8)a

9. $$\int_1^\infty x^{1-n} e^{-\alpha x^2} I_n(2\alpha x)\,dx = \frac{1}{4\alpha}\left[e^{\alpha} - e^{-\alpha}\sum_{r=1-n}^{n-1} I_r(2\alpha)\right]$$

$[n = 1, 2, \ldots]$ ET II 367(20)a

10. $$\int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu\left(2x\sqrt{z}\right)\,dx = \frac{n!}{2} e^{-z} z^{\frac{1}{2}\mu} L_n^\mu(z) \qquad [n = 0, 1, \ldots; \quad n + \operatorname{Re}\mu > -1]$$

BU 135(5)

6.632 $$\int_0^\infty x^{-\frac{1}{2}} \exp\left[-\left(x^2 + a^2 - 2ax\cos\varphi\right)^{\frac{1}{2}}\right]\left[x^2 + a^2 - 2ax\cos\varphi\right]^{-\frac{1}{2}} K_\nu(x)\,dx$$
$$= \pi a^{-\frac{1}{2}} \sec(\nu\pi)\, P_{\nu-\frac{1}{2}}\left(-\cos\varphi\right) K_\nu(a)$$

$\left[|\arg a| + |\operatorname{Re}\varphi| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 368(32)

6.633

1. $$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\gamma x)\,dx = \frac{\beta^\mu \gamma^\nu \alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1}\,\Gamma(\nu+1)} \sum_{m=0}^{\infty} \frac{\Gamma\left(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1\right)}{m!\,\Gamma(m+\mu+1)} \left(-\frac{\beta^2}{4\alpha}\right)^m$$
$$\times F\left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2}\right)$$

$[\operatorname{Re}\alpha > 0, \operatorname{Re}(\mu+\nu+\lambda) > -2, \beta > 0, \quad \gamma > 0]$ EH II 49(20)a, ET II 51(24)a

2. $$\int_0^\infty e^{-\varrho^2 x^2} J_p(\alpha x)\, J_p(\beta x) x\,dx = \frac{1}{2\varrho^2} \exp\left(-\frac{\alpha^2+\beta^2}{4\varrho^2}\right) I_p\left(\frac{\alpha\beta}{2\varrho^2}\right)$$

$\left[\operatorname{Re}p > -1, \quad |\arg\varrho| < \frac{\pi}{4}, \quad \alpha > 0, \quad \beta > 0\right]$ KU 146(16)a, WA 433(1)

3. $$\int_0^\infty x^{2\nu+1} e^{-\alpha x^2} J_\nu(x)\, Y_\nu(x)\,dx = -\frac{1}{2\sqrt{\pi}} \alpha^{-\frac{3}{2}\nu-\frac{1}{2}} \exp\left(-\frac{1}{2\alpha}\right) W_{\frac{1}{2}\nu,\frac{1}{2}\nu}\left(\frac{1}{\alpha}\right)$$

$\left[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 347(59)

4. $$\int_0^\infty x e^{-\alpha x^2} I_\nu(\beta x)\, J_\nu(\gamma x)\,dx = \frac{1}{2\alpha} \exp\left(\frac{\beta^2-\gamma^2}{4\alpha}\right) J_\nu\left(\frac{\beta\gamma}{2\alpha}\right)$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$ ET II 63(1)

5. $$\int_0^\infty x^{\lambda-1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\beta x)\,dx$$
$$= 2^{-\nu-\mu-1} \alpha^{-\frac{1}{2}(\nu+\lambda+\mu)} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$
$$\times\ {}_3F_3\left[\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{\mu}{2} + 1, \frac{\nu+\mu+\lambda}{2}; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right]$$

$[\operatorname{Re}(\nu+\lambda+\mu) > 0, \quad \operatorname{Re}\alpha > 0]$ WA 434, EH II 50(21)

6.634 $$\int_0^\infty xe^{-\frac{x^2}{2a}}\left[I_\nu(x)+I_{-\nu}(x)\right]K_\nu(x)\,dx = ae^a\,K_\nu(a) \qquad [\operatorname{Re} a>0, \quad -1<\operatorname{Re}\nu<1]$$

ET II 371(49)

6.635

1. $$\int_0^\infty x^{-1}e^{-\frac{\alpha}{x}}\,J_\nu(\beta x)\,dx = 2\,J_\nu\left(\sqrt{2\alpha\beta}\right)K_\nu\left(\sqrt{2\alpha\beta}\right)$$

$[\operatorname{Re}\alpha>0, \quad \beta>0]$ ET II 30(15)

2. $$\int_0^\infty x^{-1}e^{-\frac{\alpha}{x}}\,Y_\nu(\beta x)\,dx = 2\,Y_\nu\left(\sqrt{2\alpha\beta}\right)K_\nu\left(\sqrt{2\alpha\beta}\right)$$

$[\operatorname{Re}\alpha>0, \quad \beta>0]$ ET II 106(5)

3. $$\int_0^\infty x^{-1}e^{-\frac{\alpha}{x}-\beta x}\,J_\nu(\gamma x)\,dx = 2\,J_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2}-\beta\right]^{\frac{1}{2}}\right\}K_\nu\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2+\gamma^2}+\beta\right]^{\frac{1}{2}}\right\}$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad \gamma>0]$

ET II 30(16)

6.636 $$\int_0^\infty x^{-\frac{1}{2}}e^{-\alpha\sqrt{x}}\,J_\nu(\beta x)\,dx = \frac{\sqrt{2}}{\sqrt{\pi\beta}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)D_{-\nu-\frac{1}{2}}\left(2^{-\frac{1}{2}}\alpha e^{\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}\right)D_{-\nu-\frac{1}{2}}\left(2^{-\frac{1}{2}}\alpha e^{-\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}\right)$$

$\left[\operatorname{Re}\alpha>0, \quad \beta>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right]$

ET II 30(17)

6.637

1. $$\int_0^\infty \left(\beta^2+x^2\right)^{-\frac{1}{2}}\exp\left[-\alpha\left(\beta^2+x^2\right)^{\frac{1}{2}}\right]J_\nu(\gamma x)\,dx$$
$$= I_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac{1}{2}}-\alpha\right]\right\}K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac{1}{2}}+\alpha\right]\right\}$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad \gamma>0, \quad \operatorname{Re}\nu>-1]$ ET II 31(20)

2. $$\int_0^\infty \left(\beta^2+x^2\right)^{-\frac{1}{2}}\exp\left[-\alpha\left(\beta^2+x^2\right)^{\frac{1}{2}}\right]Y_\nu(\gamma x)\,dx$$
$$= -\sec\left(\frac{\nu\pi}{2}\right)K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac{1}{2}}+\alpha\right]\right\}$$
$$\times\left(\frac{1}{\pi}\,K_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac{1}{2}}+\alpha\right]\right\}+\sin\left(\frac{\nu\pi}{2}\right)I_{\frac{1}{2}\nu}\left\{\frac{1}{2}\beta\left[\left(\alpha^2+\gamma^2\right)^{\frac{1}{2}}-\alpha\right]\right\}\right)$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad \gamma>0, \quad |\operatorname{Re}\nu|<1]$ ET II 106(6)

3. $$\int_0^\infty \left(x^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\alpha\left(x^2+\beta^2\right)^{\frac{1}{2}}\right]K_\nu(\gamma x)\,dx$$
$$= \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right)K_{\frac{1}{2}\nu}\left(\frac{1}{2}\beta\left[\alpha+\left(\alpha^2-\gamma^2\right)^{\frac{1}{2}}\right]\right)K_{\frac{1}{2}\nu}\left(\frac{1}{2}\beta\left[\alpha-\left(\alpha^2-\gamma^2\right)^{\frac{1}{2}}\right]\right)$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}(\gamma+\beta)>0, \quad |\operatorname{Re}\nu|<1]$ ET II 132(26)

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and powers

6.641 $$\int_0^\infty \sqrt{x}e^{-\alpha x}\,J_{\pm\frac{1}{4}}\left(x^2\right)\,dx = \frac{\sqrt{\pi\alpha}}{4}\left[\mathbf{H}_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right)-Y_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right)\right]$$

MI 42

6.642

1.[10] $$\int_0^\infty x^{-1} e^{-\alpha x}\, Y_\nu\left(\frac{2}{x}\right) dx = 2\, K_\nu\left(2\sqrt{a}\right)\, Y_\nu\left(2\sqrt{a}\right)$$

$[\operatorname{Re} a > 0]$ MC

2. $$\int_0^\infty x^{-1} e^{-\alpha x}\, H_\nu^{(1,2)}\left(\frac{2}{x}\right) dx = H_\nu^{(1,2)}\left(\sqrt{\alpha}\right) K_\nu\left(\sqrt{\alpha}\right)$$ MI 44, EH II 91(26)

6.643

1. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x}\, J_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\beta\,\Gamma(2\nu+1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu}\, M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$

$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right]$

BU 14(13a), MI 42a

2. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x}\, I_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu}\, M_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$

$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right]$ MI 45

3. $$\int_0^\infty x^{\mu-\frac{1}{2}} e^{-\alpha x}\, K_{2\nu}\left(2\beta\sqrt{x}\right) dx = \frac{\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}{2\beta} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu}\, W_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right)$$

$\left[\operatorname{Re}\left(\mu+\nu+\frac{1}{2}\right) > 0\right],$ (cf. **6.631** 3)

MI 47a

4. $$\int_0^\infty x^{n+\frac{1}{2}\nu} e^{-\alpha x}\, J_\nu\left(2\beta\sqrt{x}\right) dx = n!\beta^\nu e^{-\frac{\beta^2}{\alpha}} \alpha^{-n-\nu-1}\, L_n^\nu\left(\frac{\beta^2}{\alpha}\right)$$

$[n+\nu > -1]$ MO 178a

5. $$\int_0^\infty x^{-\frac{1}{2}} e^{-\alpha x}\, Y_{2\nu}\left(\beta\sqrt{x}\right) dx = -\sqrt{\frac{\pi}{\alpha}} \frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)} \left[\sin(\nu\pi)\, I_\nu\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi} K_\nu\left(\frac{\beta^2}{8\alpha}\right)\right]$$

$\left[|\operatorname{Re}\nu| < \frac{1}{2}\right]$ MI 44

6. $$\int_0^\infty x^{\frac{1}{2}m} e^{-\alpha x}\, K_m\left(2\sqrt{x}\right) dx = \frac{\Gamma(m+1)}{2\alpha} \left(\frac{1}{\alpha}\right)^{\frac{1}{2}m-\frac{1}{2}} e^{\frac{1}{2\alpha}}\, W_{-\frac{1}{2}(m+1),-\frac{1}{2}m}\left(\frac{1}{\alpha}\right)$$ MI 48a

6.644 $$\int_0^\infty e^{-\beta x}\, J_{2\nu}\left(2a\sqrt{x}\right) J_\nu(bx)\, dx = \exp\left(-\frac{a^2\beta}{\beta^2+b^2}\right) J_\nu\left(\frac{a^2 b}{\beta^2+b^2}\right) \frac{1}{\sqrt{\beta^2+b^2}}$$

$\left[\operatorname{Re}\beta > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$

ET II 58(17)

6.645

1. $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}} e^{-\alpha x}\, J_\nu\left(\beta\sqrt{x^2-1}\right) dx = I_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right] K_{\frac{1}{2}\nu}\left[\frac{1}{2}\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)\right]$$

MO 179a

2. $$\int_1^\infty \left(x^2-1\right)^{\frac{1}{2}\nu} e^{-\alpha x}\, J_\nu\left(\beta\sqrt{x^2-1}\right) dx = \sqrt{\frac{2}{\pi}} \beta^\nu \left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} K_{\nu+\frac{1}{2}}\left(\sqrt{\alpha^2+\beta^2}\right)$$

MO 179a

3.[3] $$\int_{-1}^{1} \left(1-x^2\right)^{-1/2} e^{-ax} I_1\left(b\sqrt{1-x^2}\right) dx = \frac{2}{b}\left(\cosh\sqrt{a^2+b^2} - \cosh a\right)$$

$[a > 0, \quad b > 0]$

6.646

1. $$\int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} J_\nu\left(\beta\sqrt{x^2-1}\right) dx = \frac{\exp\left(-\sqrt{\alpha^2+\beta^2}\right)}{\sqrt{\alpha^2+\beta^2}}\left(\frac{\beta}{\alpha+\sqrt{\alpha^2+\beta^2}}\right)^\nu$$

$[\operatorname{Re}\nu > -1]$ EF 89(52), MO 179

2. $$\int_1^\infty \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} I_\nu\left(\beta\sqrt{x^2-1}\right) dx = \frac{\exp\left(-\sqrt{\alpha^2-\beta^2}\right)}{\sqrt{\alpha^2-\beta^2}}\left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^\nu$$

$[\operatorname{Re}\nu > -1, \quad \alpha > \beta]$ MO 180

3.[7] $$\int_b^\infty e^{-pt}\left(\frac{t-b}{t+b}\right)^{\nu/2} K_\nu\left[a\left(t^2-b^2\right)^{1/2}\right] dt = \frac{\Gamma(\nu+1)}{2sa^\nu}\left[x^\nu e^{-bx}\,\Gamma(-\nu,bx) - y^\nu e^{bs}\,\Gamma(-\nu,by)\right]$$

where $x = p-s, \quad y = p+s, \quad s = \left(p^2-a^2\right)^{1/2}$ $\quad [\operatorname{Re}(p+a) > 0, \quad |\operatorname{Re}(\nu)| < 1]$.

ME 39a

6.647

1. $$\int_0^\infty x^{-\lambda-\frac{1}{2}}(\beta+x)^{\lambda-\frac{1}{2}} e^{-\alpha x} K_{2\mu}\left[\sqrt{x(\beta+x)}\right] dx$$
$$= \frac{1}{\beta}e^{\frac{1}{2}\alpha\beta}\,\Gamma\left(\tfrac{1}{2}-\lambda+\mu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\mu\right) W_{\lambda,\mu}(z_1)\, W_{\lambda,\mu}(z_2)$$

$$z_1 = \tfrac{1}{2}\beta\left(\alpha+\sqrt{\alpha^2-1}\right), \quad z_2 = \tfrac{1}{2}\beta\left(\alpha-\sqrt{\alpha^2-1}\right)$$

$\left[|\arg\beta| < \pi, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\lambda + |\operatorname{Re}\mu| < \tfrac{1}{2}\right]$ ET II 377(37)

2. $$\int_0^\infty (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x\cosh t} K_\nu\left[\sqrt{x(\alpha+x)}\right] dx$$
$$= \frac{1}{2}\sec\left(\frac{\nu\pi}{2}\right) e^{\frac{1}{2}\alpha\cosh t} K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^t\right) K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^{-t}\right)$$

$[-1 < \operatorname{Re}\nu < 1]$ ET II 377(36)

3.[11] $$\int_0^\alpha x^{\lambda-\frac{1}{2}}(\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x\sinh t} I_{2\mu}\left[\sqrt{x(\alpha-x)}\right] dx$$
$$= e^{-(\alpha/2)\sinh t}\frac{2\,\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha\left[\Gamma(2\mu+1)\right]^2} M_{\lambda,\mu}\left(\frac{1}{2}\alpha e^t\right) M_{-\lambda,\mu}\left(\frac{1}{2}\alpha e^{-t}\right)$$

$\left[\operatorname{Re}\mu > |\operatorname{Re}\lambda| - \tfrac{1}{2}\right]$ ET II 377(32)

6.648 $$\int_{-\infty}^\infty e^{\varrho x}\left(\frac{\alpha+\beta e^x}{\alpha e^x+\beta}\right)^\nu K_{2\nu}\left[\left(\alpha^2+\beta^2+2\alpha\beta\cosh x\right)^{\frac{1}{2}}\right] dx = 2\,K_{\nu+\varrho}(\alpha)\,K_{\nu-\varrho}(\beta)$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0]$ ET II 379(45)

6.649

1. $$\int_0^\infty K_{\mu-\nu}\left(2z\sinh x\right) e^{(\nu+\mu)x}\,dx = \frac{\pi^2}{4\sin[(\nu-\mu)\pi]}\left[J_\nu(z)\,Y_\mu(z) - J_\mu(z)\,Y_\nu(z)\right]$$
$$\left[\operatorname{Re} z > 0, \quad -1 < \operatorname{Re}(\nu-\mu) < 1\right]$$ MO 44

2. $$\int_0^\infty J_{\nu+\mu}\left(2x\sinh t\right) e^{(\nu-\mu)t}\,dt = K_\nu(x)\,I_\mu(x)$$
$$\left[\operatorname{Re}(\nu-\mu) < \tfrac{3}{2}, \quad \operatorname{Re}(\nu+\mu) > -1, \quad x > 0\right]$$ EH II 97(68)

3. $$\int_0^\infty Y_{\nu-\mu}\left(2x\sinh t\right) e^{-(\nu+\mu)t}\,dt = \frac{1}{\sin[\pi(\mu-\nu)]}\left\{I_\mu(x)\,K_\nu(x) - \cos[(\nu-\mu)\pi]\,I_\nu(x)\,K_\mu(x)\right\}$$
$$\left[\left|\operatorname{Re}(\nu-\mu)\right| < 1, \quad \operatorname{Re}(\nu+\mu) > -\tfrac{1}{2}, \quad x > 0\right]$$ EH II 97(73)

4. $$\int_0^\infty K_0\left(2z\sinh x\right) e^{-2\nu x}\,dx = -\frac{\pi}{4}\left\{J_\nu(z)\frac{\partial\,Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial\,J_\nu(z)}{\partial\nu}\right\}$$

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.651

1. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2x^2}\,I_\mu\left(\tfrac{1}{4}\alpha^2x^2\right) J_\nu(\beta x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}} 2^{\lambda+1}\beta^{-\lambda-\frac{3}{2}}\,G^{21}_{23}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\,\begin{matrix}1-\mu, 1+\mu\\ h, \frac{1}{2}, k\end{matrix}\right)$$
$$h = \tfrac{3}{4} + \tfrac{1}{2}\lambda + \tfrac{1}{2}\nu, \quad k = \tfrac{3}{4} + \tfrac{1}{2}\lambda - \tfrac{1}{2}\nu$$
$$\left[|\arg\alpha| < \frac{\pi}{4}, \quad \beta > 0, \quad -\tfrac{3}{2} - \operatorname{Re}(2\mu+\nu) < \operatorname{Re}\lambda < 0\right]$$ ET II 68(8)

2. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2x^2}\,K_\mu\left(\tfrac{1}{4}\alpha^2x^2\right) J_\nu(\beta x)\,dx$$
$$= \sqrt{\frac{\pi}{2}}\,2^{\lambda+1}\beta^{-\lambda-\frac{3}{2}}\,G^{12}_{23}\left(\frac{\beta^2}{2\alpha^2}\,\middle|\,\begin{matrix}1-\mu, 1+\mu\\ h, \frac{1}{2}, k\end{matrix}\right)$$
$$h = \tfrac{3}{4} + \tfrac{1}{2}\lambda + \tfrac{1}{2}\nu, \quad k = \tfrac{3}{4} + \tfrac{1}{2}\lambda - \tfrac{1}{2}\nu$$
$$\left[|\arg\alpha| < \frac{\pi}{4}, \quad \operatorname{Re}\left(\lambda+\nu\pm 2\mu\right) > -\tfrac{3}{2}\right]$$ ET II 69(15)

3. $$\int_0^\infty x^{2\mu-\nu+1} e^{-\frac{1}{4}\alpha x^2}\,I_\mu\left(\tfrac{1}{4}\alpha x^2\right) J_\nu(\beta x)\,dx$$
$$= 2^{\mu-\nu+\frac{1}{2}}(\pi\alpha)^{-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+\mu\right)\frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)}\;{}_1F_1\left(\frac{1}{2}+\mu; \frac{1}{2}-\mu+\nu; -\frac{\beta^2}{2\alpha}\right)$$
$$\left[\operatorname{Re}\alpha > 0, \quad \beta > 0, \quad \operatorname{Re}\nu > 2\operatorname{Re}\mu + \tfrac{1}{2} > -\tfrac{1}{2}\right]$$ ET II 68(6)

4. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2x^2} K_\mu\left(\tfrac{1}{4}\alpha^2x^2\right) J_\nu(\beta x)\,dx$$
$$= \sqrt{\pi}2^\mu\alpha^{-2\mu-2\nu-2}\beta^\nu \frac{\Gamma(1+2\mu+\nu)}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)}\,{}_1F_1\left(1+2\mu+\nu;\mu+\nu+\frac{3}{2};-\frac{\beta^2}{2\alpha^2}\right)$$
$\left[|\arg\alpha|<\frac{1}{4}\pi,\quad \operatorname{Re}\nu>-1,\quad \operatorname{Re}(2\mu+\nu)>-1,\quad \beta>0\right]$ ET II 69(13)

5. $$\int_0^\infty x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha x^2} I_\mu\left(\tfrac{1}{2}\alpha x^2\right) K_\nu(\beta x)\,dx$$
$$= \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}}\beta^{-\mu-\frac{3}{2}}\alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}}\,\Gamma(2\mu+\nu+1)\,\Gamma\left(\mu+\tfrac{1}{2}\right)\exp\left(\frac{\beta^2}{8\alpha}\right)W_{k,m}\left(\frac{\beta^2}{4\alpha}\right)$$
$2k=-3\mu-\nu-\frac{1}{2},\quad 2m=\mu+\nu+\frac{1}{2}$
$\left[\operatorname{Re}\alpha>0,\quad \operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}(2\mu+\nu)>-1\right]$ ET II 146(53)

6. $$\int_0^\infty xe^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu}\left(\tfrac{1}{4}\beta x^2\right) J_\nu(\gamma x)\,dx = 2\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}}\exp\left(-\frac{\alpha\gamma^2}{\alpha^2+\beta^2}\right)J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{\alpha^2+\beta^2}\right)$$
$[\gamma>0,\quad \operatorname{Re}\alpha>|\operatorname{Im}\beta|,\quad \operatorname{Re}\nu>-1]$
ET II 56(2)

7. $$\int_0^\infty xe^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu}\left(\tfrac{1}{4}\alpha x^2\right) J_\nu(\beta x)\,dx = \left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{2}}\beta^{-1}\exp\left(-\frac{\beta^2}{2\alpha}\right)$$
$[\operatorname{Re}\alpha>0,\quad \beta>0,\quad \operatorname{Re}\nu>-1]$
ET II 67(3)

8. $$\int_0^\infty x^{1-\nu} e^{-\frac{1}{4}\alpha^2x^2} I_\nu\left(\tfrac{1}{4}\alpha^2x^2\right) J_\nu(\beta x)\,dx = \sqrt{\frac{2}{\pi}}\frac{\beta^{\nu-1}}{\alpha}\exp\left(-\frac{\beta^2}{4\alpha^2}\right)D_{-2\nu}\left(\frac{\beta}{\alpha}\right)$$
$\left[|\arg\alpha|<\frac{1}{4}\pi,\quad \beta>0,\quad \operatorname{Re}\nu>-\frac{1}{2}\right]$
ET II 67(1)

9. $$\int_0^\infty x^{-\nu-1} e^{-\frac{1}{4}\alpha^2x^2} I_{\nu+1}\left(\tfrac{1}{4}\alpha^2x^2\right) J_\nu(\beta x)\,dx = \sqrt{\frac{2}{\pi}}\beta^\nu\exp\left(-\frac{\beta^2}{4\alpha^2}\right)D_{-2\nu-3}\left(\frac{\beta}{\alpha}\right)$$
$\left[|\arg\alpha|<\frac{1}{4}\pi,\quad \operatorname{Re}\nu>-1,\quad \beta>0\right]$
ET II 67(2)

6.652 $$\int_0^\infty x^{2\nu} e^{-\left(\frac{x^2}{8}+\alpha x\right)} I_\nu\left(\frac{x^2}{8}\right)dx = \frac{\Gamma(4\nu+1)}{2^{4\nu}\,\Gamma(\nu+1)}\frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}}\,W_{-\frac{3}{2}\nu,\frac{1}{2}\nu}\left(\alpha^2\right)$$
$\left[\operatorname{Re}\left(\nu+\frac{1}{4}\right)>0\right]$ MI 45

6.653

1. $$\int_0^\infty \exp\left[-\frac{1}{2}x-\frac{1}{2x}\left(a^2+b^2\right)\right]I_\nu\left(\frac{ab}{x}\right)\frac{dx}{x} = 2\,I_\nu(a)\,K_\nu(b) \qquad [0<a<b]$$
$$= 2\,K_\nu(a)\,I_\nu(b) \qquad [0<b<a]$$
$[\operatorname{Re}\nu>-1]$ WA 482(2)a, EH II 53(37), WA 482(3)a

2. $$\int_0^\infty \exp\left[-\frac{1}{2}x-\frac{1}{2x}\left(z^2+w^2\right)\right]K_\nu\left(\frac{zw}{x}\right)\frac{dx}{x} = 2\,K_\nu(z)\,K_\nu(w)$$
$\left[|\arg z|<\pi,\quad |\arg w|<\pi,\quad \arg(z+w)\right]<\frac{1}{4}\pi$ WA 483(1), EH II 53(36)

6.654 $$\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x}-\alpha x} K_\nu\left(\frac{\beta^2}{8x}\right) dx = \sqrt{4\pi}\alpha^{-\frac{1}{2}} K_{2\nu}\left(\beta\sqrt{\alpha}\right)$$ ME 39

6.655 $$\int_0^\infty x\left(\beta^2+x^2\right)^{-\frac{1}{2}} \exp\left(-\frac{\alpha^2\beta}{\beta^2+x^2}\right) J_\nu\left(\frac{\alpha^2 x}{\beta^2+x^2}\right) J_\nu(\gamma x)\, dx = \gamma^{-1} e^{-\beta\gamma} J_{2\nu}\left(2\alpha\sqrt{\gamma}\right)$$

$\left[\operatorname{Re}\beta > 0, \quad \gamma > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 58(14)

6.656

1. $$\int_0^\infty e^{-(\xi-z)\cosh t} J_{2\nu}\left[2(z\xi)^{\frac{1}{2}}\sinh t\right] dt = I_\nu(z)\, K_\nu(\xi)$$

$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0\right]$ EH II 98(78)

2. $$\int_0^\infty e^{-(\xi+z)\cosh t} K_{2\nu}\left[2(z\xi)^{\frac{1}{2}}\sinh t\right] dt = \frac{1}{2} K_\nu(z)\, K_\nu(\xi) \sec(\nu\pi)$$

$\left[|\operatorname{Re}\nu| < \frac{1}{2}, \quad \operatorname{Re}\left(z^{\frac{1}{2}} + \xi^{\frac{1}{2}}\right)^2 \geq 0\right]$ EH II 98(79)

6.66 Combinations of Bessel, hyperbolic, and exponential functions

Bessel and hyperbolic functions

6.661

1. $$\int_0^\infty \sinh(ax)\, K_\nu(bx)\, dx = \frac{\pi}{2} \frac{\operatorname{cosec}\left(\frac{\nu\pi}{2}\right) \sin\left[\nu \arcsin\left(\frac{a}{b}\right)\right]}{\sqrt{b^2-a^2}}$$

$[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re}\nu| < 2]$ ET II 133(32)

2. $$\int_0^\infty \cosh(ax)\, K_\nu(bx)\, dx = \frac{\pi \cos\left[\nu \arcsin\left(\frac{a}{b}\right)\right]}{2\sqrt{b^2-a^2}\cos\left(\frac{\nu\pi}{2}\right)}$$ $[\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re}\nu| < 1]$

ET II 134(33)

6.662 Notation:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \qquad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$

1.[10] $$\int_0^\infty \cosh(\beta x)\, K_0(\alpha x)\, J_0(\gamma x)\, dx = \frac{\boldsymbol{K}(k)}{\sqrt{u+v}}$$

$$u = \frac{1}{2}\left\{\sqrt{\left(\alpha^2+\beta^2+\gamma^2\right)^2 - 4\alpha^2\beta^2}\right\} + \alpha^2 - \beta^2 - \gamma^2$$

$$v = \frac{1}{2}\left\{\sqrt{\left(\alpha^2+\beta^2+\gamma^2\right)^2 - 4\alpha^2\beta^2}\right\} - \alpha^2 + \beta^2 + \gamma^2$$

$$k^2 = v(u+v)^{-1}$$ $[\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \gamma > 0]$

ET II 15(23)

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\int_0^\infty \cosh(bx)\, K_0(cx)\, J_0(ax)\, dx = \frac{\boldsymbol{K}(k)}{\sqrt{\ell_2^2 - \ell_1^2}}$$

$$k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2}, \qquad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

2.[10] $$\int_0^\infty \sinh(\beta x)\, K_1(\alpha x)\, J_0(\gamma x)\, dx = a^{-1} \left[u\, \boldsymbol{E}(k) - \boldsymbol{K}(k)\, \boldsymbol{E}(u) + \frac{\boldsymbol{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right]$$

$$\operatorname{cn}^2 u = 2\gamma^2 \left\{ \left[\left(\alpha^2 + \beta^2 + \gamma^2\right)^2 - 4\alpha^2\beta^2 \right]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2 \right\}^{-1}$$

$$k^2 = \frac{1}{2} \left\{ 1 - \left(\alpha^2 - \beta^2 - \gamma^2\right) \left[\left(\alpha^2 + \beta^2 + \gamma^2\right)^2 - 4\alpha^2\beta^2 \right]^{-\frac{1}{2}} \right\}$$

$$[\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0]$$

ET II 15(24)

alternatively, with $a = \gamma$, $b = \beta$, $c = \alpha$,

$$\int_0^\infty \sinh(bx)\, K_1(cx)\, J_0(ax)\, dx = c^{-1} \left[u\, \boldsymbol{E}(k) - \boldsymbol{K}(k)\, \boldsymbol{E}(u) + \frac{\boldsymbol{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right]$$

$$\operatorname{cn}^2 u = \frac{a^2}{\ell_2^2 - c^2}, \quad k^2 = \frac{\ell_2^2 - c^2}{\ell_2^2 - \ell_1^2} \qquad [\operatorname{Re} c > |\operatorname{Re} b|, \quad a > 0]$$

6.663

1. $$\int_0^\infty K_{\nu \pm \mu}\left(2z \cosh t\right) \cosh\left[(\mu \mp \nu)\, t\right]\, dt = \frac{1}{2} K_\mu(z)\, K_\nu(z)$$

$$[\operatorname{Re} z > 0]$$ WA 484(1), EH II 54(39)

2. $$\int_0^\infty Y_{\mu+\nu}\left(2z \cosh t\right) \cosh[(\mu - \nu)t]\, dt = \frac{\pi}{4} \left[J_\mu(z)\, J_\nu(z) - Y_\mu(z)\, Y_\nu(z)\right]$$

$$[z > 0]$$ EH II 96(64)

3. $$\int_0^\infty J_{\mu+\nu}\left(2z \cosh t\right) \cosh[(\mu - \nu)t]\, dt = -\frac{\pi}{4} \left[J_\mu(z)\, Y_\nu(z) + J_\nu(z)\, Y_\mu(z)\right]$$

$$[z > 0]$$ EH II 97(65)

4. $$\int_0^\infty J_{\mu+\nu}\left(2z \sinh t\right) \cosh[(\mu - \nu)t]\, dt = \frac{1}{2} \left[I_\nu(z)\, K_\mu(z) + I_\mu(z)\, K_\nu(z)\right]$$

$$\left[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \tfrac{3}{2}, \quad z > 0\right]$$ EH II 97(71)

5. $$\int_0^\infty J_{\mu+\nu}\left(2z \sinh t\right) \sinh[(\mu - \nu)t]\, dt = \frac{1}{2} \left[I_\nu(z)\, K_\mu(z) - I_\mu(z)\, K_\nu(z)\right]$$

$$\left[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \tfrac{3}{2}, \quad z > 0\right]$$ EH II 97(72)

6.664

1. $$\int_0^\infty J_0\left(2z \sinh t\right) \sinh(2\nu t)\, dt = \frac{\sin(\nu\pi)}{\pi} \left[K_\nu(z)\right]^2 \qquad \left[|\operatorname{Re} \nu| < \tfrac{3}{4}, \quad z > 0\right]$$ EH II 97(69)

2. $$\int_0^\infty Y_0(2z\sinh t)\cosh(2\nu t)\,dt = -\frac{\cos(\nu\pi)}{\pi}\left[K_\nu(z)\right]^2 \qquad \left[|\operatorname{Re}\nu| < \tfrac{3}{4}, \quad z > 0\right]$$ EH II 97(70)

3. $$\int_0^\infty Y_0(2z\sinh t)\sinh(2\nu t)\,dt = \frac{1}{\pi}\left[I_\nu(z)\frac{\partial K_\nu(z)}{\partial\nu} - K_\nu(z)\frac{\partial I_\nu(z)}{\partial\nu}\right] - \frac{1}{\pi}\cos(\nu\pi)\left[K_\nu(z)\right]^2$$
$$\left[|\operatorname{Re}\nu| < \tfrac{3}{4}, \quad z > 0\right]$$ EH II 97(75)

4. $$\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t\,dt = \frac{\pi^2}{8}\left\{J_\nu^2(z) + N_\nu^2(z)\right\} \qquad [\operatorname{Re} z > 0]$$ MO 44

5. $$\int_0^\infty K_{2\mu}(z\sinh 2t)\coth^{2\nu} t\,dt = \frac{1}{4z}\Gamma\left(\frac{1}{2}+\mu-\nu\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)W_{\nu,\mu}(iz)\,W_{\nu,\mu}(-iz)$$
$$\left[|\arg z| \le \frac{\pi}{2}, \quad |\operatorname{Re}\mu| + \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ MO 119

6. $$\int_0^\infty \cosh(2\mu x)\,K_{2\nu}(2a\cosh x)\,dx = \frac{1}{2}K_{\mu+\nu}(a)\,K_{\mu-\nu}(a)$$
$$[\operatorname{Re} a > 0]$$ ET II 378(42)

6.665 $$\int_0^\infty \operatorname{sech} x\cosh(2\lambda x)\,I_{2\mu}(a\operatorname{sech} x)\,dx = \frac{\Gamma\left(\frac{1}{2}+\lambda+\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{2a\left[\Gamma(2\mu+1)\right]^2}M_{\lambda,\mu}(a)\,M_{-\lambda,\mu}(a)$$
$$\left[|\operatorname{Re}\lambda| - \operatorname{Re}\mu < \tfrac{1}{2}\right]$$ ET II 378(43)

Bessel, hyperbolic, and algebraic functions

6.666 $$\int_0^\infty x^{\nu+1}\sinh(\alpha x)\operatorname{cosech}(\pi x)\,J_\nu(\beta x)\,dx = \frac{2}{\pi}\sum_{n=1}^\infty(-1)^{n-1}n^{\nu+1}\sin(n\alpha)\,K_\nu(n\beta)$$
$$[|\operatorname{Re}\alpha| < \pi, \quad \operatorname{Re}\nu > -1]$$ ET II 41(3), WA 469(12)

6.667

1.[3] $$\int_0^a \frac{\cosh\left(\sqrt{a^2-x^2}\right)\sinh t\,I_{2\nu}(x)}{\sqrt{a^2-x^2}}\,dx = \frac{\pi}{2}I_\nu\left(\frac{1}{2}ae^t\right)I_\nu\left(\frac{1}{2}ae^{-t}\right)$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 365(10)

2. $$\int_0^a \frac{\cosh\left(\sqrt{a^2-x^2}\sinh t\right)K_{2\nu}(x)}{\sqrt{a^2-x^2}}\,dx = \frac{\pi^2}{4}\operatorname{cosec}(\nu\pi)\left[I_{-\nu}\left(ae^t\right)I_{-\nu}\left(ae^{-t}\right) - I_\nu\left(ae^t\right)I_\nu\left(ae^{-t}\right)\right]$$
$$\left[|\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$ ET II 367(25)

Exponential, hyperbolic, and Bessel functions

6.668 Notation:

$$\ell_1 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} - \sqrt{(b-c)^2+a^2}\right], \qquad \ell_2 = \frac{1}{2}\left[\sqrt{(b+c)^2+a^2} + \sqrt{(b-c)^2+a^2}\right]$$

1.[10] $$\int_0^\infty e^{-\alpha x}\sinh(\beta x)\,J_0(\gamma x)\,dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1}\left(r_2-r_1\right)^{\frac{1}{2}}\left(r_2+r_1\right)^{-\frac{1}{2}}$$
$$r_1=\sqrt{\gamma^2+(\beta-\alpha)^2},\qquad r_2=\sqrt{\gamma^2+(\beta+\alpha)^2},\qquad [\operatorname{Re}\alpha>|\operatorname{Re}\beta|,\quad \gamma>0]$$ ET II 12(52)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,
$$\int_0^\infty e^{-cx}\sinh(bx)\,J_0(ax)\,dx = \frac{\ell_1}{\ell_2^2-\ell_1^2}$$
$$[\operatorname{Re}c>|\operatorname{Re}b|,\quad a>0]$$

2.[10] $$\int_0^\infty e^{-\alpha x}\cosh(\beta x)\,J_0(\gamma x)\,dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1}\left(r_2-r_1\right)^{\frac{1}{2}}\left(r_2+r_1\right)^{-\frac{1}{2}}$$
$$r_1=\sqrt{\gamma^2+(\beta-\alpha)^2},\qquad r_2=\sqrt{\gamma^2+(\beta+\alpha)^2},\qquad [\operatorname{Re}\alpha>|\operatorname{Re}\beta|,\quad \gamma>0]$$ ET II 12(54)

alternatively, with $a=\gamma$, $b=\beta$, $c=\alpha$,
$$\int_0^\infty e^{-cx}\cosh(bx)\,J_0(ax)\,dx = \frac{\ell_2}{\ell_2^2-\ell_1^2}$$
$$[\operatorname{Re}c>|\operatorname{Re}b|,\quad a>0]$$

6.669

1. $$\int_0^\infty\left[\coth\left(\frac{1}{2}x\right)\right]^{2\lambda} e^{-\beta\cosh x}\,J_{2\mu}\left(\alpha\sinh x\right)\,dx = \frac{\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha\,\Gamma(2\mu+1)}\,M_{-\lambda,\mu}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}-\beta\right]$$
$$\times\,W_{\lambda,\mu}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}+\beta\right]$$
$$\left[\operatorname{Re}\beta>|\operatorname{Re}\alpha|,\quad \operatorname{Re}(\mu-\lambda)>-\tfrac{1}{2}\right]$$ BU 86(5b)a, ET II 363(34)

2. $$\int_0^\infty\left[\coth\left(\frac{1}{2}x\right)\right]^{2\lambda} e^{-\beta\cosh x}\,Y_{2\mu}\left(\alpha\sinh x\right)\,dx$$
$$= -\frac{\sec[(\mu+\lambda)\pi]}{\alpha}\,W_{\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}+\beta\right)W_{-\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}-\beta\right)$$
$$-\frac{\tan[(\mu+\lambda)\pi]\,\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha\,\Gamma(2\mu+1)}\,W_{\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}+\beta\right)M_{-\lambda,\mu}\left(\sqrt{\alpha^2+\beta^2}-\beta\right)$$
$$\left[\operatorname{Re}\beta>|\operatorname{Re}\alpha|,\quad \operatorname{Re}\lambda<\tfrac{1}{2}-|\operatorname{Re}\mu|\right]$$ ET II 363(35)

3. $$\int_0^\infty e^{-\frac{1}{2}(a_1a_2)t\cosh x}\left[\coth\left(\frac{1}{2}x\right)\right]^{2\nu} K_{2\mu}\left(t\sqrt{a_1a_2}\sinh x\right)\,dx$$
$$= \frac{\Gamma\left(\frac{1}{2}+\mu-\nu\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)}{2t\sqrt{a_1a_2}}\,W_{\nu,\mu}\left(a_1t\right)W_{\nu,\mu}\left(a_2t\right)$$
$$\left[\operatorname{Re}\nu<\operatorname{Re}\frac{1\pm 2\mu}{2},\quad \operatorname{Re}\left[t\left(\sqrt{a_1}+\sqrt{a_2}\right)^2\right]>0\right]$$ BU 85(4a)

4. $$\int_0^\infty e^{-\frac{1}{2}(a_1a_2)t\cosh x}\left[\coth\left(\frac{x}{2}\right)\right]^{2\nu} I_{2\mu}\left(t\sqrt{a_1a_2}\sinh x\right)\,dx = \frac{\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{t\sqrt{a_1a_2}\,\Gamma(1+2\mu)}\,W_{\nu,\mu}\left(a_1t\right)M_{\nu,\mu}\left(a_2t\right)$$
$$\left[\operatorname{Re}\left(\tfrac{1}{2}+\mu-\nu\right)>0,\quad \operatorname{Re}\mu>0,\quad a_1>a_2\right]$$ BU 86(5c)

5. $$\int_{-\infty}^\infty e^{2\nu s-\frac{x-y}{2}\tanh s}\,I_{2\mu}\left(\frac{\sqrt{xy}}{\cosh s}\right)\frac{ds}{\cosh s} = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\sqrt{xy}\left[\Gamma(1+2\mu)\right]^2}\,M_{\nu,\mu}(x)\,M_{-\nu,\mu}(y)$$
$$\left[\operatorname{Re}\left(\pm\nu+\tfrac{1}{2}+\mu\right)>0\right]$$ BU 83(3a)a

6. $$\int_{-\infty}^{\infty} e^{2\nu s - \frac{x+y}{2}\tanh s} J_{2\mu}\left(\frac{\sqrt{xy}}{\cosh s}\right) \frac{ds}{\cosh s} = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}{\sqrt{xy}\left[\Gamma(1+2\mu)\right]^2} M_{\nu,\mu}(x)\, M_{\nu,\mu}(y)$$
$\left[\operatorname{Re}\left(\mp\nu+\frac{1}{2}+\mu\right)>0\right]$ BU 84(3b)a

6.67–6.68 Combinations of Bessel and trigonometric functions

6.671

1. $$\int_0^\infty J_\nu(\alpha x)\sin\beta x\,dx = \frac{\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} \qquad [\beta<\alpha]$$
$$= \infty \text{ or } 0 \qquad [\beta=\alpha]$$
$$= \frac{\alpha^\nu\cos\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta>\alpha]$$
$[\operatorname{Re}\nu>-2]$ WA 444(4)

2. $$\int_0^\infty J_\nu(\alpha x)\cos\beta x\,dx = \frac{\cos\left(\nu\arcsin\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2-\beta^2}} \qquad [\beta<\alpha]$$
$$= \infty \text{ or } 0 \qquad [\beta=\alpha]$$
$$= \frac{-\alpha^\nu\sin\frac{\nu\pi}{2}}{\sqrt{\beta^2-\alpha^2}\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta>\alpha]$$
$[\operatorname{Re}\nu>-1]$ WA 444(5)

3. $$\int_0^\infty Y_\nu(ax)\sin(bx)\,dx$$
$$= \cot\left(\frac{\nu\pi}{2}\right)\left(a^2-b^2\right)^{-\frac{1}{2}}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a, |\operatorname{Re}\nu|<2]$$
$$= \frac{1}{2}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\left(b^2-a^2\right)^{-\frac{1}{2}}$$
$$\times\left\{a^{-\nu}\cos(\nu\pi)\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu - a^\nu\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^{-\nu}\right\} \qquad [0<a<b, \quad |\operatorname{Re}\nu|<2]$$
ET I 103(33)

4. $$\int_0^\infty Y_\nu(ax)\cos(bx)\,dx$$
$$= \frac{\tan\left(\frac{\nu\pi}{2}\right)}{\left(a^2-b^2\right)^{\frac{1}{2}}}\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right] \qquad [0<b<a, \quad |\operatorname{Re}\nu|<1]$$
$$= -\sin\left(\frac{\nu\pi}{2}\right)\left(b^2-a^2\right)^{-\frac{1}{2}}\left\{a^{-\nu}\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \cot(\nu\pi)\right.$$
$$\left. + a^\nu\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^{-\nu}\operatorname{cosec}(\nu\pi)\right\} \qquad [0<a<b, \quad |\operatorname{Re}\nu|<1]$$
ET I 47(29)

5. $$\int_0^\infty K_\nu(ax)\sin(bx)\,dx = \frac{1}{4}\pi a^{-\nu}\operatorname{cosec}\left(\frac{\nu\pi}{2}\right)\left(a^2+b^2\right)^{-\frac{1}{2}}\left\{\left[\left(b^2+a^2\right)^{\frac{1}{2}}+b\right]^\nu - \left[\left(b^2+a^2\right)^{\frac{1}{2}}-b\right]^\nu\right\}$$
$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < 2, \quad \nu \neq 0]$ ET I 105(48)

6. $$\int_0^\infty K_\nu(ax)\cos(bx)\,dx = \frac{\pi}{4}\left(b^2+a^2\right)^{-\frac{1}{2}}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\left[b+\left(b^2+a^2\right)^{\frac{1}{2}}\right]^\nu + a^\nu\left[b+\left(b^2+a^2\right)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$[\operatorname{Re} a > 0, b > 0, |\operatorname{Re}\nu| < 1]$ ET I 49(40)

7. $$\int_0^\infty J_0(ax)\sin(bx)\,dx = 0 \qquad [0 < b < a]$$
$$= \frac{1}{\sqrt{b^2-a^2}} \qquad [0 < a < b]$$
ET I 99(1)

8. $$\int_0^\infty J_0(ax)\cos(bx)\,dx = \frac{1}{\sqrt{a^2-b^2}} \qquad [0 < b < a]$$
$$= \infty \qquad [a = b]$$
$$= 0 \qquad [0 < a < b]$$
ET I 43(1)

9. $$\int_0^\infty J_{2n+1}(ax)\sin(bx)\,dx = (-1)^n\frac{1}{\sqrt{a^2-b^2}}\,T_{2n+1}\left(\frac{b}{a}\right) \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
ET I 99(2)

10. $$\int_0^\infty J_{2n}(ax)\cos(bx)\,dx = (-1)^n\frac{1}{\sqrt{a^2-b^2}}\,T_{2n}\left(\frac{b}{a}\right) \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
ET I 43(2)

11. $$\int_0^\infty Y_0(ax)\sin(bx)\,dx = \frac{2\arcsin\left(\frac{b}{a}\right)}{\pi\sqrt{a^2-b^2}} \qquad [0 < b < a]$$
$$= \frac{2}{\pi}\frac{1}{\sqrt{b^2-a^2}}\ln\left[\frac{b}{a}-\sqrt{\frac{b^2}{a^2}-1}\right] \qquad [0 < a < b]$$
ET I 103(31)

12. $$\int_0^\infty Y_0(ax)\cos(bx)\,dx = 0 \qquad [0 < b < a]$$
$$= -\frac{1}{\sqrt{b^2-a^2}} \qquad [0 < a < b]$$
ET I 47(28)

13. $$\int_0^\infty K_0(\beta x)\sin\alpha x\,dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}\ln\left(\frac{\alpha}{\beta}+\sqrt{\frac{\alpha^2}{\beta^2}+1}\right)$$
$[\alpha > 0, \quad \beta > 0]$ WA 425(11)a, MO 48

14.[8] $$\int_0^\infty K_0(\beta x)\cos\alpha x\,dx = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}}$$ $[\alpha > 0]$ WA 425(10)a, MO 48

6.672

1. $$\int_0^\infty J_\nu(ax)\,J_\nu(bx)\sin(cx)\,dx$$
$$= 0 \qquad [\operatorname{Re}\nu > -1, \quad 0 < c < b-a, \quad 0 < a < b]$$
$$= \frac{1}{2\sqrt{ab}}\,P_{\nu-\frac{1}{2}}\left(\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu > -1, \quad b-a < c < b+a, \quad 0 < a < b]$$
$$= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}}\,Q_{\nu-\frac{1}{2}}\left(-\frac{b^2+a^2-c^2}{2ab}\right) \qquad [\operatorname{Re}\nu > -1, \quad b+a < c, \quad 0 < a < b]$$
ET I 102(27)

2. $$\int_0^\infty J_\nu(x)\,J_{-\nu}(x)\cos(bx)\,dx = \frac{1}{2}\,P_{\nu-\frac{1}{2}}\left(\frac{1}{2}b^2-1\right) \qquad [0 < b < 2]$$
$$= 0 \qquad [2 < b]$$
ET I 46(21)

3. $$\int_0^\infty K_\nu(ax)\,K_\nu(bx)\cos(cx)\,dx = \frac{\pi^2}{4\sqrt{ab}}\sec(\nu\pi)\,P_{\nu-\frac{1}{2}}\left[\left(a^2+b^2+c^2\right)(2ab)^{-1}\right]$$
$\left[\operatorname{Re}(a+b) > 0, \quad c > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET I 50(51)

4. $$\int_0^\infty K_\nu(ax)\,I_\nu(bx)\cos(cx)\,dx = \frac{1}{2\sqrt{ab}}\,Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right)$$
$\left[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET I 49(47)

5. $$\int_0^\infty \sin(2ax)\left[J_\nu(x)\right]^2\,dx = \frac{1}{2}\,P_{\nu-\frac{1}{2}}\left(1-2a^2\right) \qquad [0 < a < 1, \quad \operatorname{Re}\nu > -1]$$
$$= \frac{1}{\pi}\cos(\nu\pi)\,Q_{\nu-\frac{1}{2}}\left(2a^2-1\right) \qquad [a > 1, \quad \operatorname{Re}\nu > -1]$$
ET II 343(30)

6. $$\int_0^\infty \cos(2ax)\left[J_\nu(x)\right]^2\,dx = \frac{1}{\pi}\,Q_{\nu-\frac{1}{2}}\left(1-2a^2\right) \qquad \left[0 < a < 1, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$
$$= -\frac{1}{\pi}\sin(\nu\pi)\,Q_{\nu-\frac{1}{2}}\left(2a^2-1\right) \qquad \left[a > 1, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$
ET II 344(32)

7. $$\int_0^\infty \sin(2ax)\,J_0(x)\,Y_0(x)\,dx = 0 \qquad [0 < a < 1]$$
$$= -\frac{\boldsymbol{K}\left[\left(1-a^{-2}\right)^{\frac{1}{2}}\right]}{\pi a} \qquad [a > 1]$$
ET II 348(60)

8. $$\int_0^\infty K_0(ax)\,I_0(bx)\cos(cx)\,dx = \frac{1}{\sqrt{c^2+(a+b)^2}}\,\boldsymbol{K}\left\{\frac{2\sqrt{ab}}{\sqrt{c^2+(a+b)^2}}\right\}$$

$[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0]$ ET I 49(46)

9. $$\int_0^\infty \cos(2ax)\,J_0(x)\,Y_0(x)\,dx = -\frac{1}{\pi}\,\boldsymbol{K}(a) \qquad [0 < a < 1]$$
$$= -\frac{1}{\pi a}\,\boldsymbol{K}\left(\frac{1}{a}\right) \qquad [a > 1]$$

ET II 348(61)

10. $$\int_0^\infty \cos(2ax)\,[Y_0(x)]^2\,dx = \frac{1}{\pi}\,\boldsymbol{K}\left(\sqrt{1-a^2}\right) \qquad [0 < a < 1]$$
$$= \frac{2}{\pi a}\,\boldsymbol{K}\left(\sqrt{1-\frac{1}{a^2}}\right) \qquad [a > 1]$$

ET II 348(62)

6.673

1. $$\int_0^\infty \left[J_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) - Y_\nu(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\sin(bx)\,dx$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 2]$$
$$= \frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b + \left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \left[b - \left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 2]$$

ET I 104(39)

2. $$\int_0^\infty \left[Y_\nu(ax)\cos\left(\frac{\nu\pi}{2}\right) + J_\nu(ax)\sin\left(\frac{\nu\pi}{2}\right)\right]\cos(bx)\,dx$$
$$= 0 \qquad [0 < b < a, \quad |\operatorname{Re}\nu| < 1]$$
$$= -\frac{1}{2a^\nu\sqrt{b^2-a^2}}\left\{\left[b + \left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu + \left[b - \left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu\right\} \qquad [0 < a < b, \quad |\operatorname{Re}\nu| < 1]$$

ET I 48(32)

3.* $$\int_0^{\pi/2} \left[\cos x\, I_0(a\cos x) + I_1(a\cos x)\right]\,dx = \frac{e^a-1}{a}$$

6.674

1. $$\int_0^a \sin(a-x)\,J_\nu(x)\,dx = a\,J_{\nu+1}(a) - 2\nu\sum_{n=0}^\infty (-1)^n\,J_{\nu+2n+2}(a)$$

$[\operatorname{Re}\nu > -1]$ ET II 334(12)

2. $$\int_0^a \cos(a-x)\,J_\nu(x)\,dx = a\,J_\nu(a) - 2\nu\sum_{n=0}^\infty (-1)^n\,J_{\nu+2n+1}(a)$$

$[\operatorname{Re}\nu > -1]$ ET II 336(23)

3. $$\int_0^a \sin(a-x)\,J_{2n}(x)\,dx = a\,J_{2n+1}(a) + (-1)^n 2n\left[\cos a - J_0(a) - 2\sum_{m=1}^n (-1)^m\,J_{2m}(a)\right]$$

$[n = 0, 1, 2, \dots]$ ET II 334(10)

4. $$\int_0^a \cos(a-x)\, J_{2n}(x)\, dx = a\, J_{2n}(a) - (-1)^n 2n \left[\sin a - 2\sum_{m=0}^{n-1} (-1)^m\, J_{2m+1}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 335(21)

5. $$\int_0^a \sin(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+2}(a) + (-1)^n (2n+1) \left[\sin a - 2\sum_{m=0}^{n} (-1)^m\, J_{2m+1}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 334(11)

6. $$\int_0^a \cos(a-x)\, J_{2n+1}(x)\, dx = a\, J_{2n+1}(a) + (-1)^n (2n+1) \left[\cos a - J_0(a) - 2\sum_{m=1}^{n} (-1)^m\, J_{2m}(a)\right]$$
$[n = 0, 1, 2, \ldots]$ ET II 336(22)

7. $$\int_0^z \sin(z-x)\, J_0(x)\, dx = z\, J_1(z)$$ WA 415(2)

8. $$\int_0^z \cos(z-x)\, J_0(x)\, dx = z\, J_0(z)$$ WA 415(1)

6.675

1. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -4]$
ET I 110(23)

2. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) \cos(bx)\, dx$$
$$= -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{a^2}{8b}\right) + \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{a^2}{8b}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -2]$ ET I 53(22)a

3. $$\int_0^\infty J_0\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{1}{b} \cos\left(\frac{a^2}{4b}\right)$$ $[a > 0, \quad b > 0]$ ET I 110(22)

4. $$\int_0^\infty J_0\left(a\sqrt{x}\right) \cos(bx)\, dx = \frac{1}{b} \sin\left(\frac{a^2}{4b}\right)$$ $[a > 0, \quad b > 0]$ ET I 53(21)

6.676

1. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{x}\right) \sin(cx)\, dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \cos\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -2]$
ET I 111(29)a

2. $$\int_0^\infty J_\nu\left(a\sqrt{x}\right) J_\nu\left(b\sqrt{x}\right) \cos(cx)\, dx = \frac{1}{c} J_\nu\left(\frac{ab}{2c}\right) \sin\left(\frac{a^2+b^2}{4c} - \frac{\nu\pi}{2}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -1]$
ET I 54(27)

3. $$\int_0^\infty J_0\left(a\sqrt{x}\right) K_0\left(a\sqrt{x}\right) \sin(bx)\, dx = \frac{1}{2b} K_0\left(\frac{a^2}{2b}\right)$$ $[\operatorname{Re} a > 0, \quad b > 0]$ ET I 111(31)

4. $$\int_0^\infty J_0\left(\sqrt{ax}\right) K_0\left(\sqrt{ax}\right) \cos(bx)\,dx = \frac{\pi}{4b}\left[I_0\left(\frac{a}{2b}\right) - \mathbf{L}_0\left(\frac{a}{2b}\right)\right]$$
$[\operatorname{Re} a > 0, \quad b > 0]$ ET I 54(29)

5. $$\int_0^\infty K_0\left(\sqrt{ax}\right) Y_0\left(\sqrt{ax}\right) \cos(bx)\,dx = -\frac{1}{2b} K_0\left(\frac{a}{2b}\right) \qquad \left[\operatorname{Re}\sqrt{a} > 0, \quad b > 0\right]$$
ET I 54(30)

6. $$\int_0^\infty K_0\left(\sqrt{ax}e^{\frac{1}{4}\pi i}\right) K_0\left(\sqrt{ax}e^{-\frac{1}{4}\pi i}\right) \cos(bx)\,dx = \frac{\pi^2}{8b}\left[\mathbf{H}_0\left(\frac{a}{2b}\right) - Y_0\left(\frac{a}{2b}\right)\right]$$
$[\operatorname{Re} a > 0, b > 0]$ ET I 54(31)

6.677

1. $$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\sin(cx)\,dx = 0 \qquad [0 < c < b]$$
$$= \frac{\cos\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c]$$
ET I 113(47)

2. $$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\exp\left(-a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b]$$
$$= \frac{-\sin\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}} \qquad [0 < b < c]$$
ET I 57(48)a

3.[6] $$\int_0^\infty J_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{\cos z\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-\beta^2}} \qquad [0 < \beta < \alpha, \quad z > 0]$$
$$= 0 \qquad [0 < \alpha < \beta, \quad z > 0]$$
MO 47a

4. $$\int_0^\infty Y_0\left(\alpha\sqrt{x^2+z^2}\right)\cos\beta x\,dx = \frac{1}{\sqrt{\alpha^2-\beta^2}}\sin\left(z\sqrt{\alpha^2-\beta^2}\right) \qquad [0 < \beta < \alpha, \quad z > 0]$$
$$= -\frac{1}{\sqrt{\beta^2-\alpha^2}}\exp\left(-z\sqrt{\beta^2-\alpha^2}\right) \qquad [0 < \alpha < \beta, \quad z > 0]$$
MO 47a

5. $$\int_0^\infty K_0\left[\alpha\sqrt{x^2+\beta^2}\right]\cos(\gamma x)\,dx = \frac{\pi}{2\sqrt{\alpha^2+\gamma^2}}\exp\left(-\beta\sqrt{\alpha^2+\gamma^2}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \gamma > 0]$
ET I 56(43)

6. $$\int_0^a J_0\left(b\sqrt{a^2-x^2}\right)\cos(cx)\,dx = \frac{\sin\left(a\sqrt{b^2+c^2}\right)}{\sqrt{b^2+c^2}} \qquad [b > 0]$$
MO 48a, ET I 57(47)

7. $$\int_0^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx)\,dx = \frac{\cosh\left(a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}} \qquad [0 < c < b, \quad a > 0]$$
$$= 0 \qquad [0 < b < c, \quad a > 0]$$
ET I 57(49)

8. $$\int_0^\infty H_0^{(1)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx = -i\frac{\exp\left(i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[\pi > \arg\sqrt{\beta^2-x^2} \ge 0, \quad \alpha > 0, \quad \gamma > 0\right]$$ ET I 59(59)

9. $$\int_0^\infty H_0^{(2)}\left(\alpha\sqrt{\beta^2-x^2}\right)\cos(\gamma x)\,dx = \frac{i\exp\left(-i\beta\sqrt{\alpha^2+\gamma^2}\right)}{\sqrt{\alpha^2+\gamma^2}}$$
$$\left[-\pi < \arg\sqrt{\beta^2-x^2} \le 0, \quad \alpha > 0, \quad \gamma > 0\right]$$ ET I 58(58)

6.678 $$\int_0^\infty \left[K_0\left(2\sqrt{x}\right) + \frac{\pi}{2}\,Y_0\left(2\sqrt{x}\right)\right]\sin(bx)\,dx = \frac{\pi}{2b}\sin\left(\frac{1}{b}\right) \qquad [b > 0]$$ ET I 111(34)

6.679

1. $$\int_0^\infty J_{2\nu}\left[2b\sinh\left(\frac{x}{2}\right)\right]\sin(bx)\,dx = -i\left[I_{\nu-ib}(a)\,K_{\nu+ib}(a) - I_{\nu+ib}(a)\,K_{\nu-ib}(a)\right]$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$$ ET I 115(59)

2. $$\int_0^\infty J_{2\nu}\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = I_{\nu-ib}(a)\,K_{\nu+ib}(a) + I_{\nu+ib}(a)\,K_{\nu-ib}(a)$$
$$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET I 59(64)

3. $$\int_0^\infty J_{2\nu}\left[2a\cosh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = -\frac{\pi}{2}\left[J_{\nu+ib}(a)\,Y_{\nu-ib}(a) + J_{\nu-ib}(a)\,Y_{\nu+ib}(a)\right]$$ ET I 59(63)

4. $$\int_0^\infty J_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\sin(bx)\,dx = \frac{2}{\pi}\sinh(\pi b)\left[K_{ib}(a)\right]^2$$
$$[a > 0, \quad b > 0]$$ ET I 115(58)

5. $$\int_0^\infty J_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = \left[I_{ib}(a) + I_{-ib}(a)\right]K_{ib}(a)$$
$$[a > 0, \quad b > 0]$$ ET I 59(62)

6. $$\int_0^\infty Y_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = -\frac{2}{\pi}\cosh(\pi b)\left[K_{ib}(a)\right]^2$$
$$[a > 0, \quad b > 0]$$ ET I 59(65)

7. $$\int_0^\infty K_0\left[2a\sinh\left(\frac{x}{2}\right)\right]\cos(bx)\,dx = \frac{\pi^2}{4}\left\{\left[J_{ib}(a)\right]^2 + \left[Y_{ib}(a)\right]^2\right\}$$
$$[\operatorname{Re}a > 0, \quad b > 0]$$ ET I 59(66)

6.681

1. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)\,J_{2\nu}\left(2a\cos x\right)\,dx = \frac{\pi}{2}\,J_{\nu+\mu}(a)\,J_{\nu-\mu}(a) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 361(23)

2. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)\,Y_{2\nu}\left(2a\cos x\right)\,dx = \frac{\pi}{2}\left[\cot(2\nu\pi)\,J_{\nu+\mu}(a)\,J_{\nu-\mu}(a) - \operatorname{cosec}(2\nu\pi)\,J_{\mu-\nu}(a)\,J_{-\mu-\nu}(a)\right]$$
$$\left[|\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$ ET II 361(24)

3. $$\int_0^{\frac{\pi}{2}} \cos(2\mu x)\, I_{2\nu}\left(2a\cos x\right)\, dx = \frac{\pi}{2}\, I_{\nu-\mu}(a)\, I_{\nu+\mu}(a) \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET I 59(61)

4. $$\int_0^{\frac{\pi}{2}} \cos(\nu x)\, K_\nu\left(2a\cos x\right)\, dx = \frac{\pi}{2}\, I_0(a)\, K_\nu(a) \qquad [\operatorname{Re}\nu < 1]$$ WA 484(3)

5. $$\int_0^{\pi} J_0\left(2z\cos x\right)\cos 2nx\, dx = (-1)^n \pi J_n^2(z).$$ MO 45

6. $$\int_0^{\pi} J_0\left(2z\sin x\right)\cos 2nx\, dx = \pi J_n^2(z).$$ WA 43(3), MO 45

7. $$\int_0^{\frac{\pi}{2}} \cos(2n\pi)\, Y_0\left(2a\sin x\right)\, dx = \frac{\pi}{2}\, J_n(a)\, Y_n(a) \qquad [n = 0, 1, 2, \ldots]$$ ET II 360(16)

8. $$\int_0^{\pi} \sin(2\mu x)\, J_{2\nu}\left(2a\sin x\right)\, dx = \pi\sin(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$$
$[\operatorname{Re}\nu > -1]$ ET II 360(13)

9. $$\int_0^{\pi} \cos(2\mu x)\, J_{2\nu}\left(2a\sin x\right)\, dx = \pi\cos(\mu\pi)\, J_{\nu-\mu}(a)\, J_{\nu+\mu}(a)$$
$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 360(14)

10. $$\int_0^{\frac{\pi}{2}} J_{\nu+\mu}\left(2z\cos x\right)\cos[(\nu-\mu)x]\, dx = \frac{\pi}{2}\, J_\nu(z)\, J_\mu(z) \qquad [\operatorname{Re}(\nu+\mu) > -1]$$ MO 42

11. $$\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x]\, I_{\mu+\nu}\left(2a\cos x\right)\, dx = \frac{\pi}{2}\, I_\mu(a)\, I_\nu(a) \qquad [\operatorname{Re}(\mu+\nu) > -1]$$
WA 484(2), ET II 378(39)

12. $$\int_0^{\frac{\pi}{2}} \cos[(\mu-\nu)x]\, K_{\mu+\nu}\left(2a\cos x\right)\, dx = \frac{\pi^2}{4}\operatorname{cosec}[(\mu+\nu)\pi]\left[I_{-\mu}(a)\, I_{-\nu}(a) - I_\mu(a)\, I_\nu(a)\right]$$
$[|\operatorname{Re}(\mu+\nu)| < 1]$ ET II 378(40)

13.[8] $$\int_0^{\frac{\pi}{2}} K_{\nu-m}\left(2a\cos x\right)\cos[(m+\nu)x]\, dx = (-1)^m \frac{\pi}{2}\, I_m(a)\, K_\nu(a)$$
$[|\operatorname{Re}(\nu-m)| < 1]$ WA 485(4)

6.682

1.[7] $$\int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}\left(x\sin t\right)\sin^{\nu+\frac{1}{2}} t\, dt = \sqrt{\frac{\pi}{2x}}\, J_\nu(x)$$
[ν may be zero, a natural number, one half, or a natural number plus one half; $x > 0$] MO 42a

2. $$\int_0^{\frac{\pi}{2}} J_\nu\left(z\sin x\right)\sin^\nu x\cos^{2\nu} x\, dx = 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right) z^{-\nu} J_\nu^2\left(\frac{z}{2}\right)$$
$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ MO 42a

6.683

1. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin x)\, I_\mu(z\cos x)\tan^{\nu+1} x\, dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} J_\mu(z)$$
$[\operatorname{Re}\nu > \operatorname{Re}\mu > -1]$ WA 407(4)

2. $$\int_0^{\frac{\pi}{2}} J_\nu(z_1\sin x)\, J_\mu(z_2\cos x)\sin^{\nu+1} x\cos^{\mu+1} x\, dx = \frac{z_1^\nu z_2^\mu J_{\nu+\mu+1}\left(\sqrt{z_1^2+z_2^2}\right)}{\sqrt{\left(z_1^2+z_2^2\right)^{\nu+\mu+1}}}$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ WA 410(1)

3. $$\int_0^{\frac{\pi}{2}} J_\nu\left(z\cos^2 x\right) J_\mu\left(z\sin^2 x\right)\sin x\cos x\, dx = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k J_{\nu+\mu+2k+1}(z)$$
$[\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1]$ (see also **6.513** 6) WA 414(1)

4. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}(\cos\theta)^{2\nu+1}\, d\theta = \frac{s_{\mu+\nu,\nu-\mu+1}(z)}{2^{\mu-1} z^{\nu+1}\Gamma(\mu)}$$
$[\operatorname{Re}\nu > -1]$ WA 407(2)

5. $$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}\, d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\frac{2z}{\pi}}}$$
WA 407(3)

6. $$\int_0^{\frac{\pi}{2}} J_\mu(a\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\varrho+1}\, d\theta = 2^\varrho\,\Gamma(\varrho+1) a^{-\varrho-1} J_{\varrho+\mu+1}(a)$$
$[\operatorname{Re}\varrho > -1, \quad \operatorname{Re}\mu > -1]$
WA 406(1), EH II 46(5)

7. $$\int_0^{\frac{\pi}{2}} J_\nu(2z\sin\theta)(\sin\theta)^\nu(\cos\theta)^{2\nu}\, d\theta$$
$$= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m z^{\nu+2m}\,\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{m!\,\Gamma(\nu+m+1)\,\Gamma(2\nu+m+1)}$$
$$= \frac{1}{2} z^{-\nu}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\left[J_\nu(z)\right]^2$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$
EH II 47(10)

8. $$\int_0^{\frac{\pi}{2}} J_\nu(z\sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{-2\nu}\, d\theta = 2^{-\nu}\frac{z^{\nu-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-\nu\right)\sin z$$
$\left[-1 < \operatorname{Re}\nu < \frac{1}{2}\right]$ EH II 68(39)

9. $$\int_0^{\frac{\pi}{2}} J_\nu\left(z\sin^2\theta\right) J_\nu\left(z\cos^2\theta\right)(\sin\theta)^{2\nu+1}(\cos\theta)^{2\nu+1}\, d\theta = \frac{\Gamma\left(\frac{1}{2}+\nu\right) J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}}\,\Gamma(\nu+1)\sqrt{z}}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 409(1)

10. $$\int_0^{\frac{\pi}{2}} J_\mu\left(z\sin^2\theta\right) J_\nu\left(z\cos^2\theta\right)\sin^{2\mu+1}\theta\cos^{2\nu+1}\theta\,d\theta = \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right) J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi}\,\Gamma(\mu+\nu+1)\sqrt{2z}}$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 417(1)

6.684

1.[8] $$\int_0^\pi (\sin x)^{2\nu} \frac{J_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\,dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{J_\nu(\beta)}{\beta^\nu}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 362(27)

2. $$\int_0^\pi (\sin x)^{2\nu} \frac{Y_\nu\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)}{\left(\sqrt{\alpha^2+\beta^2-2\alpha\beta\cos x}\right)^\nu}\,dx = 2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\frac{J_\nu(\alpha)}{\alpha^\nu}\frac{Y_\nu(\beta)}{\beta^\nu}$$
$\left[|\alpha| < |\beta|, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 362(28)

6.685 $$\int_0^{\frac{\pi}{2}} \sec x\cos(2\lambda x)\,K_{2\mu}\left(a\sec x\right)\,dx = \frac{\pi}{2a}\,W_{\lambda,\mu}(a)\,W_{-\lambda,\mu}(a)$$ $[\operatorname{Re}a > 0]$ ET II 378(41)

6.686

1. $$\int_0^\infty \sin\left(ax^2\right) J_\nu(bx)\,dx = -\frac{\sqrt{\pi}}{2\sqrt{a}}\sin\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)$$
$[a>0, b>0, \operatorname{Re}\nu > -3]$ ET II 34(13)

2. $$\int_0^\infty \cos\left(ax^2\right) J_\nu(bx)\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\cos\left(\frac{b^2}{8a}-\frac{\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)$$
$[a>0, \quad b>0, \quad \operatorname{Re}\nu > -1]$ ET II 38(38)

3. $$\int_0^\infty \sin\left(ax^2\right) Y_\nu(bx)\,dx$$
$$= -\frac{\sqrt{\pi}}{4\sqrt{a}}\sec\left(\frac{\nu\pi}{2}\right) \times\left[\cos\left(\frac{b^2}{8a}-\frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a}+\frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0, \quad b>0, \quad -3 < \operatorname{Re}\nu < 3]$ ET II 107(7)

4. $$\int_0^\infty \cos\left(ax^2\right) Y_\nu(bx)\,dx$$
$$= \frac{\sqrt{\pi}}{4\sqrt{a}}\sec\left(\frac{\nu\pi}{2}\right) \times\left[\sin\left(\frac{b^2}{8a}-\frac{3\nu+1}{4}\pi\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right) + \cos\left(\frac{b^2}{8a}+\frac{\nu-1}{4}\pi\right) Y_{\frac{1}{2}\nu}\left(\frac{b^2}{8a}\right)\right]$$
$[a>0, \quad b>0, \quad -1 < \operatorname{Re}\nu < 1]$ ET II 107(8)

5. $$\int_0^\infty \sin\left(ax^2\right) J_1(bx)\,dx = \frac{1}{b}\sin\frac{b^2}{4a}$$ $[a>0, \quad b>0]$ ET II 19(16)

6. $$\int_0^\infty \cos\left(ax^2\right) J_1(bx)\,dx = \frac{2}{b}\sin^2\left(\frac{b^2}{8a}\right) \qquad [a>0, \quad b>0]$$ ET II 20(20)

7. $$\int_0^\infty \sin^2\left(ax^2\right) J_1(bx)\,dx = \frac{1}{2b}\cos\left(\frac{b^2}{8a}\right) \qquad [a>0, \quad b>0]$$ ET II 19(17)

6.687 $$\int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}\left(xe^{i\frac{\pi}{4}}\right) K_{2\nu}\left(xe^{-i\frac{\pi}{4}}\right) dx = \frac{\Gamma\left(\frac{1}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)\sqrt{\pi}}{8\sqrt{a}} W_{\frac{1}{4},\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\frac{1}{4},\nu}\left(ae^{-i\frac{\pi}{2}}\right)$$
$$\left[a>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{4}\right]$$ ET II 372(1)

6.688

1. $$\int_0^{\frac{\pi}{2}} J_\nu\left(\mu z\sin t\right)\cos\left(\mu x\cos t\right)\,dt = \frac{\pi}{2} J_{\frac{\nu}{2}}\left(\mu\frac{\sqrt{x^2+z^2}+x}{2}\right) J_{\frac{\nu}{2}}\left(\mu\frac{\sqrt{x^2+z^2}-x}{2}\right)$$
$$[\operatorname{Re}\nu>-1, \quad \operatorname{Re}z>0]$$ MO 46

2. $$\int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1}\cos\left(\beta\cos x\right) J_\nu\left(\alpha\sin x\right)\,dx = 2^{-\frac{1}{2}}\sqrt{\pi}\alpha^\nu\left(\alpha^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} J_{\nu+\frac{1}{2}}\left[\left(\alpha^2+\beta^2\right)^{\frac{1}{2}}\right]$$
$$[\operatorname{Re}\nu>-1]$$ ET II 361(19)

3. $$\int_0^{\frac{\pi}{2}} \cos\left[(z-\zeta)\cos\theta\right] J_{2\nu}\left[2\sqrt{z\zeta}\sin\theta\right]\,d\theta = \frac{\pi}{2} J_\nu(z)\,J_\nu(\zeta)$$
$$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$$ EH II 47(8)

6.69–6.74 Combinations of Bessel and trigonometric functions and powers

6.691 $$\int_0^\infty x\sin(bx)\,K_0(ax)\,dx = \frac{\pi b}{2}\left(a^2+b^2\right)^{-\frac{3}{2}} \qquad [\operatorname{Re}a>0, \quad b>0]$$ ET I 105(47)

6.692

1. $$\int_0^\infty x\,K_\nu(ax)\,I_\nu(bx)\sin(cx)\,dx = -\frac{1}{2}(ab)^{-\frac{3}{2}}c\left(u^2-1\right)^{-\frac{1}{2}} Q^1_{\nu-\frac{1}{2}}(u), \qquad u=(2ab)^{-1}\left(a^2+b^2+c^2\right)$$
$$\left[\operatorname{Re}a>|\operatorname{Re}b|, \quad c>0, \quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$ ET I 106(54)

2. $$\int_0^\infty x\,K_\nu(ax)\,K_\nu(bx)\sin(cx)\,dx = \frac{\pi}{4}(ab)^{-\frac{3}{2}}c\left(u^2-1\right)^{-\frac{1}{2}}\Gamma\left(\tfrac{3}{2}+\nu\right)\Gamma\left(\tfrac{3}{2}-\nu\right) P^{-1}_{\nu-\frac{1}{2}}(u)$$
$$u=(2ab)^{-1}\left(a^2+b^2+c^2\right) \qquad \left[\operatorname{Re}(a+b)>0, \quad c>0, \quad |\operatorname{Re}\nu|<\tfrac{3}{2}\right]$$ ET I 107(61)

6.693

1. $$\int_0^\infty J_\nu(\alpha x)\sin\beta x\frac{dx}{x} = \frac{1}{\nu}\sin\left(\nu\arcsin\frac{\beta}{\alpha}\right) \qquad [\beta\le\alpha]$$
$$= \frac{\alpha^\nu\sin\frac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta\ge\alpha]$$
$$[\operatorname{Re}\nu>-1]$$ WA 443(2)

2.[8] $$\int_0^\infty J_\nu(\alpha x)\cos\beta x\frac{dx}{x} = \frac{1}{\nu}\cos\left(\nu\arcsin\frac{\beta}{\alpha}\right) \qquad [\beta \le \alpha]$$
$$= \frac{\alpha^\nu\cos\frac{\nu\pi}{2}}{\nu\left(\beta+\sqrt{\beta^2-\alpha^2}\right)^\nu} \qquad [\beta \ge \alpha] \qquad [\operatorname{Re}\nu > 0]$$

WA 443(3)

3. $$\int_0^\infty Y_\nu(ax)\sin(bx)\frac{dx}{x}$$
$$= -\frac{1}{\nu}\tan\left(\frac{\nu\pi}{2}\right)\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right]$$
$[0 < b < a, \quad |\operatorname{Re}\nu| < 1]$
$$= \frac{1}{2\nu}\sec\left(\frac{\nu\pi}{2}\right)\left\{a^{-\nu}\cos(\nu\pi)\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^\nu - a^\nu\left[b-\left(b^2-a^2\right)^{\frac{1}{2}}\right]^{-\nu}\right\}$$
$[0 < a < b, \quad |\operatorname{Re}\nu| < 1]$

ET I 103(35)

4. $$\int_0^\infty J_\nu(ax)\sin(bx)\frac{dx}{x^2}$$
$$= \frac{\sqrt{a^2-b^2}\sin\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu^2-1} - \frac{b\cos\left[\nu\arcsin\left(\frac{b}{a}\right)\right]}{\nu\left(\nu^2-1\right)} \qquad [0 < b < a, \quad \operatorname{Re}\nu > 0]$$
$$= \frac{-a^\nu\cos\left(\frac{\nu\pi}{2}\right)\left[b+\nu\sqrt{b^2-a^2}\right]}{\nu\left(\nu^2-1\right)\left[b+\sqrt{b^2-a^2}\right]^\nu} \qquad [0 < a < b, \quad \operatorname{Re}\nu > 0]$$

ET I 99(6)

5. $$\int_0^\infty J_\nu(ax)\cos(bx)\frac{dx}{x^2}$$
$$= \frac{a\cos\left[(\nu-1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu-1)} + \frac{a\cos\left[(\nu+1)\arcsin\left(\frac{b}{a}\right)\right]}{2\nu(\nu+1)} \qquad [0 < b < a, \quad \operatorname{Re}\nu > 1]$$
$$= \frac{a^\nu\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu-1)\left[b+\sqrt{b^2-a^2}\right]^{\nu-1}} - \frac{a^{\nu+2}\sin\left(\frac{\nu\pi}{2}\right)}{2\nu(\nu+1)\left[b+\sqrt{b^2-a^2}\right]^{\nu+1}} \qquad [0 < a < b, \quad \operatorname{Re}\nu > 1]$$

ET I 44(6)

6. $$\int_0^\infty J_0(\alpha x)\sin x\frac{dx}{x} = \frac{\pi}{2} \qquad [0 < \alpha < 1]$$
$$= \operatorname{arccosec}_\alpha \qquad [\alpha > 1]$$

WH

7. $$\int_0^\infty J_0(x)\sin\beta x\frac{dx}{x} = \frac{\pi}{2} \qquad [\beta > 1]$$
$$= \arcsin\beta \qquad \left[\beta^2 < 1\right]$$
$$= -\frac{\pi}{2} \qquad [\beta < -1]$$

8. $$\int_0^\infty \left[J_0(x)-\cos\alpha x\right]\frac{dx}{x} = \ln 2\alpha$$ NT 66(13)

9. $$\int_0^z J_\nu(x)\sin(z-x)\frac{dx}{x} = \frac{2}{\nu}\sum_{k=0}^\infty(-1)^k J_{\nu+2k+1}(z) \qquad [\operatorname{Re}\nu > 0]$$ WA 416(4)

10. $$\int_0^z J_\nu(x)\cos(z-x)\frac{dx}{x} = \frac{1}{\nu}\,J_\nu(z) + \frac{2}{\nu}\sum_{k=1}^{\infty}(-1)^k\,J_{\nu+2k}(z)$$
$[\operatorname{Re}\nu > 0]$ WA 416(5)

6.694[10] $$\int_0^\infty \left[\frac{J_1(ax)}{x}\right]^2 \sin(bx)\,dx$$
$$= \frac{1}{2}b - \left(\frac{4a}{3\pi}\right)\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{b}{2a}\right) + \left(1-\frac{b^2}{4a^2}\right)\boldsymbol{K}\left(\frac{b}{2a}\right)\right]$$ $[0 \le b \le 2a]$ ET I 102(22)
$$= \frac{1}{2}b - \frac{2b}{3\pi}\left[\left(1+\frac{b^2}{4a^2}\right)\boldsymbol{E}\left(\frac{2a}{b}\right) - \left(1-\left(\frac{4a^2}{b^2}\right)^{-1}\right)\boldsymbol{K}\left(\frac{2a}{b}\right)\right]$$ $[0 \le 2a \le b]$

6.695

1. $$\int_0^\infty \frac{\sin\alpha x}{\beta^2+x^2}\,J_0(ux)\,dx = \frac{\sinh\alpha\beta}{\beta}\,K_0(\beta u)$$ $[\alpha>0, \quad \operatorname{Re}\beta>0, \quad u>\alpha]$ MO 46

2. $$\int_0^\infty \frac{\cos\alpha x}{\beta^2+x^2}\,J_0(ux)\,dx = \frac{\pi}{2}\frac{e^{-\alpha\beta}}{\beta}\,I_0(\beta u)$$ $[\alpha>0, \quad \operatorname{Re}\beta>0, \quad -\alpha<u<\alpha]$ MO 46

3. $$\int_0^\infty \frac{x}{x^2+\beta^2}\sin(\alpha x)\,J_0(\gamma x)\,dx = \frac{\pi}{2}e^{-\alpha\beta}\,I_0(\gamma\beta)$$ $[\alpha>0, \quad \operatorname{Re}\beta>0, \quad 0<\gamma<\alpha]$ ET II 10(36)

4. $$\int_0^\infty \frac{x}{x^2+\beta^2}\cos(\alpha x)\,J_0(\gamma x)\,dx = \cosh(\alpha\beta)\,K_0(\beta\gamma)$$ $[\alpha>0, \quad \operatorname{Re}\beta>0, \quad \alpha<\gamma]$ ET II 11(45)

6.696 $$\int_0^\infty [1-\cos(\alpha x)]\,J_0(\beta x)\frac{dx}{x} = \operatorname{arccosh}\left(\frac{\alpha}{\beta}\right)$$ $[0<\beta<\alpha]$
$$= 0$$ $[0<\alpha<\beta]$ ET II 11(43)

6.697

1. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta}\,J_0(x)\,dx = 2\int_0^\alpha \frac{\cos\beta u}{\sqrt{1-u^2}}\,du$$ $[0\le\alpha\le1]$ WA 463(2)
$$= \pi\,J_0(\beta)$$ $[1\le\alpha<\infty]$ WA 463(1), ET II 345(42)

2. $$\int_0^\infty \frac{\sin(x+t)}{x+t}\,J_0(t)\,dt = \frac{\pi}{2}\,J_0(x)$$ $[x>0]$ WA 475(4)

3. $$\int_0^\infty \frac{\cos(x+t)}{x+t}\,J_0(t)\,dt = -\frac{\pi}{2}\,Y_0(x)$$ $[x>0]$ WA 475(5)

4. $$\int_{-\infty}^\infty \frac{|x|}{x+\beta}\sin[\alpha(x+\beta)]\,J_0(bx)\,dx = 0$$ $[0\le\alpha<b]$ WA 464(5), ET II 345(43)a

5. $$\int_{-\infty}^\infty \frac{\sin[\alpha(x+\beta)]}{x+\beta}\left[J_{n+\frac{1}{2}}(x)\right]^2\,dx = \pi\left[J_{n+\frac{1}{2}}(\beta)\right]^2$$ $[2\le\alpha<\infty, \quad n=0,1,\ldots]$ ET II 346(45)

6. $$\int_{-\infty}^{\infty} \frac{\sin[\alpha(x+\beta)]}{x+\beta} J_{n+\frac{1}{2}}(x)\, J_{-n-\frac{1}{2}}(x)\, dx = \pi\, J_{n+\frac{1}{2}}(\beta)\, J_{-n-\frac{1}{2}}(\beta)$$
$$[2 \le \alpha < \infty, \quad n = 0, 1, \ldots]$$
ET II 346(46)

7. $$\int_{-\infty}^{\infty} \frac{J_\mu[a(z+x)]}{(z+x)^\mu} \frac{J_\nu[a(\zeta+x)]}{(\zeta+x)^\nu}\, dx = \frac{\Gamma(\mu+\nu)\sqrt{\pi}\sqrt{\frac{2}{a}}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \cdot \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}}$$
$$[\operatorname{Re}(\mu+\nu) > 0]$$
WA 463(3)

6.698

1. $$\int_0^\infty \sqrt{x}\, J_{\nu+\frac{1}{4}}(ax)\, J_{-\nu+\frac{1}{4}}(ax) \sin(bx)\, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a]$$
$$= 0 \qquad [0 < 2a < b]$$
ET I 102(26)

2. $$\int_0^\infty \sqrt{x}\, J_{\nu-\frac{1}{4}}(ax)\, J_{-\nu-\frac{1}{4}}(ax) \cos(bx)\, dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \qquad [0 < b < 2a]$$
$$= 0 \qquad [0 < 2a < b]$$
ET I 46(24)

3. $$\int_0^\infty \sqrt{x}\, I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx)\, dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{\left(b+\sqrt{a^2+b^2}\right)^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu < \tfrac{5}{4}\right]$$
ET I 106(56)

4. $$\int_0^\infty \sqrt{x}\, I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx)\, dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{\left(b+\sqrt{a^2+b^2}\right)^{2\nu}}{\sqrt{a^2+b^2}}$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu < \tfrac{3}{4}\right]$$
ET I 50(49)

6.699

1. $$\int_0^\infty x^\lambda J_\nu(ax) \sin(bx)\, dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma\left(\frac{2+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda}{2}\right)} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right)$$
$$\left[0 < b < a, \quad -\operatorname{Re}\nu - 1 < 1 + \operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
$$= \left(\frac{1}{2}a\right)^\nu b^{-(\nu+\lambda+1)} \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)} \sin\left[\pi\left(\frac{1+\lambda+\nu}{2}\right)\right]$$
$$\times F\left(\frac{2+\lambda+\nu}{2}, \frac{1+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right)$$
$$\left[0 < a < b, \quad -\operatorname{Re}\nu - 1 < 1 + \operatorname{Re}\lambda < \tfrac{3}{2}\right]$$
ET I 100(11)

2. $$\int_0^\infty x^\lambda J_\nu(ax)\cos(bx)\,dx$$
$$= \frac{2^\lambda a^{-(1+\lambda)}\,\Gamma\left(\frac{1+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda+1}{2}\right)} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right)$$
$\left[0 < b < a, \quad -\operatorname{Re}\nu < 1 + \operatorname{Re}\lambda < \frac{3}{2}\right]$
$$= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)}\,\Gamma(1+\lambda+\nu)\cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right)$$
$\left[0 < a < b, \quad -\operatorname{Re}\nu < 1 + \operatorname{Re}\lambda < \frac{3}{2}\right]$

ET I 45(13)

3. $$\int_0^\infty x^\lambda K_\mu(ax)\sin(bx)dx = \frac{2^\lambda b\,\Gamma\left(\frac{2+\mu+\lambda}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(-\lambda \pm \mu) < 2, \quad \operatorname{Re} a > 0, \quad b > 0]$

ET I 106(50)

4. $$\int_0^\infty x^\lambda K_\mu(ax)\cos(bx)\,dx = 2^{\lambda-1}a^{-\lambda-1}\,\Gamma\left(\frac{\mu+\lambda+1}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right)$$
$$\times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right)$$
$[\operatorname{Re}(-\lambda \pm \mu) < 1, \quad \operatorname{Re} a > 0, \quad b > 0]$ ET I 49(42)

5. $$\int_0^\infty x^\nu \sin(ax)\,J_\nu(bx)\,dx = \frac{\sqrt{\pi}2^\nu b^\nu\left(a^2-b^2\right)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0 < b < a, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right]$$
$$= 0 \qquad \left[0 < a < b, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right]$$

ET II 32(4)

6. $$\int_0^\infty x^\nu \cos(ax)\,J_\nu(bx)\,dx = -2^\nu \frac{\sin(\nu\pi)}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+\nu\right) b^\nu\left(a^2-b^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$
$$= 2^\nu \frac{b^\nu}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+\nu\right)\left(b^2-a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

ET II 36(29)

7. $$\int_0^\infty x^{\nu+1}\sin(ax)\,J_\nu(bx)\,dx$$
$$= -2^{1+\nu}a\frac{\sin(\nu\pi)}{\sqrt{\pi}}b^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)\left(a^2-b^2\right)^{-\nu-\frac{3}{2}} \qquad \left[0 < b < a, \quad -\frac{3}{2} < \operatorname{Re}\nu < -\frac{1}{2}\right]$$
$$= -\frac{2^{1+\nu}}{\sqrt{\pi}}ab^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)\left(b^2-a^2\right)^{-\nu-\frac{3}{2}} \qquad \left[0 < a < b, \quad -\frac{3}{2} < \operatorname{Re}\nu < -\frac{1}{2}\right]$$

ET II 32(3)

8. $$\int_0^\infty x^{\nu+1}\cos(ax)\,J_\nu(bx)\,dx = 2^{1+\nu}\sqrt{\pi}ab^\nu\frac{\left(a^2-b^2\right)^{-\nu-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}-\nu\right)} \qquad \left[0 < b < a, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right]$$
$$= 0 \qquad \left[0 < a < b, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right]$$

ET II 36(28)

9. $$\int_0^1 x^\nu \sin(ax)\, J_\nu(ax)\, dx = \frac{1}{2\nu+1}\left[\sin a J_\nu(a) - \cos a\, J_{\nu+1}(a)\right]$$

$[\operatorname{Re}\nu > -1]$ ET II 334(9)a

10. $$\int_0^1 x^\nu \cos(ax)\, J_\nu(ax)\, dx = \frac{1}{2\nu+1}\left[\cos a J_\nu(a) + \sin a\, J_{\nu+1}(a)\right]$$

$\left[\operatorname{Re}\gamma > -\frac{1}{2}\right]$ ET II 335(20)

11. $$\int_0^\infty x^{1+\nu}\, K_\nu(ax) \sin(bx)\, dx = \sqrt{\pi}(2a)^\nu\, \Gamma\left(\frac{3}{2}+\nu\right) b \left(b^2+a^2\right)^{-\frac{3}{2}-\nu}$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET I 105(49)

12. $$\int_0^\infty x^\mu\, K_\mu(ax) \cos(bx)\, dx = \frac{1}{2}\sqrt{\pi}(2a)^\mu\, \Gamma\left(\mu+\frac{1}{2}\right) \left(b^2+a^2\right)^{-\mu-\frac{1}{2}}$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET I 49(41)

13. $$\int_0^\infty x^\nu\, Y_{\nu-1}(ax) \sin(bx)\, dx = 0 \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

$$= \frac{2^\nu \sqrt{\pi} a^{\nu-1} b}{\Gamma\left(\frac{1}{2}-\nu\right)} \left(b^2-a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

ET I 104(36)

14. $$\int_0^\infty x^\nu\, Y_\nu(ax) \cos(bx)\, dx = 0 \qquad \left[0 < b < a, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

$$= -2^\nu \sqrt{\pi} a^\nu \frac{\left(b^2-a^2\right)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\nu\right)} \qquad \left[0 < a < b, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$$

ET I 47(30)

6.711

1. $$\int_0^\infty x^{\nu-\mu}\, J_\mu(ax)\, J_\nu(bx) \sin(cx)\, dx = 0 \qquad [0 < c < b-a, \quad -1 < \operatorname{Re}\nu < 1 + \operatorname{Re}\mu]$$

ET I 103(28)

2. $$\int_0^\infty x^{\nu-\mu+1}\, J_\mu(ax)\, J_\nu(bx) \cos(cx)\, dx = 0$$

$[0 < c < b-a, \quad a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < \operatorname{Re}\mu]$ ET I 47(25)

3. $$\int_0^\infty x^{\nu-\mu-2}\, J_\mu(ax)\, J_\nu(bx) \sin(cx)\, dx = 2^{\nu-\mu-1} a^\mu b^{-\nu} \frac{c\,\Gamma(\nu)}{\Gamma(\mu+1)}$$

$[0 < a, \quad 0 < b, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu + 3]$ ET I 103(29)

4. $$\int_0^\infty x^{\varrho-\mu-1}\, J_\mu(ax)\, J_\varrho(bx) \cos(cx)\, dx = 2^{\varrho-\mu-1} b^{-\varrho} a^\mu \frac{\Gamma(\varrho)}{\Gamma(\mu+1)}$$

$[b > 0, \quad a > 0, \quad 0 < c < b-a, \quad 0 < \operatorname{Re}\varrho < \operatorname{Re}\mu + 2]$ ET I 47(26)

5. $$\int_0^\infty x^{1-2\nu} \sin(2ax)\, J_\nu(x)\, Y_\nu(x)\, dx = -\frac{\Gamma\left(\frac{3}{2}-\nu\right) a}{2\,\Gamma\left(2\nu-\frac{1}{2}\right)\Gamma(2-\nu)}\, F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right)$$
$$\left[0 < \operatorname{Re}\nu < \tfrac{3}{2}, \quad 0 < a < 1\right]$$
ET II 348(63)

6.[10] $$\int_0^\infty \arg\sin(zx) x^{\nu-\mu-4}\, J_\mu(ax)\, J_\nu(\rho x)\, dx = z\frac{\Gamma(\nu)\, a^\mu \rho^{-\nu}}{2^{\mu-\nu+3}\,\Gamma(\mu+1)}\left[\frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - \frac{2z^2}{3}\right]$$

7.[10] $$\int_0^\infty \cos(zx) x^{\nu-\mu-3}\, J_\mu(ax)\, J_\nu(\rho x)\, dx = \frac{\Gamma(\nu)\, a^\mu \rho^{-\nu}}{2^{\mu-\nu+3}\,\Gamma(\mu+1)}\left[\frac{\rho^2}{\nu-1} - \frac{a^2}{\mu+1} - 2z^2\right]$$

6.712

1. $$\int_0^\infty x^\nu \left[J_\nu(ax)\cos(ax) + Y_\nu(ax)\sin(ax)\right]\sin(bx)\, dx = \frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2+2ab\right)^{-\nu-\frac{1}{2}}$$
$$\left[b > 0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ ET I 104(40)

2. $$\int_0^\infty x^\nu \left[Y_\nu(ax)\cos(ax) - J_\nu(ax)\sin(ax)\right]\cos(bx)\, dx = -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2+2ab\right)^{-\nu-\frac{1}{2}}$$
ET I 48(35)

3. $$\int_0^\infty x^\nu \left[J_\nu(ax)\cos(ax) - Y_\nu(ax)\sin(ax)\right]\sin(bx)\, dx$$
$$= 0 \qquad \left[0 < b < 2a, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$
$$= \frac{2^\nu\sqrt{\pi} b^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-2ab\right)^{-\nu-\frac{1}{2}} \qquad \left[2a < b, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$
ET I 104(41)

4. $$\int_0^\infty x^\nu \left[J_\nu(ax)\sin(ax) + Y_\nu(ax)\cos(ax)\right]\cos(bx)\, dx$$
$$= 0 \qquad \left[0 < b < 2a, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= -\frac{\sqrt{\pi}(2a)^\nu}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-2ab\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < 2a < b, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET I 48(33)

6.713

1. $$\int_0^\infty x^{1-2\nu} \sin(2ax)\left\{\left[J_\nu(x)\right]^2 - \left[Y_\nu(x)\right]^2\right\} dx$$
$$= \frac{\sin(2\nu\pi)\,\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{3}{2}-2\nu\right) a}{\pi\,\Gamma(2-\nu)}\, F\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right)$$
$$\left[0 < \operatorname{Re}\nu < \tfrac{3}{4}, \quad 0 < a < 1\right]$$ ET II 348(64)

2. $$\int_0^\infty x^{2-2\nu} \sin(2ax)\left[J_\nu(x)\, J_{\nu-1}(x) - Y_\nu(x)\, Y_{\nu-1}(x)\right] dx$$
$$= -\frac{\sin(2\nu\pi)\,\Gamma\left(\frac{3}{2}-\nu\right)\Gamma\left(\frac{5}{2}-2\nu\right) a}{\pi\,\Gamma(2-\nu)}\, F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right)$$
$$\left[\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{5}{4}, \quad 0 < a < 1\right]$$ ET II 348(65)

3. $$\int_0^\infty x^{2-2\nu} \sin(2ax) \left[J_\nu(x)\, Y_{\nu-1}(x) + Y_\nu(x)\, J_{\nu-1}(x)\right] dx = -\frac{\Gamma\left(\frac{3}{2}-\nu\right) a}{\Gamma\left(2\nu-\frac{3}{2}\right)\Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right)$$

$\left[\frac{1}{2} < \operatorname{Re}\nu < \frac{5}{2}, \quad 0 < a < 1\right]$ ET II 349(66)

6.714

1. $$\int_0^\infty \sin(2ax) \left[x^\nu J_\nu(x)\right]^2 dx$$
$$= \frac{a^{-2\nu}\,\Gamma\left(\frac{1}{2}+\nu\right)}{2\sqrt{\pi}\,\Gamma(1-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}; 1-\nu; a^2\right) \qquad \left[0 < a < 1, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= \frac{a^{-4\nu-1}\,\Gamma\left(\frac{1}{2}+\nu\right)}{2\,\Gamma(1+\nu)\,\Gamma\left(\frac{1}{2}-2\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \qquad \left[a > 1, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$

ET II 343(31)

2. $$\int_0^\infty \cos(2ax) \left[x^\nu J_\nu(x)\right]^2 dx$$
$$= \frac{a^{-2\nu}\,\Gamma(\nu)}{2\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\nu+\frac{1}{2}, \frac{1}{2}; 1-\nu; a^2\right)$$
$$+ \frac{\Gamma(-\nu)\,\Gamma\left(\frac{1}{2}+2\nu\right)}{2\pi\,\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^2\right) \qquad \left[0 < a < 1, \quad -\tfrac{1}{4} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$
$$= -\frac{\sin(\nu\pi) a^{-4\nu-1}\,\Gamma\left(\frac{1}{2}+2\nu\right)}{\Gamma(1+\nu)\,\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \qquad \left[a > 1, \quad -\tfrac{1}{4} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$

ET II 344(33)

6.715

1. $$\int_0^\infty \frac{x^\nu}{x+\beta} \sin(x+\beta)\, J_\nu(x)\, dx = \frac{\pi}{2} \sec(\nu\pi) \beta^\nu\, J_{-\nu}(\beta)$$

$\left[|\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 340(8)

2. $$\int_0^\infty \frac{x^\nu}{x+\beta} \cos(x+\beta)\, J_\nu(x)\, dx = -\frac{\pi}{2} \sec(\nu\pi) \beta^\nu\, Y_{-\nu}(\beta)$$

$\left[|\arg\beta| < \pi, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$ ET II 340(9)

6.716

1. $$\int_0^a x^\lambda \sin(a-x)\, J_\nu(x)\, dx = 2a^{\lambda+1} \sum_{n=0}^\infty \frac{(-1)^n\,\Gamma(\nu-\lambda+2n)\,\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\,\Gamma(\nu+\lambda+3+2n)} (\nu+2n+1)\, J_{\nu+2n+1}(a)$$

$[\operatorname{Re}(\lambda+\nu) > -1]$ ET II 335(16)

2. $$\int_0^a x^\lambda \cos(a-x)\, J_\nu(x)\, dx = \frac{a^{\lambda+1}\, J_\nu(a)}{\lambda+\nu+1} + 2a^{\lambda+1} \times \sum_{n=1}^\infty \frac{(-1)^n\,\Gamma(\nu-\lambda+2n-1)\,\Gamma(\nu+\lambda+1)}{\Gamma(\nu-\lambda)\,\Gamma(\nu+\lambda+2n+2)} (\nu+2n)\, J_{\nu+2n}(a)$$

$[\operatorname{Re}(\lambda+\nu) > -1]$ ET II 335(26)

6.717 $$\int_{-\infty}^{\infty} \frac{\sin[a(x+\beta)]}{x^\nu (x+\beta)} J_{\nu+2n}(x)\,dx = \pi\beta^{-\nu} J_{\nu+2n}(\beta)$$
$\left[1 \le a < \infty, n = 0, 1, 2, \ldots; \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET II 345(44)

6.718

1. $$\int_0^\infty \frac{x^\nu}{x^2+\beta^2} \sin(\alpha x)\, J_\nu(\gamma x)\,dx = \beta^{\nu-1} \sinh(\alpha\beta)\, K_\nu(\beta\gamma)$$
$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \frac{3}{2}\right]$ ET II 33(8)

2. $$\int_0^\infty \frac{x^{\nu+1}}{x^2+\beta^2} \cos(\alpha x)\, J_\nu(\gamma x)\,dx = \beta^\nu \cosh(\alpha\beta)\, K_\nu(\beta\gamma)$$
$\left[0 < \alpha \le \gamma, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right]$ ET II 37(33)

3. $$\int_0^\infty \frac{x^{1-\nu}}{x^2+\beta^2} \sin(\alpha x)\, J_\nu(\gamma x)\,dx = \frac{\pi}{2}\beta^{-\nu} e^{-\alpha\beta}\, I_\nu(\beta\gamma)$$
$\left[0 < \gamma \le \alpha, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 33(9)

4. $$\int_0^\infty \frac{x^{-\nu}}{x^2+\beta^2} \cos(\alpha x)\, J_\nu(\gamma x)\,dx = \frac{\pi}{2}\beta^{-\nu-1} e^{-\alpha\beta}\, I_\nu(\beta\gamma)$$
$\left[0 < \gamma \le \alpha, \operatorname{Re}\beta > 0, \operatorname{Re}\nu > -\frac{3}{2}\right]$
ET II 37(34)

6.719

1.[6] $$\int_0^\alpha \frac{\sin(\beta x)}{\sqrt{\alpha^2 - x^2}} J_\nu(x)\,dx = \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\alpha\beta)\, J_{\frac{1}{2}\nu+n+\frac{1}{2}}\left(\tfrac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n-\frac{1}{2}}\left(\tfrac{1}{2}\alpha\right)$$
$[\operatorname{Re}\nu > -2]$ ET II 335(17)

2. $$\int_0^\alpha \frac{\cos(\beta x)}{\sqrt{\alpha^2 - x^2}} J_\nu(x)\,dx = \frac{\pi}{2} J_0(\alpha\beta) \left[J_{\frac{1}{2}\nu}\left(\tfrac{1}{2}\alpha\right)\right]^2 + \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(\alpha\beta)\, J_{\frac{1}{2}\nu+n}\left(\tfrac{1}{2}\alpha\right) J_{\frac{1}{2}\nu-n}\left(\tfrac{1}{2}\alpha\right)$$
$[\operatorname{Re}\nu > -1]$ ET II 336(27)

6.721

1. $$\int_0^\infty \sqrt{x}\, J_{\frac{1}{4}}\left(a^2x^2\right) \sin(bx)\,dx = 2^{-3/2} a^{-2} \sqrt{\pi b}\, J_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$[b > 0]$ ET I 108(1)

2. $$\int_0^\infty \sqrt{x}\, J_{-\frac{1}{4}}\left(a^2x^2\right) \cos(bx)\,dx = 2^{-3/2} a^{-2} \sqrt{\pi b}\, J_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$[b > 0]$ ET I 51(1)

3. $$\int_0^\infty \sqrt{x}\, Y_{\frac{1}{4}}\left(a^2x^2\right) \sin(bx)\,dx = -2^{-3/2} \sqrt{\pi b} a^{-2}\, \mathbf{H}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$ ET I 108(7)

4. $$\int_0^\infty \sqrt{x}\, Y_{-\frac{1}{4}}\left(a^2x^2\right) \cos(bx)\,dx = -2^{-3/2} \sqrt{\pi b} a^{-2}\, \mathbf{H}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$ ET I 52(7)

5. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{4}}\left(a^2x^2\right)\sin(bx)\,dx = 2^{-5/2}\sqrt{\pi^3 b}\,a^{-2}\left[I_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right]$$

$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0\right]$ ET I 109(11)

6. $$\int_0^\infty \sqrt{x}\, K_{-\frac{1}{4}}\left(a^2x^2\right)\cos(bx)\,dx = 2^{-5/2}\sqrt{\pi^3 b}\,a^{-2}\left[I_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right) - \mathbf{L}_{-\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)\right]$$

$[b > 0]$ ET I 52(10)

6.722

1. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{8}+\nu}\left(a^2x^2\right) I_{\frac{1}{8}-\nu}\left(a^2x^2\right)\sin(bx)\,dx = \sqrt{2\pi}\,b^{-3/2}\frac{\Gamma\left(\frac{5}{8}-\nu\right)}{\Gamma\left(\frac{5}{4}\right)}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$

$\left[\operatorname{Re}\nu < \frac{5}{8}, \quad |\arg a| < \frac{\pi}{4}, \quad b > 0\right]$

ET I 109(13)

2.[10] $$\int_0^\infty \sqrt{x}\, J_{-\frac{1}{8}-\nu}\left(a^2x^2\right) J_{-\frac{1}{8}+\nu}\left(a^2x^2\right)\cos(bx)\,dx$$

$$= \frac{\sqrt{\pi}}{2^{3/4}a^{3/2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{8}-\nu\right)\Gamma\left(\frac{5}{8}+\nu\right)}\ {}_2F_3\left(\frac{3}{8}-\nu, \frac{3}{8}+\nu;\ \frac{3}{8}, \frac{3}{4}, \frac{7}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$
$$-\frac{1}{a^2}\sqrt{\frac{2b}{\pi}}\cos(\pi\nu)\ {}_2F_3\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu;\ \frac{1}{2}, \frac{7}{8}, \frac{9}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$
$$-\frac{b^{5/2}\nu}{15a^4}\sqrt{\frac{2}{\pi}}\sin(\pi\nu)\ {}_2F_3\left(1-\nu, 1+\nu;\ \frac{11}{8}, \frac{3}{2}, \frac{13}{8};\ -\left(\frac{b}{4a}\right)^4\right)$$

$\left[a^2 > 0, \quad \operatorname{Im} b = 0\right]$ MC

3. $$\int_0^\infty \sqrt{x}\, J_{\frac{1}{8}-\nu}\left(a^2x^2\right) J_{\frac{1}{8}+\nu}\left(a^2x^2\right)\sin(bx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,b^{-3/2}\left[e^{\pi i/8}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2e^{\pi i/2}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2e^{\pi i/2}}{8a^2}\right)\right.$$
$$\left. + e^{-i\pi/8}\, W_{\nu,\frac{1}{8}}\left(\frac{b^2e^{-\pi i/2}}{8a^2}\right) W_{-\nu,\frac{1}{8}}\left(\frac{b^2e^{-\frac{\pi i}{2}}}{8a^2}\right)\right]$$

$[b > 0]$ ET I 108(6)

4. $$\int_0^\infty \sqrt{x}\, K_{\frac{1}{8}-\nu}\left(a^2x^2\right) I_{-\frac{1}{8}-\nu}\left(a^2x^2\right)\cos(bx)\,dx$$

$$= \sqrt{2\pi}\,b^{-3/2}\frac{\Gamma\left(\frac{3}{8}-\nu\right)}{\Gamma\left(\frac{3}{4}\right)}\, W_{\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu,-\frac{1}{8}}\left(\frac{b^2}{8a^2}\right)$$

$\left[\operatorname{Re}\nu < \frac{3}{8}, \quad b > 0\right]$ ET I 52(12)

6.723 $$\int_0^\infty x\, J_\nu\left(x^2\right)\left[\sin(\nu\pi)\, J_\nu\left(x^2\right) - \cos(\nu\pi)\, Y_\nu\left(x^2\right)\right] J_{4\nu}(4ax)\,dx = \frac{1}{4}\, J_\nu\left(a^2\right) J_{-\nu}\left(a^2\right)$$

$[a > 0, \quad \operatorname{Re}\nu > -1]$ ET II 375(20)

6.724

1. $$\int_0^\infty x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \sin(bx)\,dx$$
$$= \frac{\sqrt{\pi} a^{2\nu}\,\Gamma(\lambda-\nu+1) b^{2\nu-2\lambda-1}}{4^{2\nu-\lambda}\,\Gamma(2\nu+1)\,\Gamma\left(\nu-\lambda+\frac{1}{2}\right)}\ {}_0F_3\left(2\nu+1, \nu-\lambda, \nu-\lambda+\frac{1}{2}; \frac{a^2b^2}{16}\right)$$
$$+\frac{a^{2\lambda+2}\,\Gamma(\nu-\lambda-1)b}{2^{2\lambda+3}\,\Gamma(\nu+\lambda+2)}\ {}_0F_3\left(\frac{3}{2}, \lambda-\nu+2, \lambda+\nu+2; \frac{a^2b^2}{16}\right)$$
$\left[-\frac{5}{4} < \operatorname{Re}\lambda < \operatorname{Re}\nu, \quad a>0, \quad b>0\right]$ ET I 109(15)

2. $$\int_0^\infty x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \cos(bx)\,dx$$
$$= 4^{\lambda-2\nu}\sqrt{\pi} a^{2\nu} b^{2\nu-2\lambda-1} \frac{\Gamma\left(\lambda-\nu+\frac{1}{2}\right)}{\Gamma(2\nu+1)\,\Gamma(\nu-\lambda)}\ {}_0F_3\left(2\nu+1, \nu-\lambda+\frac{1}{2}, \nu-\lambda; \frac{a^2b^2}{16}\right)$$
$$+4^{-\lambda-1} a^{2\lambda+1} \frac{\Gamma\left(\nu-\lambda-\frac{1}{2}\right)}{\Gamma\left(\nu+\lambda+\frac{3}{2}\right)}\ {}_0F_3\left(\frac{1}{2}, \lambda-\nu+\frac{3}{2}, \nu+\lambda+\frac{3}{2}; \frac{a^2b^2}{16}\right)$$
$\left[-\frac{3}{4} < \operatorname{Re}\lambda < \operatorname{Re}\nu - \frac{1}{2}, \quad a>0, \quad b>0\right]$ ET I 53(14)

6.725

1. $$\int_0^\infty \frac{\sin(bx)}{\sqrt{x}} J_\nu\left(a\sqrt{x}\right)\,dx = -\sqrt{\frac{\pi}{b}} \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{\nu}{2}}\left(\frac{a^2}{8b}\right)$$
$[\operatorname{Re}\nu > -3, \quad a>0, \quad b>0]$ ET I 110(27)

2. $$\int_0^\infty \frac{\cos(bx)}{\sqrt{x}} J_\nu\left(a\sqrt{x}\right)\,dx = \sqrt{\frac{\pi}{b}} \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{1}{2}\nu}\left(\frac{a^2}{8b}\right)$$
$[\operatorname{Re}\nu > -1, \quad a>0, \quad b>0]$ ET I 54(25)

3. $$\int_0^\infty x^{\frac{1}{2}\nu} J_\nu\left(a\sqrt{x}\right) \sin(bx)\,dx = 2^{-\nu} a^\nu b^{-\nu-1} \cos\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right)$$
$\left[-2 < \operatorname{Re}\nu < \frac{1}{2}, \quad a>0, \quad b>0\right]$ ET I 110(28)

4. $$\int_0^\infty x^{\frac{1}{2}\nu} J_\nu\left(a\sqrt{x}\right) \cos(bx)\,dx = 2^{-\nu} b^{-\nu-1} a^\nu \sin\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right)$$
$\left[-1 < \operatorname{Re}\nu < \frac{1}{2}, \quad a>0, \quad b>0\right]$ ET I 54(26)

6.726

1. $$\int_0^\infty x\left(x^2+b^2\right)^{-\frac{1}{2}\nu} J_\nu\left(a\sqrt{x^2+b^2}\right) \sin(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{3}{2}} c\left(a^2-c^2\right)^{\frac{1}{2}\nu-\frac{3}{4}} J_{\nu-\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0<c<a, \quad \operatorname{Re}\nu > \frac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<c, \quad \operatorname{Re}\nu > \frac{1}{2}\right]$$
ET I 111(37)

2. $$\int_0^\infty \left(x^2+b^2\right)^{-\frac{1}{2}\nu} J_\nu\left(a\sqrt{x^2+b^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} \left(a^2-c^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0<c<a, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0<a<c, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 55(37)

3. $$\int_0^\infty x\left(x^2+b^2\right)^{\frac{1}{2}\nu} K_{\pm\nu}\left(a\sqrt{x^2+b^2}\right)\sin(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}} a^{\nu} b^{\nu+\frac{3}{2}} c\left(a^2+c^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} K_{-\nu-\frac{3}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$[\operatorname{Re}a>0, \quad \operatorname{Re}b>0, \quad c>0]$ ET I 113(45)

4.[11] $$\int_0^\infty \left(x^2+b^2\right)^{\mp\frac{1}{2}\nu} K_\nu\left(a\sqrt{x^2+b^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}} a^{\mp\nu} b^{\frac{1}{2}\mp\nu} \left(a^2+c^2\right)^{\pm\frac{1}{2}\nu-\frac{1}{4}} K_{\pm\nu-\frac{1}{2}}\left(b\sqrt{a^2+c^2}\right)$$
$[\operatorname{Re}a>0, \quad \operatorname{Re}b>0, \quad c \text{ is real}]$ ET I 56(45)

5. $$\int_0^\infty \left(x^2+a^2\right)^{-\frac{1}{2}\nu} Y_\nu\left(b\sqrt{x^2+a^2}\right)\cos(cx)\,dx$$
$$= \sqrt{\frac{a\pi}{2}}(ab)^{-\nu}\left(b^2-c^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} Y_{\nu-\frac{1}{2}}\left(a\sqrt{b^2-c^2}\right) \qquad \left[0<c<b, \quad a>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
$$= -\sqrt{\frac{2a}{\pi}}(ab)^{-\nu}\left(c^2-b^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} K_{\nu-\frac{1}{2}}\left(a\sqrt{c^2-b^2}\right) \qquad \left[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET I 56(41)

6.727

1.[9] $$\int_0^a \frac{\cos(cx)}{\sqrt{a^2-x^2}} J_\nu\left(b\sqrt{a^2-x^2}\right)dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}-c\right)\right] J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2}+c\right)\right]$$
$[\operatorname{Re}\nu>-1, \quad c>0, \quad a>0]$
ET I 113(48)

2. $$\int_a^\infty \frac{\sin(cx)}{\sqrt{x^2-a^2}} J_\nu\left(b\sqrt{x^2-a^2}\right)dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2+b^2}\right)\right] J_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2+b^2}\right)\right]$$
$[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-1]$
ET I 113(49)

3. $$\int_a^\infty \frac{\cos(cx)}{\sqrt{x^2-a^2}} J_\nu\left(b\sqrt{x^2-a^2}\right)dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(c-\sqrt{c^2-b^2}\right)\right] Y_{-\frac{1}{2}\nu}\left[\frac{a}{2}\left(c+\sqrt{c^2-b^2}\right)\right]$$
$[0<b<c, \quad a>0, \quad \operatorname{Re}\nu>-1]$
ET I 58(54)

4.[8] $$\int_0^a \left(a^2-x^2\right)^{\frac{1}{2}\nu}\cos x\, I_\nu\left(\sqrt{a^2-x^2}\right)dx = \frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}$$
$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ WA 409(2)

6.728

1. $$\int_0^\infty x \sin\left(ax^2\right) J_\nu(bx)\, dx = \frac{\sqrt{\pi} b}{8a^{3/2}} \left[\cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right) - \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -4]$ ET II 34(14)

2. $$\int_0^\infty x \cos\left(ax^2\right) J_\nu(bx)\, dx = \frac{\sqrt{\pi} b}{8a^{3/2}} \left[\cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right) + \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -2]$ ET II 38(39)

3. $$\int_0^\infty J_0(\beta x) \sin\left(\alpha x^2\right) x\, dx = \frac{1}{2\alpha} \cos\frac{\beta^2}{4\alpha}$$ $[\alpha > 0, \quad \beta > 0]$ MO 47

4. $$\int_0^\infty J_0(\beta x) \cos\left(\alpha x^2\right) x\, dx = \frac{1}{2\alpha} \sin\frac{\beta^2}{4\alpha}$$ $[\alpha > 0, \quad \beta > 0]$ MO 47

5. $$\int_0^\infty x^{\nu+1} \sin\left(ax^2\right) J_\nu(bx)\, dx = \frac{b^\nu}{2^{\nu+1} a^{\nu+1}} \cos\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right)$$
$\left[a > 0, \quad b > 0, \quad -2 < \operatorname{Re}\nu < \frac{1}{2}\right]$
ET II 34(15)

6. $$\int_0^\infty x^{\nu+1} \cos\left(ax^2\right) J_\nu(bx)\, dx = \frac{b^\nu}{2^{\nu+1} a^{\nu+1}} \sin\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right)$$
$\left[a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < \frac{1}{2}\right]$
ET II 38(40)

6.729

1. $$\int_0^\infty x \sin\left(ax^2\right) J_\nu(bx) J_\nu(cx)\, dx = \frac{1}{2a} \cos\left(\frac{b^2+c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -2]$
ET II 51(26)

2. $$\int_0^\infty x \cos\left(ax^2\right) J_\nu(bx) J_\nu(cx)\, dx = \frac{1}{2a} \sin\left(\frac{b^2+c^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{bc}{2a}\right)$$
$[a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re}\nu > -1]$
ET II 51(27)

6.731

1.[11] $$\int_0^\infty x \sin\left(ax^2\right) J_\nu\left(bx^2\right) J_{2\nu}(2cx)\, dx$$
$$= \frac{1}{2\sqrt{b^2-a^2}} \sin\left(\frac{ac^2}{b^2-a^2}\right) J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad [0 < a < b, \quad \operatorname{Re}\nu > -1]$$
$$= \frac{1}{2\sqrt{a^2-b^2}} \cos\left(\frac{ac^2}{a^2-b^2}\right) J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad [0 < b < a, \quad \operatorname{Re}\nu > -1]$$
ET II 356(41)a

2.[10] $$\int_0^\infty x \cos\left(ax^2\right) J_\nu\left(bx^2\right) J_{2\nu}(2cx)\,dx$$

$$= \frac{1}{2\sqrt{b^2-a^2}} \cos\left(\frac{ac^2}{b^2-a^2}\right) J_\nu\left(\frac{bc^2}{b^2-a^2}\right) \qquad \left[0<a<b, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$

$$= \frac{1}{2\sqrt{a^2-b^2}} \sin\left(\frac{ac^2}{a^2-b^2}\right) J_\nu\left(\frac{bc^2}{a^2-b^2}\right) \qquad \left[0<b<a, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$

ET II 356(42)a

6.732 $$\int_0^\infty x^2 \cos\left(\frac{x^2}{2a}\right) Y_1(x)\, K_1(x)\,dx = -a^3\, K_0(a) \qquad [a>0]$$ ET II 371(52)

6.733

1. $$\int_0^\infty \sin\left(\frac{a}{2x}\right) \left[\sin x\, J_0(x) + \cos x\, Y_0(x)\right] \frac{dx}{x} = \pi\, J_0\left(\sqrt{a}\right) Y_0\left(\sqrt{a}\right)$$

$[a>0]$ ET II 346(51)

2. $$\int_0^\infty \cos\left(\frac{a}{2x}\right) \left[\sin x\, Y_0(x) - \cos x\, J_0(x)\right] \frac{dx}{x} = \pi\, J_0\left(\sqrt{a}\right) Y_0\left(\sqrt{a}\right)$$

$[a>0]$ ET II 347(52)

3. $$\int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x)\,dx = \frac{\pi a}{2} J_1\left(\sqrt{a}\right) K_1\left(\sqrt{a}\right) \qquad [a>0]$$ ET II 368(34)

4. $$\int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x)\,dx = -\frac{\pi a}{2} Y_1\left(\sqrt{a}\right) K_1\left(\sqrt{a}\right) \qquad [a>0]$$ ET II 369(35)

6.734 $$\int_0^\infty \cos\left(a\sqrt{x}\right) K_\nu(bx) \frac{dx}{\sqrt{x}}$$

$$= \frac{\pi}{2\sqrt{b}} \sec(\nu\pi) \left[D_{\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) + D_{\nu-\frac{1}{2}}\left(-\frac{a}{\sqrt{2b}}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{2b}}\right)\right]$$

$\left[\operatorname{Re} b>0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$ ET II 132(27)

6.735

1. $$\int_0^\infty x^{1/4} \sin\left(2a\sqrt{x}\right) J_{-\frac{1}{4}}(x)\,dx = \sqrt{\pi} a^{3/2}\, J_{\frac{3}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(10)

2. $$\int_0^\infty x^{1/4} \cos\left(2a\sqrt{x}\right) J_{\frac{1}{4}}(x)\,dx = \sqrt{\pi} a^{3/2}\, J_{-\frac{3}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(12)

3. $$\int_0^\infty x^{1/4} \sin\left(2a\sqrt{x}\right) J_{\frac{3}{4}}(x)\,dx = \sqrt{\pi} a^{3/2}\, J_{-\frac{1}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(11)

4. $$\int_0^\infty x^{1/4} \cos\left(2a\sqrt{x}\right) J_{-\frac{3}{4}}(x)\,dx = \sqrt{\pi} a^{3/2}\, J_{\frac{1}{4}}\left(a^2\right) \qquad [a>0]$$ ET II 341(13)

6.736

1.[11] $$\int_0^\infty x^{-1/2} \sin x \cos\left(4a\sqrt{x}\right) J_0(x)\,dx = -2^{-3/2}\sqrt{\pi} \left[\cos\left(a^2-\frac{\pi}{4}\right) J_0\left(a^2\right) - \sin\left(a^2-\frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$

$[a>0]$ ET II 341(18)

2. $$\int_0^\infty x^{-1/2} \cos x \cos\left(4a\sqrt{x}\right) J_0(x)\,dx = -2^{-3/2}\sqrt{\pi} \left[\sin\left(a^2-\frac{\pi}{4}\right) J_0\left(a^2\right) + \cos\left(a^2-\frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$

$[a>0]$ ET II 342(22)

3. $$\int_0^\infty x^{-1/2} \sin x \sin\left(4a\sqrt{x}\right) J_0(x)\,dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 + \frac{\pi}{4}\right) J_0\left(a^2\right)$$
$[a > 0]$ ET II 341(16)

4. $$\int_0^\infty x^{-1/2} \cos x \sin\left(4a\sqrt{x}\right) J_0(x)\,dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 - \frac{\pi}{4}\right) J_0\left(a^2\right)$$
$[a > 0]$ ET II 342(20)

5. $$\int_0^\infty x^{-1/2} \sin x \cos\left(4a\sqrt{x}\right) Y_0(x)\,dx = 2^{-3/2}\sqrt{\pi}\left[3\sin\left(a^2 - \frac{\pi}{4}\right) J_0\left(a^2\right) - \cos\left(a^2 - \frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$
$[a > 0]$ ET II 347(55)

6. $$\int_0^\infty x^{-1/2} \cos x \cos\left(4a\sqrt{x}\right) Y_0(x)\,dx$$
$$= -2^{-3/2}\sqrt{\pi}\left[3\cos\left(a^2 - \frac{\pi}{4}\right) J_0\left(a^2\right) + \sin\left(a^2 - \frac{\pi}{4}\right) Y_0\left(a^2\right)\right]$$
$[a > 0]$ ET II 347(56)

6.737

1. $$\int_0^\infty \frac{\sin\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} J_\nu(cx)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{b}{2}\left(a - \sqrt{a^2-c^2}\right)\right] J_{-\frac{1}{2}\nu}\left[\frac{b}{2}\left(a + \sqrt{a^2-c^2}\right)\right]$$
$[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re}\nu > -1]$ ET II 35(19)

2. $$\int_0^\infty \frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} J_\nu(cx)\,dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{b}{2}\left(a - \sqrt{a^2-c^2}\right)\right] Y_{-\frac{1}{2}\nu}\left[\frac{b}{2}\left(a + \sqrt{a^2-c^2}\right)\right]$$
$[a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re}\nu > -1]$ ET II 39(44)

3. $$\int_0^a \frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} J_\nu(cx)\,dx = \frac{\pi}{2} J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2} - b\right)\right] J_{\frac{1}{2}\nu}\left[\frac{a}{2}\left(\sqrt{b^2+c^2} + b\right)\right]$$
$[c > 0, \quad \operatorname{Re}\nu > -1]$ ET II 39(47)

4. $$\int_0^a x^{\nu+1} \frac{\cos\left(\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} I_\nu(x)\,dx = \frac{\sqrt{\pi}a^{2\nu+1}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}$$
$[\operatorname{Re}\nu > -1]$ ET II 365(9)

5. $$\int_0^\infty x^{\nu+1} \frac{\sin\left(a\sqrt{b^2+x^2}\right)}{\sqrt{b^2+x^2}} J_\nu(cx)\,dx$$
$$= \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu \left(a^2-c^2\right)^{-\frac{1}{4}-\frac{1}{2}\nu} J_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \qquad \left[0 < c < a, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0 < a < c, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$
ET II 35(20)

6. $$\int_0^\infty x^{\nu+1}\frac{\cos\left(a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}J_\nu(cx)\,dx = -\sqrt{\frac{\pi}{2}}b^{\frac{1}{2}+\nu}c^\nu\left(a^2-c^2\right)^{-\frac{1}{4}-\frac{1}{2}\nu}Y_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right)$$
$$\left[0<c<a,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\frac{1}{2}\right]$$
$$=\sqrt{\frac{2}{\pi}}b^{\frac{1}{2}+\nu}c^\nu\left(c^2-a^2\right)^{-\frac{1}{4}-\frac{1}{2}\nu}K_{\nu+\frac{1}{2}}\left(b\sqrt{c^2-a^2}\right)$$
$$\left[0<a<c,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<\frac{1}{2}\right]$$
ET II 39(45)

6.738

1. $$\int_0^a x^{\nu+1}\sin\left(b\sqrt{a^2-x^2}\right)J_\nu(x)\,dx = \sqrt{\frac{\pi}{2}}a^{\nu+\frac{3}{2}}b\left(1+b^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}}J_{\nu+\frac{3}{2}}\left(a\sqrt{1+b^2}\right)$$
$[\operatorname{Re}\nu>-1]$ ET II 335(19)

2. $$\int_0^\infty x^{\nu+1}\cos\left(a\sqrt{x^2+b^2}\right)J_\nu(cx)\,dx$$
$$=\sqrt{\frac{\pi}{2}}ab^{\nu+\frac{3}{2}}c^\nu\left(a^2-c^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}}\left[\cos(\pi\nu)\,J_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right)-\sin(\pi\nu)\,Y_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right)\right]$$
$$\left[0<c<a,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
$$=0$$
$$\left[0<a<c,\quad \operatorname{Re}b>0,\quad -1<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$
ET II 39(43)

6.739 $$\int_0^t x^{-1/2}\frac{\cos\left(b\sqrt{t-x}\right)}{\sqrt{t-x}}J_{2\nu}\left(a\sqrt{x}\right)\,dx = \pi J_\nu\left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}+b\right)\right]J_\nu\left[\frac{\sqrt{t}}{2}\left(\sqrt{a^2+b^2}-b\right)\right]$$
$\left[\operatorname{Re}\nu>-\tfrac{1}{2}\right]$ EH II 47(7)

6.741

1. $$\int_0^1 \frac{\cos(\mu\arccos x)}{\sqrt{1-x^2}}J_\nu(ax)\,dx = \frac{\pi}{2}J_{\frac{1}{2}(\mu+\nu)}\left(\frac{a}{2}\right)J_{\frac{1}{2}(\nu-\mu)}\left(\frac{a}{2}\right)$$
$[\operatorname{Re}(\mu+\nu)>-1,\quad a>0]$ ET II 41(54)

2. $$\int_0^1 \frac{\cos\left[(\nu+1)\arccos x\right]}{\sqrt{1-x^2}}J_\nu(ax)\,dx = \sqrt{\frac{\pi}{a}}\cos\left(\frac{a}{2}\right)J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right)$$
$[\operatorname{Re}\nu>-1,\quad a>0]$ ET II 40(53)

3. $$\int_0^1 \frac{\cos\left[(\nu-1)\arccos x\right]}{\sqrt{1-x^2}}J_\nu(ax)\,dx = \sqrt{\frac{\pi}{a}}\sin\left(\frac{a}{2}\right)J_{\nu-\frac{1}{2}}\left(\frac{a}{2}\right)$$
$[\operatorname{Re}\nu>0,\quad a>0]$ ET II 40(52)a

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

6.751 **Notation**: $\ell_1=\frac{1}{2}\left[\sqrt{(b+c)^2+a^2}-\sqrt{(b-c)^2+a^2}\right]$, $\ell_2=\frac{1}{2}\left[\sqrt{(b+c)^2+a^2}+\sqrt{(b-c)^2+a^2}\right]$

1. $$\int_0^\infty e^{-\frac{1}{2}ax} \sin(bx)\, I_0\left(\frac{1}{2}ax\right) dx = \frac{1}{\sqrt{2b}} \frac{1}{\sqrt{b^2+a^2}} \sqrt{b+\sqrt{b^2+a^2}}$$
$[\operatorname{Re} a > 0, \quad b > 0]$ ET I 105(44)

2. $$\int_0^\infty e^{-\frac{1}{2}ax} \cos(bx)\, I_0\left(\frac{1}{2}ax\right) dx = \frac{a}{\sqrt{2b}} \frac{1}{\sqrt{a^2+b^2}\sqrt{b+\sqrt{a^2+b^2}}}$$
$[\operatorname{Re} a > 0, \quad b > 0]$ ET I 48(38)

3.[10] $$\int_0^\infty e^{-bx} \cos(ax)\, J_0(cx)\, dx = \frac{\left[\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}+b^2+c^2-a^2\right]^{1/2}}{\sqrt{2}\sqrt{\left(b^2+c^2-a^2\right)^2+4a^2b^2}}$$
$[c > 0]$ ET II 11(46)

alternatively, with a and b interchanged,

$$\int_0^\infty e^{-ax} \cos(bx)\, J_0(cx)\, dx = \frac{\sqrt{\ell_2^2-b^2}}{\ell_2^2-\ell_1^2} \qquad [c > 0]$$

6.752

1.[10] $$\int_0^\infty e^{-ax} J_0(bx) \sin(cx) \frac{dx}{x} = \arcsin\left(\frac{2c}{\sqrt{a^2+(c+b)^2}+\sqrt{a^2+(c-b)^2}}\right) = \arcsin\left(\frac{c}{\ell_2}\right)$$
$[\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0]$ ET I 101(17)

2.[10] $$\int_0^\infty e^{-ax} J_1(cx) \sin(bx) \frac{dx}{x} = \frac{b}{c}(1-r) = \frac{b-\sqrt{b^2-\ell_1^2}}{c},$$
$$\left[b^2 = \frac{c^2}{1-r^2} - \frac{a^2}{r^2}, \quad c > 0\right]$$
ET II 19(15)

Notation: For integrals 6.752 3–6.752 5 we define the auxiliary functions

$$\ell_1(a) \equiv \ell_1(a,\rho,z) = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} - \sqrt{(a-\rho)^2+z^2}\right]$$
$$\ell_2(a) \equiv \ell_1(a,\rho,z) = \frac{1}{2}\left[\sqrt{(a+\rho)^2+z^2} + \sqrt{(a-\rho)^2+z^2}\right]$$

when $a \ge 0$, $\rho \ge 0$, and $z \ge 0$.

3.[10] $$\sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zx} J_{\nu+1/2}(ax)\, J_{\nu+1}(\rho x) \sqrt{x}\, dx$$
$$= a^{-\nu-3/2} \rho^{-\nu-1} \frac{\ell_1^{2\nu+2}}{\sqrt{\rho^2-\ell_1^2}} \frac{a\left(\rho^2-\ell_1^2\right)}{\ell_1\left(\ell_2^2-\ell_1^2\right)}$$
$$= a^{\nu+1/2} \frac{\rho^{\nu+1}}{\ell_2^{2\nu+2}} \frac{\sqrt{\ell_2^2-a^2}}{\ell_2^2-\ell_1^2} \qquad [\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im} \rho|]$$

4.[10] $$\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-zx}\, J_{\nu+1/2}(ax)\, J_\nu(\rho x)\frac{dx}{\sqrt{x}}$$
$$= a^{\nu+1/2}\rho^\nu\int_0^{1/\ell_2}\frac{1}{\ell_2^{2\nu}}\frac{1}{\sqrt{1-a^2/\ell_2^2}}\,d\left(\frac{1}{\ell_2}\right)$$
$$= a^{-\nu-1/2}\rho^\nu\int_0^{a/\ell_2}x^{2\nu}\frac{dx}{\sqrt{1-x^2}} \qquad \left[\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im}\rho|\right]$$

5.[10] $$\int_0^\infty e^{-zx}\sin(ax)\, J_1(\rho x)\frac{dx}{x^2} = \frac{\sqrt{\ell_2^2-a^2}\left(a-\sqrt{a^2-\ell_1^2}\right)^2}{2a\rho} + \frac{\rho}{2}\arcsin\left(\frac{a}{\ell_2}\right)$$
$$[\operatorname{Re} z > |\operatorname{Im} a| + |\operatorname{Im}\rho|]$$

6.753

1.[8] $$\int_0^\infty \frac{\sin(xa\sin\psi)}{x}e^{-xa\cos\varphi\cos\psi}\, J_\nu(xa\sin\varphi)\, dx = \nu^{-1}\left(\tan\frac{\varphi}{2}\right)^\nu\sin(\nu\psi)$$
$$\left[\operatorname{Re}\nu > -1, \quad a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}\right]$$ ET II 33(10)

2. $$\int_0^\infty \frac{\cos(xa\sin\psi)}{x}e^{-xa\cos\varphi\cos\psi}\, J_\nu(xa\sin\varphi)\, dx = \nu^{-1}\left(\tan\frac{\varphi}{2}\right)^\nu\cos(\nu\psi)$$
$$\left[\operatorname{Re}\nu > 0, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}\right]$$
ET II 38(35)

3.[8] $$\int_0^\infty x^{\nu+1}e^{-sx}\sin(bx)\, J_\nu(ax)\, dx = -\frac{2(2a)^\nu}{\sqrt{\pi}}\,\Gamma(\nu+\tfrac{3}{2})R^{-2\nu-3}\left[b\cos(\nu+\tfrac{3}{2})\varphi + s\sin(\nu+\tfrac{3}{2})\varphi\right]$$
$$\left[\operatorname{Re}\nu > -\tfrac{3}{2}, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\right.$$
$$\left.R^4 = \left(s^2+a^2-b^2\right)^2 + 4b^2s^2, \quad \varphi = \arg\left(s^2+a^2-b^2-2ibs\right)\right]$$

4.[8] $$\int_0^\infty x^{\nu+1}e^{-sx}\cos(bx)\, J_\nu(ax)\, dx = \frac{2(2a)^\nu}{\sqrt{\pi}}\,\Gamma(\nu+\tfrac{3}{2})R^{-2\nu-3}\left[s\cos(\nu+\tfrac{3}{2})\varphi - b\sin(\nu+\tfrac{3}{2})\varphi\right],$$
$$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|,\right.$$
$$\left.R^4 = \left(s^2+a^2-b^2\right)^2 + 4b^2s^2, \quad \varphi = \arg\left(s^2+a^2-b^2-2ibs\right)\right]$$

5.[10] $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi}\sin(ax\sin\psi)\, J_\nu(ax\sin\varphi)\, dx$$
$$= 2^\nu\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}}a^{-\nu-1}(\sin\varphi)^\nu\left(\cos^2\psi+\sin^2\psi\cos^2\varphi\right)^{-\nu-\frac{1}{2}}\sin\left[\left(\nu+\tfrac{1}{2}\right)\beta\right]$$
$$\tan\frac{\beta}{2} = \tan\psi\cos\varphi \qquad \left[a > 0, \quad 0 < \varphi < \frac{\pi}{2}, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re}\nu > -1\right]$$ ET II 34(12)

6. $$\int_0^\infty x^\nu e^{-ax\cos\varphi\cos\psi} \cos(ax\sin\psi)\, J_\nu(ax\sin\varphi)\, dx$$
$$= 2^\nu \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1} (\sin\varphi)^\nu \left(\cos^2\psi + \sin^2\psi\cos^2\varphi\right)^{-\nu-\frac{1}{2}} \cos\left[\left(\nu+\tfrac{1}{2}\right)\beta\right]$$
$$\tan\frac{\beta}{2} = \tan\psi\cos\varphi \qquad \left[a>0, \quad 0<\varphi, \quad \psi<\frac{\pi}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$ ET II 38(37)

6.754

1. $$\int_0^\infty e^{-x^2} \sin(bx)\, I_0\left(x^2\right) dx = \frac{\sqrt{\pi}}{2^{3/2}} e^{-\frac{b^2}{8}}\, I_0\left(\frac{b^2}{8}\right) \qquad [b>0]$$ ET I 108(9)

2. $$\int_0^\infty e^{-ax} \cos\left(x^2\right) J_0\left(x^2\right) dx = \frac{1}{4}\sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} + \frac{\pi}{4}\right)\right]$$
$$[a>0]$$ MI 42

3. $$\int_0^\infty e^{-ax} \sin\left(x^2\right) J_0\left(x^2\right) dx = \frac{1}{4}\sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - Y_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} + \frac{\pi}{4}\right)\right]$$
$$[a>0]$$ MI 42

6.755

1. $$\int_0^\infty x^{-\nu} e^{-x} \sin\left(4a\sqrt{x}\right) I_\nu(x)\, dx = \left(2^{3/2}a\right)^{\nu-1} e^{-a^2}\, W_{\frac{1}{2}-\frac{3}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu}\left(2a^2\right)$$
$$[a>0, \quad \operatorname{Re}\nu>0]$$ ET II 366(14)

2. $$\int_0^\infty x^{-\nu-\frac{1}{2}} e^{-x} \cos\left(4a\sqrt{x}\right) I_\nu(x)\, dx = 2^{\frac{3}{2}\nu-1} a^{\nu-1} e^{-a^2}\, W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 366(16)

3. $$\int_0^\infty x^{-\nu} e^{x} \sin\left(4a\sqrt{x}\right) K_\nu(x)\, dx = \left(2^{3/2}a\right)^{\nu-1} \pi \frac{\Gamma\left(\frac{3}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} e^{a^2}\, W_{\frac{3}{2}\nu-\frac{1}{2}, \frac{1}{2}-\frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0, \quad 0<\operatorname{Re}\nu<\tfrac{3}{4}\right]$$ ET II 369(38)

4. $$\int_0^\infty x^{-\nu-\frac{1}{2}} e^{x} \cos\left(4a\sqrt{x}\right) K_\nu(x)\, dx = 2^{\frac{3}{2}\nu-1} \pi a^{\nu-1} \frac{\Gamma\left(\frac{1}{2}-2\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} e^{a^2}\, W_{\frac{3}{2}\nu, -\frac{1}{2}\nu}\left(2a^2\right)$$
$$\left[a>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{4}\right]$$ ET II 369(42)

5. $$\int_0^\infty x^{\varrho-\frac{3}{2}} e^{-x} \sin\left(4a\sqrt{x}\right) K_\nu(x)\, dx = \frac{\sqrt{\pi}a\,\Gamma(\varrho+\nu)\,\Gamma(\varrho-\nu)}{2^{\varrho-2}\,\Gamma\left(\varrho+\frac{1}{2}\right)}\, {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{3}{2}, \varrho+\frac{1}{2}; -2a^2\right)$$
$$[\operatorname{Re}\varrho > |\operatorname{Re}\nu|]$$ ET II 369(39)

6. $$\int_0^\infty x^{\varrho-1} e^{-x} \cos\left(4a\sqrt{x}\right) K_\nu(x)\, dx = \frac{\sqrt{\pi}\,\Gamma(\varrho+\nu)\,\Gamma(\varrho-\nu)}{2^{\varrho}\,\Gamma\left(\varrho+\frac{1}{2}\right)}\, {}_2F_2\left(\varrho+\nu, \varrho-\nu; \frac{1}{2}, \varrho+\frac{1}{2}; -2a^2\right)$$
$$[\operatorname{Re}\varrho > |\operatorname{Re}\nu|]$$ ET II 370(43)

7. $$\int_0^\infty x^{-1/2} e^{-x} \cos\left(4a\sqrt{x}\right) I_0(x)\, dx = \frac{1}{\sqrt{2\pi}} e^{-a^2}\, K_0\left(a^2\right)$$
$$[a>0]$$ ET II 366(15)

8. $$\int_0^\infty x^{-1/2} e^{x} \cos\left(4a\sqrt{x}\right) K_0(x)\,dx = \sqrt{\frac{\pi}{2}} e^{a^2}\, K_0\left(a^2\right) \qquad [a > 0]$$ ET II 369(40)

9. $$\int_0^\infty x^{-1/2} e^{-x} \cos\left(4a\sqrt{x}\right) K_0(x)\,dx = \frac{1}{\sqrt{2}} \pi^{3/2} e^{-a^2}\, I_0\left(a^2\right)$$ ET II 369(41)

6.756

1. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{i}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$
$[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -1]$ ET II 34(17)

2. $$\int_0^\infty x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_\nu(bx)\,dx = \frac{1}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 39(42)

3. $$\int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \sin\left(a\sqrt{x}\right) J_0(bx)\,dx = \frac{1}{2b} a\, I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right)$$
$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0\right]$ ET II 11(40)

4. $$\int_0^\infty x^{-1/2} e^{-a\sqrt{x}} \cos\left(a\sqrt{x}\right) J_0(bx)\,dx = \frac{a}{2b}\, I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right)$$
$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0\right]$ ET II 12(49)

6.757

1. $$\int_0^\infty e^{-bx} \sin\left[a\left(1 - e^{-x}\right)\right] J_\nu\left(ae^{-x}\right)\,dx = 2\sum_{n=0}^\infty \frac{(-1)^n\, \Gamma(\nu - b + 2n + 1)\, \Gamma(\nu + b)}{\Gamma(\nu - b + 1)\, \Gamma(\nu + b + 2n + 2)} (\nu + 2n - 1)\, J_{\nu+2n+1}(a)$$
$[\operatorname{Re} b > -\operatorname{Re}\nu]$ ET I 193(26)

2. $$\int_0^\infty e^{-bx} \cos\left[a\left(1 - e^{-x}\right)\right] J_\nu\left(ae^{-x}\right)\,dx = \frac{J_\nu(a)}{\nu + b} + \sum_{n=0}^\infty 2(-1)^n \frac{\Gamma(\nu - b + 2n)\, \Gamma(\nu + b)}{\Gamma(\nu - b + 1)\, \Gamma(\nu + b + 2n + 1)} (\nu + 2n)\, J_{\nu+2n}(a)$$
$[\operatorname{Re} b > -\operatorname{Re}\nu]$ ET I 193(27)

6.758 $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta} (\cos\theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu}\, J_{\nu+\mu}(\lambda z)\,d\theta = \pi(2az)^{-\mu}(2bz)^{-\nu}\, J_\mu(az)\, J_\nu(bz); \lambda = \sqrt{2\cos\theta\left(a^2 e^{i\theta} + b^2 e^{-i\theta}\right)}$$
$$\lambda = \sqrt{2\cos\theta\left(a^2 e^{i\theta} + b^2 e^{-i\theta}\right)}$$
$[\operatorname{Re}(\nu + \mu) > -1]$ EH II 48(12)

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

6.761 $$\int_0^\infty \cosh x \cos(2a\sinh x)\, J_\nu(be^x)\, J_\nu\left(be^{-x}\right) dx = \frac{J_{2\nu}\left(2\sqrt{b^2-a^2}\right)}{2\sqrt{b^2-a^2}} \qquad [0<a<b, \quad \operatorname{Re}\nu > -1]$$
$$= 0 \qquad [0<b<a, \quad \operatorname{Re}\nu > -1]$$

ET II 359(10)

6.762 $$\int_0^\infty \cosh x \sin(2a\sinh x)\left[J_\nu(be^x)\, Y_\nu\left(be^{-x}\right) - Y_\nu(be^x)\, J_\nu\left(be^{-x}\right)\right] dx$$
$$= 0 \qquad \left[0<a<b, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
$$= -\frac{2}{\pi}\cos(\nu\pi)\left(a^2-b^2\right)^{-1/2} K_{2\nu}\left[2\left(a^2-b^2\right)^{1/2}\right] \qquad \left[0<b<a, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$

ET II 360(12)

6.763 $$\int_0^\infty \cosh x \cos(2a\sinh x)\, Y_\nu(be^x)\, Y_\nu\left(be^{-x}\right) dx$$
$$= -\frac{1}{2}\left(b^2-a^2\right)^{-1/2} J_{2\nu}\left[2\left(b^2-a^2\right)^{1/2}\right] \qquad [0<a<b, \quad |\operatorname{Re}\nu| < 1]$$
$$= \frac{2}{\pi}\cos(\nu\pi)\left(a^2-b^2\right)^{-1/2} K_{2\nu}\left[2\left(a^2-b^2\right)^{1/2}\right] \qquad [0<b<a, \quad |\operatorname{Re}\nu| < 1]$$

ET II 360(11)

6.77 Combinations of Bessel functions and the logarithm, or arctangent

6.771 $$\int_0^\infty x^{\mu+\frac{1}{2}} \ln x\, J_\nu(ax)\, dx = \frac{2^{\mu-\frac{1}{2}}\,\Gamma\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right) a^{\mu+\frac{3}{2}}}\left[\psi\left(\frac{\mu+\nu}{2}+\frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2}+\frac{1}{4}\right) - \ln\frac{a^2}{4}\right]$$
$$\left[a>0, \quad -\operatorname{Re}\nu - \tfrac{3}{2} < \operatorname{Re}\mu < 0\right]$$

ET II 32(25)

6.772

1. $$\int_0^\infty \ln x\, J_0(ax)\, dx = -\frac{1}{a}\left[\ln(2a) + \boldsymbol{C}\right]$$ WA 430(4)a, ET II 10(27)

2. $$\int_0^\infty \ln x\, J_1(ax)\, dx = -\frac{1}{a}\left[\ln\left(\frac{a}{2}\right) + \boldsymbol{C}\right]$$ ET II 19(11)

3. $$\int_0^\infty \ln\left(a^2+x^2\right) J_1(bx)\, dx = \frac{2}{b}\left[K_0(ab) + \ln a\right]$$ ET II 19(12)

4. $$\int_0^\infty J_1(tx) \ln\sqrt{1+t^4}\, dt = \frac{2}{x}\operatorname{ker} x$$ MO 46

6.773 $$\int_0^\infty \frac{\ln\left(x+\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}} J_0(bx)\, dx = \left[\frac{1}{2}K_0^2\left(\frac{ab}{2}\right) + \ln a\, I_0\left(\frac{ab}{2}\right) K_0\left(\frac{ab}{2}\right)\right]$$
$$[a>0, \quad b>0]$$ ET II 10(28)

6.774 $$\int_0^\infty \ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} J_0(bx)\frac{dx}{\sqrt{x^2+a^2}} = K_0^2\left(\frac{ab}{2}\right) \qquad [\operatorname{Re} a>0, \quad b>0]$$ ET II 10(29)

6.775 $$\int_0^\infty x\left[\ln\left(1+\sqrt{a^2+x^2}\right) - \ln x\right] J_0(bx)\, dx = \frac{1}{b^2}\left(1-e^{-ab}\right)$$
$$[\operatorname{Re} a>0, \quad b>0]$$ ET II 12(55)

6.776 $$\int_0^\infty x \ln\left(1+\frac{a^2}{x^2}\right) J_0(bx)\,dx = \frac{2}{b}\left[\frac{1}{b} - a\,K_1(ab)\right] \qquad [\operatorname{Re} a > 0, \quad b > 0]$$ ET II 10(30)

6.777 $$\int_0^\infty J_1(tx) \arctan t^2\,dt = -\frac{2}{x}\operatorname{kei} x$$ MO 46

6.78 Combinations of Bessel and other special functions

6.781 $$\int_0^\infty \operatorname{si}(ax)\,J_0(bx)\,dx = -\frac{1}{b}\arcsin\left(\frac{b}{a}\right) \qquad [0 < b < a]$$
$$= 0 \qquad [0 < a < b]$$
ET II 13(6)

6.782

1. $$\int_0^\infty \operatorname{Ei}(-x)\,J_0\left(2\sqrt{zx}\right)\,dx = \frac{e^{-z}-1}{z}$$ NT 60(4)

2. $$\int_0^\infty \operatorname{si}(x)\,J_0\left(2\sqrt{zx}\right)\,dx = -\frac{\sin z}{z}$$ NT 60(6)

3. $$\int_0^\infty \operatorname{ci}(x)\,J_0\left(2\sqrt{zx}\right)\,dx = \frac{\cos z - 1}{z}$$ NT 60(5)

4. $$\int_0^\infty \operatorname{Ei}(-x)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = \frac{\operatorname{Ei}(-z) - \boldsymbol{C} - \ln z}{\sqrt{z}}$$ NT 60(7)

5. $$\int_0^\infty \operatorname{si}(x)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \operatorname{si}(z)}{\sqrt{z}}$$ NT 60(9)

6. $$\int_0^\infty \operatorname{ci}(z)\,J_1\left(2\sqrt{zx}\right)\frac{dx}{\sqrt{x}} = \frac{\operatorname{ci}(z) - \boldsymbol{C} - \ln z}{\sqrt{z}}$$ NT 60(8)

7. $$\int_0^\infty \operatorname{Ei}(-x)\,Y_0\left(2\sqrt{zx}\right)\,dx = \frac{\boldsymbol{C} + \ln z - e^2\operatorname{Ei}(-z)}{\pi z}$$ NT 63(5)

6.783

1. $$\int_0^\infty x \operatorname{si}\left(a^2x^2\right) J_0(bx)\,dx = -\frac{2}{b^2}\sin\left(\frac{b^2}{4a^2}\right) \qquad [a > 0]$$ ET II 13(7)a

2. $$\int_0^\infty x \operatorname{ci}\left(a^2x^2\right) J_0(bx)\,dx = \frac{2}{b^2}\left[1 - \cos\left(\frac{b^2}{4a^2}\right)\right] \qquad [a > 0]$$ ET II 13(8)a

3. $$\int_0^\infty \operatorname{ci}\left(a^2x^2\right) J_0(bx)\,dx = \frac{1}{b}\left[\operatorname{ci}\left(\frac{b^2}{4a^2}\right) + \ln\left(\frac{b^2}{4a^2}\right) + 2\boldsymbol{C}\right]$$
$$[a > 0]$$ ET II 13(8)a

4. $$\int_0^\infty \operatorname{si}\left(a^2x^2\right) J_1(bx)\,dx = \frac{1}{b}\left[-\operatorname{si}\left(\frac{b^2}{4a^2}\right) - \frac{\pi}{2}\right] \qquad [a > 0]$$ ET II 20(25)a

6.784

1. $$\int_0^\infty x^{\nu+1}\left[1 - \Phi(ax)\right] J_\nu(bx)\,dx = a^{-\nu}\frac{\Gamma\left(\nu + \frac{3}{2}\right)}{b^2\,\Gamma(\nu+2)}\exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re}\nu > -1\right]$$
ET II 92(22)

2. $$\int_0^\infty x^\nu \left[1-\Phi(ax)\right] J_\nu(bx)\,dx = \sqrt{\frac{2}{\pi}} \frac{a^{\frac{1}{2}-\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)}{b^{3/2}\,\Gamma\left(\nu+\frac{3}{2}\right)} \exp\left(-\frac{b^2}{8a^2}\right) M_{\frac{1}{2}\nu-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}\left(\frac{b^2}{4a^2}\right)$$
$$\left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad b > 0\right]$$
ET II 92(23)

6.785 $$\int_0^\infty \frac{\exp\left(\frac{a^2}{2x}-x\right)}{x}\left[1-\Phi\left(\frac{a}{\sqrt{2x}}\right)\right] K_\nu(x)\,dx = \frac{\pi^{5/2}}{4}\sec(\nu\pi)\left\{\left[J_\nu(a)\right]^2+\left[Y_\nu(a)\right]^2\right\}$$
$$\left[\operatorname{Re} a > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$
ET II 370(46)

6.786 $$\int_0^\infty x^{\nu-2\mu+2n+2} e^{x2}\,\Gamma\left(\mu,x^2\right) Y_\nu(bx)\,dx$$
$$= (-1)^n \frac{\Gamma\left(\frac{3}{2}-\mu+\nu+n\right)\Gamma\left(\frac{3}{2}-\mu+n\right)}{b\,\Gamma(1-\mu)} \exp\left(\frac{b^2}{8}\right) W_{\mu-\frac{1}{2}\nu-n-1,\frac{1}{2}\nu}\left(\frac{b^2}{4}\right)$$
$$\left[n \text{ is an integer}, \quad b > 0, \quad \operatorname{Re}(\nu-\mu+n) > -\tfrac{3}{2}, \quad \operatorname{Re}(-\mu+n) > -\tfrac{3}{2}, \quad \operatorname{Re}\nu < \tfrac{1}{2}-2n\right]$$
ET II 108(2)

6.787 $$\int_0^\infty \frac{x^{\nu+2n-\frac{1}{2}}}{\mathrm{B}(a+x,a-x)} J_\nu(bx)\,dx = 0$$
$$\left[\pi \le b < \infty, \quad -1 < \operatorname{Re}\nu < 2a-2n-\tfrac{7}{2}\right]$$
ET II 92(21)

6.79 Integration of Bessel functions with respect to the order

6.791

1. $$\int_{-\infty}^\infty K_{ix+iy}(a)\,K_{ix+iz}(b)\,dx = \pi\,K_{iy-iz}(a+b) \qquad \left[|\arg a|+|\arg b| < \pi\right]$$
ET II 382(21)

2. $$\int_{-\infty}^\infty J_{\nu-x}(a)\,J_{\mu+x}(a)\,dx = J_{\mu+\nu}(2a) \qquad \left[\operatorname{Re}(\mu+\nu) > 1\right]$$
ET II 379(1)

3. $$\int_{-\infty}^\infty J_{\kappa+x}(a)\,J_{\lambda-x}(a)\,J_{\mu+x}(a)\,J_{\nu-x}(a)\,dx$$
$$= \frac{\Gamma(\kappa+\lambda+\mu+\nu+1)}{\Gamma(\kappa+\lambda+1)\,\Gamma(\lambda+\mu+1)\,\Gamma(\mu+\nu+1)\,\Gamma(\nu+\kappa+1)}$$
$$\times {}_4F_5\left(\frac{\kappa+\lambda+\mu+\nu+1}{2}, \frac{\kappa+\lambda+\mu+\nu+1}{2}, \frac{\kappa+\lambda+\mu+\nu}{2}+1, \frac{\kappa+\lambda+\mu+\nu}{2}+1;\right.$$
$$\left.\kappa+\lambda+\mu+\nu+1, \kappa+\lambda+1, \lambda+\mu+1, \mu+\nu+1, \nu+\kappa+1; -4a^2\right)$$
$$\left[\operatorname{Re}(\kappa+\lambda+\mu+\nu) > -1\right]$$
ET II 379(3)

6.792

1. $$\int_{-\infty}^\infty e^{\pi x}\,K_{ix+iy}(a)\,K_{ix+iz}(b)\,dx = \pi e^{-\pi z}\,K_{i(y-z)}(a-b)$$
$$\left[a > b > 0\right]$$
ET II 382(22)

2. $$\int_{-\infty}^{\infty} e^{i\varrho x} K_{\nu+ix}(\alpha) K_{\nu-ix}(\beta)\,dx = \pi\left(\frac{\alpha e^{\rho}+\beta}{\alpha+\beta e^{\rho}}\right)^{\nu} K_{2\nu}\left(\sqrt{\alpha^2+\beta^2+2\alpha\beta\cosh\varrho}\right)$$

$[|\arg\alpha| + |\arg\beta| + |\operatorname{Im}\varrho| < \pi]$ ET II 382(23)

3. $$\int_{-\infty}^{\infty} e^{(\pi-\gamma)x} K_{ix+iy}(a) K_{ix+iz}(b)\,dx = \pi e^{-\beta y-\alpha z} K_{iy-iz}(c)$$

$[0<\gamma<\pi,\quad a>0,\quad b>0,\quad c>0,\quad \alpha,\beta,\gamma$—the angles of the triangle with sides a, b, $c]$

ET II 382(24), EH II 55(44)a

4.[11] $$\int_{-\infty}^{\infty} e^{-cxi} H^{(2)}_{\nu-ix}(a) H^{(2)}_{\nu+ix}(b)\,dx = 2i\left(\frac{h}{k}\right)^{2\nu} H^{(2)}_{2\nu}(hk)$$

$h=\sqrt{ae^{\frac{1}{2}c}+be^{-\frac{1}{2}c}},\quad k=\sqrt{ae^{-\frac{1}{2}c}+be^{\frac{1}{2}c}}\qquad [a,b>0,\quad c \text{ is real}]$ ET II 380(11)

5. $$\int_{-\infty}^{\infty} a^{-\mu-x}b^{-\nu+x}e^{cxi} J_{\mu+x}(a) J_{\nu-x}(b)\,dx$$

$$= \left[\frac{2\cos\left(\frac{c}{2}\right)}{a^2e^{-\frac{1}{2}ci}+b^2e^{\frac{1}{2}ci}}\right]^{\frac{1}{2}\mu+\frac{1}{2}\nu} \exp\left[\frac{c}{2}(\nu-\mu)i\right] J_{\mu+\nu}\left\{\left[2\cos\left(\frac{c}{2}\right)\left(a^2e^{-\frac{1}{2}ci}+b^2e^{\frac{1}{2}ci}\right)\right]^{1/2}\right\}$$

$[a>0,\quad b>0,\quad |c|<\pi,\quad \operatorname{Re}(\mu+\nu)>1]$

$$=0$$

$[a>0,\quad b>0,\quad |c|\ge\pi,\quad \operatorname{Re}(\mu+\nu)>1]$

EH II 54(41), ET II 379(2)

6.793

1. $$\int_{-\infty}^{\infty} e^{-cxi}\left[J_{\nu-ix}(a)\,Y_{\nu+ix}(b) + Y_{\nu-ix}(a)\,J_{\nu+ix}(b)\right]dx = -2\left(\frac{h}{k}\right)^{2\nu} J_{2\nu}(hk)$$

$h=\sqrt{ae^{\frac{1}{2}c}+be^{-\frac{1}{2}c}},\qquad k=\sqrt{ae^{-\frac{1}{2}c}+be^{\frac{1}{2}c}}\qquad [a,b>0,\quad \operatorname{Im}c=0]$ ET II 380(9)

2. $$\int_{-\infty}^{\infty} e^{-cxi}\left[J_{\nu-ix}(a)\,J_{\nu+ix}(b) - Y_{\nu-ix}(a)\,Y_{\nu+ix}(b)\right]dx = 2\left(\frac{h}{k}\right)^{2\nu} Y_{2\nu}(hk)$$

$h=\sqrt{ae^{\frac{1}{2}c}+be^{-\frac{1}{2}c}},\qquad k=\sqrt{ae^{-\frac{1}{2}c}+be^{\frac{1}{2}c}}\qquad [a,b>0,\quad \operatorname{Im}c=0]$ ET II 380(10)

3.[10] $$\int_{-\infty}^{\infty} e^{i\gamma x}\operatorname{sech}(\pi x)\left[J_{-ix}(\alpha)\,J_{ix}(\beta) - J_{ix}(\alpha)\,J_{-ix}(\beta)\right]dx = 2i\,\mathrm{H}(\sigma)\operatorname{sign}(\beta-\alpha)\,J_0\left(\sigma^{1/2}\right)$$

$\left[\alpha,\beta,\gamma\in\mathbb{R},\quad \alpha,\beta>0,\quad \sigma=\alpha^2+\beta^2-2\alpha\beta\cosh\gamma,\quad \mathrm{H}(\sigma) \text{ the Heaviside step function}\right]$

6.794

1. $$\int_0^{\infty} K_{ix}(a)\,K_{ix}(b)\cosh[(\pi-\varphi)x]\,dx = \frac{\pi}{2}K_0\left(\sqrt{a^2+b^2-2ab\cos\varphi}\right)$$ EH II 55(42)

2. $$\int_0^{\infty}\cosh\left(\frac{\pi}{2}x\right)K_{ix}(a)\,dx = \frac{\pi}{2}$$ $[a>0]$ ET II 382(19)

3. $$\int_0^\infty \cosh(\varrho x)\, K_{ix+\nu}(a)\, K_{-ix+\nu}(a)\, dx = \frac{\pi}{2}\, K_{2\nu}\left[2a\cos\left(\frac{\varrho}{2}\right)\right]$$
$[2|\arg a| + |\operatorname{Re}\varrho| < \pi]$ ET II 383(28)

4. $$\int_{-\infty}^\infty \operatorname{sech}\left(\frac{\pi}{2}x\right) J_{ix}(a)\, dx = 2\sin a$$ $[a > 0]$ ET II 380(6)

5. $$\int_{-\infty}^\infty \operatorname{cosech}\left(\frac{\pi}{2}x\right) J_{ix}(a)\, dx = -2i\cos a$$ $[a > 0]$ ET II 380(7)

6. $$\int_0^\infty \operatorname{sech}(\pi x)\left\{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\right\} dx = -\,Y_0(2a) - \mathbf{E}_0(2a)$$
$[a > 0]$ ET II 380(12)

7. $$\int_0^\infty x\sinh\left(\frac{\pi}{2}x\right) K_{ix}(a)\, dx = \frac{\pi a}{2}$$ $[a > 0]$ ET II 382(20)

8. $$\int_0^\infty x\tanh(\pi x)\, K_{ix}(\beta)\, K_{ix}(\alpha)\, dx = \frac{\pi}{2}\sqrt{\alpha\beta}\frac{\exp(-\beta-\alpha)}{\alpha+\beta}$$
$[|\arg\beta| < \pi, \quad |\arg\alpha| < \pi]$ ET II 175(4)

9. $$\int_0^\infty x\sinh(\pi x)\, K_{2ix}(\alpha)\, K_{ix}(\beta)\, dx = \frac{\pi^{3/2}\alpha}{2^{5/2}\sqrt{\beta}}\exp\left(-\beta - \frac{\alpha^2}{8\beta}\right)$$
$\left[\beta > 0, \quad |\arg\alpha| < \frac{\pi}{4}\right]$ ET II 175(5)

10. $$\int_0^\infty \frac{x\sinh(\pi x)}{x^2+n^2}\, K_{ix}(\alpha)\, K_{ix}(\beta)\, dx = \frac{\pi^2}{2}\, I_n(\beta)\, K_n(\alpha) \qquad [0 < \beta < \alpha; \quad n = 0, 1, 2, \dots]$$
$$= \frac{\pi^2}{2}\, I_n(\alpha)\, K_n(\beta) \qquad [0 < \alpha < \beta; \quad n = 0, 1, 2, \dots]$$
ET II 176(8)

11. $$\int_0^\infty x\sinh(\pi x)\, K_{ix}(\alpha)\, K_{ix}(\beta)\, K_{ix}(\gamma)\, dx = \frac{\pi^2}{4}\exp\left[-\frac{\gamma}{2}\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\gamma^2}\right)\right]$$
$\left[|\arg\alpha| + |\arg\beta| < \frac{\pi}{2}, \quad \gamma > 0\right]$
ET II 176(9)

12. $$\int_0^\infty x\sinh\left(\frac{\pi}{2}x\right) K_{\frac{1}{2}ix}(\alpha)\, K_{\frac{1}{2}ix}(\beta)\, K_{ix}(\gamma)\, dx = \frac{\pi^2\gamma}{2\sqrt{\gamma^2+4\alpha\beta}}\exp\left[-\frac{(\alpha+\beta)\sqrt{\gamma^2+4\alpha\beta}}{2\sqrt{\alpha\beta}}\right]$$
$[|\arg\alpha| + |\arg\beta| < \pi, \quad \gamma > 0]$
ET II 176(10)

13. $$\int_0^\infty x\sinh(\pi x)\, K_{\frac{1}{2}ix+\lambda}(\alpha)\, K_{\frac{1}{2}ix-\lambda}(\alpha)\, K_{ix}(\gamma)\, dx = 0 \qquad [0 < \gamma < 2\alpha]$$
$$= \frac{\pi^2\gamma}{2^{2\lambda+1}\alpha^{2\lambda}z}\left[(\gamma+z)^{2\lambda} + (\gamma-z)^{2\lambda}\right]$$
$$z = \sqrt{\gamma^2 - 4\alpha^2} \qquad [0 < 2\alpha < \gamma]$$ ET II 176(11)

6.795

1. $$\int_0^\infty \cos(bx)\, K_{ix}(a)\, dx = \frac{\pi}{2} e^{-a\cosh b} \qquad \left[|\operatorname{Im} b| < \frac{\pi}{2}, \quad a > 0\right]$$ EH II 55(46), ET II 175(2)

2. $$\int_0^\infty J_x(ax)\, J_{-x}(ax) \cos(\pi x)\, dx = \frac{1}{4}\left(1 - a^2\right)^{-1/2} \qquad [|a| < 1]$$ ET II 380(4)

3. $$\int_0^\infty x \sin(ax)\, K_{ix}(b)\, dx = \frac{\pi b}{2} \sinh a \exp\left(-b \cosh a\right) \qquad \left[|\operatorname{Im} a| < \frac{\pi}{2}, \quad b > 0\right]$$ ET II 175(1)

4. $$\int_{-\infty}^{-\infty} \frac{\sin[(\nu + ix)\pi]}{n + \nu + ix} K_{\nu+ix}(a)\, K_{\nu-ix}(b)\, dx = \pi^2 I_n(a)\, K_{n+2\nu}(b) \qquad [0 < a < b; \quad n = 0, 1, \ldots]$$
$$= \pi^2 K_{n+2\nu}(a)\, I_n(b) \qquad [0 < b < a; \quad n = 0, 1, \ldots]$$ ET II 382(25)

5. $$\int_0^\infty x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a)\, K_{ix}(b)\, dx = \frac{\pi^{3/2} b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right)$$
$$\left[|\arg a| < \frac{\pi}{2}, \quad b > 0\right]$$ ET II 175(6)

6.796

1. $$\int_{-\infty}^\infty \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)} J_{ix}(a)\, dx = -i \exp\left(ia \cosh b\right) \qquad [a > 0, \quad b > 0]$$ ET II 380(8)

2. $$\int_0^\infty \cos(bx) \cosh\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2} \cos\left(a \sinh b\right)$$ EH II 55(47)

3. $$\int_0^\infty \sin(bx) \sinh\left(\frac{1}{2}\pi x\right) K_{ix}(a)\, dx = \frac{\pi}{2} \sin\left(a \sinh b\right)$$ EH II 55(48)

4. $$\int_0^\infty \cos(bx) \cosh(\pi x) \left[K_{ix}(a)\right]^2\, dx = -\frac{\pi^2}{4}\, Y_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$$
$$[a > 0, \quad b > 0]$$ ET II 383(27)

5. $$\int_0^\infty \sin(bx) \sinh(\pi x) \left[K_{ix}(a)\right]^2\, dx = \frac{\pi^2}{4}\, J_0\left[2a \sinh\left(\frac{b}{2}\right)\right]$$
$$[a > 0, \quad b > 0]$$ ET II 382(26)

6.797

1. $$\int_0^\infty x e^{\pi x} \sinh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx$$
$$= i 2^\nu \sqrt{\pi}\, \Gamma\left(\tfrac{1}{2} + \nu\right) (ab)^\nu (a+b)^{-\nu} K_\nu(a+b)$$
$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0]$$ ET II 381(14)

2. $$\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x)\, \Gamma(\nu + ix)\, \Gamma(\nu - ix)\, H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\, dx = \frac{i \pi^{3/2} 2^\nu}{\Gamma\left(\frac{1}{2} - \nu\right)} (b-a)^{-\nu} H_\nu^{(2)}(b-a)$$
$$\left[0 < a < b, \quad 0 < \operatorname{Re} \nu < \tfrac{1}{2}\right]$$ ET II 381(15)

3. $$\int_0^\infty x e^{\pi x} \sinh(\pi x)\,\Gamma\left(\frac{\nu+ix}{2}\right)\Gamma\left(\frac{\nu-ix}{2}\right) H_{ix}^{(2)}(a)\, H_{ix}^{(2)}(b)\,dx = i\pi 2^{2-\nu}(ab)^\nu \left(a^2+b^2\right)^{-\frac{1}{2}\nu} H_\nu^{(2)}\left(\sqrt{a^2+b^2}\right)$$
$[a>0, \quad b>0, \quad \operatorname{Re}\nu>0]$ ET II 381(16)

4.[11] $$\int_0^\infty x\sinh(\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,dx = 2^{\lambda-1}\pi^{3/2}(ab)^\lambda(a+b)^{-\lambda}\,\Gamma\left(\lambda+\tfrac{1}{2}\right)K_\lambda(a+b)$$
$[|\arg a|<\pi, \quad \operatorname{Re}\lambda>0, \quad b>0]$
ET II 176(12)

5. $$\int_0^\infty x\sinh(2\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,dx = \frac{2^\lambda\pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2}-\lambda\right)}\left(\frac{ab}{|b-a|}\right)^\lambda K_\lambda(|b-a|)$$
$\left[a>0, \quad 0<\operatorname{Re}\lambda<\tfrac{1}{2}, \quad b>0\right]$
ET II 176(13)

6. $$\int_0^\infty x\sinh(\pi x)\,\Gamma\left(\lambda+\tfrac{1}{2}ix\right)\Gamma\left(\lambda-\tfrac{1}{2}ix\right)K_{ix}(a)\,K_{ix}(b)\,dx = 2\pi^2\left(\frac{ab}{2\sqrt{a^2+b^2}}\right)K_{2\lambda}\left(\sqrt{a^2+b^2}\right)$$
$\left[|\arg a|<\frac{\pi}{2}, \quad \operatorname{Re}\lambda>0, \quad b>0\right]$
ET II 177(14)

7. $$\int_0^\infty \frac{x\tanh(\pi x)\,K_{ix}(a)\,K_{ix}(b)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}ix\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}ix\right)}\,dx = \frac{1}{2}\sqrt{\frac{\pi ab}{a^2+b^2}}\exp\left(-\sqrt{a^2+b^2}\right)$$
$\left[|\arg a|<\frac{\pi}{2}, \quad b>0\right],$ ET II 177(15)

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.811

1. $$\int_0^\infty \mathbf{H}_\nu(bx)\,dx = -\frac{\cot\left(\frac{\nu\pi}{2}\right)}{b}$$ $[-2<\operatorname{Re}\nu<0, \quad b>0]$ ET II 158(1)

2. $$\int_0^\infty \mathbf{H}_\nu\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{J_{2\nu}\left(2a\sqrt{b}\right)}{b}$$ $\left[a>0, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$
ET II 170(37)

3. $$\int_0^\infty \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right)\mathbf{H}_\nu(bx)\frac{dx}{x} = -\frac{1}{a\sqrt{b}}\,J_{2\nu-1}\left(2a\sqrt{b}\right)$$ $\left[a>0, \quad b>0, \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$
ET II 170(38)

6.812

1. $$\int_0^\infty \frac{\mathbf{H}_1(bx)\,dx}{x^2+a^2} = \frac{\pi}{2a}\left[I_1(ab)-\mathbf{L}_1(ab)\right]$$ $[\operatorname{Re}a>0, \quad b>0]$ ET II 158(6)

2. $$\int_0^\infty \frac{\mathbf{H}_\nu(bx)}{x^2+a^2}\,dx = -\frac{\pi}{2a\sin\left(\frac{\nu\pi}{2}\right)}\,\mathbf{L}_\nu(ab) + \frac{b\cot\left(\frac{\nu\pi}{2}\right)}{1-\nu^2}\;{}_1F_2\left(1;\frac{3-\nu}{2};\frac{3+\nu}{2};\frac{a^2b^2}{2}\right)$$

$[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < 2]$ ET II 159(7)

6.813

1. $$\int_0^\infty x^{s-1}\,\mathbf{H}_\nu(ax)\,dx = \frac{2^{s-1}\,\Gamma\left(\frac{s+\nu}{2}\right)}{a^s\,\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}\tan\left(\frac{s+\nu}{2}\pi\right)$$

$\left[a > 0, \quad -1-\operatorname{Re}\nu < \operatorname{Re}s < \min\left(\frac{3}{2}, 1-\operatorname{Re}\nu\right)\right]$ WA 429(2), ET I 335(52)

2. $$\int_0^\infty x^{-\nu-1}\,\mathbf{H}_\nu(x)\,dx = \frac{2^{-\nu-1}\pi}{\Gamma(\nu+1)}$$ $\left[\operatorname{Re}\nu > -\frac{3}{2}\right]$ ET II 383(2)

3. $$\int_0^\infty x^{-\mu-\nu}\,\mathbf{H}_\mu(x)\,\mathbf{H}_\nu(x)\,dx = \frac{2^{-\mu-\nu}\sqrt{\pi}\,\Gamma(\mu+\nu)}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\mu+\nu+\frac{1}{2}\right)}$$

$[\operatorname{Re}(\mu+\nu) > 0]$ WA 435(2), ET II 384(8)

4. $$\int_0^1 x^{\nu+1}\,\mathbf{H}_\nu(ax)\,dx = \frac{1}{a}\,\mathbf{H}_{\nu+1}(a)$$ $\left[a > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET II 158(2)a

5. $$\int_0^1 x^{1-\nu}\,\mathbf{H}_\nu(ax)\,dx = \frac{a^{\nu-1}}{2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \frac{1}{a}\,\mathbf{H}_{\nu-1}(a)$$

$[a > 0]$ ET II 158(3)a

6.814

1. $$\int_0^\infty \frac{x^{\nu+1}\,\mathbf{H}_\nu(bx)}{(x^2+a^2)^{1-\mu}}\,dx = \frac{2^{\mu-1}\pi a^{\mu+\nu}b^{-\mu}}{\Gamma(1-\mu)\cos[(\mu+\nu)\pi]}\left[I_{-\mu-\nu}(ab) - \mathbf{L}_{\mu+\nu}(ab)\right]$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) < \frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) < \frac{3}{2}\right]$ ET II 159(8)

6.815

1. $$\int_0^1 x^{\frac{1}{2}\nu}(1-x)^{\mu-1}\,\mathbf{H}_\nu\left(a\sqrt{x}\right)\,dx = 2^\mu a^{-\mu}\,\Gamma(\mu)\,\mathbf{H}_{\mu+\nu}(a)$$

$\left[\operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET II 199(88)a

2. $$\int_0^1 x^{\lambda-\frac{1}{2}\nu-\frac{3}{2}}(1-x)^{\mu-1}\,\mathbf{H}_\nu\left(a\sqrt{x}\right)\,dx = \frac{\mathrm{B}(\lambda,\mu)a^{\nu+1}}{2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_2F_3\left(1,\lambda;\frac{3}{2},\nu+\frac{3}{2},\lambda+\mu;-\frac{a^2}{4}\right)$$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 199(89)a

6.82 Combinations of Struve functions, exponentials, and powers

6.821

1.[6] $$\int_0^\infty e^{-\alpha x}\,\mathbf{H}_{-n-\frac{1}{2}}(\beta x)\,dx = (-1)^n\beta^{n+\frac{1}{2}}\left(\alpha+\sqrt{\alpha^2+\beta^2}\right)^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2+\beta^2}}$$

$[\operatorname{Re}\alpha > |\operatorname{Im}\beta|]$ ET I 206(6)

2.[6] $$\int_0^\infty e^{-\alpha x}\,\mathbf{L}_{-n-\frac{1}{2}}(\beta x)\,dx = \beta^{n+\frac{1}{2}}\left(\alpha+\sqrt{\alpha^2-\beta^2}\right)^{-n-\frac{1}{2}}\frac{1}{\sqrt{\alpha^2-\beta^2}}$$

$[\operatorname{Re}\alpha > |\operatorname{Re}\beta|]$ ET I 208(26)

3. $$\int_0^\infty e^{-\alpha x}\,\mathbf{H}_0(\beta x)\,dx = \frac{2}{\pi}\frac{\ln\left(\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}}$$ $[\operatorname{Re}\alpha > |\operatorname{Im}\beta|]$ ET II 205(1)

4. $$\int_0^\infty e^{-\alpha x}\,\mathbf{L}_0(\beta x)\,dx = \frac{2}{\pi}\frac{\arcsin\left(\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2+\beta^2}}$$ $[\operatorname{Re}\alpha > |\operatorname{Re}\beta|]$ ET II 207(18)

6.822 $$\int_0^\infty e^{(\nu+1)x}\,\mathbf{H}_\nu\left(a\sinh x\right)\,dx = \sqrt{\frac{\pi}{a}}\operatorname{cosec}(\nu\pi)\left[\sinh\left(\frac{a}{2}\right)I_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right)-\cosh\left(\frac{a}{2}\right)I_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$

$[\operatorname{Re}a > 0, \quad -2 < \operatorname{Re}\nu < 0]$

ET II 385(11)

6.823

1. $$\int_0^\infty x^\lambda e^{-\alpha x}\,\mathbf{H}_\nu(bx)\,dx = \frac{b^{\nu+1}\,\Gamma(\lambda+\nu+2)}{2^\nu a^{\lambda+\nu+2}\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_3F_2\left(1,\frac{\lambda+\nu}{2}+1,\frac{\lambda+\nu+3}{2};\frac{3}{2},\nu+\frac{3}{2};-\frac{b^2}{a^2}\right)$$

$[\operatorname{Re}a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda+\nu) > -2]$

ET II 161(19)

2. $$\int_0^\infty x^\nu e^{-\alpha x}\,\mathbf{L}_\nu(\beta x)\,dx = \frac{(2\beta)^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left(\sqrt{\alpha^2-\beta^2}\right)^{2\nu+1}} - \frac{\Gamma(2\nu+1)\left(\frac{\beta}{\alpha}\right)^\nu}{\sqrt{\frac{\pi}{2}}\alpha\left(\beta^2-\alpha^2\right)^{\frac{1}{2}\nu+\frac{1}{4}}}\,P_{-\nu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)$$

$\left[\operatorname{Re}\alpha > |\operatorname{Re}\beta|, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$

ET I 209(35)a

6.824

1. $$\int_0^\infty t^\nu e^{-at}\,\mathbf{L}_{2\nu}\left(2\sqrt{t}\right)\,dt = \frac{1}{a^{2\nu+1}}e^{\frac{1}{a}}\,\Phi\left(\frac{1}{\sqrt{a}}\right)$$ MI 51

2. $$\int_0^\infty t^\nu e^{-at}\,\mathbf{L}_{-2\nu}\left(\sqrt{t}\right)\,dt = \frac{1}{\Gamma\left(\frac{1}{2}-2\nu\right)a^{2\nu+1}}e^{\frac{1}{a}}\gamma\left(\frac{1}{2}-2\nu,\frac{1}{a}\right)$$ MI 51

6.825 $$\int_0^\infty x^{s-1}e^{-\alpha^2x^2}\,\mathbf{H}_\nu(\beta x)\,dx = \frac{\beta^{\nu+1}\,\Gamma\left(\frac{1}{2}+\frac{s}{2}+\frac{\nu}{2}\right)}{2^{\nu+1}\sqrt{\pi}\alpha^{\nu+s+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}\;{}_2F_2\left(1,\frac{\nu+s+1}{2};\frac{3}{2},\nu+\frac{3}{2};-\frac{\beta^2}{4\alpha^2}\right)$$

$\left[\operatorname{Re}s > -\operatorname{Re}\nu - 1, \quad |\arg\alpha| < \frac{\pi}{4}\right]$

ET I 335(51)a, ET II 162(20)

6.83 Combinations of Struve and trigonometric functions

6.831 $$\int_0^\infty x^{-\nu}\sin(ax)\,\mathbf{H}_\nu(bx)\,dx = 0 \qquad \left[0 < b < a, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$

$$= \sqrt{\pi}2^{-\nu}b^{-\nu}\frac{\left(b^2-a^2\right)^{\nu-\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \qquad \left[0 < a < b, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$$

ET II 162(21)

6.832 $$\int_0^\infty \sqrt{x}\sin(ax)\,\mathbf{H}_{\frac{1}{4}}\left(b^2x^2\right)\,dx = -2^{-3/2}\sqrt{\pi}\frac{\sqrt{a}}{b^2}\,Y_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right)$$
$[a > 0]$ ET I 109(14)

6.84–6.85 Combinations of Struve and Bessel functions

6.841 $$\int_0^\infty \mathbf{H}_{\nu-1}(ax)\,Y_\nu(bx)\,dx = -a^{\nu-1}b^{-\nu} \qquad \left[0 < b < a, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$$
$$= 0 \qquad \left[0 < a < b, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$$
ET II 114(36)

6.842 $$\int_0^\infty \left[\mathbf{H}_0(ax) - Y_0(ax)\right] J_0(bx)\,dx = \frac{4}{\pi(a+b)}\,\boldsymbol{K}\left(\frac{|a-b|}{a+b}\right)$$
$[a > 0, \quad b > 0]$ ET II 15(22)

6.843

1. $$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = -\frac{1}{b}\,Y_\nu\left(\frac{a^2}{4b}\right) \qquad \left[a > 0, \quad b > 0, \quad -1 < \mathrm{Re}\,\nu < \tfrac{5}{4}\right]$$
ET II 164(10)

2. $$\int_0^\infty K_{2\nu}\left(2a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{2^\nu}{\pi b}\,\Gamma(\nu+1)\,S_{-\nu-1,\nu}\left(\frac{a^2}{b}\right)$$
$[\mathrm{Re}\,a > 0, \quad b > 0, \quad \mathrm{Re}\,\nu > -1]$ ET II 168(27)

6.844 $$\int_0^\infty \left[\cos\left(\frac{\mu-\nu}{2}\pi\right) J_\mu\left(a\sqrt{x}\right) - \sin\left(\frac{\mu-\nu}{2}\pi\right) Y_\mu\left(a\sqrt{x}\right)\right] K_\mu\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{1}{a^2}\,W_{\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) W_{-\frac{1}{2}\nu,\frac{1}{2}\mu}\left(\frac{a^2}{2b}\right)$$
$\left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \mathrm{Re}\,\nu > |\mathrm{Re}\,\mu| - 2\right]$ ET II 169(35)

6.845

1. $$\int_0^\infty \left[\mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - Y_{-\nu}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx = \frac{4}{\pi b}\cos(\nu\pi)\,K_{2\nu}\left(2\sqrt{ab}\right)$$
$\left[|\arg a| < \pi, \quad b > 0, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$
ET II 73(7)

2. $$\int_0^\infty \left[J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi)\,\mathbf{H}_\nu\left(\frac{a^2}{x}\right)\right]\mathbf{H}_\nu(bx)\,dx = \frac{1}{b}\left[\frac{2}{\pi}K_{2\nu}\left(2a\sqrt{b}\right) - Y_{2\nu}\left(2a\sqrt{b}\right)\right]$$
$\left[a > 0, \quad b > 0, \quad -\tfrac{3}{2} < \mathrm{Re}\,\nu < 0\right]$
ET II 170(39)

6.846 $$\int_0^\infty \left[\frac{2}{\pi}K_{2\nu}\left(2a\sqrt{x}\right) + Y_{2\nu}\left(2a\sqrt{x}\right)\right]\mathbf{H}_\nu(bx)\,dx = \frac{1}{b}\,J_\nu\left(\frac{a^2}{b}\right)$$
$\left[a > 0, \quad b > 0, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}\right]$
ET II 169(30)

6.847 $$\int_0^\infty \left[\cos\frac{\nu\pi}{2}\,J_\nu(ax) + \sin\frac{\nu\pi}{2}\,\mathbf{H}_\nu(ax)\right]\frac{dx}{x^2+k^2} = \frac{\pi}{2k}\left[I_\nu(ak) - \mathbf{L}_\nu(ak)\right]$$
$\left[a > 0, \quad \mathrm{Re}\,k > 0, \quad -\tfrac{1}{2} < \mathrm{Re}\,\nu < 2\right]$
ET II 384(5)a, WA 467(8)

6.848

1. $$\int_0^\infty x\left[I_\nu(ax) - \mathbf{L}_{-\nu}(ax)\right] J_\nu(bx)\,dx = \frac{2}{\pi}\left(\frac{a}{b}\right)^{\nu-1}\cos(\nu\pi)\frac{1}{a^2+b^2}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2}\right]$ ET II 74(12)

2. $$\int_0^\infty x\left[\mathbf{H}_{-\nu}(ax) - Y_{-\nu}(ax)\right] J_\nu(bx)\,dx = 2\frac{\cos(\nu\pi)}{a^\nu\pi}b^{\nu-1}\frac{1}{a+b}$$
$\left[|\arg a| < \pi, \quad -\frac{1}{2} < \operatorname{Re}\nu, \quad b > 0\right]$ ET II 73(5)

6.849

1. $$\int_0^\infty x\,K_\nu(ax)\,\mathbf{H}_\nu(bx)\,dx = a^{-\nu-1}b^{\nu+1}\frac{1}{a^2+b^2} \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$$
ET II 164(12)

2. $$\int_0^\infty x\left[K_\mu(ax)\right]^2\mathbf{H}_0(bx)\,dx = -2^{-\mu-1}\pi a^{-2\mu}\frac{\left[(z+b)^{2\mu} + (z-b)^{2\mu}\right]}{bz}\sec(\mu\pi),$$
$$z = \sqrt{4a^2+b^2} \qquad \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\mu| < \frac{3}{2}\right]$$
ET II 166(18)

6.851

1. $$\int_0^\infty x\left\{\left[J_{\frac{1}{2}\nu}(ax)\right]^2 - \left[Y_{\frac{1}{2}\nu}(ax)\right]^2\right\}\mathbf{H}_\nu(bx)\,dx$$
$$= 0 \qquad \left[0 < b < 2a, \quad -\frac{3}{2} < \operatorname{Re}\nu < 0\right]$$
$$= \frac{4}{\pi b}\frac{1}{\sqrt{b^2-4a^2}} \qquad \left[0 < 2a < b, \quad -\frac{3}{2} < \operatorname{Re}\nu < 0\right]$$
ET II 164(7)

2. $$\int_0^\infty x^{\nu+1}\left\{\left[J_\nu(ax)\right]^2 - \left[Y_\nu(ax)\right]^2\right\}\mathbf{H}_\nu(bx)\,dx$$
$$= 0 \qquad \left[0 < b < 2a, \quad -\frac{3}{4} < \operatorname{Re}\nu < 0\right]$$
$$= \frac{2^{3\nu+2}a^{2\nu}b^{-\nu-1}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\left(b^2-4a^2\right)^{-\nu-\frac{1}{2}} \qquad \left[0 < 2a < b, \quad -\frac{3}{4} < \operatorname{Re}\nu < 0\right]$$
ET II 163(6)

6.852

1. $$\int_0^\infty x^{1-\mu-\nu}\,J_\nu(x)\,\mathbf{H}_\mu(x)\,dx = \frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1)\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}$$
$\left[\operatorname{Re}\nu > \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > 1\right]$ ET II 383(4)

2. $$\int_0^\infty x^{\mu-\nu+1}\,Y_\mu(ax)\,\mathbf{H}_\nu(bx)\,dx$$
$$= 0 \qquad \left[0 < b < a, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re}\mu < \frac{1}{2}\right]$$
$$= \frac{2^{1+\mu-\nu}a^\mu b^{-\nu}}{\Gamma(\nu-\mu)}\left(b^2-a^2\right)^{\nu-\mu-1} \qquad \left[0 < a < b, \quad \operatorname{Re}(\nu-\mu) > 0, \quad -\frac{3}{2} < \operatorname{Re}\mu < \frac{1}{2}\right]$$
ET II 163(3)

3. $$\int_0^\infty x^{\mu+\nu+1} K_\mu(ax) \mathbf{H}_\nu(bx)\,dx = \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}} \Gamma\left(\mu+\nu+\frac{3}{2}\right) F\left(1, \mu+\nu+\frac{3}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) > -\frac{3}{2}\right]$ ET II 165(13)

6.853

1. $$\int_0^\infty x^{1-\mu} \left[\sin(\mu\pi) J_{\mu+\nu}(ax) + \cos(\mu\pi) Y_{\mu+\nu}(ax)\right] \mathbf{H}_\nu(bx)\,dx$$
$$= 0 \qquad \left[0 < b < a, \quad 1 < \operatorname{Re}\mu < \frac{3}{2}, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \frac{1}{2}\right]$$
$$= \frac{b^\nu \left(b^2-a^2\right)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)} \qquad \left[0 < a < b, \quad 1 < \operatorname{Re}\mu < \frac{3}{2}, \quad \operatorname{Re}\nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu-\mu) < \frac{1}{2}\right]$$
ET II 163(4)

2. $$\int_0^\infty x^{\lambda+\frac{1}{2}} \left[I_\mu(ax) - \mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\,dx$$
$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}} G_{33}^{22}\left(\frac{b^2}{a^2}\middle| \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu+\lambda) > -\frac{3}{2}, \quad -\operatorname{Re}\nu - \frac{5}{2} < \operatorname{Re}(\lambda-\mu) < 1\right]$ ET II 76(21)

3. $$\int_0^\infty x^{\lambda+\frac{1}{2}} \left[\mathbf{H}_\mu(ax) - Y_\mu(ax)\right] J_\nu(bx)\,dx$$
$$= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}} G_{33}^{23}\left(\frac{b^2}{a^2}\middle| \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4}+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix}\right)$$
$\left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda+\mu) < 1, \quad \operatorname{Re}(\lambda+\nu) + \frac{3}{2} > |\operatorname{Re}\mu|\right]$ ET II 73(6)

4. $$\int_0^\infty \sqrt{x} \left[I_{\nu-\frac{1}{2}}(ax) - \mathbf{L}_{\nu-\frac{1}{2}}(ax)\right] J_\nu(bx)\,dx = \sqrt{\frac{2}{\pi}} a^{\nu-\frac{1}{2}} b^{-\nu} \frac{1}{\sqrt{a^2+b^2}}$$
$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET II 74(11)

5. $$\int_0^\infty x^{\mu-\nu+1} \left[I_\mu(ax) - \mathbf{L}_\mu(ax)\right] J_\nu(bx)\,dx = \frac{2^{\mu-\nu+1} a^{\mu-1} b^{\nu-2\mu-1}}{\sqrt{\pi}\,\Gamma\left(\nu-\mu+\frac{1}{2}\right)} F\left(1, \frac{1}{2}; \nu-\mu+\frac{1}{2}; -\frac{b^2}{a^2}\right)$$
$\left[-1 < 2\operatorname{Re}\mu + 1 < \operatorname{Re}\nu + \frac{1}{2}, \quad \operatorname{Re} a > 0, \quad b > 0\right]$ ET II 74(13)

6. $$\int_0^\infty x^{\mu-\nu+1} \left[I_\mu(ax) - \mathbf{L}_{-\mu}(ax)\right] J_\nu(bx)\,dx = \frac{2^{\mu-\nu+1} a^{-\mu-1} b^{\nu-1}}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma\left(\frac{1}{2}+\nu\right)} F\left(1, \frac{1}{2}+\mu; \frac{1}{2}+\nu; -\frac{b^2}{a^2}\right)$$
$\left[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\mu > -1, \quad b > 0\right]$ ET II 75(18)

6.854

1. $$\int_0^\infty x\,\mathbf{H}_{\frac{1}{2}\nu}\left(ax^2\right) K_\nu(bx)\,dx = \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{2^{1-\frac{1}{2}\nu} a\pi} S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$\left[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re}\nu > -2\right]$
ET II 150(75)

2. $$\int_0^\infty x\,\mathbf{H}_{\frac{1}{2}\nu}\left(ax^2\right) J_\nu(bx)\,dx = -\frac{1}{2a}\,Y_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \qquad \left[a>0, \quad b>0, \quad -2<\operatorname{Re}\nu<\tfrac{3}{2}\right]$$
ET II 73(3)

6.855

1. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[I_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)-\mathbf{L}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx = 2^{\frac{3}{2}}\frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi}b^{\nu+1}}\,J_{2\nu+1}\left(\sqrt{2ab}\right) K_{2\nu+1}\left(\sqrt{2ab}\right)$$
$$\left[\operatorname{Re}a>0, \quad b>0, \quad -1<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$
ET II 76(22)

2. $$\int_0^\infty \left[\mathbf{H}_{-\nu-1}\left(\frac{a}{x}\right)-Y_{-\nu-1}\left(\frac{a}{x}\right)\right] J_\nu(bx)\frac{dx}{x} = -\frac{4}{\pi\sqrt{ab}}\cos(\nu\pi)\,K_{-2\nu-1}\left(2\sqrt{ab}\right)$$
$$\left[|\arg a|<\pi, \quad b>0, \quad |\operatorname{Re}\nu|<\tfrac{1}{2}\right]$$
ET II 74(8)

3. $$\int_0^\infty x^{2\nu+\frac{1}{2}}\left[\mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)-Y_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right)\right] J_\nu(bx)\,dx$$
$$= -2^{5/2}\pi^{-3/2}a^{\nu+\frac{1}{2}}b^{-\nu-1}\sin(\nu\pi)\,K_{2\nu+1}\left(\sqrt{2ab}e^{\frac{1}{4}\pi i}\right) K_{2\nu+1}\left(\sqrt{2ab}e^{-\frac{1}{4}\pi i}\right)$$
$$\left[|\arg a|<\pi, \quad b>0, \quad -1<\operatorname{Re}\nu<-\tfrac{1}{6}\right]$$ ET II 74(9)

6.856 $$\int_0^\infty x\,Y_\nu\left(a\sqrt{x}\right) K_\nu\left(a\sqrt{x}\right)\mathbf{H}_\nu(bx)\,dx = \frac{1}{2b^2}\exp\left(-\frac{a^2}{2b}\right)$$
$$\left[b>0, \quad |\arg a|<\frac{\pi}{4}, \quad \operatorname{Re}\nu>-\tfrac{3}{2}\right]$$
ET II 169(32)

6.857

1. $$\int_0^\infty x\exp\left(\frac{a^2x^2}{8}\right) K_{\frac{1}{2}\nu}\left(\frac{a^2x^2}{8}\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{2}{\sqrt{\pi}}a^{-\frac{\nu}{2}-1}b^{\frac{\nu}{2}-1}\cos\left(\frac{\nu\pi}{2}\right)\Gamma\left(-\frac{1}{2}\nu\right)\exp\left(\frac{b^2}{2a^2}\right) W_{k,m}\left(\frac{b^2}{a^2}\right)$$
$$k=\tfrac{1}{4}\nu, \qquad m=\tfrac{1}{2}+\tfrac{1}{4}\nu \qquad \left[|\arg a|<\tfrac{3}{4}\pi, \quad b>0, \quad -\tfrac{3}{2}<\operatorname{Re}\nu<0\right]$$ ET II 167(24)

2. $$\int_0^\infty x^{\sigma-2}\exp\left(-\frac{1}{2}a^2x^2\right) K_\mu\left(\frac{1}{2}a^2x^2\right)\mathbf{H}_\nu(bx)\,dx$$
$$= \frac{\sqrt{\pi}}{2^{\nu+2}}a^{-\nu-\sigma}b^{\nu+1}\frac{\Gamma\left(\frac{\nu+\sigma}{2}+\mu\right)\Gamma\left(\frac{\nu+\sigma}{2}-\mu\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{\nu+\sigma}{2}\right)}$$
$$\times\,{}_3F_3\left(1,\frac{\nu+\sigma}{2}+\mu,\frac{\nu+\sigma}{2}-\mu;\frac{3}{2},\nu+\frac{3}{2},\frac{\nu+\sigma}{2};-\frac{b^2}{4a^2}\right)$$
$$\left[b>0, \quad |\arg a|<\frac{\pi}{4}, \quad \operatorname{Re}(\sigma+\nu)>2|\operatorname{Re}\mu|\right]$$ ET II 167(23)

6.86 Lommel functions

6.861

1. $$\int_0^\infty x^{\lambda-1}\, S_{\mu,\nu}(x)\, dx = \frac{\Gamma\left[\frac{1}{2}(1+\lambda+\mu)\right]\Gamma\left[\frac{1}{2}(1-\lambda-\mu)\right]\Gamma\left[\frac{1}{2}(1+\mu+\nu)\right]\Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]}{2^{2-\lambda-\mu}\,\Gamma\left[\frac{1}{2}(\nu-\lambda)+1\right]\Gamma\left[1-\frac{1}{2}(\lambda+\nu)\right]}$$
$\left[-\operatorname{Re}\mu < \operatorname{Re}\lambda + 1 < \frac{5}{2}\right]$ ET II 385(17)

6.862

1. $$\int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}}(u-x)^{\sigma-1}\, s_{\mu,\nu}\left(a\sqrt{x}\right)\, dx = \Gamma(\sigma)\frac{a^{\mu+1}u^{\lambda+\sigma}\,\Gamma(\lambda+1)}{(\mu-\nu+1)\,(\mu+\nu+1)\,\Gamma(\lambda+\sigma+1)} \times {}_2F_3\left(1, 1+\lambda; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}, \lambda+\sigma+1; -\frac{a^2u}{4}\right)$$
$[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\sigma > 0]$ ET II 199(92)

2. $$\int_u^\infty x^{\frac{1}{2}\nu}(x-u)^{\mu-1}\, s_{\lambda,\nu}\left(a\sqrt{x}\right)\, dx = \frac{\mathrm{B}\left[\mu, \frac{1}{2}(1-\lambda-\nu)-\mu\right]u^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{a^\mu}\, S_{\lambda+\mu,\mu+\nu}\left(a\sqrt{u}\right)$$
$\left[\left|\arg\left(a\sqrt{u}\right)\right| < \pi, \quad 0 < 2\operatorname{Re}\mu < 1 - \operatorname{Re}(\lambda+\nu)\right]$ ET II 211(71)

6.863 $$\int_0^\infty \sqrt{x}e^{-\alpha x}\, s_{\mu,\frac{1}{4}}\left(\frac{x^2}{2}\right)\, dx = 2^{-2\mu-1}\sqrt{\alpha}\,\Gamma\left(2\mu+\frac{3}{2}\right) S_{-\mu-1,\frac{1}{4}}\left(\frac{\alpha^2}{2}\right)$$
$\left[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -\frac{3}{4}\right]$ ET I 209(38)

6.864 $$\int_0^\infty \exp[(\mu+1)x]\, s_{\mu,\nu}\,(a\sinh x)\, dx = 2^{\mu-2}\pi\operatorname{cosec}(\mu\pi)\,\Gamma(\varrho)\,\Gamma(\sigma) \times \left[I_\varrho\left(\frac{a}{2}\right)I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right)I_{-\sigma}\left(\frac{a}{2}\right)\right]$$
$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1$ $[a > 0, \quad -2 < \operatorname{Re}\mu < 0]$ ET II 386(22)

6.865 $$\int_0^\infty \sqrt{\sinh x}\cosh(\nu x)\, S_{\mu,\frac{1}{2}}\,(a\cosh x)\, dx = \frac{\mathrm{B}\left(\frac{1}{4}-\frac{\mu+\nu}{2}, \frac{1}{4}-\frac{\mu-\nu}{2}\right)}{\sqrt{a}2^{\mu+\frac{3}{2}}}\, S_{\mu+\frac{1}{2},\nu}(a)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu + |\operatorname{Re}\nu| < \frac{1}{2}\right]$
ET II 388(31)

6.866

1. $$\int_0^\infty x^{-\mu-1}\cos(ax)\, s_{\mu,\nu}(x)\, dx$$
$$= 0 \qquad [a > 1]$$
$$= 2^{\mu-\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)\left(1-a^2\right)^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \qquad [0 < a < 1]$$
ET II 386(18)

2. $$\int_0^\infty x^{-\mu}\sin(ax)\, S_{\mu,\nu}(x)\, dx = 2^{-\mu-\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(1-\frac{\mu+\nu}{2}\right)\Gamma\left(1-\frac{\mu-\nu}{2}\right)\left(a^2-1\right)^{\frac{1}{2}\mu-\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a)$$
$[a > 1, \quad \operatorname{Re}\mu < 1 - |\operatorname{Re}\nu|]$
ET II 387(23)

6.867

1. $$\int_0^{\pi/2} \cos(2\mu x)\, S_{2\mu-1,2\nu}\,(a\cos x)\; dx = \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1-\mu-\nu)\,\Gamma\,(1-\mu+\nu)} \left[J_{\mu+\nu}\left(\frac{a}{2}\right) Y_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) Y_{\mu+\nu}\left(\frac{a}{2}\right)\right]$$
$[\operatorname{Re}\mu > -2, \quad |\operatorname{Re}\nu| < 1]$ ET II 388(29)

2. $$\int_0^{\pi/2} \cos\left[(\mu+1)\,x\right] s_{\mu,\nu}\,(a\cos x)\; dx = 2^{\mu-2}\pi\,\Gamma(\varrho)\,\Gamma(\sigma)\,J_{\varrho}\left(\frac{a}{2}\right) J_{\sigma}\left(\frac{a}{2}\right)$$
$2\varrho = \mu+\nu+1, \quad 2\sigma = \mu-\nu+1 \qquad [\operatorname{Re}\mu > -2]$ ET II 386(21)

6.868 $$\int_0^{\pi/2} \frac{\cos(2\mu x)}{\cos x}\, S_{2\mu,2\nu}\,(a\sec x)\; dx = \frac{\pi 2^{2\mu-1}}{a}\, W_{\mu,\nu}\left(ae^{i\frac{\pi}{2}}\right) W_{\mu,\nu}\left(ae^{-i\frac{\pi}{2}}\right)$$
$[|\arg a| < \pi, \quad \operatorname{Re}\mu < 1]$ ET II 388(30)

6.869

1. $$\int_0^{\infty} x^{1-\mu-\nu}\, J_\nu(ax)\, S_{\mu,-\mu-2\nu}(x)\, dx = \frac{\sqrt{\pi} a^{\nu-1}\,\Gamma(1-\mu-\nu)}{2^{\mu+2\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(a^2-1\right)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$$
$\left[a > 1, \quad \operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) < 1\right]$
ET II 388(28)

2. $$\int_0^{\infty} x^{-\mu}\, J_\nu(ax)\, s_{\nu+\mu,-\nu+\mu+1}(x)\, dx$$
$$= 2^{\nu-1}\,\Gamma(\nu) a^{-\nu} \left(1-a^2\right)^{\mu} \qquad \left[0 < a < 1, \quad \operatorname{Re}\mu > -1, \quad -1e < \operatorname{Re}\nu < \tfrac{3}{2}\right]$$
$$= 0 \qquad \left[1 < a, \quad \operatorname{Re}\mu > -1, \quad -1 < \operatorname{Re}\nu < \tfrac{3}{2}\right]$$
ET II 388(28)

3. $$\int_0^{\infty} x\, K_\nu(bx)\, s_{\mu,\frac{1}{2}\nu}\left(ax^2\right)\; dx = \frac{1}{4a}\,\Gamma\left(\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\mu-\frac{1}{2}\nu+1\right) S_{-\mu-1,\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$
$\left[\operatorname{Re}\mu > \frac{1}{2}|\operatorname{Re}\nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0\right]$ ET II 151(78)

6.87 Thomson functions

6.871

1. $$\int_0^{\infty} e^{-\beta x} \operatorname{ber} x\, dx = \frac{\left(\sqrt{\beta^4+1}+\beta^2\right)^{1/2}}{\sqrt{2\left(\beta^4+1\right)}}$$ ME 40

2. $$\int_0^{\infty} e^{-\beta x} \operatorname{bei} x\, dx = \frac{\left(\sqrt{\beta^4+1}-\beta^2\right)^{1/2}}{\sqrt{2\left(\beta^4+1\right)}}$$ ME 40

6.872

1. $$\int_0^\infty e^{-\beta x} \operatorname{ber}_\nu\left(2\sqrt{x}\right) dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu\pi}{4}\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\cos\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right]$$ MI 49

2. $$\int_0^\infty e^{-\beta x} \operatorname{bei}_\nu\left(2\sqrt{x}\right) dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}\left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu}{4}\pi\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right)\sin\left(\frac{1}{2\beta}+\frac{3\nu+6}{4}\pi\right)\right]$$ MI 49

3. $$\int_0^\infty e^{-\beta x} \operatorname{ber}\left(2\sqrt{x}\right) dx = \frac{1}{\beta}\cos\frac{1}{\beta}$$ ME 40

4. $$\int_0^\infty e^{-\beta x} \operatorname{bei}\left(2\sqrt{x}\right) dx = \frac{1}{\beta}\sin\frac{1}{\beta}$$ ME 40

5. $$\int_0^\infty e^{-\beta x} \operatorname{ker}\left(2\sqrt{x}\right) dx = -\frac{1}{2\beta}\left[\cos\frac{1}{\beta}\operatorname{ci}\frac{1}{\beta} + \sin\frac{1}{\beta}\operatorname{si}\frac{1}{\beta}\right]$$ MI 50

6. $$\int_0^\infty e^{-\beta x} \operatorname{kei}\left(2\sqrt{x}\right) dx = -\frac{1}{2\beta}\left[\sin\frac{1}{\beta}\operatorname{ci}\frac{1}{\beta} - \cos\frac{1}{\beta}\operatorname{si}\frac{1}{\beta}\right]$$ MI 50

7. $$\int_0^\infty e^{-\beta x} \operatorname{ber}_\nu\left(2\sqrt{x}\right)\operatorname{bei}_\nu\left(2\sqrt{x}\right) dx = \frac{1}{2\beta} J_\nu\left(\frac{2}{\beta}\right)\sin\left(\frac{2}{\beta}+\frac{3\nu\pi}{2}\right)$$ $[\operatorname{Re}\nu > -1]$ MI 49

6.873 $$\int_0^\infty \left[\operatorname{ber}_\nu^2\left(2\sqrt{x}\right) + \operatorname{bei}_\nu^2\left(2\sqrt{x}\right)\right] e^{-\beta x}\, dx = \frac{1}{\beta} I_\nu\left(\frac{2}{\beta}\right)$$ $[\operatorname{Re}\nu > -1]$ ME 40

6.874

1. $$\int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{ber}_{2\nu}\left(2\sqrt{2x}\right) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right)\cos\left(\frac{1}{\beta}-\frac{3\pi}{4}+\frac{3\nu\pi}{2}\right)$$ $[\operatorname{Re}\nu > -\frac{1}{2}]$ MI 49

2. $$\int_0^\infty \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{bei}_{2\nu}\left(2\sqrt{2x}\right) dx = \sqrt{\frac{\pi}{\beta}} J_\nu\left(\frac{1}{\beta}\right)\sin\left(\frac{1}{\beta}-\frac{3\pi}{4}+\frac{3\nu\pi}{2}\right)$$ $[\operatorname{Re}\nu > -\frac{1}{2}]$ MI 49

3. $$\int_0^\infty x^{\frac{\nu}{2}} \operatorname{ber}_\nu\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{2^{-\nu}}{\beta^{1+\nu}}\cos\left(\frac{1}{4\beta}+\frac{3\nu\pi}{4}\right)$$ $[\operatorname{Re}\nu > -1]$ ME 40

4. $$\int_0^\infty x^{\frac{\nu}{2}} \operatorname{bei}_\nu\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{2^{-\nu}}{\beta^{1+\nu}}\sin\left(\frac{1}{4\beta}+\frac{3\nu\pi}{4}\right)$$ $[\operatorname{Re}\nu > -1]$ ME 40

6.875

1. $$\int_0^\infty e^{-\beta x}\left[\operatorname{ker}\left(2\sqrt{x}\right)-\frac{1}{2}\ln x \operatorname{ber}\left(2\sqrt{x}\right)\right]dx=\frac{1}{\beta}\left[\ln\beta\cos\frac{1}{\beta}+\frac{\pi}{4}\sin\frac{1}{\beta}\right]$$ MI 50

2. $$\int_0^\infty e^{-\beta x}\left[\operatorname{kei}\left(2\sqrt{x}\right)-\frac{1}{2}\ln x \operatorname{bei}\left(2\sqrt{x}\right)\right]dx=\frac{1}{\beta}\left[\ln\beta\sin\frac{1}{\beta}-\frac{\pi}{4}\cos\frac{1}{\beta}\right]$$ MI 50

6.876

1. $$\int_0^\infty x\operatorname{kei}x\,J_1(ax)\,dx=-\frac{1}{2a}\arctan a^2 \qquad [a>0]$$ ET II 21(32)

2. $$\int_0^\infty x\operatorname{ker}x\,J_1(ax)\,dx=\frac{1}{2a}\ln\sqrt{(1+a^4)} \qquad [a>0]$$ ET II 21(33)

6.9 Mathieu Functions

Notation: $k^2=q$. For definition of the coefficients $A_p^{(m)}$ and $B_p^{(m)}$, see section 8.6.

6.91 Mathieu functions

6.911

1. $$\int_0^{2\pi}\operatorname{ce}_m(z,q)\operatorname{ce}_p(z,q)\,dz=0 \qquad [m\neq p]$$ MA

2. $$\int_0^{2\pi}\left[\operatorname{ce}_{2n}(z,q)\right]^2dz=2\pi\left[A_0^{(2n)}\right]^2+\pi\sum_{r=1}^\infty\left[A_{2r}^{(2n)}\right]^2=\pi$$ MA

3. $$\int_0^{2\pi}\left[\operatorname{ce}_{2n+1}(z,q)\right]^2dz=\pi\sum_{r=0}^\infty\left[A_{2r+1}^{(2n+1)}\right]^2=\pi$$ MA

4. $$\int_0^{2\pi}\operatorname{se}_m(z,q)\operatorname{se}_p(z,q)\,dz=0 \qquad [m\neq p]$$ MA

5. $$\int_0^{2\pi}\left[\operatorname{se}_{2n+1}(z,q)\right]^2dz=\pi\sum_{r=0}^\infty\left[B_{2r+1}^{(2n+1)}\right]^2=\pi$$ MA

6. $$\int_0^{2\pi}\left[\operatorname{se}_{2n+2}(z,q)\right]^2dz=\pi\sum_{r=0}^\infty\left[B_{2r+2}^{(2n+2)}\right]^2=\pi$$ MA

7. $$\int_0^{2\pi}\operatorname{se}_m(z,q)\operatorname{ce}_p(z,q)\,dz=0 \qquad [m=1,2,\ldots;\quad p=1,2,\ldots]$$ MA

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.921

1. $$\int_0^\pi\cosh\left(2k\cos u\sinh z\right)\operatorname{ce}_{2n}(u,q)\,du=\frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{\pi}{2},q\right)}(-1)^n\operatorname{Ce}_{2n}(z,-q)$$
$[q>0]$ MA

2. $$\int_0^\pi \cosh(2k\sin u\cosh z)\,\mathrm{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\mathrm{ce}_{2n}(0,q)}(-1)^n\,\mathrm{Ce}_{2n}(z,-q)$$
$[q>0]$ MA

3. $$\int_0^\pi \sinh(2k\sin u\cosh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0,q)}(-1)^n\,\mathrm{Ce}_{2n+1}(z,-q)$$
$[q>0]$ MA

4. $$\int_0^\pi \sinh(2k\cos u\sinh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi k A_1^{(2n+1)}}{\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}(-1)^{n+1}\,\mathrm{Se}_{2n+1}(z,-q)$$
$[q>0]$ MA

5. $$\int_0^\pi \sinh(2k\sin u\sin z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0,q)}\,\mathrm{se}_{2n+1}(z,q)$$
$[q>0]$ MA

6.922

1. $$\int_0^\pi \cos u\cosh z\cos(2k\sin u\sinh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi A_1^{(2n+1)}}{2\,\mathrm{ce}_{2n+1}(0,q)}\,\mathrm{Ce}_{2n+1}(z,q)$$
$[q>0]$ MA

2. $$\int_0^\pi \sin u\sinh z\cos(2k\cos u\cosh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi {B_1}^{(2n+1)}}{2\,\mathrm{se}_{2n+1}\left(\dfrac{\pi}{2},q\right)}\,\mathrm{Se}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^\pi \sin u\sinh z\sin(2k\cos u\cosh z)\,\mathrm{se}_{2n+2}(u,q)\,du = -\frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}\,\mathrm{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

4. $$\int_0^\pi \cos u\cosh z\sin(2k\sin u\sinh z)\,\mathrm{se}_{2n+2}(u,q)\,du = \frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}(0,q)}\,\mathrm{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

5. $$\int_0^\pi \sin u\cosh z\cosh(2k\cos u\sinh z)\,\mathrm{se}_{2n+1}(u,q)\,du = \frac{\pi {B_1}^{(2n+1)}}{2\,\mathrm{se}_{2n+1}\left(\dfrac{\pi}{2},q\right)}(-1)^n\,\mathrm{Ce}_{2n+1}(z,-q)$$
$[q>0]$ MA

6. $$\int_0^\pi \cos u\sinh z\cosh(2k\sin u\cosh z)\,\mathrm{ce}_{2n+1}(u,q)\,du = \frac{\pi A_1^{(2n+1)}}{2\,\mathrm{ce}_{2n+1}(0,q)}(-1)^n\,\mathrm{Se}_{2n+1}(z,-q)$$
$[q>0]$ MA

7. $$\int_0^\pi \sin u\cosh z\sinh(2k\cos u\sinh z)\,\mathrm{se}_{2n+2}(u,q)\,du = \frac{\pi k B_2^{(2n+2)}}{2\,\mathrm{se}'_{2n+2}\left(\dfrac{\pi}{2},q\right)}(-1)^{n+1}\,\mathrm{Se}_{2n+2}(z,-q)$$
$[q>0]$ MA

8. $$\int_0^{\pi} \cos u \sinh z \sinh\left(2k \sin u \cosh z\right) \operatorname{se}_{2n+2}(u,q)\,du = \frac{\pi k B_2^{(2n+2)}}{2\operatorname{se}'_{2n+2}(0,q)}(-1)^n \operatorname{Se}_{2n+2}(z,-q)$$
$[q>0]$ MA

6.923

1. $$\int_0^{\infty} \sin\left(2k\cosh z\cosh u\right)\sinh z\sinh u \operatorname{Se}_{2n+1}(u,q)\,du = -\frac{\pi {B_1}^{(2n+1)}}{4\operatorname{se}_{2n+1}\left(\frac{1}{2}\pi,q\right)}\operatorname{Se}_{2n+1}(z,q)$$
$[q>0]$ MA

2. $$\int_0^{\infty} \cos\left(2k\cosh z\cosh u\right)\sinh z\sinh u \operatorname{Se}_{2n+1}(u,q)\,du = -\frac{\pi {B_1}^{(2n+1)}}{4\operatorname{se}_{2n+1}\left(\frac{1}{2}\pi,q\right)}\operatorname{Gey}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^{\infty} \sin\left(2k\cosh z\cosh u\right)\sinh z\sinh u \operatorname{Se}_{2n+2}(u,q)\,du = -\frac{k\pi {B_2}^{(2n+2)}}{4\operatorname{se}'_{2n+2}\left(\frac{1}{2}\pi,q\right)}\operatorname{Gey}_{2n+2}(z,q)$$
$[q>0]$ MA

4. $$\int_0^{\infty} \cos\left(2k\cosh z\cosh u\right)\sinh z\sinh u \operatorname{Se}_{2n+2}(u,q)\,du = -\frac{k\pi {B_2}^{(2n+2)}}{4\operatorname{se}_{2n+2}\left(\frac{1}{2}\pi,q\right)}\operatorname{Se}_{2n+2}(z,q)$$
$[q>0]$ MA

5. $$\int_0^{\infty} \sin\left(2k\cosh z\cosh u\right)\operatorname{Ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{2\operatorname{ce}_{2n}\left(\frac{1}{2}\pi,q\right)}\operatorname{Ce}_{2n}(z,q)$$
$[q>0]$ MA

6. $$\int_0^{\infty} \cos\left(2k\cosh z\cosh u\right)\operatorname{Ce}_{2n}(u,q)\,du = -\frac{\pi A_0^{(2n)}}{2\operatorname{ce}_{2n}\left(\frac{1}{2}\pi,q\right)}\operatorname{Fey}_{2n}(z,q)$$
$[q>0]$ MA

7. $$\int_0^{\infty} \sin\left(2k\cosh z\cosh u\right)\operatorname{Ce}_{2n+1}(u,q)\,du = \frac{k\pi A_1^{(2n+1)}}{2\operatorname{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)}\operatorname{Fey}_{2n+1}(z,q)$$
$[q>0]$ MA

8. $$\int_0^{\infty} \cos\left(2k\cosh z\cosh u\right)\operatorname{Ce}_{2n+1}(u,q)\,du = \frac{k\pi A_1^{(2n+1)}}{2\operatorname{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)}\operatorname{Ce}_{2n+1}(z,q)$$
$[q>0]$ MA

6.924

1. $$\int_0^{\pi} \cos\left(2k\cos u\cos z\right)\operatorname{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{1}{2}\pi,q\right)}\operatorname{ce}_{2n}(z,q)$$
$[q>0]$ MA

2. $$\int_0^{\pi} \sin\left(2k\cos u\cos z\right)\operatorname{ce}_{2n+1}(u,q)\,du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)}\operatorname{ce}_{2n+1}(z,q)$$
$[q>0]$ MA

3. $$\int_0^{\pi} \cos\left(2k\cos u\cosh z\right) \operatorname{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \operatorname{Ce}_{2n}(z,q)$$

$[q > 0]$ MA

4. $$\int_0^{\pi} \cos\left(2k\sin u\sinh z\right) \operatorname{ce}_{2n}(u,q)\,du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0,q)} \operatorname{Ce}_{2n}(z,q)$$

$[q > 0]$ MA

5. $$\int_0^{\pi} \sin\left(2k\cos u\cosh z\right) \operatorname{ce}_{2n+1}(u,q)\,du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}\left(\frac{1}{2}\pi, q\right)} \operatorname{Ce}_{2n+1}(z,q)$$

$[q > 0]$ MA

6. $$\int_0^{\pi} \sin\left(2k\sin u\sinh z\right) \operatorname{se}_{2n+1}(u,q)\,du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}''_{2n+1}(0,q)} \operatorname{Se}_{2n+1}(z,q)$$

$[q > 0]$ MA

6.925 **Notation**: $z_1 = 2k\sqrt{\cosh^2\xi - \sin^2\eta}$, and $\tan\alpha = \tanh\xi\tan\eta$

1. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \operatorname{ce}_{2n}(\theta,q)\,d\theta = 0.$$ MA

2. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right] \operatorname{ce}_{2n}(\theta,q)\,d\theta = \frac{2\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0,q)\operatorname{ce}_{2n}\left(\frac{1}{2}\pi,q\right)} \operatorname{Ce}_{2n}(\xi,q)\operatorname{ce}_{2n}(\eta,q)$$ MA

3. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \operatorname{ce}_{2n+1}(\theta,q)\,d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}(0,q)\operatorname{ce}'_{2n+1}\left(\frac{1}{2}\pi,q\right)} \operatorname{Ce}_{2n+1}(\xi,q)\operatorname{ce}_{2n+1}(\eta,q)$$ MA

4. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right] \operatorname{ce}_{2n+1}(\theta,q)\,d\theta = 0$$ MA

5. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \operatorname{se}_{2n+1}(\theta,q)\,d\theta = \frac{2\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}(0,q)\operatorname{se}_{2n+1}\left(\frac{1}{2}\pi,q\right)} \operatorname{Se}_{2n+1}(\xi,q)\operatorname{se}_{2n+1}(\eta,q)$$ MA

6. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right] \operatorname{se}_{2n+1}(\theta,q)\,d\theta = 0$$ MA

7. $$\int_0^{2\pi} \sin\left[z_1\cos(\theta-\alpha)\right] \operatorname{se}_{2n+2}(\theta,q)\,d\theta = 0$$ MA

8. $$\int_0^{2\pi} \cos\left[z_1\cos(\theta-\alpha)\right] \operatorname{se}_{2n+2}(\theta,q)\,d\theta = \frac{2\pi k^2 {B_2}^{(2n+2)}}{\operatorname{se}'_{2n+2}(0,q)\operatorname{se}'_{2n+2}\left(\frac{1}{2}\pi,q\right)} \operatorname{Se}_{2n+2}(\xi,q)\operatorname{se}_{2n+2}(\eta,q)$$ MA

6.926 $$\int_0^{\pi} \sin u\sin z\sin\left(2k\cos u\cos z\right) \operatorname{se}_{2n+2}(u,q)\,du = -\frac{\pi k B_2^{(2n+2)}}{2\operatorname{se}'_{2n+2}\left(\frac{\pi}{2},q\right)} \operatorname{se}_{2n+2}(z,q)$$

$[q > 0]$ MA

6.93 Combinations of Mathieu and Bessel functions

6.931

1. $$\int_0^{\pi} J_0\left\{k\left[2\left(\cos 2u+\cos 2z\right)\right]^{1/2}\right\} \operatorname{ce}_{2n}(u,q)\,du = \frac{\pi\left[A_0^{(2n)}\right]^2}{\operatorname{ce}_{2n}(0,q)\operatorname{ce}_{2n}\left(\frac{\pi}{2},q\right)} \operatorname{ce}_{2n}(z,q)$$ MA

2. $$\int_0^{2\pi} Y_0\left\{k\left[2\left(\cos 2u+\cosh 2z\right)\right]^{1/2}\right\} \operatorname{ce}_{2n}(u,q)\,du = \frac{2\pi\left[A_0^{(2n)}\right]^2}{\operatorname{ce}_{2n}(0,q)\operatorname{ce}_{2n}\left(\frac{\pi}{2},q\right)} \operatorname{Fey}_{2n}(z,q)$$ MA

6.94 Relationships between eigenfunctions of the Helmholtz equation in different coordinate systems

Notation: Particular solutions of the Helmholtz equation in three-dimensional infinite space

$$\nabla^2\,\Psi + k^2\Psi = 0$$

in Cartesian (x,y,z), spherical (r,θ,ϕ), and cylindrical (ρ,z,ϕ) coordinates are

$$\Psi_{k_xk_yk_z}(x,y,z) \propto e^{i(k_xx+k_yy+k_zz)} \qquad \text{with} \qquad k^2 = k_x^2+k_y^2+k_z^2$$

$$\Psi_{lm}(r,\theta,\phi) \propto e^{im\phi}\sqrt{\frac{k}{r}}\, Z_{l+1/2}(kr)\, P_l^m(\cos\theta)$$

$$\Psi_{mk_z}(\rho,z,\phi) \propto e^{i(m\phi+k_zz)}\, Z_{l+1/2}\left(\rho\sqrt{k^2-k_z^2}\right)$$

with $P_l^m(\cos\theta)$ the associated Legendre function, Z is any Bessel function, $m = 0,1,\ldots,l$; $l \in \mathbb{N}$, $r^2 = \rho^2+z^2$, $\rho = r\sin\theta$, $z = r\cos\theta$, $\phi = \operatorname{arccot}(x/y)$, and $k_t^2 = k^2-k_z^2$.

6.941

1. $$\int_{-k}^{k} e^{i\rho z}\, J_m\left(\rho\sqrt{k^2-\rho^2}\right) P_l^m\left(\frac{p}{k}\right) dp = i^{l-m}\sqrt{\frac{2\pi k}{r}}\, J_{l+1/2}(kr)\, P_l^m\left(\frac{z}{r}\right)$$

$[\rho > 0, \quad l \ge m \ge 0]$

2. $$\int_{-\infty}^{\infty} e^{-i\rho z}\, J_{l+1/2}(kr)\, P_l^m\left(\frac{z}{r}\right) dz = i^{m-l}\sqrt{\frac{2\pi r}{k}}\, J_m\left(\rho\sqrt{k^2-\rho^2}\right) P_l^m\left(\frac{\rho}{k}\right)$$

$[\rho > 0, \quad l \ge m \ge 0]$

3. $$\int_0^{\infty} J_m\left(\rho k_t\right)\cos\left[k_xx + m\arcsin\left(\frac{x}{\rho}\right)\right] dx$$
$$= \frac{(-1)^m}{\sqrt{k_t^2-k_x^2}}\cos\left[y\sqrt{k_t^2-k_x^2} + m\arccos\left(\frac{k_x}{k_t}\right)\right] \qquad \left[k_x^2 < k_t^2\right]$$
$$= 0 \qquad \left[k_x^2 > k_t^2\right]$$

4. $$\int_0^\infty Y_m(\rho k_t)\cos\left[k_x x + m\arcsin\left(\frac{x}{\rho}\right)\right]dx$$
$$= \frac{(-1)^m}{\sqrt{k_t^2-k_x^2}}\sin\left[y\sqrt{k_t^2-k_x^2} + m\arccos\left(\frac{k_x}{k_t}\right)\right] \qquad [k_x^2 < k_t^2]$$
$$= \frac{(-1)^m}{\sqrt{k_x^2-k_t^2}}\exp\left[-y\sqrt{k_x^2-k_t^2} - m\,\mathrm{sign}\,(k_x)\,\mathrm{arccosh}\left(\frac{|k_x|}{k_t}\right)\right] \qquad [k_x^2 > k_t^2]$$

5. $$\int_{-\infty}^\infty H^{(j)}_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dx = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,H^{(j)}_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)$$
$$[\rho > 0]$$
The result is true for $j=1$ if $\pi > \arg\sqrt{k^2-k_z^2} \geq 0$, for $j=2$ if $-\pi < \arg\sqrt{k^2-k_z^2} \leq 0$.

6. $$\int_{-\infty}^\infty H^{(j)}_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z = i^{l-m}\sqrt{\frac{2\pi k}{r}}\,H^{(j)}_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$
The result is true for $j=1$ if $\pi > \arg\sqrt{k^2-k_z^2} \geq 0$, for $j=2$ if $-\pi < \arg\sqrt{k^2-k_z^2} \leq 0$.

7. $$\int_{-\infty}^\infty J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dz = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,J_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2 < k^2]$$
$$= 0 \qquad [k_z^2 > k^2]$$

8. $$\int_{-k}^k J_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z = i^{l-m}\sqrt{\frac{2\pi k}{r}}\,J_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$

9. $$\int_{-\infty}^\infty Y_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)e^{-ik_z z}\,dz = i^{m-l}\sqrt{\frac{2\pi r}{k}}\,Y_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2 < k^2]$$
$$= -2i^{m-l}\sqrt{\frac{2r}{k\pi}}\,K_m\left(\rho\sqrt{k_z^2-k^2}\right)P_l^m\left(\frac{k_z}{k}\right) \qquad [k_z^2 > k^2]$$

10. $$i^{l-m}\int_{-k}^k Y_m\left(\rho\sqrt{k^2-k_z^2}\right)P_l^m\left(\frac{k_z}{k}\right)e^{ik_z z}\,dk_z$$
$$-\frac{4}{\pi}\int_k^\infty \cos\left[k_z z + \tfrac{1}{2}\pi(m-l)\right]P_l^m\left(\frac{k_z}{k}\right)K_m\left(\rho\sqrt{k_z^2-k^2}\right)e^{ik_z z}\,dk_z$$
$$= \sqrt{\frac{2\pi k}{r}}\,Y_{l+1/2}(kr)\,P_l^m\left(\frac{z}{r}\right)$$

7.1–7.2 Associated Legendre Functions

7.11 Associated Legendre functions

7.111 $\int_{\cos\varphi}^{1} P_\nu(x)\,dx = \sin\varphi\, P_\nu^{-1}(\cos\varphi)$ MO 90

7.112

1. $$\int_{-1}^{1} P_n^m(x)\,P_k^m(x)\,dx = 0 \qquad [n \neq k]$$
$$= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \qquad [n = k]$$
SM III 185, WH

2. $$\int_{-1}^{1} Q_n^m(x)\,P_k^m(x)\,dx = (-1)^m \frac{1-(-1)^{n+k}(n+m)!}{(k-n)(k+n+1)(n-m)!}$$ EH I 171(18)

3. $$\int_{-1}^{1} P_\nu(x)\,P_\sigma(x)\,dx$$
$$= \frac{2\pi\sin\pi(\sigma-\nu) + 4\sin(\pi\nu)\sin(\pi\sigma)\left[\psi(\nu+1)-\psi(\sigma+1)\right]}{\pi^2(\sigma-\nu)\,(\sigma+\nu+1)} \qquad [\sigma+\nu+1 \neq 0]$$ EH I 170(7)
$$= \frac{\pi^2 - 2\left(\sin\pi\nu\right)^2\psi'(\nu+1)}{\pi^2\left(\nu+\frac{1}{2}\right)} \qquad [\sigma = \nu]$$ EH I 170(9)a

4. $$\int_{-1}^{1} Q_\nu(x)\,Q_\sigma(x)\,dx = \frac{\left[\psi(\nu+1)-\psi(\sigma+1)\right]\left[1+\cos(\pi\sigma)\cos(\nu\pi)\right] - \frac{\pi}{2}\sin\pi(\nu-\sigma)}{(\sigma-\nu)\,(\sigma+\nu+1)}$$
$[\sigma+\nu+1 \neq 0; \quad \nu, \quad \sigma \neq -1,-2,-3,\dots]$
EH I 170(11)
$$= \frac{\frac{1}{2}\pi^2 - \psi'(\nu+1)\left[1+\left(\cos\nu\pi\right)^2\right]}{2\nu+1}$$
$[\nu = \sigma, \quad \nu \neq -1,-2,-3,\dots]$
EH I 170(12)

5. $$\int_{-1}^{1} P_\nu(x)\,Q_\sigma(x)\,dx = \frac{1-\cos\pi(\sigma-\nu) - 2\pi^{-1}\sin(\pi\nu)\cos(\pi\sigma)\left[\psi(\nu+1)-\psi(\sigma+1)\right]}{(\nu-\sigma)\,(\nu+\sigma+1)}$$
$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0, \quad \sigma \neq \nu]$
EH I 170(13)
$$= -\frac{\sin(2\nu\pi)\,\psi'(\nu+1)}{\pi(2\nu+1)}$$
$[\operatorname{Re}\nu > 0, \quad \sigma = \nu]$
EH I 171(14)

7.113 **Notation**: $A = \dfrac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}$

1. $$\int_{0}^{1} P_\nu(x)\,P_\sigma(x)\,dx = \frac{A\sin\frac{\pi\sigma}{2}\cos\frac{\pi\nu}{2} - A^{-1}\sin\frac{\pi\nu}{2}\cos\frac{\pi\sigma}{2}}{\frac{1}{2}\pi(\sigma-\nu)(\sigma+\nu+1)}$$ EH I 171(15)

2. $$\int_0^1 Q_\nu(x)\, Q_\sigma(x)\, dx = \frac{\psi(\nu+1) - \psi(\sigma+1) - \frac{\pi}{2}\left[\left(A - A^{-1}\right)\sin\frac{\pi(\sigma+\nu)}{2}\left(A + A^{-1}\right)\sin\frac{\pi(\sigma-\nu)}{2}\right]}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0]$ EH I 171(16)

3. $$\int_0^1 P_\nu(x)\, Q_\sigma(x)\, dx = \frac{A^{-1}\cos\frac{\pi(\nu-\sigma)}{2} - 1}{(\sigma-\nu)(\sigma+\nu+1)}$$ $[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0]$ EH I 171(17)

7.114

1. $$\int_1^\infty P_\nu(x)\, Q_\sigma(x)\, dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)}$$ $[\operatorname{Re}(\sigma-\nu) > 0, \quad \operatorname{Re}(\sigma+\nu) > -1]$
ET II 324(19)

2. $$\int_1^\infty Q_\nu(x)\, Q_\sigma(x)\, dx = \frac{\psi(\sigma+1) - \psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)}$$
$[\operatorname{Re}(\nu+\sigma) > -1; \quad \sigma, \nu \neq -1, -2, -3, \ldots]$ EH I 170(5)

3. $$\int_1^\infty \left[Q_\nu(x)\right]^2 dx = \frac{\psi'(\nu+1)}{2\nu+1}$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ EH I 170(6)

7.115 $$\int_1^\infty Q_\nu(x)\, dx = \frac{1}{\nu(\nu+1)}$$ $[\operatorname{Re}\nu > 0]$ ET II 324(18)

7.12–7.13 Combinations of associated Legendre functions and powers

7.121 $$\int_{\cos\varphi}^1 x\, P_\nu(x)\, dx = \frac{-\sin\varphi}{(\nu-1)(\nu+2)}\left[\sin\varphi P_\nu\left(\cos\varphi\right) + \cos\varphi\, P_\nu^1\left(\cos\varphi\right)\right]$$ MO 90

7.122

1. $$\int_0^1 \frac{\left[P_n^m(x)\right]^2}{1-x^2}\, dx = \frac{1}{2m}\frac{(n+m)!}{(n-m)!}$$ $[0 < m \leq n]$ MO 74

2. $$\int_0^1 \left[P_\nu^\mu(x)\right]^2 \frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu\,\Gamma(1-\mu+\nu)}$$ $[\operatorname{Re}\mu < 0, \quad \nu+\mu \text{ is a positive integer}]$
EH I 172(26)

3. $$\int_0^1 \left[P_\nu^{n-\nu}(x)\right]^2 \frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu)\,\Gamma(1-n+2\nu)}$$ $[n = 0, 1, 2, \ldots; \quad \operatorname{Re}\nu > n]$
ET II 315(9)

7.123 $$\int_{-1}^1 P_n^m(x)\, P_n^k(x) \frac{dx}{1-x^2} = 0$$ $[0 \leq m \leq n, \quad 0 \leq k \leq n; \quad m \neq k]$
MO 74

7.124 $$\int_{-1}^1 x^k (z-x)^{-1} \left(1-x^2\right)^{\frac{1}{2}m} P_n^m(x)\, dx = (-2)^m \left(z^2-1\right)^{\frac{1}{2}m} Q_n^m(z) \cdot z^k$$
$[m \leq n; k = 0, 1, \ldots, n-m;$
z is in the complex plane with a cut along the interval $(-1, 1)$ on the real axis$]$
ET II 279(26)

7.125 $$\int_{-1}^{1} \left(1-x^2\right)^{\frac{1}{2}m} P_k^m(x)\, P_l^m(x)\, P_n^m(x)\, dx = (-1)^m \pi^{-3/2} \frac{(k+m)!(l+m)!\,(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!} \times \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(t-k+\frac{1}{2}\right)\Gamma\left(t-l+\frac{1}{2}\right)\Gamma\left(t-n+\frac{1}{2}\right)}{(s-l)!(s-n)!\,\Gamma\left(s+\frac{3}{2}\right)}$$

[$2s = k+l+n+m$ and $2t = k+l-n-m$ are both even
$l \geq m, \quad m \leq k-l-m \leq n \leq k+l+m$]
ET II 280(32)

7.126

1. $$\int_0^1 P_\nu(x) x^\sigma\, dx = \frac{\sqrt{\pi} 2^{-\sigma-1}\,\Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)} \qquad [\operatorname{Re}\sigma > -1]$$ EH I 171(23)

2. $$\int_0^1 x^\sigma\, P_\nu^m(x)\, dx = \frac{(-1)^m \pi^{1/2} 2^{-2m-1}\,\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma(1+m+\nu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}m\right)\Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}\right)\Gamma(1-m+\nu)} \times {}_3F_2\left(\frac{m+\nu+1}{2}, \frac{m-\nu}{2}, \frac{m}{2}+1; m+1, \frac{3+\sigma+m}{2}; 1\right)$$
[$\operatorname{Re}\sigma > -1; \quad m = 0, 1, 2, \ldots$] ET II 313(2)

3. $$\int_0^1 x^\sigma\, P_\nu^\mu(x)\, dx = \frac{\pi^{1/2} 2^{2\mu-1}\,\Gamma\left(\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{3+\sigma-\mu}{2}\right)}\ {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}; 1\right)$$
[$\operatorname{Re}\sigma > -1, \quad \operatorname{Re}\mu < 2$] ET II 313(3)

4. $$\int_1^\infty x^{\mu-1}\, Q_\nu(ax)\, dx = e^{\mu\pi i}\,\Gamma(\mu) a^{-\mu} \left(a^2-1\right)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a)$$
[$|\arg(a-1)| < \pi, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}(\nu-\mu) > -1$] ET II 325(26)

7.127 $$\int_{-1}^{1} (1+x)^\sigma\, P_\nu(x)\, dx = \frac{2^{\sigma+1}\left[\Gamma(\sigma+1)\right]^2}{\Gamma(\sigma+\nu+2)\,\Gamma(1+\sigma-\nu)} \qquad [\operatorname{Re}\sigma > -1]$$ ET II 316(15)

7.128

1. $$\int_{-1}^{1} (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}}\, P_\nu^\mu(x)\, dx = -\frac{\Gamma\left(\mu-\frac{1}{2}\right)(z-1)^{\mu-\frac{1}{2}}(z+1)^{-1/2}}{\pi^{1/2} e^{2\mu\pi i}\,\Gamma(\mu+\nu)\,\Gamma(\mu-\nu-1)} \times \left\{ Q_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right] Q_{-\nu-1}^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right] + Q_\nu^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right] Q_{-\nu-1}^{\mu}\left[\left(\frac{1+z}{2}\right)^{1/2}\right] \right\}$$
[$-\frac{1}{2} < \operatorname{Re}\mu < 1$,
z is in the complex plane with a cut along the interval $(-1, 1)$ of the real axis]
ET II 317(20)

2. $$\int_{-1}^{1} (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}}\, P_\nu^\mu(x)\, dx = \frac{2e^{-2\mu\pi i}\,\Gamma\left(\frac{1}{2}+\mu\right)}{\pi^{1/2}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)} (z-1)^\mu\, Q_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right] Q_{-\nu-1}^{\mu}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]$$
[$-\frac{1}{2} < \operatorname{Re}\mu < 1$,
z is in the complex plane with a cut along the interval $(-1, 1)$ of the real axis]
ET II 316(18)

7.129 $$\int_{-1}^{1} P_\nu(x)\, P_\lambda(x)(1+x)^{\lambda+\nu}\, dx = \frac{2^{\lambda+\nu+1}\, [\Gamma(\lambda+\nu+1)]^4}{[\Gamma(\lambda+1)\,\Gamma(\nu+1)]^2\,\Gamma(2\lambda+2\nu+2)}$$
$[\operatorname{Re}(\nu+\lambda+1) > 0]$ EH I 172(30)

7.131

1. $$\int_1^\infty (x-1)^{-\frac{1}{2}\mu}\,(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{1}{2}}\,P_\nu^\mu(x)\,dx = \pi^{1/2}\frac{\Gamma(-\mu-\nu)\,\Gamma(1-\mu+\nu)}{\Gamma\left(\frac{1}{2}-\mu\right)}(z-1)^\mu \left\{P_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]\right\}^2$$
$[\operatorname{Re}(\mu+\nu) < 0, \quad \operatorname{Re}(\mu-\nu) < 1, \quad |\arg(z+1)| < \pi]$ ET II 321(6)

2. $$\int_1^\infty (x-1)^{-\frac{1}{2}\mu}\,(x+1)^{\frac{1}{2}\mu-\frac{1}{2}}\,(z+x)^{\mu-\frac{3}{2}}\,P_\nu^\mu(x)\,dx = \frac{\pi^{1/2}\,\Gamma(1-\mu-\nu)\,\Gamma(2-\mu+\nu)\,(z-1)^{\mu-\frac{1}{2}}\,(z+1)^{-1/2}}{\Gamma\left(\frac{3}{2}-\mu\right)}\,P_\nu^\mu\left[\left(\frac{1+z}{2}\right)^{1/2}\right]P_\nu^{\mu-1}\left[\left(\frac{1+z}{2}\right)^{1/2}\right]$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu+\nu) < 1, \quad \operatorname{Re}(\mu-\nu) < 2, \quad |\arg(1+z)| < \pi]$ ET II 321(7)

7.132

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-1} P_\nu^\mu(x)\,dx = \frac{\pi 2^\mu\,\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{\Gamma\left(\lambda+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(-\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$
$[2\operatorname{Re}\lambda > |\operatorname{Re}\mu|]$ ET II 316(16)

2. $$\int_1^\infty \left(x^2-1\right)^{\lambda-1} P_n^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\lambda-\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\lambda-\frac{1}{2}\mu\right)}$$
$[\operatorname{Re}\lambda > \operatorname{Re}\mu, \quad \operatorname{Re}(1-2\lambda-\nu) > 0, \quad \operatorname{Re}(2-2\lambda+\nu) > 0]$ ET II 320(2)

3.[9] $$\int_1^\infty \left(x^2-1\right)^{\lambda-1} Q_\nu^\mu(x)\,dx = e^{\mu\pi i}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1-\lambda+\frac{1}{2}\nu\right)\Gamma\left(\lambda+\frac{1}{2}\mu\right)\Gamma\left(\lambda-\frac{1}{2}\mu\right)}{2^{2-\mu}\,\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\lambda+\frac{1}{2}\nu\right)}$$
$[|\operatorname{Re}\mu| < 2\operatorname{Re}\lambda < \operatorname{Re}\nu + 2]$
ET II 324(23)

4. $$\int_0^1 x^\sigma \left(1-x^2\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{3}{2}\right)}$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\sigma > -1]$ EH I 172(24)

5. $$\int_0^1 x^\sigma \left(1-x^2\right)^{\frac{1}{2}m} P_\nu^m(x)\,dx = \frac{(-1)^m 2^{-m-1}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)\Gamma\left(1+m+\nu\right)}{\Gamma(1-m+\nu)\,\Gamma\left(1+\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}m+\frac{1}{2}\nu\right)}$$
$[\operatorname{Re}\sigma > -1, \quad m \text{ is a positive integer}]$ EH I 172(25), ET II 313(4)

6. $$\int_0^1 \left(1-x^2\right)^\eta P_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(1+\eta-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)}{\Gamma(1-\mu)\,\Gamma\left(\frac{3}{2}+\eta+\frac{1}{2}\sigma-\frac{1}{2}\mu\right)} \times {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1+\eta-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}+\eta; 1\right)$$
$[\operatorname{Re}\left(\eta-\frac{1}{2}\mu\right) > -1, \operatorname{Re}\sigma > -1]$ ET II 314(6)

7. $$\int_1^\infty x^{-\rho}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{2^{\rho+\mu-2}\,\Gamma\left(\frac{\rho+\mu+\nu}{2}\right)\Gamma\left(\frac{\rho+\mu-\nu-1}{2}\right)}{\sqrt{\pi}\,\Gamma(\rho)}$$
$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\rho+\mu+\nu) > 0, \quad \operatorname{Re}(\rho+\mu-\nu) > 1]$ ET II 320(3)

7.133

1. $$\int_u^\infty Q_\nu(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}\left(u^2-1\right)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(u)$$
$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1+\operatorname{Re}\nu]$ MO 90a

2. $$\int_u^\infty \left(x^2-1\right)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x)(x-u)^{\mu-1}\,dx = \Gamma(\mu)e^{\mu\pi i}\left(u^2-1\right)^{\frac{1}{2}\lambda+\frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(u)$$
$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1+\operatorname{Re}(\nu-\lambda)]$ ET II 204(30)

7.134

1. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{2^{\lambda+\mu}\,\Gamma(\lambda)\,\Gamma(-\lambda-\mu-\nu)\,\Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)\,\Gamma(1-\lambda-\mu)}$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu+\nu) < 0, \quad \operatorname{Re}(\lambda+\mu-\nu) < 1]$ ET II 321(4)

2. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = -\frac{2^{\lambda-\mu}\sin\pi\nu\,\Gamma(\lambda-\mu)\,\Gamma(-\lambda+\mu-\nu)\,\Gamma(1-\lambda+\mu+\nu)}{\pi\,\Gamma(1-\lambda)}$$
$[\operatorname{Re}(\lambda-\mu) > 0, \quad \operatorname{Re}(\mu-\lambda-\nu) > 0, \quad \operatorname{Re}(\mu-\lambda+\nu) > -1]$ ET II 321(5)

7.135

1. $$\int_{-1}^1 \left(1-x^2\right)^{-\frac{1}{2}\mu}(z-x)^{-1} P_{\mu+n}^\mu(x)\,dx = 2e^{-i\mu\pi}\left(z^2-1\right)^{-\frac{1}{2}\mu} Q_{\mu+n}^\mu(z)$$
$[n = 0, 1, 2, \ldots, \quad \operatorname{Re}\mu + n > -1,$ z is in the complex plane with a cut along the interval $(-1, 1)$ of the real axis.] ET II 316(17)

2. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{\mu/2}(x+z)^{-\rho} P_\nu^\mu(x)\,dx$$
$$= \frac{2^{\lambda+\mu-\rho}\,\Gamma(\lambda-\rho)\,\Gamma(\rho-\lambda-\mu-\nu)\,\Gamma(\rho-\lambda-\mu+\nu+1)}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)\,\Gamma(1+\rho-\lambda-\mu)}$$
$$\times\ {}_3F_2\left(\rho, \rho-\lambda-\mu-\nu, \rho-\lambda-\mu+\nu+1; \rho-\lambda+1, \rho-\lambda-\mu+1; \frac{1+z}{2}\right)$$
$$+\frac{\Gamma(\rho-\lambda)\,\Gamma(\lambda)}{\Gamma(\rho)\,\Gamma(1-\mu)}2^\mu(z+1)^{\lambda-\rho}\ {}_3F_2\left(\lambda, -\mu-\nu, 1-\mu+\nu; 1-\mu, 1-\rho+\lambda; \frac{1+z}{2}\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\rho-\lambda-\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda-\mu+\nu+1) > 0, \quad |\arg(z+1)| < \pi]$
ET II 322(9)

3. $$\int_1^\infty (x-1)^{\lambda-1}\left(x^2-1\right)^{-\mu/2}(x+z)^{-\rho}\,P_\nu^\mu(x)\,dx$$
$$= -\frac{\sin(\nu\pi)\,\Gamma(\lambda-\mu-\rho)\,\Gamma\left(\rho-\lambda+\mu-\nu\right)\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\,\Gamma\left(1+\rho-\lambda\right)}$$
$$\times\ {}_3F_2\left(\rho,\rho-\lambda+\mu-\nu,\rho-\lambda+\mu+\nu+1;1+\rho-\lambda,1+\rho-\lambda+\mu;\frac{1+z}{2}\right)$$
$$+\frac{\Gamma(\lambda-\mu)\,\Gamma\left(\rho-\lambda+\mu\right)}{\Gamma(\rho)\,\Gamma(1-\mu)}(z+1)^{\lambda-\rho-\mu}$$
$$\times\ {}_3F_2\left(\lambda-\mu,-\nu,\nu+1;1+\lambda-\mu-\rho,1-\mu;\frac{1+z}{2}\right)$$
$[\operatorname{Re}(\lambda-\mu)>0, \quad \operatorname{Re}\left(\rho-\lambda+\mu-\nu\right)>0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1)>0, \quad |\arg(z+1)|<\pi]$

ET II 322(10)

7.136

1. $$\int_{-1}^1 \left(1-x^2\right)^{\lambda-1}\left(1-a^2x^2\right)^{\mu/2}\,P_\nu(ax)\,dx$$
$$= \frac{\pi 2^\mu\,\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\ {}_2F_1\left(-\frac{\mu+\nu}{2},\frac{1-\mu+\nu}{2};\frac{1}{2}+\lambda;a^2\right)$$
$[\operatorname{Re}\lambda>0, \quad -1<a<1]$ ET II 318(31)

2. $$\int_1^\infty \left(x^2-1\right)^{\lambda-1}\left(a^2x^2-1\right)^{\mu/2}\,P_\nu^\mu(ax)\,dx$$
$$= \frac{\Gamma(\lambda)\,\Gamma\left(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma(1-\lambda-\mu)}$$
$$\times 2^{\mu-1}a^{\mu-\nu-1}\ {}_2F_1\left(\frac{1-\mu+\nu}{2},1-\lambda-\frac{\mu-\nu}{2};1-\lambda-\mu;1-\frac{1}{a^2}\right)$$
$[\operatorname{Re}a>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(\nu-\mu-2\lambda)>-2, \quad \operatorname{Re}(2\lambda+\mu+\nu)<1]$ ET II 325(25)

3. $$\int_1^\infty \left(x^2-1\right)^{\lambda-1}\left(a^2x^2-1\right)^{-\frac{1}{2}\mu}\,Q_\nu^\mu(ax)\,dx = \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma(\lambda)\,\Gamma\left(1-\lambda+\frac{\mu+\nu}{2}\right)2^{\mu-2}e^{\mu\pi i}a^{-\mu-\nu-1}}{\Gamma\left(\nu+\frac{3}{2}\right)}$$
$$\times\ {}_2F_1\left(\frac{\mu+\nu+1}{2},1-\lambda+\frac{\mu+\nu}{2};\nu+\frac{3}{2};a^{-2}\right)$$
$[|\arg(a-1)|<\pi, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(2\lambda-\mu-\nu)<2]$ ET II 325(27)

7.137

1. $$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{-1/2}e^{-\mu\pi i}\,\Gamma\left(\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\left\{Q_\nu^\mu\left[(1+a)^{1/2}\right]\right\}^2$$
$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu<\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu)>-1\right]$ ET II 325(28)

2. $$\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{3}{2}}\left(1+ax\right)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= -\pi^{-1/2}e^{-\mu\pi i}\,\Gamma\left(-\mu-\frac{1}{2}\right)a^{\frac{1}{2}\mu+\frac{1}{2}}\left(1+a^2\right)^{-1/2}Q_\nu^{\mu+1}\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]$$
$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu<-\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu+2)>0\right]$ ET II 326(29)

3. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx = \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\left\{P_\nu^\mu\left[(1+a)^{1/2}\right]\right\}^2$$
$\left[\operatorname{Re}\mu < \tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 319(32)

4. $$\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}\,(1+ax)^{\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(-\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu+\frac{1}{2}}\,P_\nu^{\mu+1}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^2\right]$$
$\left[\operatorname{Re}\mu < -\tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 319(33)

5. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}+\mu\right)a^{-\frac{1}{2}\mu}\,P_\nu^\mu\left[(1+a)^{1/2}\right]P_\nu^{-\mu}\left[(1+a)^{1/2}\right]$$
$\left[\operatorname{Re}\mu > -\tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 319(34)

6. $$\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}\,(1+ax)^{-\frac{1}{2}\mu}\,P_\nu^\mu(1+2ax)\,dx$$
$$= \frac{1}{2}\pi^{1/2}\,\Gamma\left(\mu-\tfrac{1}{2}\right)a^{\frac{1}{2}-\frac{1}{2}\mu}\,(1+a)^{-1/2}\left\{P_\nu^{1-\mu}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^{1/2}\right]\right\}$$
$$+(\mu+\nu)(1-\mu+\nu)\,P_\nu^{-\mu}\left[(1+a)^{1/2}\right]P_\nu^\mu\left[(1+a)^{1/2}\right]$$
$\left[\operatorname{Re}\mu > \tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 319(35)

7. $$\int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= \pi^{1/2}\,\Gamma\left(\tfrac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\,P_\nu^\mu\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]$$
$\left[\operatorname{Re}\mu < \tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 320(38)

8. $$\int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}\,(1+ax)^{\frac{1}{2}\mu}\,Q_\nu^\mu(1+2ax)\,dx$$
$$= \frac{1}{2}\pi^{1/2}\,\Gamma\left(-\mu-\tfrac{1}{2}\right)(1+a)^{-1/2}\,a^{\frac{1}{2}\mu+\frac{1}{2}}$$
$$\times\left\{P_\nu^{\mu+1}\left[(1+a)^{1/2}\right]Q_\nu^\mu\left[(1+a)^{1/2}\right]+P_\nu^\mu\left[(1+a)^{1/2}\right]Q_\nu^{\mu+1}\left[(1+a)^{1/2}\right]\right\}$$
$\left[\operatorname{Re}\mu < -\tfrac{1}{2}, \quad |\arg a| < \pi\right]$ ET II 320(39)

9. $$\int_0^y (y-x)^{\mu-1}\left[x\left(1+\tfrac{1}{2}\gamma x\right)\right]^{-\frac{1}{2}\lambda}P_\nu^\lambda(1+\gamma x)\,dx$$
$$= \Gamma(\mu)\left(\frac{2}{\gamma}\right)^{\frac{1}{2}\mu}\left[y\left(1+\frac{1}{2}\gamma y\right)\right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}P_\nu^{\lambda-\mu}(1+\gamma y)$$
$[\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0, \quad |\arg\gamma y| < \pi]$ ET II 193(52)

10. $$\int_0^y (y-x)^{\mu-1}x^{\sigma+\frac{1}{2}\lambda-1}\left(1+\tfrac{1}{2}\gamma x\right)^{-\frac{1}{2}\lambda}P_\nu^\lambda(1+\gamma x)\,dx$$
$$= \frac{\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}\lambda}\Gamma(\sigma)\,\Gamma(\mu)y^{\sigma+\mu-1}}{\Gamma(1-\lambda)\,\Gamma(\sigma+\mu)}\,{}_3F_2\left(-\nu,1+\nu,\sigma;1-\lambda,\sigma+\mu;-\frac{1}{2}\gamma y\right)$$
$[\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad |\gamma y| < 1]$ ET II 193(53)

11. $$\int_0^y (y-x)^{\mu-1}[x(1-x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x)\,dx = \Gamma(\mu)[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1-2y)$$
$[\operatorname{Re}\lambda < 1, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1]$
ET II 193(54)

12. $$\int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1}(1-x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x)\,dx = \frac{\Gamma(\mu)\,\Gamma(\sigma) y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\,\Gamma(1-\lambda)}\, {}_3F_2(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; y)$$
$[\operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \quad 0 < y < 1]$ ET II 193(155)

7.138 $$\int_0^\infty (a+x)^{-\mu-\nu-2} P_\mu\left(\frac{a-x}{a+x}\right) P_\nu\left(\frac{a-x}{a+x}\right) dx = \frac{a^{-\mu-\nu-1}\left[\Gamma(\mu+\nu+1)\right]^4}{\left[\Gamma(\mu+1)\,\Gamma(\nu+1)\right]^2 \Gamma(2\mu+2\nu+2)}$$
$[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\nu) > -1]$
ET II 326(3)

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

1. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = \frac{a^{-\lambda-\mu}e^{-a}}{\Gamma(1-\mu+\nu)\,\Gamma(-\mu-\nu)}\, G_{23}^{31}\left(2a\,\middle|\,\begin{matrix}1+\mu, 1\\ \lambda+\mu, -\nu, 1+\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0]$ ET II 323(13)

2. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{\frac{1}{2}\mu} Q_\nu^\mu(x)\,dx = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\,\Gamma(\nu-\mu+1)} a^{-\lambda-\mu}e^{-a}\, G_{23}^{22}\left(2a\,\middle|\,\begin{matrix}1+\mu, 1\\ \lambda+\mu, \nu+1, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0]$ ET II 325(24)

3. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = -\pi^{-1}\sin(\nu\pi)a^{\mu-\lambda}e^{-a}\, G_{23}^{31}\left(2a\,\middle|\,\begin{matrix}1, 1-\mu\\ \lambda-\mu, 1+\nu, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda-\mu) > 0]$
ET II 323(15)

4. $$\int_1^\infty e^{-ax}(x-1)^{\lambda-1}\left(x^2-1\right)^{-\frac{1}{2}\mu} Q_\nu^\mu(x)\,dx = \frac{1}{2}e^{\mu\pi i}a^{\mu-\lambda}e^{-a}\, G_{23}^{22}\left(2a\,\middle|\,\begin{matrix}1-\mu, 1\\ \lambda-\mu, \nu+1, -\nu\end{matrix}\right)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu) > 0]$
ET II 323(14)

5. $$\int_1^\infty e^{-ax}\left(x^2-1\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\,dx = 2^{1/2}\pi^{-1/2}a^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(a)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\mu < 1]$
ET II 323(11), MO 90

7.142 $$\int_1^\infty e^{-\frac{1}{2}ax}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\,dx = \frac{2}{a}\, W_{\mu,\nu}(a) \qquad \left[\operatorname{Re}\mu < 1, \quad \nu - \tfrac{1}{2} \neq 0, \pm 1, \pm 2, \dots\right]$$
BU 79(34), MO 118

7.143

1. $$\int_0^\infty [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x)\,dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta} K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right)$$

$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 179(1)

2. $$\int_0^\infty \left(1+\frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x} P_\nu^\mu(1+2x)\,dx = \frac{e^{\frac{1}{2}\beta}}{\beta} W_{\mu,\nu+\frac{1}{2}}(\beta)$$

$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 179(2)

7.144

1. $$\int_0^\infty e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x)\,dx$$
$$= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left\{ \frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu}\sin(\mu\pi)} E\left(-\nu,\nu+1,\lambda+\mu;\mu+1:2\beta\right) - \frac{\sin\left[(\mu+\nu)\pi\right]}{2^{1-\mu}\beta^\lambda \sin(\mu\pi)} E\left(\nu-\mu+1,-\nu-\mu,\lambda:1-\mu:2\beta\right) \right\}$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0]$ ET I 181(16)

2. $$\int_0^\infty e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1}(x+2)^{\frac{1}{2}\mu} Q_\nu^\mu(1+x)\,dx = -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu}\sin(\mu\pi)} E(-\nu,\nu+1,\lambda-\mu:1-\mu:2\beta) - \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu}\beta^\lambda\sin(\mu\pi)} E(\mu+\nu+1,\mu-\nu,\lambda:1+\mu:2\beta)$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}(\lambda-\mu)] > 0$ ET I 181(17)

7.145

1. $$\int_0^\infty \frac{e^{-\beta x}}{1+x} P_\nu\left[\frac{1}{(1+x)^2}-1\right] dx = \frac{e^\beta}{\beta} W_{\nu+\frac{1}{2},0}(\beta)\, W_{-\nu-\frac{1}{2},0}(\beta)$$

$[\operatorname{Re}\beta > 0]$ ET I 180(6)

2. $$\int_0^\infty x^{-1}e^{-\beta x} Q_{-\frac{1}{2}}\left(1+2x^{-2}\right) dx = \frac{\pi^2}{8}\left\{\left[J_0\left(\frac{1}{2}\beta\right)\right]^2 + \left[Y_0\left(\frac{1}{2}\beta\right)\right]^2\right\}$$

$[\operatorname{Re}\beta > 0]$ ET II 327(5)

3. $$\int_0^\infty x^{-1}e^{-ax} Q_\nu\left(1+2x^{-2}\right) dx = \frac{1}{2}\left[\Gamma(\nu+1)\right]^2 a^{-1}\, W_{-\nu-\frac{1}{2},0}(ai)\, W_{-\nu-\frac{1}{2},0}(-ai)$$

$[\operatorname{Re}a > 0, \quad \operatorname{Re}\nu > -1]$ ET II 327(6)

7.146

1. $$\int_0^\infty x^{-\frac{1}{2}\mu}e^{-\beta x} P_\nu^\mu\left(\sqrt{1+x}\right) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{5}{4}} e^{\frac{\beta}{2}}\, W_{\frac{1}{2}\mu+\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(\beta)$$

$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}\beta > 0]$ ET I 180(7)

2. $$\int_0^\infty x^{-\frac{1}{2}\mu}\frac{e^{-\beta x}}{\sqrt{1+x}} P_\nu^\mu\left(\sqrt{1+x}\right) dx = 2^\mu \beta^{\frac{1}{2}\mu-\frac{3}{4}} e^{\frac{\beta}{2}}\, W_{\frac{1}{2}\mu+\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(\beta)$$

$[\operatorname{Re}\mu < 1, \operatorname{Re}\beta > 0]$ ET I 180(8)a

3. $$\int_0^\infty \sqrt{x} e^{-\beta x} P_\nu^{1/4}\left(\sqrt{1+x^2}\right) P_\nu^{-1/4}\left(\sqrt{1+x^2}\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2\beta}} H^{(1)}_{\nu+\frac{1}{2}}\left(\frac{1}{2}\beta\right) H^{(2)}_{\nu+\frac{1}{2}}\left(\frac{1}{2}\beta\right)$$
$[\operatorname{Re}\beta > 0]$ ET I 180(9)

7.147 $$\int_0^\infty x^{\lambda-1}\left(x^2+a^2\right)^{\frac{1}{2}\nu} e^{-\beta x} P_\nu^\mu\left[\frac{x}{\left(x^2+a^2\right)^{1/2}}\right] dx = \frac{2^{-\nu-2}a^{\lambda+\nu}}{\pi\,\Gamma(-\mu-\nu)} G^{32}_{24}\left(\frac{a^2\beta^2}{4}\middle| \begin{matrix} 1-\frac{\lambda}{2}, \frac{1-\lambda}{2} \\ 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2}\end{matrix}\right)$$
$[a > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\lambda > 0]$ ET II 327(7)

7.148 $$\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu}(1+x)^{\frac{1}{2}\mu+\nu-1}\exp\left(-\frac{1-x}{1+x}y\right) P_\nu^\mu(x)\,dx = 2^\nu y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2},\frac{1}{2}\mu}(y)$$
$[\operatorname{Re} y > 0]$ ET II 317(21)

7.149 $$\int_1^\infty \left(\alpha^2+\beta^2+2\alpha\beta x\right)^{-1/2}\exp\left[-\left(\alpha^2+\beta^2+2\alpha\beta x\right)^{1/2}\right] P_\nu(x)\,dx = 2\pi^{-1}(\alpha\beta)^{-1/2} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0]$ ET II 323(16)

7.15 Combinations of associated Legendre and hyperbolic functions

7.151

1. $$\int_0^\infty (\sinh x)^{\alpha-1} P_\nu^{-\mu}(\cosh x)\,dx = \frac{2^{-1-\mu}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\alpha+1\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1+\frac{1}{2}\mu-\frac{1}{2}\alpha\right)}$$
$[\operatorname{Re}(\alpha+\mu) > 0, \quad \operatorname{Re}(\nu-\alpha+2) > 0, \quad \operatorname{Re}(1-\alpha-\nu) > 0]$ EH I 172(28)

2. $$\int_0^\infty (\sinh x)^{\alpha-1} Q_\nu^\mu(\cosh x)\,dx = \frac{e^{i\mu\pi}2^{\mu-\alpha}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\alpha\right)}{\Gamma\left(1+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha\right)} \times\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\mu\right)$$
$[\operatorname{Re}(\alpha\pm\mu) > 0, \quad \operatorname{Re}(\nu-\alpha+2) > 0]$ EH I 172(29)

7.152 $$\int_0^\infty e^{-\alpha x}\sinh^{2\mu}\left(\tfrac{1}{2}x\right) P_{2n}^{-2\mu}\left[\cosh\left(\tfrac{1}{2}x\right)\right] dx = \frac{\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\alpha-n-\mu)\,\Gamma\left(\alpha+n-\mu+\frac{1}{2}\right)}{4^\mu\sqrt{\pi}\,\Gamma(\alpha+n+\mu+1)\,\Gamma\left(\alpha-n+\mu+\frac{1}{2}\right)}$$
$\left[\operatorname{Re}\alpha > n+\operatorname{Re}\mu, \quad \operatorname{Re}\mu > -\frac{1}{4}\right]$
ET I 181(15)

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.161

1. $$\int_0^1 x^{\lambda-1}\left(1-x^2\right)^{-\frac{1}{2}\mu}\sin(ax)\,P_\nu^\mu(x)\,dx = \frac{\pi^{1/2}2^{\mu-\lambda-1}\,\Gamma(\lambda+1)\,a}{\Gamma\left(1+\frac{\lambda-\mu-\nu}{2}\right)\Gamma\left(\frac{3+\lambda-\mu+\nu}{2}\right)} \times {}_2F_3\left(\frac{1+\lambda}{2}, 1+\frac{\lambda}{2}; \frac{3}{2}, 1+\frac{\lambda-\mu-\nu}{2}, \frac{3+\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right)$$
$[\operatorname{Re}\lambda > -1, \quad \operatorname{Re}\mu < 1]$ ET II 314(7)

2. $$\int_0^1 x^{\lambda-1}\left(1-x^2\right)^{-\frac{1}{2}\mu}\cos(ax)\,P_\nu^\mu(x)\,dx = \frac{\pi^{1/2}2^{\mu-\lambda}\,\Gamma(\lambda)}{\Gamma\left(1+\frac{\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)} \times {}_2F_3\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{2}, \frac{1+\lambda-\mu-\nu}{2}, 1+\frac{\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu < 1]$ ET II 314(8)

3. $$\int_0^\infty \left(x^2-1\right)^{\frac{1}{2}\mu}\sin(ax)\,P_\nu^\mu(x)\,dx = \frac{2^\mu\pi^{1/2}a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\,S_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a)$$
$\left[a > 0, \quad \operatorname{Re}\mu < \frac{3}{2}, \quad \operatorname{Re}(\mu+\nu) < 1\right]$ ET II 320(1)

7.162

1. $$\int_a^\infty P_\nu\left(2x^2a^{-2}-1\right)\sin(bx)\,dx = -\frac{\pi a}{4\cos(\nu\pi)}\left\{\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2 - \left[J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]^2\right\}$$
$[a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < 0]$ ET II 326(1)

2. $$\int_a^\infty P_\nu\left(2x^2a^{-2}-1\right)\cos(bx)\,dx = -\frac{\pi}{4}a\left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) - Y_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right)Y_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right)\right]$$
$[a > 0, \quad b > 0, \quad -1 < \operatorname{Re}\nu < 0]$ ET II 326(2)

3. $$\int_0^\infty \left(x^2+2\right)^{-1/2}\sin(ax)\,P_\nu^{-1}\left(x^2+1\right)\,dx = 2^{-1/2}\pi^{-1}a\sin(\nu\pi)\left[K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)\right]^2$$
$[a > 0, \quad -2 < \operatorname{Re}\nu < 1]$ ET I 98(22)

4. $$\int_0^\infty \left(x^2+2\right)^{-1/2}\sin(ax)\,Q_\nu^1\left(x^2+1\right)\,dx = -2^{-3/2}\pi a\,K_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)I_{\nu+\frac{1}{2}}\left(2^{-1/2}a\right)$$
$\left[a > 0, \quad \operatorname{Re}\nu > -\frac{3}{2}\right]$ ET 98(23)

5. $$\int_0^\infty \cos(ax)\, P_\nu\left(1+x^2\right)\, dx = -\frac{\sqrt{2}}{\pi}\sin(\nu\pi)\left[K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right)\right]^2$$
$[a>0, \quad -1<\operatorname{Re}\nu<0]$ ET I 42(23)

6. $$\int_0^\infty \cos(ax)\, Q_\nu\left(1+x^2\right)\, dx = \frac{\pi}{\sqrt{2}}\, K_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right) I_{\nu+\frac{1}{2}}\left(\frac{a}{\sqrt{2}}\right)$$
$[a>0, \quad \operatorname{Re}\nu>-1]$ ET I 42(24)

7. $$\int_0^1 \cos(ax)\, P_\nu\left(2x^2-1\right)\, dx = \frac{\pi}{2}\, J_{\nu+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{a}{2}\right)$$
$[a>0]$ ET I 42(25)

7.163

1. $$\int_a^\infty \left(x^2-a^2\right)^{\frac{1}{2}\nu-\frac{1}{4}} \sin(bx)\, P_0^{\frac{1}{2}-\nu}\left(ax^{-1}\right)\, dx = b^{-\nu-\frac{1}{2}}\cos\left(ab-\frac{\nu\pi}{2}+\frac{\pi}{4}\right)$$
$\left[a>0, \quad |\operatorname{Re}\nu|<\frac{1}{2}\right]$ ET I 98(24)

2. $$\int_0^1 x^{-1}\cos(ax)\, P_\nu\left(2x^{-2}-1\right)\, dx = -\frac{1}{2}\pi\operatorname{cosec}(\nu\pi)\; {}_1F_1\left((\nu+1;1;ai)\right)\; {}_1F_1\left(\nu+1;1;-ai\right)$$
$[a>0, \quad -1<\operatorname{Re}\nu<0]$ ET II 327(4)

7.164

1. $$\int_0^\infty x^{1/2}\sin(bx)\left[P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right)\right]^2 dx = \frac{\sqrt{\frac{2}{\pi}}a^{-1}b^{-1/2}}{\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{5}{4}<\operatorname{Re}\nu<\frac{1}{4}\right]$ ET II 327(8)

2. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right) Q_{\nu-1}^{-1/4}\left(\sqrt{1+a^2x^2}\right) dx$$
$$= \frac{\sqrt{\frac{\pi}{2}}e^{-\frac{1}{4}\pi i}\,\Gamma\left(\nu+\frac{5}{4}\right)}{ab^{\frac{1}{2}}\,\Gamma\left(\nu+\frac{3}{4}\right)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad \operatorname{Re}\nu>-\frac{5}{4}\right]$ ET II 328(9)

3. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right) P_{\nu-1}^{-1/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{5}{4}-\nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{5}{4}<\operatorname{Re}\nu<\frac{5}{4}\right]$ ET II 328(10)

4. $$\int_0^\infty x^{1/2}\sin(bx)\, P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right) P_\nu^{-3/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{7}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$\left[\operatorname{Re}a>0, \quad b>0, \quad -\frac{7}{4}<\operatorname{Re}\nu<\frac{3}{4}\right]$ ET II 328(11)

5. $$\int_0^\infty x^{1/2}\cos(bx)\left[P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)\right]^2 dx = \frac{a^{-1}\left(\frac{\pi b}{2}\right)^{-1/2}}{\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(-\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\tfrac{3}{4} < \operatorname{Re}\nu < -\tfrac{1}{4}\right]$$
ET II 328(12)

6. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)Q_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)dx$$
$$= \frac{\sqrt{\frac{\pi}{2}}e^{\frac{1}{4}\pi i}\,\Gamma\left(\nu+\frac{3}{4}\right)}{ab^{1/2}\,\Gamma\left(\nu+\frac{5}{4}\right)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\nu > -\tfrac{3}{4}\right]$$ ET II 328(13)

7. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{-1/4}\left(\sqrt{1+a^2x^2}\right)P_\nu^{3/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)}\left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)\right]^2$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\tfrac{5}{4} < \operatorname{Re}\nu < \tfrac{1}{4}\right]$$ ET II 328(14)

8. $$\int_0^\infty x^{1/2}\cos(bx)\,P_\nu^{1/4}\left(\sqrt{1+a^2x^2}\right)P_{\nu-1}^{1/4}\left(\sqrt{1+a^2x^2}\right)\frac{dx}{\sqrt{1+a^2x^2}}$$
$$= \frac{a^{-2}b^{1/2}}{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right)K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$$
$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re}\nu| < \tfrac{3}{4}\right]$$ ET II 329(15)

7.165 $$\int_0^\infty \cos(ax)\,P_\nu(\cosh x)\,dx$$
$$= -\frac{\sin(\nu\pi)}{4\pi^2}\Gamma\left(\frac{1+\nu+i\alpha}{2}\right)\Gamma\left(\frac{1+\nu-i\alpha}{2}\right)\Gamma\left(-\frac{\nu+i\alpha}{2}\right)\Gamma\left(-\frac{\nu-i\alpha}{2}\right)$$
$$[a > 0, \quad -1 < \operatorname{Re}\nu < 0]$$ ET II 329(18)

7.166 $$\int_0^\pi P_\nu^{-\mu}(\cos\varphi)\sin^{\alpha-1}\varphi\,d\varphi = \frac{2^{-\mu}\pi\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\alpha-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}$$
$$[\operatorname{Re}(\alpha\pm\mu) > 0]$$ MO 90, EH I 172(27)

7.167 $$\int_0^a P_\nu^{-\mu}(\cos x)\,P_\nu^{-\eta}[\cos(a-x)]\left[\frac{\sin(a-x)}{\sin x}\right]^\eta\frac{dx}{\sin x} = \frac{2^\eta\,\Gamma(\mu-\eta)\,\Gamma\left(\eta+\frac{1}{2}\right)(\sin a)^\eta}{\sqrt{\pi}\,\Gamma(\eta+\mu+1)}P_\nu^{-\mu}(\cos a)$$
$$\left[\operatorname{Re}\mu > \operatorname{Re}\eta > -\tfrac{1}{2}\right]$$ ET II 329(16)

7.17 A combination of an associated Legendre function and the probability integral

7.171 $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}\mu}\exp\left(a^2x^2\right)[1-\Phi(ax)]\,P_\nu^\mu(x)\,dx$$
$$= \pi^{-1}2^{\mu-1}\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)a^{\mu-\frac{3}{2}}e^{\frac{a^2}{2}}\,W_{\frac{1}{4}-\frac{1}{2}\mu,\frac{1}{4}+\frac{1}{2}\nu}\left(a^2\right)$$
$$[\operatorname{Re} a > 0, \quad \operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu+\nu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$$
ET II 324(17)

7.18 Combinations of associated Legendre and Bessel functions

7.181

1. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{1/2}\, Y_\nu(ax)\, dx = 2^{-1/2} a^{-1} \left[\cos\left(\tfrac{1}{2}a\right) J_\nu\left(\tfrac{1}{2}a\right) - \sin\left(\tfrac{1}{2}a\right) Y_\nu\left(\tfrac{1}{2}a\right)\right]$$ $$\left[a > 0, \quad \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ ET II 108(3)a

2. $$\int_1^\infty P_{\nu-\frac{1}{2}}(x) x^{1/2}\, J_\nu(ax)\, dx = -\frac{1}{\sqrt{2}a} \left[\cos\left(\tfrac{1}{2}a\right) Y_\nu\left(\tfrac{1}{2}a\right) + \sin\left(\tfrac{1}{2}a\right) J_\nu\left(\tfrac{1}{2}a\right)\right]$$ $$\left[|\operatorname{Re}\nu| < \tfrac{1}{2}\right]$$ ET II 344(36)a

7.182

1. $$\int_1^\infty x^\nu \left(x^2-1\right)^{\frac{1}{2}\lambda-\frac{1}{2}} P_\lambda^{\lambda-1}(x)\, J_\nu(ax)\, dx = \frac{2^{\lambda+\nu} a^{-\lambda}\, \Gamma\left(\frac{1}{2}+\nu\right)}{\pi^{1/2}\, \Gamma(1-\lambda)} S_{\lambda-\nu,\lambda+\nu}(a)$$ $$\left[a > 0, \quad \operatorname{Re}\nu < \tfrac{5}{2}, \quad \operatorname{Re}(2\lambda+\nu) < \tfrac{3}{2}\right]$$ ET II 345(38)a

2. $$\int_1^\infty x^{\frac{1}{2}-\mu} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, J_\nu(ax)\, dx$$ $$= -2^{-3/2}\pi^{1/2} a^{\mu-\frac{1}{2}} \left[J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) Y_\nu\left(\frac{a}{2}\right) + Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) J_\nu\left(\frac{a}{2}\right)\right]$$ $$\left[-\tfrac{1}{4} < \operatorname{Re}\mu < 1, \quad a > 0, \quad |\operatorname{Re}\nu| < \tfrac{1}{2} + 2\operatorname{Re}\mu\right]$$ ET II 344(37)a

3. $$\int_1^\infty x^{\frac{1}{2}-\mu} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, Y_\nu(ax)\, dx$$ $$= 2^{-3/2}\pi^{1/2} a^{\mu-\frac{1}{2}} \left[J_\nu\left(\frac{a}{2}\right) J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - Y_\nu\left(\frac{a}{2}\right) Y_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$ $$\left[-\tfrac{1}{4} < \operatorname{Re}\mu < 1, \quad a > 0, \quad \operatorname{Re}(2\mu-\nu) > -\tfrac{1}{2}\right]$$ ET II 349(67)a

4. $$\int_0^1 x^{\frac{1}{2}-\mu} \left(1-x^2\right)^{-\frac{1}{2}\mu} P_\nu^\mu(x)\, J_{\nu+\frac{1}{2}}(ax)\, dx = \sqrt{\frac{\pi}{2}} a^{\mu-\frac{1}{2}}\, J_{\frac{1}{2}-\mu}\left(\tfrac{1}{2}a\right) J_{\nu+\frac{1}{2}}\left(\tfrac{1}{2}a\right)$$ $$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}(\mu-\nu) < 2]$$ ET II 337(33)a

5. $$\int_1^\infty x^{\frac{1}{2}-\mu} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, K_\nu(ax)\, dx = (2\pi)^{-1/2} a^{\mu-\frac{1}{2}}\, K_\nu\left(\tfrac{1}{2}a\right) K_{\mu-\frac{1}{2}}\left(\tfrac{1}{2}a\right)$$ $$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0]$$ ET II 135(5)a

6. $$\int_1^\infty x^{\mu+\frac{1}{2}} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, K_\nu(ax)\, dx = \sqrt{\frac{\pi}{2}} a^{-3/2} e^{-\frac{1}{2}a}\, W_{\mu,\nu}(a)$$ $$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0]$$ ET II 135(3)a

7. $$\int_1^\infty x^{\mu-\frac{3}{2}} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x)\, K_\nu(ax)\, dx = \sqrt{\frac{\pi}{2}} a^{-1/2} e^{-\frac{1}{2}a}\, W_{\mu-1,\nu}(a)$$ $$[\operatorname{Re}\mu < 1, \quad \operatorname{Re}a > 0]$$ ET II 135(4)a

8. $$\int_1^\infty x^{\mu-\frac{1}{2}} \left(x^2-1\right)^{-\frac{1}{2}\mu} P_{\nu-\frac{3}{2}}^\mu(x)\, K_\nu(ax)\, dx = \sqrt{\frac{\pi}{2}} a^{-1} e^{-\frac{1}{2}a}\, W_{\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$ $$[\operatorname{Re}\mu < 1]$$ ET II 135(6)a

9. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) K_\nu(ax)\,dx = \pi^{-1/2}a^{-\nu}2^{\nu-1}\left[K_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}a > 0\right]$ ET II 136(11)a

10. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) Y_\nu(ax)\,dx$$
$$= \pi^{1/2}2^{\nu-2}a^{-\nu}\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - Y_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) Y_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu + |2\operatorname{Re}\mu + 1| < \frac{3}{2}\right]$ ET II 108(5)a

11. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\frac{1}{2}-\nu}\left(2x^2-1\right) J_\nu(ax)\,dx$$
$$= -2^{\nu-2}a^{-\nu}\pi^{1/2}\sec(\mu\pi)\left\{\left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 - \left[J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2\right\}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad a > 0, \quad \operatorname{Re}\nu - \frac{3}{2} < 2\operatorname{Re}\mu < \frac{1}{2} - \operatorname{Re}\nu\right]$ ET II 345(39)a

12. $$\int_1^\infty x\left(x^2-1\right)^{-\frac{1}{2}\nu} P_\mu^\nu\left(2x^2-1\right) K_\nu(ax)\,dx = 2^{-\nu}a^{\nu-1} K_{\mu+1}(a)$$
$\left[\operatorname{Re}a > 0, \quad \operatorname{Re}\nu < 1\right]$ ET II 136(10)a

13. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu} P_\mu^\nu\left(1+2x^2a^{-2}\right) K_\nu(xy)\,dx = 2^{-\nu}ay^{-\nu-1} S_{2\nu,2\mu+1}(ay)$$
$\left[\operatorname{Re}a > 0, \quad \operatorname{Re}y > 0, \quad \operatorname{Re}\nu < 1\right]$
ET II 135(7)

14. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu}\left[(\mu-\nu) P_\mu^\nu\left(1+2x^2a^{-2}\right) + (\mu+\nu) P_{-\mu}^\nu\left(1+2x^2a^{-2}\right)\right] K_\nu(xy)\,dx$$
$$= 2^{1-\nu}\mu y^{-\nu-2} S_{2\nu+1,2\mu}(ay)$$
$\left[\operatorname{Re}a > 0, \quad \operatorname{Re}y > 0, \quad \operatorname{Re}\nu < 1\right]$ ET II 136(8)

15. $$\int_0^\infty x\left(x^2+a^2\right)^{\frac{1}{2}\nu-1}\left[P_\mu^\nu\left(1+2x^2a^{-2}\right) + P_{-\mu}^\nu\left(1+2x^2a^{-2}\right)\right] K_\nu(xy)\,dx = 2^{1-\nu}y^{-\nu} S_{2\nu-1,2\mu}(ay)$$
$\left[\operatorname{Re}a > 0, \quad \operatorname{Re}y > 0, \quad \operatorname{Re}\nu < 1\right]$
ET II 136(9)

16. $$\int_0^\infty x^{1/2}\left(x^2+2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}\left(x^2+1\right) J_\nu(xy)\,dx = \frac{y^{-1/2}2^{\frac{1}{2}-\nu}\pi^{-1/2}\left[K_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right)\right]^2}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}$$
$\left[-\frac{3}{2} - \operatorname{Re}\nu < \operatorname{Re}\mu < \operatorname{Re}\nu + \frac{1}{2}, \quad y > 0\right]$
ET II 44(1)

17. $$\int_0^\infty x^{1/2}\left(x^2+2\right)^{-\frac{1}{2}\nu-\frac{1}{4}} Q_\mu^{\nu+\frac{1}{2}}\left(x^2+1\right) J_\nu(xy)\,dx$$
$$= 2^{-\nu-\frac{1}{2}}\pi^{1/2}e^{\left(\nu+\frac{1}{2}\right)\pi i}y^\nu K_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right) I_{\mu+\frac{1}{2}}\left(2^{-1/2}y\right)$$
$\left[\operatorname{Re}\nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -\frac{5}{2}, \quad y > 0\right]$ ET II 46(12)

7.183 $$\int_0^\infty x^{1-\mu}\left(1+a^2x^2\right)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax)\, J_\nu(xy)\,dx = i(2\pi)^{1/2} e^{i\pi\left(\mu\mp\frac{1}{2}\nu\mp\frac{1}{4}\right)} a^{-1} y^{\mu-1} I_\nu\left(\tfrac{1}{2}a^{-1}y\right) K_\mu\left(\tfrac{1}{2}a^{-1}y\right)$$

$$\left[-\tfrac{3}{4}-\tfrac{1}{2}\operatorname{Re}\nu<\operatorname{Re}\mu<1+\operatorname{Re}\nu, \quad y>0, \quad \operatorname{Re}a>0\right]$$ ET II 46(11)

7.184

1. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}\left(x^{-1}\right) J_\nu(xa)\,dx = 2^{1/2}a^{-1-\mu}\pi^{-1/2}\cos\left[a+\tfrac{1}{2}(\nu-\mu)\pi\right]$$

$$\left[|\operatorname{Re}\mu|<\tfrac{1}{2}, \quad \operatorname{Re}\nu>-1, \quad a>0\right]$$ ET II 44(2)a

2. $$\int_1^\infty x^{-\nu}\left(x^2-1\right)^{\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu-\frac{1}{2}}\left(2x^{-2}-1\right) K_\nu(ax)\,dx = \pi^{1/2}2^{-\nu}a^{-2+\nu} W_{\mu+\frac{1}{2},\nu-\frac{1}{2}}(a)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$

$$\left[\operatorname{Re}\nu<\tfrac{3}{2}, \quad a>0\right]$$ ET II 370(45)a

3. $$\int_0^\infty x^\nu\left(1+x^2\right)^{\frac{1}{4}+\frac{\nu}{2}} Q_\mu^{\nu+\frac{1}{2}}\left(1+\frac{2}{x^2}\right) J_\nu(ax)\,dx = -ie^{i\pi\nu}\pi^{-\frac{1}{2}}2^\nu a^{-\nu-2}\left[\Gamma\left(\tfrac{3}{2}+\mu+\nu\right)\right]^2\Gamma\left(\tfrac{1}{2}+\nu-\mu\right) \times W_{-\mu-\frac{1}{2},\nu+\frac{1}{2}}(a)\left[\frac{\cos(\mu\pi)}{\Gamma(2+2\nu)} M_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a) + \frac{\sin(\mu\pi)}{\Gamma\left(\nu+\mu+\frac{3}{2}\right)} W_{\mu+\frac{1}{2},\nu+\frac{1}{2}}(a)\right]$$

$$\left[a>0, \quad \operatorname{Re}(\mu+\nu)>-\tfrac{3}{2}, \quad \operatorname{Re}(\mu-\nu)<\tfrac{1}{2}\right]$$ ET II 46(14)

4. $$\int_0^1 x^\nu\left(1-x^2\right)^{\frac{1}{2}\nu+\frac{1}{4}} P_\mu^{-\nu-\frac{1}{2}}\left(2x^{-2}-1\right) J_\nu(xy)\,dx = 2^{\nu+\frac{1}{2}}y^\nu\frac{\Gamma\left(\frac{3}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}{(2\pi)^{1/2}\left[\Gamma\left(\frac{3}{2}+\nu\right)\right]^2} \times {}_1F_1\left(\nu+\mu+\frac{3}{2};2\nu+2;iy\right)\, {}_1F_1\left(\nu+\mu+\frac{3}{2};2\nu+2;-iy\right)$$

$$\left[y>0, \quad -\tfrac{3}{2}-\operatorname{Re}\nu<\operatorname{Re}\mu<\operatorname{Re}\nu+\tfrac{1}{2}\right]$$ ET II 45(3)

5. $$\int_0^\infty x^{-\nu}\left(x^2+a^2\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}\left(1+2a^2x^{-2}\right) K_\nu(xy)\,dx = ie^{-i\pi\nu}\pi^{1/2}2^{-\nu-1}a^{-\nu-\frac{1}{2}}y^{\nu-2}\left[\Gamma\left(\tfrac{3}{2}+\mu-\nu\right)\right]^2 W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(iay)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(-iay)$$

$$\left[\operatorname{Re}a>0, \quad \operatorname{Re}y>0, \quad \operatorname{Re}\mu>-\tfrac{3}{2}, \quad \operatorname{Re}(\mu-\nu)>-\tfrac{3}{2}\right]$$ ET II 137(13)

6. $$\int_0^\infty x^{-\nu}\left(x^2+1\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_\mu^{\frac{1}{2}-\nu}\left(1+2x^{-2}\right) J_\nu(ax)\,dx = 2^{-\nu}a^{-\nu-2}\frac{ie^{-i\nu\pi}\pi^{1/2}\,\Gamma\left(\frac{3}{2}+\mu-\nu\right)}{\Gamma(2\nu)} M_{\mu+\frac{1}{2},\nu-\frac{1}{2}}(a)\, W_{-\mu-\frac{1}{2},\nu-\frac{1}{2}}(a)$$

$$\left[a>0, \quad 0<\operatorname{Re}\nu<\operatorname{Re}\mu+\tfrac{3}{2}\right]$$ ET II 47(15)a

7. $$\int_0^\infty x^{-\nu}\left(x^2+a^2\right)^{\frac{1}{4}-\frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu}\left(1+2a^2x^{-2}\right) K_\nu(xy)\,dx$$
$$= ie^{-i\pi\nu}\pi^{3/2}2^{-\nu-3}a^{\frac{1}{2}-\nu}y^{\nu-1}\left[\Gamma(1-\nu)\right]^2 \times \left\{\left[J_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2+\left[Y_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)\right]^2\right\}$$
$[\operatorname{Re} a>0, \quad \operatorname{Re} y>0, \quad \operatorname{Re}\nu<1]$ ET II 136(12)

7.185 $$\int_0^\infty x^{1/2}\, Q_{\nu-\frac{1}{2}}\left[\left(a^2+x^2\right)x^{-1}\right] J_\nu(xy)\,dx = 2^{-1/2}\pi y^{-1}\exp\left[-\left(a^2-\tfrac{1}{4}\right)^{1/2}y\right] J_\nu\left(\tfrac{1}{2}y\right)$$
$\left[\operatorname{Re}\nu>-\frac{1}{2}, \quad y>0\right]$ ET II 46(10)

7.186 $$\int_0^\infty x\left(1+x^2\right)^{-\nu-1} P_\nu\left(\frac{1-x^2}{1+x^2}\right) J_0(xy)\,dx = y^{2\nu}\left[2^\nu\,\Gamma(\nu+1)\right]^{-2} K_0(y)$$
$[\operatorname{Re}\nu>0]$ ET II 13(10)

7.187

1. $$\int_0^\infty x\, P_\mu^\nu\left(\sqrt{1+x^2}\right) K_\nu(xy)\,dx = y^{-3/2}\, S_{\nu+\frac{1}{2},\mu+\frac{1}{2}}(y)$$
$[\operatorname{Re}\nu<1, \quad \operatorname{Re} y>0]$ ET II 137(14)

2. $$\int_0^\infty x\left[P_{\lambda-\frac{1}{2}}\left(\sqrt{1+a^2x^2}\right)\right]^2 J_0(xy)\,dx = 2\pi^{-2}y^{-1}a^{-1}\cos(\lambda\pi)\left[K_\lambda\left(\frac{y}{2a}\right)\right]^2$$
$\left[\operatorname{Re} a>0, \quad |\operatorname{Re}\lambda|<\frac{1}{4}, \quad y>0\right]$ ET II 13(11)

3. $$\int_0^\infty x\left(1+x^2\right)^{-1/2} P_\mu^\nu\left(\sqrt{1+x^2}\right) K_\nu(xy)\,dx = y^{-1/2}\, S_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(y)$$
$[\operatorname{Re}\nu<1, \quad \operatorname{Re} y>0]$ ET II 137(15)

4. $$\int_0^\infty x\, P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) Q_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx$$
$$= \frac{y^{-1}e^{-\frac{1}{2}\nu\pi i}\,\Gamma\left(1+\mu+\frac{1}{2}\nu\right)}{a\,\Gamma\left(1+\mu-\frac{1}{2}\nu\right)} I_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)$$
$\left[\operatorname{Re} a>0, \quad y>0, \quad \operatorname{Re}\mu>-\frac{3}{4}, \quad \operatorname{Re}\nu>-1\right]$ ET II 47(16)

5. $$\int_0^\infty x\, P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) Q_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) J_0(xy)\,dx$$
$$= y^{-2}e^{\mu\pi i}\frac{\Gamma\left(\frac{1}{2}+\sigma-\mu\right)}{\Gamma(1+2\sigma)} W_{\mu,\sigma}\left(\frac{y}{a}\right) M_{-\mu,\sigma}\left(\frac{y}{a}\right)$$
$\left[\operatorname{Re} a>0, \quad y>0, \quad \operatorname{Re}\sigma>-\frac{1}{4}, \quad \operatorname{Re}\mu<1\right]$ ET II 14(15)

6. $$\int_0^\infty x\, P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right) P_{\sigma-\frac{1}{2}}^{-\mu}\left(\sqrt{1+a^2x^2}\right) J_0(xy)\,dx$$
$$= 2\pi^{-1}y^{-2}\cos(\sigma\pi)\, W_{\mu,\sigma}\left(\frac{y}{a}\right) W_{-\mu,\sigma}\left(\frac{y}{a}\right)$$
$\left[\operatorname{Re} a>0, \quad y>0, \quad |\operatorname{Re}\sigma|<\frac{1}{4}\right]$ ET II 14(14)

7. $$\int_0^\infty x\left\{P_{\sigma-\frac{1}{2}}^\mu\left(\sqrt{1+a^2x^2}\right)\right\}^2 J_0(xy)\,dx = -i\pi^{-1}y^{-2}\, W_{\mu,\sigma}\left(\frac{y}{a}\right)\left[W_{\mu,\sigma}\left(e^{\pi i}\frac{y}{a}\right) - W_{\mu,\sigma}\left(e^{-\pi i}\frac{y}{a}\right)\right]$$
$\left[\operatorname{Re} a>0, \quad y>0, \quad |\operatorname{Re}\sigma|<\frac{1}{4}, \quad \operatorname{Re}\mu<1\right]$ ET II 14(13)

8. $$\int_0^\infty x\left(1+a^2x^2\right)^{-1/2} P_\mu^{-\frac{1}{2}-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) P_\mu^{\frac{1}{2}-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx = \frac{\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2}{\pi a^2\,\Gamma\left(\frac{\nu}{2}+\mu+\frac{3}{2}\right)\Gamma\left(\frac{\nu}{2}-\mu+\frac{1}{2}\right)}$$
$$\left[\operatorname{Re} a>0, \quad y>0, \quad -\tfrac{5}{4}<\operatorname{Re}\mu<\tfrac{1}{4}\right]$$ ET II 46(9)

9. $$\int_0^\infty x\left\{P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right)\right\}^2 J_\nu(xy)\,dx = \frac{2\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right)\right]^2 y^{-1}}{\pi a\,\Gamma\left(1+\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)}$$
$$\left[\operatorname{Re} a>0, \quad y>0, \quad -\tfrac{3}{4}<\operatorname{Re}\mu<-\tfrac{1}{4}, \quad \operatorname{Re}\nu>-1\right]$$ ET II 45(7)

10. $$\int_0^\infty x\left(1+a^2x^2\right)^{-1/2} P_\mu^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) P_{\mu+1}^{-\frac{1}{2}\nu}\left(\sqrt{1+a^2x^2}\right) J_\nu(xy)\,dx = \frac{K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{3}{2}}\left(\frac{y}{2a}\right)}{\pi a^2\,\Gamma\left(2+\frac{1}{2}\nu+\mu\right)\Gamma\left(\frac{1}{2}\nu-\mu\right)}$$
$$\left[\operatorname{Re} a>0, \quad y>0, \quad -\tfrac{7}{4}<\operatorname{Re}\mu<-\tfrac{1}{4}\right]$$ ET II 45(8)

7.188

1. $$\int_0^\infty x\left(a^2+x^2\right)^{-\frac{1}{2}\mu} P_{\mu-1}^{-\nu}\left[\frac{a}{\sqrt{a^2+x^2}}\right] J_\nu(xy)\,dx = \frac{y^{\mu-2}e^{-ay}}{\Gamma(\mu+\nu)}$$
$$\left[\operatorname{Re} a>0, \quad y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>\tfrac{1}{2}\right]$$ ET II 45(4)

2. $$\int_0^\infty x^{\nu+1}\left(x^2+a^2\right)^{\frac{1}{2}\nu} P_\nu\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{\nu+1}y^{-\nu-1}}{\pi\,\Gamma(-\nu)}\left[K_{\nu+\frac{1}{2}}\left(\frac{ya}{2}\right)\right]^2$$
$$[\operatorname{Re} a>0, \quad -1<\operatorname{Re}\nu<0, \quad y>0]$$ ET II 45(5)

3. $$\int_0^\infty x^{1-\nu}\left(x^2+a^2\right)^{-\frac{1}{2}\nu} P_{\nu-1}\left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}}\right) J_\nu(xy)\,dx = \frac{(2a)^{1-\nu}y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) K_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right)$$
$$[\operatorname{Re} a>0, \quad y>0, \quad 0<\operatorname{Re}\nu<1]$$ ET II 45(6)

7.189

1. $$\int_0^\infty (a+x)^\mu e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = 0$$
$$\left[-\tfrac{1}{2}<\operatorname{Re}\mu<0, \quad -\tfrac{1}{2}+\operatorname{Re}\mu<\operatorname{Re}\nu<-\tfrac{1}{2}-\operatorname{Re}\mu\right]$$ ET II 366(18)

2. $$\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu}\left(1+\frac{2x}{a}\right) I_\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu-\frac{1}{2}\right)e^a}{\pi^{1/2}\,\Gamma\left(2\mu+\nu+1\right)\Gamma(2\mu-\nu)} W_{\frac{1}{2}-\mu,\frac{1}{2}+\nu}(2a)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>\left|\operatorname{Re}\nu+\tfrac{1}{2}\right|\right]$$ ET II 367(19)

3. $$\int_0^\infty x^{-\mu} e^x P_\nu^{2\mu}\left(1+\frac{2x}{a}\right) K_\mu(x+a)\,dx$$
$$= \pi^{-1/2} 2^{\mu-1} \cos(\mu\pi)\,\Gamma\left(\mu+\nu+\tfrac{1}{2}\right)\Gamma\left(\mu-\nu+\tfrac{1}{2}\right) W_{\frac{1}{2}-\mu,\frac{1}{2}+\nu}(2a)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\mu > \left|\operatorname{Re}\nu + \tfrac{1}{2}\right|\right]$ ET II 373(11)

4. $$\int_0^\infty x^{-\frac{1}{2}\mu}(x+a)^{-1/2} e^{-x} P_{\nu-\frac{1}{2}}^{\mu}\left(\frac{a-x}{a+x}\right) K_\nu(a+x)\,dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}\mu}\,\Gamma(\mu, 2a)$$
$[a > 0, \quad \operatorname{Re}\mu < 1]$ ET II 374(12)

5. $$\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{-2\mu-\frac{3}{2}} P_\nu^{-\mu}[\cosh(2x)]\, I_{\mu-\frac{1}{2}}(a \operatorname{sech} x)\,dx$$
$$= \frac{2^{\mu-\frac{1}{2}}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu+1)}{\pi^{1/2} a^{\mu+\frac{3}{2}} \left[\Gamma(\mu+1)\right]^2} M_{\nu+\frac{1}{2},\mu}(a)\, M_{-\nu-\frac{1}{2},\mu}(a)$$
$[\operatorname{Re}\mu > \operatorname{Re}\nu, \quad \operatorname{Re}\mu > -\operatorname{Re}\nu - 1]$ ET II 378(44)

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.191

1. $$\int_a^\infty x^{1/2}\left(x^2-a^2\right)^{-\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}}\left(2x^2a^{-2}-1\right)\left[\mathbf{H}_\nu(x) - Y_\nu(x)\right]dx$$
$$= 2^{-\nu-2}\pi^{1/2} a \operatorname{cosec}(\mu\pi)\cos(\nu\pi)\left\{\left[Y_\nu\left(\tfrac{1}{2}a\right)\right]^2 - \left[J_\nu\left(\tfrac{1}{2}a\right)\right]^2\right\}$$
$\left[-1 < \operatorname{Re}\mu < 0, \quad \operatorname{Re}\nu < \tfrac{1}{2}\right]$ ET II 384(6)

2. $$\int_0^\infty x^{1/2}\left(x^2-a^2\right)^{-1/4-\nu/2} P_\mu^{\nu+1/2}\left(2x^2a^{-2}-1\right)\left[I_{-\nu}(x) - \mathbf{L}_\nu(x)\right]dx$$
$$= 2^{-\nu-1}\pi^{1/2} a \operatorname{cosec}(2\mu\pi)\cos(\nu\pi)\left\{\left[I_\nu\left(\tfrac{1}{2}a\right)\right]^2 - \left[I_{-\nu}\left(\tfrac{1}{2}a\right)\right]^2\right\}$$
$\left[-1 < \operatorname{Re}\mu < 0, \quad \operatorname{Re}\nu < \tfrac{1}{2}\right]$ ET II 385(15)

7.192

1. $$\int_0^1 x^{(\nu-\mu-1)/2}\left(1-x^2\right)^{(\nu-\mu-2)/4} P_{\nu-1/2}^{(\mu-\nu+2)/2}(x)\, S_{\mu,\nu}(ax)\,dx$$
$$= 2^{\mu-3/2}\pi^{1/2} a^{-(\nu-\mu-1)/2}\,\Gamma\left(\frac{\mu+\nu+3}{4}\right)\Gamma\left(\frac{\mu-3\nu+3}{4}\right)\cos\left(\frac{\mu-\nu}{2}\pi\right)$$
$$\times\left[J_\nu\left(\tfrac{1}{2}a\right) Y_{-(\mu-\nu+1)/2}\left(\tfrac{1}{2}a\right) - Y_\nu\left(\tfrac{1}{2}a\right) J_{-(\mu-\nu+1)/2}\left(\tfrac{1}{2}a\right)\right]$$
$[\operatorname{Re}(\mu-\nu) < 0, \quad a > 0, \quad |\operatorname{Re}(\mu+\nu)| < 1, \quad \operatorname{Re}(\mu-3\nu) < 1]$ ET II 387(24)a

2. $$\int_1^\infty x^{1/2}\left(x^2-1\right)^{-\beta/2} P_\nu^\beta(x)\, S_{\mu,1/2}(ax)\,dx = \frac{2^{-3/2+\beta-\mu}a^{\beta-1}\,\Gamma\left(\frac{\beta-\mu+\nu}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\beta-\mu-\nu}{2}-\frac{1}{4}\right)}{\pi^{1/2}\,\Gamma\left(\frac{1}{2}-\mu\right)} S_{\mu-\beta+1,\nu+1/2}(a)$$
$\left[\operatorname{Re}\beta<1, \quad a>0, \quad \operatorname{Re}(\mu+\nu-\beta)<-\frac{1}{2}, \quad \operatorname{Re}(\mu-\nu-\beta)<\frac{1}{2}\right]$ ET II 387(25)a

7.193

1. $$\int_1^\infty x^{-\nu}\left(x^2-1\right)^{1/4-\nu/2} P_{\mu/2-\nu/2}^{\nu-1/2}\left(2x^{-2}-1\right) S_{\mu,\nu}(ax)\,dx = \frac{2^{\mu-\nu}a^{\nu-2}\pi^{1/2}\,\Gamma\left(\frac{3\nu-\mu-1}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} W_{\rho,\sigma}\left(ae^{i\pi/2}\right) W_{\rho,\sigma}\left(ae^{-i\pi/2}\right)$$
$\rho=\frac{1}{2}(\mu+1-\nu), \quad \sigma=\nu-\frac{1}{2}, \qquad \left[\operatorname{Re}(\mu-\nu)<0, \quad a>0, \quad \operatorname{Re}\nu<\frac{3}{2}, \quad \operatorname{Re}(3\nu-\mu)>1\right]$
ET II 387(27)a

2. $$\int_1^\infty x\left(x^2-1\right)^{-\nu/2} P_\lambda^\nu\left(2x^2-1\right) S_{\mu,\nu}(ax)\,dx = \frac{a^{\nu-1}\,\Gamma\left(\frac{\nu-\mu+1}{2}+\lambda\right)\Gamma\left(\frac{\nu-\mu-1}{2}-\lambda\right)}{2\,\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)} S_{\mu-\nu+1,2\lambda+1}(a)$$
$[\operatorname{Re}\nu<1, \quad a>0, \quad \operatorname{Re}(\mu-\nu+\lambda)<-1, \quad \operatorname{Re}(\mu-\nu+\lambda)<0]$ ET II 387(26)a

7.21 Integration of associated Legendre functions with respect to the order

7.211

1. $$\int_0^\infty P_{-x-\frac{1}{2}}(\cos\theta)\,dx = \frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\theta\right)$$ $[0<\theta<\pi]$ ET II 329(19)

2. $$\int_{-\infty}^\infty P_x(\cos\theta)\,dx = \operatorname{cosec}\left(\frac{1}{2}\theta\right)$$ $[0<\theta<\pi]$ ET II 329(20)

7.212 $$\int_0^\infty x^{-1}\tanh(\pi x)\,P_{-\frac{1}{2}+ix}(\cosh a)\,dx = 2e^{-\frac{1}{2}a}\,\boldsymbol{K}\left(e^{-a}\right)$$
$[a>0]$ ET II 330(22)

7.213 $$\int_0^\infty \frac{x\tanh(\pi x)}{a^2+x^2} P_{-\frac{1}{2}+ix}(\cosh b)\,dx = Q_{a-\frac{1}{2}}(\cosh b)$$ $[\operatorname{Re}a>0]$ ET II 387(23)

7.214 $$\int_0^\infty \sinh(\pi x)\cos(ax)\,P_{-\frac{1}{2}+ix}(b)\,dx = \frac{1}{\sqrt{2\,(b+\cosh a)}}$$
$[a>0, \quad |b|<1]$ ET I 42(27)

7.215 $$\int_0^\infty \cos(bx)\,P_{-\frac{1}{2}+ix}^\mu(\cosh a)\,dx = 0 \qquad [0<a<b]$$
$$= \frac{\sqrt{\frac{\pi}{2}}\,(\sinh a)^\mu}{\Gamma\left(\frac{1}{2}-\mu\right)(\cosh a-\cosh b)^{\mu+\frac{1}{2}}} \qquad [0<b<a]$$
ET II 330(21)

7.216 $$\int_0^\infty \cos(bx)\,\Gamma(\mu+ix)\,\Gamma(\mu-ix)\,P^{\frac{1}{2}-\mu}_{-\frac{1}{2}+ix}(\cosh a)\;dx = \frac{\sqrt{\frac{\pi}{2}}\,\Gamma(\mu)\,(\sinh a)^{\mu-\frac{1}{2}}}{(\cosh a+\cosh b)^{\mu}}$$
$[a>0, \quad b>0, \quad \operatorname{Re}\mu>0]$

ET II 330(24)

7.217

1. $$\int_{-\infty}^{\infty}\left(\nu-\frac{1}{2}+ix\right)\Gamma\left(\frac{1}{2}-ix\right)\Gamma\left(2\nu-\frac{1}{2}+ix\right)P^{\frac{1}{2}-\nu}_{\nu+ix-1}(\cos\theta)\,I_{\nu-\frac{1}{2}+ix}(a)\,K_{\nu-\frac{1}{2}+ix}(b)\,dx = \sqrt{2\pi}\,(\sin\theta)^{\nu-\frac{1}{2}}\left(\frac{ab}{\omega}\right)^{\nu}K_\nu(\omega)$$
$\left[\omega=\left(a^2+b^2+2ab\cos\theta\right)^{1/2}\right]$ ET II 383(29)

2. $$\int_0^\infty xe^{\pi x}\tanh(\pi x)\,P_{-\frac{1}{2}+ix}(-\cos\theta)\,H^{(2)}_{ix}(ka)\,H^{(2)}_{ix}(kb)\,dx = -\frac{2(ab)^{1/2}}{\pi R}e^{-ikR};$$
$R=\left(a^2+b^2-2ab\cos\theta\right)^{1/2}$ $[a>0, \quad b>0, \quad 0<\theta<\pi, \quad \operatorname{Im}k\le 0]$ ET II 381(17)

3. $$\int_0^\infty xe^{\pi x}\sinh(\pi x)\,\Gamma(\nu+ix)\,\Gamma(\nu-ix)\,P^{\frac{1}{2}-\nu}_{-\frac{1}{2}+ix}(-\cos\theta)\,H^{(2)}_{ix}(a)\,H^{(2)}_{ix}(b)\,dx = i(2\pi)^{1/2}(\sin\theta)^{\nu-\frac{1}{2}}\left(\frac{ab}{R}\right)^{\nu}H^{(2)}_{\nu}(R)$$
$R=\left(a^2+b^2-2ab\cos\theta\right)^{1/2}$ $[a>0, \quad b>0, \quad 0<\theta<\pi, \quad \operatorname{Re}\nu>0]$ ET II 381 (18)

4. $$\int_0^\infty x\sinh(\pi x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\,K_{ix}(a)\,K_{ix}(b)\,P^{\frac{1}{2}-\lambda}_{-\frac{1}{2}+ix}(\beta)\,dx = \frac{\pi^{1/2}}{\sqrt{2}}\left(\frac{ab}{z}\right)^{\lambda}\left(\beta^2-1\right)^{\frac{1}{2}\lambda-\frac{1}{4}}K_\lambda(z)$$
$z=\sqrt{a^2+b^2+2ab\beta}$ $\left[|\arg a|<\frac{\pi}{2}, \quad |\arg(\beta-1)|<\pi, \quad \operatorname{Re}\lambda>0\right]$ ET II 177(16)

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.221

1. $$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0 \qquad [m\neq n]$$
$$= \frac{2}{2n+1} \qquad [m=n]$$
WH, EH I 170(8, 10)

2.[6] $$\int_0^1 P_n(x)\,P_m(x)\,dx = \frac{1}{2n+1} \qquad [m=n]$$
$$= 0 \qquad [n-m \text{ is even}, \quad m\neq n]$$
$$= \frac{(-1)^{\frac{1}{2}(m+n-1)}m!n!}{2^{m+n-1}(m-n)\,(n+m+1)\left[\left(\frac{n}{2}\right)!\left(\frac{m-1}{2}\right)!\right]^2} \qquad [n \text{ is even}, m \text{ is odd}]$$
WH

3. $$\int_0^{2\pi} P_{2n}(\cos\varphi)\;d\varphi = 2\pi\left[\binom{2n}{n}2^{-2n}\right]^2.$$
MO 70, EH II 183(50)

7.222

1. $$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \qquad [m < n]$$

2. $$\int_{-1}^{1} (1+x)^{m+n} P_m(x) P_n(x)\,dx = \frac{2^{m+n+1}\left[(m+n)!\right]^4}{(m!n!)^2 (2m+2n+1)!}$$ ET II 277(15)

3. $$\int_{-1}^{1} (1+x)^{m-n-1} P_m(x) P_n(x)\,dx = 0 \qquad [m > n]$$ ET II 278(16)

4. $$\int_{-1}^{1} \left(1-x^2\right)^n P_{2m}(x)\,dx = \frac{2n^2}{(n-m)(2m+2n+1)} \int_{-1}^{1} \left(1-x^2\right)^{n-1} P_{2m}(x)\,dx \qquad [m < n]$$ WH

5. $$\int_{0}^{1} x^2 P_{n+1}(x) P_{n-1}(x)\,dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}$$ WH

7.223 $$\int_{-1}^{1} \frac{1}{z-x} \left\{P_n(x) P_{n-1}(x) - P_{n-1}(x) P_n(z)\right\}\,dx = -\frac{2}{n}$$ WH

7.224 [z belongs to the complex plane with a discontinuity along the interval from -1 to $+1$.]

1. $$\int_{-1}^{1} (z-x)^{-1} P_n(x)\,dx = 2\,Q_n(z)$$ ET II 277(7)

2. $$\int_{-1}^{1} x(z-x)^{-1} P_0(x)\,dx = 2\,Q_1(z)$$ ET II 277(8)

3. $$\int_{-1}^{1} x^{n+1}(z-x)^{-1} P_n(x)\,dx = 2z^{n+1}\,Q_n(z) - \frac{2^{n+1}\,(n!)^2}{(2n+1)!}$$ ET II 277(9)

4. $$\int_{-1}^{1} x^m (z-x)^{-1} P_n(x)\,dx = 2z^m\,Q_n(z) \qquad [m \le n]$$ ET II 277(10)a

5. $$\int_{-1}^{1} (z-x)^{-1} P_m(x) P_n(x)\,dx = 2\,P_m(z)\,Q_n(z) \qquad [m \le n]$$ ET II 278(18)a

6. $$\int_{-1}^{1} (z-x)^{-1} P_n(x) P_{n+1}(x)\,dx = 2\,P_{n+1}(z)\,Q_n(z) - \frac{2}{n+1}$$ ET II 278(19)

7. $$\int_{-1}^{1} x(z-x)^{-1} P_m(x) P_n(x)\,dx = 2z\,P_m(z)\,Q_n(z) \qquad [m < n]$$ ET II 278(21)

8. $$\int_{-1}^{1} x(z-x)^{-1} \left[P_n(x)\right]^2\,dx = 2z\,P_n(z)\,Q_n(z) - \frac{2}{2n+1}$$ ET II 278(20)

7.225

1. $$\int_{-1}^{x} (x-t)^{-1/2} P_n(t)\,dt = \left(n+\frac{1}{2}\right)^{-1} (1+x)^{-1/2} \left[T_n(x) + T_{n+1}(x)\right]$$ EH II 187(43)

2. $$\int_{x}^{1} (t-x)^{-1/2} P^{-1/2} P_n(t)\,dt = \left(n+\frac{1}{2}\right)^{-1} (1-x)^{-1/2} \left[T_n(x) - T_{n+1}(x)\right]$$ EH II 187(44)

3. $$\int_{-1}^{1}(1-x)^{-1/2}\,P_n(x)\,dx=\frac{2^{3/2}}{2n+1}$$ EH II 183(49)

4. $$\int_{-1}^{1}(\cosh 2p-x)^{-1/2}\,P_n(x)\,dx=\frac{2\sqrt{2}}{2n+1}\exp[-(2n+1)p]$$ $[p>0]$ WH

5.[10] $$\frac{1}{2}\int_{-1}^{1}\frac{P_\ell(z)\,dz}{\sqrt{(xy-z)^2-\left(x^2-1\right)\left(y^2-1\right)}}=P_\ell(x)\,Q_\ell(y)\qquad(1<x\le y)$$
$$=P_\ell(y)\,Q_\ell(x)\qquad(1<y\le x)$$

7.226

1. $$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}P_{2m}(x)\,dx=\left[\frac{\Gamma\left(\frac{1}{2}+m\right)}{m!}\right]^2$$ ET II 276(4)

2. $$\int_{-1}^{1}x\left(1-x^2\right)^{-1/2}P_{2m+1}(x)\,dx=\frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}$$ ET II 276(5)

3. $$\int_{-1}^{1}\left(1+px^2\right)^{-m-3/2}P_{2m}(x)\,dx=\frac{2}{2m+1}(-p)^m(1+p)^{-m-1/2}$$ $[|p|<1]$ MO 71

7.227 $$\int_0^1 x\left(a^2+x^2\right)^{-1/2}P_n\left(1-2x^2\right)\,dx=\frac{\left[a+\left(a^2+1\right)^{1/2}\right]^{-2n-1}}{2n+1}$$ $[\operatorname{Re}a>0]$ ET II 278(23)

7.228[6] $$\frac{1}{2}\Gamma(1+\mu)\int_{-1}^{1}P_l(x)(z-x)^{-\mu-1}\,dx=\left(z^2-1\right)^{-\mu/2}e^{-i\pi\mu}\,Q_l^\mu(z)$$ $[l=0,1,2,\ldots,\quad|\arg(z-1)|<\pi]$

7.23 Combinations of Legendre polynomials and powers

7.231

1. $$\int_0^1 x^\lambda\,P_{2m}(x)\,dx=\frac{(-1)^m\,\Gamma\left(m-\frac{1}{2}\lambda\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\lambda\right)}{2\,\Gamma\left(-\frac{1}{2}\lambda\right)\Gamma\left(m+\frac{3}{2}+\frac{1}{2}\lambda\right)}$$ $[\operatorname{Re}\lambda>-1]$ EH II 183(51)

2.[6] $$\int_0^1 x^\lambda\,P_{2m+1}(x)\,dx=\frac{(-1)^m\,\Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)}$$ $[\operatorname{Re}\lambda>-2]$ EH II 183(52)

7.232

1. $$\int_{-1}^{1}(1-x)^{a-1}\,P_m(x)\,P_n(x)\,dx$$
$$=\frac{2^a\,\Gamma(a)\,\Gamma(n-a+1)}{\Gamma(1-a)\,\Gamma(n+a+1)}\;{}_4F_3\left(-m,m+1,a,a;1,a+n+1,a-n;1\right)$$ $[\operatorname{Re}a>0]$ ET II 278(17)

2. $$\int_{-1}^{1} (1-x)^{a-1}(1+x)^{b-1}\, P_n(x)\, dx = \frac{2^{a+b-1}\,\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\; {}_3F_2(-n, 1+n, a; 1, a+b; 1)$$

$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0]$ ET II 276(6)

3. $$\int_0^1 (1-x)^{\mu-1}\, P_n(1-\gamma x)\, dx = \frac{\Gamma(\mu) n!}{\Gamma(\mu+n+1)}\, P_n^{(\mu,-\mu)}(1-\gamma)$$

$[\operatorname{Re} \mu > 0]$ ET II 190(37)a

4. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, P_n(1-\gamma x)\, dx = \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\; {}_3F_2\left(-n, n+1, \nu; 1, \mu+\nu; \frac{1}{2}\gamma\right)$$

$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]$ ET II 190(38)

7.233 $$\int_0^1 x^{2\mu-1}\, P_n\left(1-2x^2\right)\, dx = \frac{(-1)^n\, [\Gamma(\mu)]^2}{2\,\Gamma(\mu+n+1)\,\Gamma(\mu-n)}$$

$[\operatorname{Re} \mu > 0]$ ET II 278(22)

7.24 Combinations of Legendre polynomials and other elementary functions

7.241 $$\int_0^\infty P_n(1-x) e^{-ax}\, dx = e^{-a} a^n \left(\frac{1}{a}\frac{d}{da}\right)^n \left(\frac{e^a}{a}\right) = a^n \left(1 + \frac{1}{2}\frac{d}{da}\right)^n \left(\frac{1}{a^{n+1}}\right)$$

$[\operatorname{Re} a > 0]$ ET I 171(2)

7.242 $$\int_0^\infty P_n\left(e^{-x}\right) e^{-ax}\, dx = \frac{(a-1)(a-2)\cdots(a-n+1)}{(a+n)(a+n-2)\cdots(a-n+2)}$$

$[n \geq 2, \quad \operatorname{Re} a > 0]$ ET I 171(3)

7.243

1. $$\int_0^\infty P_{2n}(\cosh x)\, e^{-ax}\, dx = \frac{\left(a^2-1^2\right)\left(a^2-3^2\right)\cdots\left[a^2-(2n-1)^2\right]}{a\left(a^2-2^2\right)\left(a^2-4^2\right)\cdots\left[a^2-(2n)^2\right]}$$

$[\operatorname{Re} a > 2n]$ ET I 171(6)

2. $$\int_0^\infty P_{2n+1}(\cosh x)\, e^{-ax}\, dx = \frac{a\left(a^2-2^2\right)\left(a^2-4^2\right)\cdots\left[a^2-(2n)^2\right]}{\left(a^2-1\right)\left(a^2-3^2\right)\cdots\left[a^2-(2n+1)^2\right]}$$

$[\operatorname{Re} a > 2n+1]$ ET I 171(7)

3. $$\int_0^\infty P_{2n}(\cos x)\, e^{-ax}\, dx = \frac{\left(a^2+1^2\right)\left(a^2+3^2\right)\cdots\left[a^2+(2n-1)^2\right]}{a\left(a^2+2^2\right)\left(a^2+4^2\right)\cdots\left[a^2+(2n)^2\right]}$$

$[\operatorname{Re} a > 0]$ ET I 171(4)

4. $$\int_0^\infty P_{2n+1}(\cos x)\, e^{-ax}\, dx = \frac{a\left(a^2+2^2\right)\left(a^2+4^2\right)\cdots\left[a^2+(2n)^2\right]}{\left(a^2+1^2\right)\left(a^2+3^2\right)\cdots\left[a^2+(2n+1)^2\right]}$$

$[\operatorname{Re} a > 0]$ ET I 171(5)

5.[11] $$\int_{-1}^{1} e^{ix\alpha}\, P_n(x)\, dx = i^n \sqrt{\frac{2\pi}{\alpha}}\, J_{n+\frac{1}{2}}(\alpha)$$ $[n = 0, 1, 2, \ldots, \quad a > 0]$

GH2 24 (171.10)

7.244

1. $$\int_0^1 P_n\left(1-2x^2\right)\sin ax\,dx = \frac{\pi}{2}\left[J_{n+\frac{1}{2}}\left(\frac{a}{2}\right)\right]^2 \qquad [a>0]$$ ET I 94(2)

2. $$\int_0^1 P_n\left(1-2x^2\right)\cos ax\,dx = \frac{\pi}{2}(-1)^n J_{n+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-n-\frac{1}{2}}\left(\frac{a}{2}\right) \qquad [a>0]$$ ET I 38(1)

7.245

1. $$\int_0^{2\pi} P_{2m+1}(\cos\theta)\cos\theta\,d\theta = \frac{\pi}{2^{4m+1}}\binom{2m}{m}\binom{2m+2}{m+1}$$ MO 70, EH II 183(5)

2. $$\int_0^{\pi} P_m(\cos\theta)\sin n\theta\,d\theta = \frac{2(n-m+1)(n-m+3)\cdots(n+m-1)}{(n-m)(n-m+2)\cdots(n+m)} \qquad [n>m \text{ and } n+m \text{ is odd}]$$
$$= 0 \qquad [n\le m \text{ or } n+m \text{ is even}]$$ MO 71

3.[10] $$\int_0^{2\pi} P_{2n+1}(\sin\alpha\sin\phi)\sin\phi\,d\phi = (-1)^{n+1}\frac{2\sqrt{\pi}\,\Gamma\left(n+\frac{3}{2}\right)}{(2n+1)\,\Gamma(n+2)}P^1_{2n+1}(\cos\alpha) \qquad \left[\alpha\ne\tfrac{1}{2}(2n+1)\pi, \quad n \text{ an integer}\right]$$

4. $$\int_{-1}^1 \cos(\alpha x)\,P_n(x)\,dx = 0 \qquad [n \text{ is odd}]$$
$$= (-1)^v\sqrt{\frac{2\pi}{\alpha}}\,J_{2v+\frac{1}{2}}(\alpha) \qquad [n=2v \text{ is even}]$$ GH2 24 (171.10a)

7.246 $$\int_0^{\pi} P_n\left(1-2\sin^2 x\sin^2\theta\right)\sin x\,dx = \frac{2\sin(2n+1)\theta}{(2n+1)\sin\theta}$$ MO 71

7.247 $$\int_0^1 P_{2n+1}(x)\sin ax\frac{dx}{\sqrt{x}} = (-1)^{n+1}\sqrt{\frac{\pi}{2a}}\,J_{2n+\frac{3}{2}}(a) \qquad [a>0]$$ ET I 94(1)

7.248

1. $$\int_{-1}^1 \left(a^2+b^2-2abx\right)^{-1/2}\sin\left[\lambda\left(a^2+b^2-2abx\right)^{1/2}\right]P_n(x)\,dx = \pi(ab)^{-1/2}J_{n+\frac{1}{2}}(a\lambda)\,J_{n+\frac{1}{2}}(b\lambda) \qquad [a>0, \quad b>0]$$ ET II 277(11)

2. $$\int_{-1}^1 \left(a^2+b^2-2abx\right)^{-1/2}\cos\left[\lambda\left(a^2+b^2-2abx\right)^{1/2}\right]P_n(x)\,dx = -\pi(ab)^{-1/2}J_{n+\frac{1}{2}}(a\lambda)\,Y_{n+\frac{1}{2}}(b\lambda) \qquad [0\le a\le b]$$ ET II 277(12)

7.249

1. $$\int_{-1}^1 P_n(x)\arcsin x\,dx = 0 \qquad [n \text{ is even}]$$
$$= \pi\left\{\frac{(n-2)!!}{2^{\frac{1}{2}(n+1)}\left(\frac{n+1}{2}\right)!}\right\}^2 \qquad [n \text{ is odd}]$$ WH

2. $$P_n(x) = \frac{1}{t}\sum_{t=0}^{t-1}\left(x + \sqrt{x^2-1}\cos\frac{2\pi r}{t}\right)^n \qquad [t > n]$$

7.25 Combinations of Legendre polynomials and Bessel functions

7.251

1. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,Y_\nu(xy)\,dx = \pi^{-1}y^{-1}\left[S_{2n+1}(y) + \pi\,Y_{2n+1}(y)\right]$$
$[n = 0, 1, \ldots; \quad y > 0, \quad \nu > 0]$ ET II 108(1)

2. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,K_0(xy)\,dx = y^{-1}\left[(-1)^{n+1}\,K_{2n+1}(y) + \frac{i}{2}\,S_{2n+1}(iy)\right]$$
$[y > 0]$ ET II 134(1)

3. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,J_0(xy)\,dx = y^{-1}\,J_{2n+1}(y) \qquad [y > 0]$$ ET II 13(1)

4. $$\int_0^1 x\,P_n\left(1-2x^2\right)\left[J_0(ax)\right]^2\,dx = \frac{1}{2(2n+1)}\left\{\left[J_n(a)\right]^2 + \left[J_{n+1}(a)\right]^2\right\}$$ ET II 338(39)a

5. $$\int_0^1 x\,P_n\left(1-2x^2\right)\,J_0(ax)\,Y_0(ax)\,dx = \frac{1}{2(2n+1)}\left[J_n(a)\,Y_n(a) + J_{n+1}(a)\,Y_{n+1}(a)\right]$$
ET II 339(48)a

6. $$\int_0^1 x^2\,P_n\left(1-2x^2\right)\,J_1(xy)\,dx = y^{-1}(2n+1)^{-1}\left[(n+1)\,J_{2n+2}(y) - n\,J_{2n}(y)\right]$$
$[y > 0]$ ET II 20(23)

7. $$\int_0^1 x^{\mu-1}\,P_n\left(2x^2-1\right)\,J_\nu(ax)\,dx = \frac{2^{-\nu-1}a^\nu\left[\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu\right)\right]^2}{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+n+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-n\right)}$$
$$\times\,{}_2F_3\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu}{2}; \nu+1, \frac{\mu+\nu}{2}+n+1, \frac{\mu+\nu}{2}-n; -\frac{a^2}{4}\right)$$
$[a > 0, \quad \operatorname{Re}(\mu+\nu) > 0]$ ET II 337(32)a

7.252 $$\int_0^1 e^{-ax}\,P_n(1-2x)\,I_0(ax)\,dx = \frac{e^{-a}}{2n+1}\left[I_n(a) + I_{n+1}(a)\right]$$
$[a > 0]$ ET II 366(11)a

7.253 $$\int_0^{\pi/2} \sin(2x)\,P_n\left(\cos 2x\right)\,J_0\left(a\sin x\right)\,dx = a^{-1}\,J_{2n+1}(a)$$ ET II 361(20)

7.254 $$\int_0^1 x\,P_n\left(1-2x^2\right)\left[I_0(ax) - \mathbf{L}_0(ax)\right]\,dx = (-1)^n\left[I_{2n+1}(a) - \mathbf{L}_{2n+1}(a)\right]$$
$[a > 0]$ ET II 385(14)a

7.3–7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.311

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx = 0 \qquad \left[n>0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 280(1)

2. $$\int_0^1 x^{n+2\rho}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx = \frac{\Gamma(2\nu+n)\,\Gamma(2\rho+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\rho+\frac{1}{2}\right)}{2^{n+1}\,\Gamma(2\nu)\,\Gamma(2\rho+1)n!\,\Gamma(n+\nu+\rho+1)}$$ $$\left[\operatorname{Re}\rho > -\tfrac{1}{2}, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 280(2)

3. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{\beta} C_n^\nu(x)\,dx = \frac{2^{\beta+\nu+\frac{1}{2}}\,\Gamma(\beta+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)\,\Gamma\left(\beta-\nu+\frac{3}{2}\right)}{n!\,\Gamma(2\nu)\,\Gamma\left(\beta-\nu-n+\frac{3}{2}\right)\Gamma\left(\beta+\nu+n+\frac{3}{2}\right)}$$ $$\left[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 280(3)

4. $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} C_n^\nu(x)\,dx = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)\,\Gamma(\alpha+\beta+2)} \times {}_3F_2\left(-n, n+2\nu, \alpha+1; \nu+\frac{1}{2}, \alpha+\beta+2; 1\right)$$ $$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$$ ET II 281(4)

7.312 In the following integrals, z belongs to the complex plane with a cut along the interval of the real axis from -1 to 1.

1. $$\int_{-1}^{1} x^m (z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx = \frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-\left(\nu-\frac{1}{2}\right)\pi i} z^m \left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$$ $$\left[m \le n, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 281(5)

2. $$\int_{-1}^{1} x^{n+1} (z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_n^\nu(x)\,dx = \frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-\left(\nu-\frac{1}{2}\right)\pi i} z^{n+1} \left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) - \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu)\,\Gamma(\nu+n+1)}$$ $$\left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 281(6)

3.[6] $$\int_{-1}^{1} (z-x)^{-1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_m^\nu(x)\, C_n^\nu(x)\,dx = \frac{\pi^{1/2}2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-\left(\nu-\frac{1}{2}\right)\pi i} \left(z^2-1\right)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z)\, Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$$ $$\left[m \le n, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 283(17)

7.313

1. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} C_m^\nu(x)\, C_n^\nu(x)\,dx = 0 \qquad \left[m \ne n, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 282(12), MO 98a, EH I 177(16)

2. $$\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}} \left[C_n^\nu(x)\right]^2 dx = \frac{\pi 2^{1-2\nu}\,\Gamma(2\nu+n)}{n!(n+\nu)\left[\Gamma(\nu)\right]^2} \qquad \left[\operatorname{Re}\nu > -\tfrac{1}{2}\right]$$ ET II 281(8), MO 98a, EH I 177(17)

7.314

1. $$\int_{-1}^{1}(1-x)^{\nu-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}\left[C_n^\nu(x)\right]^2\,dx=\frac{\pi^{1/2}\,\Gamma\left(\nu-\frac{1}{2}\right)\Gamma(2\nu+n)}{n!\,\Gamma(\nu)\,\Gamma(2\nu)}$$
$\left[\operatorname{Re}\nu>\frac{1}{2}\right]$ ET II 281(9)

2. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{2\nu-1}\left[C_n^\nu(x)\right]^2\,dx=\frac{2^{3\nu-\frac{1}{2}}\left[\Gamma(2\nu+n)\right]^2\Gamma\left(2n+\nu+\frac{1}{2}\right)}{(n!)^2\,\Gamma(2\nu)\,\Gamma\left(3\nu+2n+\frac{1}{2}\right)}$$
$[\operatorname{Re}\nu>0]$ ET II 282(10)

3. $$\int_{-1}^{1}(1-x)^{3\nu+2n-\frac{3}{2}}(1+x)^{\nu-\frac{1}{2}}\left[C_n^\nu(x)\right]^2\,dx$$
$$=\frac{\pi^{1/2}\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2\Gamma\left(\nu+2n+\frac{1}{2}\right)\Gamma\left(2\nu+2n\right)\Gamma\left(3\nu+2n-\frac{1}{2}\right)}{2^{2\nu+2n}\left[n!\,\Gamma\left(\nu+n+\frac{1}{2}\right)\Gamma(2\nu)\right]^2\Gamma\left(2\nu+2n+\frac{1}{2}\right)}$$
$\left[\operatorname{Re}\nu>\frac{1}{6}\right]$ ET II 282(11)

4. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{\nu+m-n-\frac{3}{2}}\,C_m^\nu(x)\,C_n^\nu(x)\,dx$$
$$=(-1)^m\frac{2^{2-2\nu-m+n}\pi^{3/2}\,\Gamma(2\nu+n)}{m!(n-m)!\left[\Gamma(\nu)\right]^2\Gamma\left(\frac{1}{2}+\nu+m\right)}\,\frac{\Gamma\left(\nu-\frac{1}{2}+m-n\right)\Gamma\left(\frac{1}{2}-\nu+m-n\right)}{\Gamma\left(\frac{1}{2}-\nu-n\right)\Gamma\left(\frac{1}{2}+m-n\right)}$$
$\left[\operatorname{Re}\nu>-\frac{1}{2};\quad n\geq m\right]$ ET II 282(13)a

5. $$\int_{-1}^{1}(1-x)^{2\nu-1}\,(1+x)^{\nu-\frac{1}{2}}\,C_m^\nu(x)\,C_n^\nu(x)\,dx$$
$$=\frac{2^{3\nu-\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+m\right)\Gamma(2\nu+n)}{m!n!\,\Gamma(2\nu)\,\Gamma\left(\frac{1}{2}-\nu\right)}\,\frac{\Gamma\left(\nu+\frac{1}{2}+m+n\right)\Gamma\left(\frac{1}{2}-\nu+n-m\right)}{\Gamma\left(\nu+\frac{1}{2}+n-m\right)\Gamma\left(3\nu+\frac{1}{2}+m+n\right)}$$
$[\operatorname{Re}\nu>0]$ ET II 282(14)

6. $$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^{3\nu+m+n-\frac{3}{2}}\,C_m^\nu(x)\,C_n^\nu(x)\,dx$$
$$=\frac{2^{4\nu+m+n-1}\left[\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+m+n)\right]^2}{\Gamma\left(\nu+m+\frac{1}{2}\right)\Gamma\left(\nu+n+\frac{1}{2}\right)\Gamma\left(2\nu+m\right)}\,\frac{\Gamma\left(\nu+m+n+\frac{1}{2}\right)\Gamma\left(3\nu+m+n-\frac{1}{2}\right)}{\Gamma(2\nu+n)\,\Gamma(4\nu+2m+2n)}$$
$\left[\operatorname{Re}\nu>\frac{1}{6}\right]$ ET II 282(15)

7. $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\nu-\frac{1}{2}}\,C_m^\mu(x)\,C_n^\nu(x)\,dx$$
$$=\frac{2^{\alpha+\nu+\frac{1}{2}}\,\Gamma(\alpha+1)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n-\frac{1}{2}\right)}{m!n!\,\Gamma\left(\nu-\alpha-\frac{1}{2}\right)\Gamma\left(\nu-\alpha+n+\frac{3}{2}\right)}\,\frac{\Gamma(2\mu+m)\,\Gamma\left(2\nu+n\right)}{\Gamma(2\mu)\,\Gamma(2\nu)}$$
$$\times\;{}_4F_3\left(-m,m+2\mu,\alpha+1,\alpha-\nu+\frac{3}{2};\mu+\frac{1}{2},\nu+\alpha+n+\frac{3}{2},\alpha-\nu-n+\frac{3}{2};1\right)$$
$\left[\operatorname{Re}\alpha>-1,\quad\operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 283(16)

7.315 $$\int_{-1}^{1}\left(1-x^2\right)^{\frac{1}{2}\nu-1}\,C_{2n}^\nu(ax)\,dx=\frac{\pi^{1/2}\,\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)}\,C_n^{\frac{1}{2}\nu}\left(2a^2-1\right)$$
$[\operatorname{Re}\nu>0]$ ET II 283(19)

7.316 $$\int_{-1}^{1} \left(1-x^2\right)^{\nu-1} C_n^\nu\left(\cos\alpha\cos\beta + x\sin\alpha\sin\beta\right) dx = \frac{2^{2\nu-1} n! \left[\Gamma(\nu)\right]^2}{\Gamma(2\nu+n)} C_n^\nu(\cos\alpha)\, C_n^\nu(\cos\beta)$$
$[\operatorname{Re}\nu > 0]$ ET II 283(20)

7.317

1. $$\int_0^1 (1-x)^{\mu-1} x^{\lambda-\frac{1}{2}} C_n^\lambda(1-\gamma x)\, dx = \frac{\Gamma(2\lambda+n)\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(\mu)}{\Gamma(2\lambda)\,\Gamma\left(\lambda+\mu+n+\frac{1}{2}\right)} P_n^{(\alpha,\beta)}(1-\gamma)$$
$\alpha = \lambda+\mu-\frac{1}{2}, \qquad \beta = \lambda-\mu-\frac{1}{2}$ $\left[\operatorname{Re}\lambda > -1, \quad \lambda \neq 0, \quad -\frac{1}{2}, \quad \operatorname{Re}\mu > 0\right]$ ET II 190(39)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_n^\lambda(1-\gamma x)\, dx = \frac{\Gamma(2\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(2\lambda)\,\Gamma(\mu+\nu)} \times {}_3F_2\left(-n, n+2\lambda, \nu; \lambda+\frac{1}{2}, \mu+\nu; \frac{\gamma}{2}\right)$$
$[2\lambda \neq 0, -1, -2, \ldots, \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(40)a

7.318 $$\int_0^1 x^{2\nu}\left(1-x^2\right)^{\sigma-1} C_n^\nu\left(1-x^2 y\right) dx = \frac{\Gamma(2\nu+n)\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\sigma)}{2\,\Gamma(2\nu)\,\Gamma\left(n+\nu+\sigma+\frac{1}{2}\right)} P_n^{(\alpha,\beta)}(1-y),$$
$\alpha = \nu+\sigma-\frac{1}{2}, \qquad \beta = \nu-\sigma-\frac{1}{2}$ $\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re}\sigma > 0\right]$ ET II 283(21)

7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\ {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$
$\left[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

7.321 $$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x)\, dx = \frac{\pi 2^{1-\nu} i^n\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab)$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ ET I 171(9)

7.323

1. $$\int_0^\pi C_n^\nu(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0 \qquad [n = 1, 2, 3, \ldots]$$
$$= 2^{-2\nu}\pi\,\Gamma(2\nu+1)\left[\Gamma(1+\nu)\right]^{-2} \qquad [n = 0]$$
EH I 177(18)

2.[11] $$\int_0^\pi C_n^\nu\left(\cos\psi\cos\psi' + \sin\psi\sin\psi'\cos\varphi\right)(\sin\varphi)^{2\nu-1}\,d\varphi = 2^{2\nu-1}n!\,[\Gamma(\nu)]^2\,C_n^\nu(\cos\psi)\,C_n^\nu(\cos\psi')\,[\Gamma(2\nu+n)]^{-1}$$
$[\operatorname{Re}\nu > 0]$ EH I 177(20)

7.324

1. $$\int_0^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} C_{2n+1}^\nu(x)\sin ax\,dx = (-1)^n\pi\frac{\Gamma(2n+2\nu+1)\,J_{2n+\nu+1}(a)}{(2n+1)!\,\Gamma(\nu)(2a)^\nu}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2},\quad a>0\right]$ ET I 94(4)

2. $$\int_0^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} C_{2n}^\nu(x)\cos ax\,dx = \frac{(-1)^n\pi\,\Gamma(2n+2\nu)\,J_{\nu+2n}(a)}{(2n)!\,\Gamma(\nu)(2a)^\nu}$$
$\left[\operatorname{Re}\nu > -\frac{1}{2},\quad a>0\right]$ ET I 38(3)a

7.325* Complete System of Orthogonal Step Functions

Let $s_j(x) = (-1)^{\lfloor 2jx\rfloor}$ for $j\in\mathbb{N}$ and $c_j(x) = (-1)^{\lfloor 2jx+1/2\rfloor}$ for $j\in 0+\mathbb{N}$ where $\lfloor z\rfloor$ denotes the integer part of z. Thus, $c_j(z)$ and $s_j(z)$ have minimal period j^{-1} and manifest even and odd symmetry about $x=1/2$, respectively, and so are the discrete analogues of $\cos 2\pi jx$ and $\sin 2\pi jx$. Furthermore, for $j\in\mathbb{N}$ let $\underline{j}$ denote its odd part: the quotient of j by its highest power-of-two factor. Then for all j and $k\in\mathbb{N}$, if (j,k) denotes their highest common factor and $[j,k]$ denotes their lowest common multiple:

1. $$\int_0^1 s_j(x)s_k(x)\,dx = \begin{cases}\frac{(j,k)}{[j,k]} & \text{if } j/\underline{j} = k/\underline{k}\\ 0 & \text{otherwise}\end{cases}$$

2. $$\int_0^1 c_j(x)c_k(x)\,dx = \begin{cases}(-1)^{(j+k)/2+1}\frac{(j,k)}{[j,k]} & \text{if } j/\underline{j} = k/\underline{k}\\ 0 & \text{otherwise}\end{cases}$$

7.33 Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions; Integration of Gegenbauer functions with respect to the index

7.331

1. $$\int_1^\infty x^{2n+1-\nu}\left(x^2-1\right)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n}\left(\frac{1}{x}\right) J_\nu(xy)\,dx = (-1)^n 2^{2n-\nu+1}y^{-\nu+2n-1}\left[(2n)!\right]^{-1}\Gamma(2\nu-2n)\left[\Gamma(\nu-2n)\right]^{-1}\cos y$$
$\left[y>0,\quad 2n-\frac{1}{2} < \operatorname{Re}\nu < 2n+\frac{1}{2}\right]$ ET II 44(10)a

7.332

1. $$\int_0^\infty x^{\nu+1}\left(x^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}}\left[\left(x^2+\beta^2\right)^{-1/2}\beta\right] J_{\nu+\frac{3}{2}+2n}\left[\left(x^2+\beta^2\right)^{1/2}a\right] J_\nu(xy)\,dx$$
$$= (-1)^n 2^{1/2}\pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu \left(a^2-y^2\right)^{-1/2}\sin\left[\beta\left(a^2-y^2\right)^{1/2}\right] C_{2n+1}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{1/2}\right] \qquad [0<y<a]$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1]$$
ET II 59(23)

2. $$\int_0^\infty x^{\nu+1}\left(x^2+\beta^2\right)^{-\frac{1}{2}\nu-\frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}}\left[\beta\left(x^2+\beta^2\right)^{-1/2}\right] J_{\nu+\frac{1}{2}+2n}\left[\left(x^2+\beta^2\right)^{1/2}a\right] J_\nu(xy)\,dx$$
$$= (-1)^n 2^{1/2}\pi^{-1/2} a^{\frac{1}{2}-\nu} y^\nu \left(a^2-y^2\right)^{-1/2}\cos\left[\beta\left(a^2-y^2\right)^{1/2}\right] C_{2n}^{\nu+\frac{1}{2}}\left[\left(1-\frac{y^2}{a^2}\right)^{1/2}\right] \qquad [0<y<a]$$
$$= 0 \qquad [a<y<\infty] \qquad [a>0, \quad \operatorname{Re}\beta>0, \quad \operatorname{Re}\nu>-1]$$
ET II 59(24)

7.333

1. $$\int_0^\pi (\sin x)^{\nu+1}\cos\left(a\cos\theta\cos x\right) C_n^{\nu+\frac{1}{2}}(\cos x)\, J_\nu\left(a\sin\theta\sin x\right)\,dx$$
$$= (-1)^{\frac{n}{2}}\left(\frac{2\pi}{a}\right)^{1/2}(\sin\theta)^\nu\, C_n^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+\frac{1}{2}+n}(a) \qquad [n=0,2,4,\ldots]$$
$$= 0 \qquad [n=1,3,5,\ldots]$$
$$[\operatorname{Re}\nu>-1]$$
WA 414(2)a

2. $$\int_0^\pi (\sin x)^{\nu+1}\sin\left(a\cos\theta\cos x\right) C_n^{\nu+\frac{1}{2}}(\cos x)\, J_\nu\left(a\sin\theta\sin x\right)\,dx$$
$$= 0 \qquad [n=0,2,4,\ldots]$$
$$= (-1)^{\frac{n-1}{2}}\left(\frac{2\pi}{a}\right)^{1/2}(\sin\theta)^\nu\, C_n^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+\frac{1}{2}+n}(a) \qquad [n=1,3,5,\ldots]$$
$$[\operatorname{Re}\nu>-1]$$
WA 414(3)a

7.334

1. $$\int_0^\pi (\sin x)^{2\nu}\, C_n^\nu(\cos x)\,\frac{J_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\,\Gamma(2\nu+n)}{2^{\nu-1}n!\,\Gamma(\nu)}\,\frac{J_{\nu+n}(\alpha)}{\alpha^\nu}\,\frac{J_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = \left(\alpha^2+\beta^2-2\alpha\beta\cos x\right)^{1/2} \qquad \left[n=0,1,2,\ldots; \quad \operatorname{Re}\nu>-\tfrac{1}{2}\right]$$
ET II 362(29)

2. $$\int_0^\pi (\sin x)^{2\nu}\, C_n^\nu(\cos x)\,\frac{Y_\nu(\omega)}{\omega^\nu}\,dx = \frac{\pi\,\Gamma(2\nu+n)}{2^{\nu-1}n!\,\Gamma(\nu)}\,\frac{J_{\nu+n}(\alpha)}{\alpha^\nu}\,\frac{Y_{\nu+n}(\beta)}{\beta^\nu},$$
$$\omega = \left(\alpha^2+\beta^2-2\alpha\beta\cos x\right)^{1/2} \qquad \left[|\alpha|<|\beta|, \quad \operatorname{Re}\nu-\tfrac{1}{2}\right]$$
ET II 362(30)

Integration of Gegenbauer functions with respect to the index

7.335 $$\int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1} t^{\alpha} C_{\alpha}^{\nu}(z)\, d\alpha = -2i\left(1+2tz+t^2\right)^{-\nu}$$

$[-2 < \operatorname{Re}\nu < c < 0, \quad |\arg(z \pm 1)| < \pi]$

EH I 178(25)

7.336 $$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x)\left(\nu - \frac{1}{2} + ix\right) K_{\nu-\frac{1}{2}+ix}(a)\, I_{\nu-\frac{1}{2}+ix}(b)\, C_{-\frac{1}{2}+ix}^{\nu}(-\cos\varphi)\, dx = \frac{2^{-\nu+1}(ab)^{\nu}}{\Gamma(\nu)}\omega^{-\nu} K_{\nu}(\omega)$$

$\omega = \sqrt{a^2+b^2-2ab\cos\varphi}$ EH II 55(45)

7.34 Combinations of Chebyshev polynomials and powers

7.341 $$\int_{-1}^{1} [T_n(x)]^2\, dx = 1 - \left(4n^2-1\right)^{-1}$$ ET II 271(6)

7.342 $$\int_{-1}^{1} U_n\left[x\left(1-y^2\right)^{1/2}\left(1-z^2\right)^{1/2} + yz\right] dx = \frac{2}{n+1} U_n(y)\, U_n(z)$$

$[|y| < 1, \quad |z| < 1]$ ET II 275(34)

7.343

1. $$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \qquad [m \neq n]$$
$$= \frac{\pi}{2} \qquad [m = n \neq 0]$$
$$= \pi \qquad [m = n = 0]$$

MO 104

2. $$\int_{-1}^{1} \sqrt{1-x^2}\, U_n(x)\, U_m(x)\, dx = 0 \qquad [m \neq n]$$ ET II 274(28)
$$= \frac{\pi}{2} \qquad [m = n]$$ ET II 274(27), MO 105a

7.344

1. $$\int_{-1}^{1} (y-x)^{-1}\left(1-y^2\right)^{-1/2} T_n(y)\, dy = \pi\, U_{n-1}(x) \qquad [n = 1, 2, \ldots]$$ EH II 187(47)

2. $$\int_{-1}^{1} (y-x)^{-1}\left(1-y^2\right)^{1/2} U_{n-1}(y)\, dy = -\pi\, T_n(x) \qquad [n = 1, 2, \ldots]$$ EH II 187(48)

7.345

1. $$\int_{-1}^{1} (1-x)^{-1/2}(1+x)^{m-n-\frac{3}{2}}\, T_m(x)\, T_n(x)\, dx = 0 \qquad [m > n]$$ ET II 272(10)

2. $$\int_{-1}^{1} (1-x)^{-1/2}(1+x)^{m+n-\frac{3}{2}}\, T_m(x)\, T_n(x)\, dx = \frac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!(2n-1)!}$$

$[m + n \neq 0]$ ET II 272(11)

3. $$\int_{-1}^{1} (1-x)^{1/2}(1+x)^{m+n+\frac{3}{2}}\, U_m(x)\, U_n(x)\, dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}$$ ET II 274(31)

4. $$\int_{-1}^{1} (1-x)^{1/2}(1+x)^{m-n-\frac{1}{2}}\, U_m(x)\, U_n(x)\, dx = 0 \qquad [m > n]$$ ET II 274(30)

5. $$\int_{-1}^{1} (1-x)(1+x)^{1/2}\, U_m(x)\, U_n(x)\, dx = \frac{2^{5/2}(m+1)(n+1)}{\left(m+n+\frac{3}{2}\right)\left(m+n+\frac{5}{2}\right)\left[1-4(m-n)^2\right]}$$ ET II 274(29)

6. $$\int_{-1}^{1} (1+x)^{-1/2}\,(1-x)^{\alpha-1}\, T_m(x)\, T_n(x)\, dx = \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}}\,\Gamma(\alpha)\,\Gamma\left(n-\alpha+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\alpha\right)\Gamma\left(\alpha+n+\frac{1}{2}\right)}\ {}_4F_3\left(-m, m, \alpha, \alpha+\frac{1}{2}; \frac{1}{2}, \alpha+n+\frac{1}{2}, \alpha-n+\frac{1}{2}; 1\right)$$ $[\operatorname{Re}\alpha > 0]$ ET II 272(12)

7. $$\int_{-1}^{1} (1+x)^{1/2}\,(1-x)^{\alpha-1}\, U_m(x)\, U_n(x)\, dx = \frac{\pi^{1/2} 2^{\alpha-\frac{1}{2}}(m+1)(n+1)\,\Gamma(\alpha)\,\Gamma\left(n-\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\alpha\right)\Gamma\left(\frac{3}{2}+\alpha+n\right)} \times {}_4F_3\left(-m, m+2, \alpha, \alpha-\frac{1}{2}; \frac{3}{2}, \alpha+n+\frac{3}{2}, \alpha-n-\frac{1}{2}; 1\right)$$ $[\operatorname{Re}\alpha > 0]$ ET II 275(32)

7.346 $$\int_{0}^{1} x^{s-1}\, T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s 2^s\, \mathrm{B}\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)}$$ $[\operatorname{Re} s > 0]$ ET II 324(2)

7.347

1. $$\int_{-1}^{1} (1-x)^{\alpha}\,(1+x)^{\beta}\, T_n(x)\, dx = \frac{2^{\alpha+\beta+2n+1}\,(n!)^2\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(2n)!\,\Gamma(\alpha+\beta+2)} \times {}_3F_2\left(-n, n, \alpha+1; \frac{1}{2}, \alpha+\beta+2; 1\right)$$ $[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$ ET II 271(2)

2. $$\int_{-1}^{1} (1-x)^{\alpha}\,(1+x)^{\beta}\, U_n(x)\, dx = \frac{2^{\alpha+\beta+2n+2}\,[(n+1)!]^2\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{(2n+2)!\,\Gamma(\alpha+\beta+2)} \times {}_3F_2\left(-n, n+1, \alpha+1; \frac{3}{2}, \alpha+\beta+2; 1\right)$$ ET II 273(22)

7.348 $$\int_{-1}^{1} \left(1-x^2\right)^{-1/2}\, U_{2n}(xz)\, dx = \pi\, P_n\left(2z^2-1\right) \qquad [|z| < 1]$$ ET II 275(33)

7.349 $$\int_{-1}^{1} \left(1-x^2\right)^{-1/2}\, T_n\left(1-x^2 y\right)\, dx = \frac{1}{2}\pi\left[P_n(1-y) + P_{n-1}(1-y)\right]$$ ET II 222(14)

7.35 Combinations of Chebyshev polynomials and elementary functions

7.351 $$\int_0^1 x^{-1/2}\left(1-x^2\right)^{-\frac{1}{2}} e^{-\frac{2a}{x}}\, T_n(x)\,dx = \pi^{1/2} D_{n-\frac{1}{2}}\left(2a^{1/2}\right) D_{-n-\frac{1}{2}}\left(2a^{1/2}\right)$$

$[\operatorname{Re} a > 0]$ ET II 272(13)

7.352

1. $$\int_0^\infty \frac{x\,U_n\left[a\left(a^2+x^2\right)^{-1/2}\right]}{\left(a^2+x^2\right)^{\frac{1}{2}n+1}\left(e^{\pi x}+1\right)}\,dx = \frac{a^{-n}}{2n} - 2^{-n-1}\zeta\left(n+1, \frac{a+1}{2}\right)$$

$[\operatorname{Re} a > 0]$ ET II 275(39)

2. $$\int_0^\infty \frac{x\,U_n\left[a\left(a^2+x^2\right)^{-1/2}\right]}{\left(a^2+x^2\right)^{\frac{1}{2}n+1}\left(e^{2\pi x}-1\right)}\,dx = \frac{1}{2}\zeta(n+1,a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n}$$

$[\operatorname{Re} a > 0]$ ET II 276(40)

7.353

1. $$\int_0^\infty \left(a^2+x^2\right)^{-\frac{1}{2}n} \operatorname{sech}\left(\frac{1}{2}\pi x\right) T_n\left[a\left(a^2+x^2\right)^{-1/2}\right] dx = 2^{1-2n}\left[\zeta\left(n, \frac{a+1}{4}\right) - \zeta\left(n, \frac{a+3}{4}\right)\right]$$
$$= 2^{1-n}\,\Phi\left(-1, n, \frac{a+1}{2}\right)$$

$[\operatorname{Re} a > 0]$ ET II 273(19)

2. $$\int_0^\infty \left(a^2+x^2\right)^{-\frac{1}{2}n}\left[\cosh\left(\frac{1}{2}\pi x\right)\right]^{-2} T_n\left[a\left(a^2+x^2\right)^{-1/2}\right] dx = \pi^{-1} n 2^{1-n}\zeta\left(n+1, \frac{a+1}{2}\right)$$

$[\operatorname{Re} a > 0]$ ET II 273(20)

7.354

1. $$\int_{-1}^1 \sin(xyz)\cos\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] T_{2n+1}(x)\,dx = (-1)^n \pi\, T_{2n+1}(y)\, J_{2n+1}(x)$$

ET II 271(4)

2. $$\int_{-1}^1 \sin(xyz)\sin\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] U_{2n+1}(x)\,dx = (-1)^n \pi\left(1-y^2\right)^{1/2} U_{2n+1}(y)\, J_{2n+2}(z)$$

ET II 274(25)

3. $$\int_{-1}^1 \cos(xyz)\cos\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] T_{2n}(x)\,dx = (-1)^n \pi\, T_{2n}(y)\, J_{2n}(z)$$ ET II 271(5)

4. $$\int_{-1}^1 \cos(xyz)\sin\left[\left(1-x^2\right)^{1/2}\left(1-y^2\right)^{1/2} z\right] U_{2n}(x)\,dx = (-1)^n \pi\left(1-y^2\right)^{1/2} U_{2n}(y)\, J_{2n+1}(z)$$

ET II 274(24)

7.355

1. $$\int_0^1 T_{2n+1}(x)\sin ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(a)$$ $[a > 0]$ ET I 94(3)a

2. $$\int_0^1 T_{2n}(x)\cos ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n}(a)$$ $[a > 0]$ ET I 38(2)a

7.36 Combinations of Chebyshev polynomials and Bessel functions

7.361 $$\int_0^1 \left(1-x^2\right)^{-1/2} T_n(x)\, J_\nu(xy)\, dx = \frac{1}{2}\pi\, J_{\frac{1}{2}(\nu+n)}\left(\frac{1}{2}y\right) J_{\frac{1}{2}(\nu-n)}\left(\frac{1}{2}y\right)$$
$[y>0, \quad \operatorname{Re}\nu > -n-1]$ ET II 42(1)

7.362 $$\int_1^\infty \left(x^2-1\right)^{-\frac{1}{2}} T_n\left(\frac{1}{x}\right) K_{2\mu}(ax)\, dx = \frac{\pi}{2a}\, W_{\frac{1}{2}n,\mu}(a)\, W_{-\frac{1}{2}n,\mu}(a)$$
$[\operatorname{Re} a > 0]$ ET II 366(17)a

7.37–7.38 Hermite polynomials

7.371 $$\int_0^x H_n(y)\, dy = [2(n+1)]^{-1}\left[H_{n+1}(x) - H_{n+1}(0)\right]$$ EH II 194(27)

7.372 $$\int_{-1}^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} H_{2n}\left(\sqrt{x}t\right)\, dx = \frac{(-1)^n \pi^{1/2} (2n)!\, \Gamma\left(\alpha+\frac{1}{2}\right) L_n^\alpha(x)}{\Gamma(n+\alpha+1)}$$
$\left[\operatorname{Re} a > -\frac{1}{2}\right]$ EH II 195(34)

7.373

1. $$\int_0^x e^{-y^2} H_n(y)\, dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$ [see **8.956**] EH II 194(26)

2. $$\int_{-\infty}^\infty e^{-x^2} H_{2m}(xy)\, dx = \sqrt{\pi}\frac{(2m)!}{m!}\left(y^2-1\right)^m$$ EH II 195(28)

7.374

1. $$\int_{-\infty}^\infty e^{-x^2} H_n(x)\, H_m(x)\, dx = 0 \qquad [m \neq n]$$
$$= 2^n \cdot n!\sqrt{\pi} \qquad [m = n]$$ SM II 567

SM II 568

2.[11] $$\int_{-\infty}^\infty e^{-2x^2} H_m(x)\, H_n(x)\, dx = (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} 2^{\frac{m+n-1}{2}}\, \Gamma\left(\frac{m+n+1}{2}\right) \qquad [m+n \text{ is even}]$$
$$= 0 \qquad [m+n \text{ is odd}]$$
ET II 289(10)a

3. $$\int_{-\infty}^\infty e^{-x^2} H_m(ax)\, H_n(x)\, dx = 0$$ $[m<n]$ ET II 290(20)a

4. $$\int_{-\infty}^\infty e^{-x^2} H_{2m+n}(ax)\, H_n(x)\, dx = \sqrt{\pi} 2^n \frac{(2m+n)!}{m!}\left(a^2-1\right)^m a^n$$ ET II 291(21)a

5. $$\int_{-\infty}^\infty e^{-2\alpha^2 x^2} H_m(x)\, H_n(x)\, dx = 2^{\frac{m+n-1}{2}} \alpha^{-m-n-1}\left(1-2\alpha^2\right)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \times {}_2F_1\left(-m, n; \frac{1-m-n}{2}; \frac{\alpha^2}{2\alpha^2-1}\right)$$
$\left[\operatorname{Re}\alpha^2 > 0, \quad \alpha^2 \neq \frac{1}{2}, \quad m+n \text{ is even}\right]$ ET II 289(12)a

6. $$\int_{-\infty}^\infty e^{-(x-y)^2} H_n(x)\, dx = \pi^{1/2} y^n 2^n$$ ET II 288(2)a, EH II 195(31)

7. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x)\, dx = 2^n \pi^{1/2} m! y^{n-m} L_m^{n-m}\left(-2y^2\right)$$

$[m \leq n]$ BU 148(15), ET II 289(13)a

8. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x)\, dx = \pi^{1/2}\left(1-\alpha^2\right)^{\frac{n}{2}} H_n\left[\frac{\alpha y}{\left(1-\alpha^2\right)^{1/2}}\right]$$ ET II 290(17)a

9. $$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x)\, dx$$
$$= \pi^{1/2} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k}\binom{n}{k}\left(1-\alpha^2\right)^{\frac{m+n}{2}-k} H_{m+n-2k}\left[\frac{\alpha y}{\left(1-\alpha^2\right)^{1/2}}\right]$$
ET II 291(26)a

10. $$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x)\, dx = (2\pi u)^{1/2}(1-2u)^{\frac{n}{2}} H_n\left[y(1-2u)^{-1/2}\right]$$

$\left[0 \leq u < \frac{1}{2}\right]$ EH II 195(30)

7.375

1. $$\int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x)\, dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)} \Gamma(s-k)\Gamma(s-m)\Gamma(s-n)$$

$2s = k+m+n+1$ $[k+m+n \text{ is even}]$ ET II 290(14)a

2. $$\int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x)\, dx = \frac{2^{\frac{m+n+k}{2}} \pi^{1/2} k! m! n!}{(s-k)!(s-m)!(s-n)!},$$

$2s = m+n+k$ $[k+m+n \text{ is even}]$
ET II 290(15)a

7.376

1. $$\int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x)\, dx = (2\pi)^{1/2} e^{-\frac{y^2}{2}} H_n(y) i^n$$ MO 165a

2. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n}(x)\, dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}(\nu+1)}} F\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right)$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1]$ BU 150(18a)

3. $$\int_0^{\infty} e^{-2\alpha x^2} x^{\nu} H_{2n+1}(x)\, dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\alpha^{\frac{1}{2}\nu+1}} F\left(-n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha}\right)$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -2]$ BU 150(18b)

7.377[8] $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z)\, dx = 2^n \pi^{1/2} m! z^{n-m} L_m^{n-m}(-2yz)$$

$[m \leq n]$ ET II 292(30)a

7.378 $$\int_0^{\infty} x^{\alpha-1} e^{-\beta x} H_n(x)\, dx = 2^n \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!\,\Gamma(\alpha+n-2m)}{m!(n-2m)!}(-1)^m 2^{-2m} \beta^{2m-\alpha-n}$$

$[\operatorname{Re}\alpha > 0, \text{ if } n \text{ is even}; \quad \operatorname{Re}\alpha > -1, \text{ if } n \text{ is odd}; \quad \operatorname{Re}\beta > 0]$ ET I 172(11)a

7.379

1. $$\int_{-\infty}^{\infty} xe^{-x^2} H_{2m+1}(xy)\,dx = \pi^{1/2}\frac{(2m+1)!}{m!}y\left(y^2-1\right)^m$$ EH II 195(28)

2. $$\int_{-\infty}^{\infty} x^n e^{-x^2} H_n(xy)\,dx = \pi^{1/2} n! P_n(y)$$ EH II 195(29)

7.381 $$\int_{-\infty}^{\infty} (x \pm ic)^\nu e^{-x^2} H_n(x)\,dx = 2^{n-1-\nu}\pi^{1/2}\frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma(-\nu)}\exp\left[\pm\tfrac{1}{2}\pi(\nu+n)i\right]$$
$[c > 0]$ ET II 288(3)a

7.382 $$\int_0^\infty x^{-1}\left(x^2+a^2\right)^{-1} e^{-x^2} H_{2n+1}(x)\,dx = (-2)^n\pi^{1/2}a^{-2}\left[2^\nu n! - (2n+1)! e^{\frac{1}{2}a^2} D_{-2n-2}\left(a\sqrt{2}\right)\right]$$
ET II 288(4)a

7.383

1. $$\int_0^\infty e^{-xp} H_{2n+1}\left(\sqrt{x}\right)\,dx = (-1)^n 2^n (2n+1)!!\pi^{1/2}(p-1)^n p^{-n-\frac{3}{2}}$$
$[\operatorname{Re} p > 0]$ EF 151(261)a, ET I 172(12)a

2. $$\int_0^\infty e^{-(b-\beta x)} H_{2n+1}\left(\sqrt{(\alpha-\beta)x}\right)\,dx = (-1)^n\sqrt{\pi}\sqrt{\alpha-\beta}\frac{(2n+1)!}{n!}\frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}}$$
$[\operatorname{Re}(b-\beta) > 0]$ ET I 172(15)a

3. $$\int_0^\infty \frac{1}{\sqrt{x}} e^{-(b-\beta)x} H_{2n}\left(\sqrt{(\alpha-\beta)x}\right)\,dx = (-1)^n\sqrt{\pi}\frac{(2n)!}{n!}\frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}}$$
$[\operatorname{Re}(b-\beta) > 0]$ ET I 172(16)a

4. $$\int_0^\infty x^{a-\frac{1}{2}n-1} e^{-bx} H_n\left(\sqrt{x}\right)\,dx = 2^n\,\Gamma(a) b^{-a}\ {}_2F_1\left(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; 1-a; b\right)$$
$\Big[\operatorname{Re} a > \frac{1}{2}n$, if n is even, $\operatorname{Re} a > \frac{1}{2}n - \frac{1}{2}$, if n is odd, $\operatorname{Re} b > 0$,
If a is even, only the first $1 + \left\lfloor \frac{n}{2} \right\rfloor$ terms are kept in the series for ${}_2F_1\Big]$
ET I 172(14)a

5. $$\int_0^\infty x^{-1/2} e^{-px} H_{2n}\left(\sqrt{x}\right)\,dx = (-1)^n 2^n (2n-1)!!\pi^{1/2}(p-1)^n p^{-n-\frac{1}{2}}$$ MO 177a

7.384 $$\int_0^\infty \frac{1}{\sqrt{x}} e^{-bx}\left[H_n\left(\frac{\alpha+\sqrt{x}}{\lambda}\right) + H_n\left(\frac{a-\sqrt{x}}{\lambda}\right)\right]\,dx = \sqrt{\frac{2\pi}{b}}\left(1-\lambda^{-2}b^{-1}\right)^{\frac{n}{2}} H_n\left(\frac{\alpha}{\sqrt{\lambda^2-\frac{1}{b}}}\right)$$
$[\operatorname{Re} b > 0]$ ET I 173(17)a

7.385

1. $$\int_0^\infty \frac{e^{-bx}}{\sqrt{e^x-1}} H_{2n}\left[\sqrt{s\left(1-e^{-x}\right)}\right]\,dx = (-1)^n 2^{2n}\sqrt{\pi}\frac{(2n)!\,\Gamma\left(b+\frac{1}{2}\right)}{\Gamma(n+b+1)} L_n^n(s)$$
$\left[\operatorname{Re} b > -\frac{1}{2}\right]$ ET I 174(23)a

2. $$\int_0^\infty e^{-bx} H_{2n+1}\left[\sqrt{s}\sqrt{1-e^{-x}}\right] dx = (-1)^n 2^{2n} \sqrt{\pi s} \frac{(2n+1)!\,\Gamma(b)}{\Gamma\left(n+b+\frac{3}{2}\right)} L_n^b(s)$$
$[\operatorname{Re} b > 0]$ ET I 174(24)a

7.386 $$\int_0^\infty x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}} H_n\left(\frac{q}{2\sqrt{x}}\right) e^{-px}\, dx = 2^n \pi^{1/2} p^{\frac{n-1}{2}} e^{-q\sqrt{p}}$$ EF 129(117)

7.387

1. $$\int_0^\infty e^{-x^2} \sinh\left(\sqrt{2}\beta x\right) H_{2n+1}(x)\, dx = 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{\frac{1}{2}\beta^2}$$ ET II 289(7)a

2. $$\int_0^\infty e^{-x^2} \cosh\left(\sqrt{2}\beta x\right) H_{2n}(x)\, dx = 2^{n-1} \pi^{1/2} \beta^{2n} e^{\frac{1}{2}\beta^2}$$ ET II 289(8)a

7.388

1. $$\int_0^\infty e^{-x^2} \sin\left(\sqrt{2}\beta x\right) H_{2n+1}(x)\, dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{1/2} \beta^{2n+1} e^{-\frac{1}{2}\beta^2}$$ ET II 288(5)a

2. $$\int_0^\infty e^{-x^2} \sin\left(\sqrt{2}\beta x\right) H_{2n+1}(ax)\, dx = (-1)^n 2^{-1} \pi^{1/2} \left(a^2-1\right)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1}\left(\frac{a\beta}{\sqrt{2}\left(a^2-1\right)^{1/2}}\right)$$ ET II 290(18)a

3. $$\int_0^\infty e^{-x^2} \cos\left(\sqrt{2}\beta x\right) H_{2n}(x)\, dx = (-1)^n 2^{n-1} \pi^{1/2} \beta^{2n} e^{-\frac{1}{2}\beta^2}$$ ET II 289(6)a

4. $$\int_0^\infty e^{-x^2} \cos\left(\sqrt{2}\beta x\right) H_{2n}(ax)\, dx = 2^{-1} \pi^{1/2} \left(1-a^2\right)^n e^{-\frac{1}{2}\beta^2} H_{2n}\left[\frac{a\beta}{\sqrt{2}\left(a^2-1\right)^{1/2}}\right]$$ ET II 290(19)a

5. $$\int_0^\infty e^{-y^2} \left[H_n(y)\right]^2 \cos\left(\sqrt{2}\beta y\right)\, dy = \pi^{1/2} 2^{n-1} n! e^{-\frac{\beta^2}{2}} L_n\left(\beta^2\right)$$ EH II 195(33)

6.[11] $$\int_0^\infty e^{-x^2} \sin(bx)\, H_n(x)\, H_{n+2m+1}(x)\, dx = 2^{n-1} (-1)^m \sqrt{\pi} n! b^{2m+1} e^{-\frac{b^2}{4}} L_n^{2m+1}\left(\frac{b^2}{2}\right)$$
$[b > 0]$ ET I 39(11)a

7. $$\int_0^\infty e^{-x^2} \cos(bx)\, H_n(x)\, H_{n+2m}(x)\, dx = 2^{n-\frac{1}{2}} \sqrt{\frac{\pi}{2}} n! (-1)^m b^{2m} e^{-\frac{b^2}{4}} L_n^{2m}\left(\frac{b^2}{2}\right)$$
$[b > 0]$ ET I 39(11)a

7.389 $$\int_0^\pi (\cos x)^n H_{2n}\left[a\left(1-\sec x\right)^{1/2}\right] dx = 2^{-n} (-1)^n \pi \frac{(2n)!}{(n!)^2} \left[H_n(a)\right]^2$$ ET II 292(31)

7.39 Jacobi polynomials

7.391

1. $$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\, dx$$
$$= 0 \qquad [m \neq n, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$$
$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+1+2n)\,\Gamma(\alpha+\beta+n+1)} \qquad [m = n, \quad \operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$$
ET II 285(5, 9)

2. $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\rho+\sigma+1}\,\Gamma(\rho+1)\,\Gamma(\sigma+1)\,\Gamma(n+1+\alpha)}{n!\,\Gamma(\rho+\sigma+2)\,\Gamma(1+\alpha)}\times {}_3F_2\left(-n,\alpha+\beta+n+1,\rho+1;\alpha+1,\rho+\sigma+2;1\right)$$

$[\operatorname{Re}\rho>-1, \quad \operatorname{Re}\sigma>-1]$ ET II 284(3)

3.[6] $$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\alpha+\sigma+1}\,\Gamma(\sigma+1)\,\Gamma(\alpha+1)\,\Gamma(\sigma-\beta+1)}{n!\,\Gamma(\sigma-\beta-n+1)\,\Gamma(\alpha+\sigma+n+2)}$$

$[\operatorname{Re}\alpha>-1, \quad \operatorname{Re}\sigma>-1]$ ET II 284(1)

4. $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,dx=\frac{2^{\beta+\rho+1}\,\Gamma(\rho+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha-\rho+n)}{n!\,\Gamma(\alpha-\rho)\,\Gamma(\beta+\rho+n+2)}$$

$[\operatorname{Re}\rho>-1, \quad \operatorname{Re}\beta>-1]$ ET II 284(2)

5. $$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2\,dx=\frac{2^{\alpha+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\alpha\,\Gamma(\alpha+\beta+n+1)}$$

$[\operatorname{Re}\alpha>0, \quad \operatorname{Re}\beta>-1]$ ET II 285(6)

6. $$\int_{-1}^{1}(1-x)^{2\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2\,dx=\frac{2^{4\alpha+\beta+1}\,\Gamma\left(\alpha+\frac{1}{2}\right)\left[\Gamma(\alpha+n+1)\right]^2\Gamma(\beta+2n+1)}{\sqrt{\pi}\,(n!)^2\,\Gamma(\alpha+1)\,\Gamma(2\alpha+\beta+2n+2)}$$

$\left[\operatorname{Re}\alpha>-\frac{1}{2}, \quad \operatorname{Re}\beta>-1\right]$ ET II 285(7)

7. $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\rho,\beta)}(x)\,dx =\frac{2^{\rho+\beta+1}\,\Gamma(\rho+n+1)\,\Gamma(\beta+n+1)\,\Gamma\left(\alpha+\beta+2n+1\right)}{n!\,\Gamma(\beta+\rho+2n+2)\,\Gamma(\alpha+\beta+n+1)}$$

$[\operatorname{Re}\rho>-1, \quad \operatorname{Re}\beta>-1]$ ET II285(10)

8. $$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\rho,\beta)}(x)\,dx=\frac{2^{\rho+\beta}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\rho)}{n!\,\Gamma(\alpha+1)\,\Gamma(\rho+\beta+n+1)}$$

$[\operatorname{Re}\beta>-1, \quad \operatorname{Re}\rho>0]$ ET II 286(11)

9.[7] $$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\sigma)}(x)\,dx =\frac{2^{\alpha+\sigma+1}\,\Gamma(\alpha+n+1)\,\Gamma\left(\alpha+\beta+m+n+1\right)\Gamma(\sigma+m+1)\,\Gamma(\sigma-\beta+1)}{m!(n-m)!\,\Gamma\left(\alpha+\beta+n+1\right)\Gamma(\alpha+\sigma+m+n+2)\,\Gamma\left(\alpha-\beta+m-n+1\right)}$$

$[\operatorname{Re}\alpha>-1, \quad \operatorname{Re}\sigma>-1]$ ET II 286(12)

10.[6] $$\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,P_m^{(\rho,\beta)}(x)\,dx =\frac{2^{\beta+\rho+1}\,\Gamma(\alpha+\beta+m+n+1)\,\Gamma\left(\beta+n+1\right)\Gamma(\rho+m+1)}{m!(n-m)!\,\Gamma(\alpha+\beta+n+1)\,\Gamma\left(\beta+\rho+m+n+2\right)}\frac{\Gamma\left(\alpha-\rho-m+n\right)}{\Gamma(\alpha-\rho)}$$

$[\operatorname{Re}\beta>-1, \quad \operatorname{Re}\rho>-1]$ ET II 287(16)

11. $$\int_{0}^{x}(1-y)^{\alpha}(1+y)^{\beta}\,P_n^{(\alpha,\beta)}(y)\,dy=\frac{1}{2n}\left[P_{n-1}^{(\alpha+1,\beta+1)}(0)-(1-x)^{\alpha+1}(1+x)^{\beta+1}\,P_{n-1}^{(\alpha+1,\beta+1)}(x)\right]$$

EH II 173(38)

7.392

1. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,P_n^{(\alpha,\beta)}(1-\gamma x)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\;{}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \frac{1}{2}\gamma\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(46)a

2. $$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,P_n^{(\alpha,\beta)}(\gamma x-1)\,dx = (-1)^n\frac{\Gamma(\beta+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\beta+1)\,\Gamma(\lambda+\mu)}\;{}_3F_2\left(-n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \frac{1}{2}\gamma\right) a$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(47)a

3. $$\int_0^1 x^{\alpha}(1-x)^{\mu-1}\,P_n^{(\alpha,\beta)}(1-\gamma x)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)}\,P_n^{(\alpha+\mu,\beta-\mu)}(1-\gamma)$$
$[\operatorname{Re} a > -1, \quad \operatorname{Re}\mu > 0]$ ET II 191(43)a

4. $$\int_0^1 x^{\beta}(1-x)^{\mu-1}\,P_n^{(\alpha,\beta)}(\gamma x-1)\,dx = \frac{\Gamma(\beta+n+1)\,\Gamma(\mu)}{\Gamma(\beta+\mu+n+1)}\,P_n^{(\alpha-\mu,\beta+\mu)}(\gamma-1)$$
$[\operatorname{Re}\beta > -1, \quad \operatorname{Re}\mu > 0]$ ET II 191(44)a

7.393

1. $$\int_0^1 \left(1-x^2\right)^{\nu}\sin bx\,P_{2n+1}^{(\nu,\nu)}(x)\,dx = \frac{(-1)^n\sqrt{\pi}\,\Gamma(2n+\nu+2)\,J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu}(2n+1)!b^{\nu+\frac{1}{2}}}$$
$[b > 0, \quad \operatorname{Re}\nu > -1]$ ET I 94(5)

2. $$\int_0^1 \left(1-x^2\right)^{\nu}\cos bx\,P_{2n}^{(\nu,\nu)}(x)\,dx = \frac{(-1)^n 2^{\nu-\frac{1}{2}}\sqrt{\pi}\,\Gamma(2n+\nu+1)\,J_{2n+\nu+\frac{1}{2}}(b)}{(2n)!b^{\nu+\frac{1}{2}}}$$
$[b > 0, \quad \operatorname{Re}\nu > -1]$ ET I 38(4)

7.41–7.42 Laguerre polynomials

7.411

1. $$\int_0^t L_n(x)\,dx = L_n(t) - L_{n+1}(t)/(n+1)$$ MO 110

2. $$\int_0^t L_n^{\alpha}(x)\,dx = L_n^{\alpha}(t) - L_{n+1}^{\alpha}(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}$$ EH II 189(16)a

3. $$\int_0^t L_{n-1}^{\alpha+1}(x)\,dx = -\,L_n^{\alpha}(t) + \binom{n+\alpha}{n}$$ EH II 189(15)a

4. $$\int_0^t L_m(x)\,L_n(t-x)\,dx = L_{m+n}(t) - L_{m+n+1}(t)$$ EH II 191(31)

5. $$\sum_{k=0}^{\infty}\left[\int_0^t \frac{L_k(x)}{k!}\,dx\right]^2 = e^t - 1$$ $[t \geq 0]$ MO 110

7.412

1. $$\int_0^1 (1-x)^{\mu-1} x^\alpha L_n^\alpha(ax)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} L_n^{\alpha+\mu}(a)$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\mu > 0]$
EH II 191(30)a, BU 129(14c)

2. $$\int_0^1 (1-x)^{\mu-1} x^{\lambda-1} L_n^\alpha(\beta x)\,dx = \frac{\Gamma(\alpha+n+1)\,\Gamma(\lambda)\,\Gamma(\mu)}{n!\,\Gamma(\alpha+1)\,\Gamma(\lambda+\mu)}\, {}_2F_2(-n,\lambda;\alpha+1,\lambda+\mu:\beta)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 192(50)a

7.413 $$\int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy)\, L_n^\beta[(1-x)y]\,dx = \frac{(m+n)!\,\Gamma(\alpha+m+1)\,\Gamma(\beta+n+1)}{m!n!\,\Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y)$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re}\beta > -1]$ ET II 293(7)

7.414

1.[11] $$\int_y^\infty e^{-x} L_n^\alpha(x)\,dx = e^{-y}\left[L_n^\alpha(y) - L_{n-1}^\alpha(y)\right]$$
EH II 191(29)

2. $$\int_0^\infty e^{-bx} L_n(\lambda x)\, L_n(\mu x)\,dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n\left[\frac{b^2-(\lambda+\mu)b+2\lambda\mu}{b(b-\lambda-\mu)}\right]$$
$[\operatorname{Re} b > 0]$ ET I 175(34)

3.[8] $$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x)\, L_m^\alpha(x)\,dx = 0 \qquad [m \neq n, \quad \operatorname{Re}\alpha > -1]$$ BU 115(8), ET II 293(3)
$$= \frac{\Gamma(\alpha+n+1)}{n!} \qquad [m = n, \quad \operatorname{Re}\alpha > 0]$$ BU 115(8), ET II 292(2)

4. $$\int_0^\infty e^{-bx} x^\alpha L_n^\alpha(\lambda x)\, L_m^\alpha(\mu x)\,dx = \frac{\Gamma(m+n+\alpha+1)}{m!n!} \frac{(b-\lambda)^n (b-\mu)^m}{b^{m+n+\alpha+1}} \times F\left[-m,-n;-m-n-\alpha,\frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right]$$
$[\operatorname{Re}\alpha > -1, \quad \operatorname{Re} b > 0]$ ET I 175(35)

4(1)[9]. $$\int_0^\infty e^{-x} x^{\alpha+1/2} L_n^\alpha(x)\, L_m^\alpha(x)\,dx = \frac{\Gamma(\alpha+n+1)^2\,\Gamma(\alpha+m+1)\,\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma\left(m-\frac{1}{2}\right)}{n!m!\,\Gamma(\alpha+1)\,\Gamma\left(-\frac{1}{2}\right)} \times {}_3F_2\left(-n,\alpha+\tfrac{3}{2},\tfrac{3}{2};\alpha+1,\tfrac{3}{2}-m;1\right)$$

5. $$\int_0^\infty e^{-bx} L_n^a(x)\,dx = \sum_{m=0}^n \binom{a+m-1}{m} \frac{(b-1)^{n-m}}{b^{n-m+1}} \qquad [\operatorname{Re} b > 0]$$ ET I 174(27)

6. $$\int_0^\infty e^{-bx} L_n(x)\,dx = (b-1)^n b^{-n-1} \qquad [\operatorname{Re} b > 0]$$ ET I 174(25)

7. $$\int_0^\infty e^{-st} t^\beta L_n^\alpha(t)\,dt = \frac{\Gamma(\beta+1)\,\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)} s^{-\beta-1} F\left(-n,\beta+1;\alpha+1;\frac{1}{s}\right)$$
$[\operatorname{Re}\beta > -1, \quad \operatorname{Re} s > 0]$
BU 119(4b), EH II 191(133)

8. $$\int_0^\infty e^{-st} t^\alpha L_n^\alpha(t)\,dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n! s^{\alpha+n+1}} \qquad [\operatorname{Re}\alpha > -1, \quad \operatorname{Re} s > 0]$$
EH II 191(32), MO 176a

9. $$\int_0^\infty e^{-x}x^{\alpha+\beta}\,L_m^\alpha(x)\,L_n^\beta(x)\,dx = (-1)^{m+n}(\alpha+\beta)!\binom{\alpha+m}{n}\binom{\beta+n}{m}$$
$[\operatorname{Re}(\alpha+\beta) > -1]$ ET II 293(4)

10.[6] $$\int_0^\infty e^{-bx}x^{2a}\left[L_n^a(x)\right]^2\,dx = \frac{2^{2a}\,\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\pi\,(n!)^2\,b^{2a+1}} \times F\left(-n, a+\frac{1}{2}; \frac{1}{2}-n; \left(1-\frac{2}{b}\right)^2\right)\Gamma(a+n+1)$$
$\left[\operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re} b > 0\right]$ ET I 174(30)

11. $$\int_0^\infty e^{-x}x^{\gamma-1}\,L_n^\mu(x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(1+\mu+n-\gamma)}{n!\,\Gamma(1+\mu-\gamma)}$$ $[\operatorname{Re}\gamma > 0]$ BU 120(4b)

12. $$\int_0^\infty e^{-x\left(s+\frac{a_1+a_2}{2}\right)}x^{\mu+\beta}\,L_k^\mu\left(a_1x\right)L_k^\mu\left(a_2x\right)\,dx$$
$$= \frac{\Gamma(1+\mu+\beta)\,\Gamma(1+\mu+k)}{k!k!\,\Gamma\left(1+\mu\right)}\left\{\frac{d^k}{dh^k}\left[\frac{F\left(\frac{1+\mu+\beta}{2}, 1+\frac{\mu+\beta}{2}; 1+\mu; \frac{A^2}{B^2}\right)}{(1-h)^{1+\mu}B^{1+\mu+\beta}}\right]\right\}_{h=0}$$
$$A^2 = \frac{4a_1a_2h}{(1-h)^2}; \qquad B = s+\frac{a_1+a_2}{2}\,\frac{1+h}{1-h}$$
$\left[\operatorname{Re}\left(s+\frac{a_1+a_2}{2}\right) > 0, \quad a_1 > 0, \quad a_2 > 0, \quad \operatorname{Re}(\mu+\beta) > -1\right]$ BU 142(19)

13. $$\int_0^\infty \exp\left[-x\left(s+\frac{a_1+a_2}{2}\right)\right]x^\mu\,L_k^\mu(a_1x)\,L_k^\mu(a_2x)\,dx = \frac{\Gamma(1+\mu+k)}{b_0^{1+\mu+k}}\cdot\frac{b_0^k}{k!}\cdot P_k^{(\mu,0)}\left(\frac{b_1^2}{b_0b_2}\right)$$
$$b_0 = s+\frac{a_1+a_2}{2}, \quad b_1^2 = b_0b_2+2a_1a_2, \quad b_2 = s-\frac{a_1+a_2}{2}$$
$\left[\operatorname{Re}\mu > -1, \quad \operatorname{Re}\left(s+\frac{a_1+a_2}{2}\right) > 0\right]$
BU 144(22)

7.415 $$\int_0^1 (1-x)^{\mu-1}x^{\lambda-1}e^{-\beta x}\,L_n^\alpha(\beta x)\,dx = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}\,\mathrm{B}(\lambda,\mu)\;{}_2F_2\left(\alpha+n+1, \lambda; \alpha+1, \lambda+\mu; -\beta\right)$$
$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0]$ ET II 193(51)a

7.416 $$\int_{-\infty}^\infty x^{m-n}\exp\left[-\frac{1}{2}(x-y)^2\right]L_n^{m-n}\left(x^2\right)\,dx = \frac{(2\pi)^{1/2}}{n!}i^{n-m}2^{-\frac{n+m}{2}}\,H_n\left(\frac{iy}{\sqrt{2}}\right)H_m\left(\frac{iy}{\sqrt{2}}\right)$$
BU 149(15b), ET II 293(8)a

7.417

1. $$\int_0^\infty x^{\nu-2n-1}e^{-ax}\sin(bx)\,L_{2n}^{\nu-2n-1}(ax)\,dx = (-1)^n i\,\Gamma(\nu)\frac{b^{2n}\left[(a-ib)^{-\nu}-(a+ib)^{-\nu}\right]}{2(2n)!}$$
$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 2n]$
ET I 95(12)

2. $$\int_0^\infty x^{\nu-2n-2}e^{-ax}\sin(bx)\,L_{2n+1}^{\nu-2n-2}(ax)\,dx = (-1)^{n+1}\,\Gamma(\nu)\frac{b^{2n+1}\left[(a+ib)^{-\nu}+(a-ib)^{-\nu}\right]}{2(2n+1)!}$$
$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 2n+1]$
ET I 95(13)

3. $$\int_0^\infty x^{\nu-2n} e^{-ax} \cos(bx)\, L_{\nu-2n}^{2n-1}(ax)\, dx = i(-1)^{n+1}\,\Gamma(\nu) \frac{b^{2n-1}\left[(a-ib)^{-\nu} - (a+ib)^{-\nu}\right]}{2(2n-1)!}$$
$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 2n-1]$
ET I 39(12)

4. $$\int_0^\infty x^{\nu-2n-1} e^{-ax} \cos(bx)\, L_{2n}^{\nu-2n-1}(ax)\, dx = (-1)^n\,\Gamma(\nu) \frac{b^{2n}\left[(a+ib)^{-\nu} + (a-ib)^{-\nu}\right]}{2(2n)!}$$
$[b > 0, \quad \operatorname{Re}\nu > 2n, \quad \operatorname{Re} a > 0]$
ET I 39(13)

7.418

1. $$\int_0^\infty e^{-\frac{1}{2}x^2} \sin(bx)\, L_n\left(x^2\right)\, dx = (-1)^n \frac{i}{2} n! \frac{1}{\sqrt{2\pi}} \left\{\left[D_{-n-1}(ib)\right]^2 - \left[D_{-n-1}(-ib)\right]^2\right\}$$
$[b > 0]$ ET I 95(14)

2. $$\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx)\, L_n\left(x^2\right)\, dx = \sqrt{\frac{\pi}{2}}\,(n!)^{-1}\, e^{-\frac{1}{2}b^2} 2^{-n} \left[H_n\left(\frac{b}{\sqrt{2}}\right)\right]^2$$
$[b > 0]$ ET I 39(14)

3. $$\int_0^\infty x^{2n+1} e^{-\frac{1}{2}x^2} \sin(bx)\, L_n^{n+\frac{1}{2}}\left(\frac{1}{2}x^2\right)\, dx = \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2}\, L_n^{n+\frac{1}{2}}\left(\frac{b^2}{2}\right)$$
$[b > 0]$ ET I 95(15)

4. $$\int_0^\infty x^{2n} e^{-\frac{1}{2}x^2} \cos(bx)\, L_n^{n-\frac{1}{2}}\left(\frac{1}{2}x^2\right)\, dx = \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2}\, L_n^{n+\frac{1}{2}}\left(\frac{1}{2}b^2\right)$$
$[b > 0]$ ET I 39(16)

5. $$\int_0^\infty x e^{-\frac{1}{2}x^2}\, L_n^{\alpha}\left(\frac{1}{2}x^2\right) L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right) \sin(xy)\, dx = \left(\frac{\pi}{2}\right)^{1/2} y e^{-\frac{1}{2}y^2}\, L_n^{\alpha}\left(\frac{1}{2}y^2\right) L_n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}y^2\right)$$
ET II 294(11)

6. $$\int_0^\infty e^{-\frac{1}{2}x^2}\, L_n^{\alpha}\left(\frac{1}{2}x^2\right) L_n^{-\frac{1}{2}-\alpha}\left(\frac{1}{2}x^2\right) \cos(xy)\, dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2}\, L_n^{\alpha}\left(\frac{1}{2}y^2\right) L_n^{-\alpha-\frac{1}{2}}\left(\frac{1}{2}y^2\right)$$
ET II 294(12)

7.419 $$\int_0^\infty x^{n+2\nu-\frac{1}{2}} \exp[-(1+a)x]\, L_n^{2\nu}(ax)\, K_\nu(x)\, dx$$
$$= \frac{\pi^{1/2}\,\Gamma\left(n+\nu+\frac{1}{2}\right)\Gamma\left(n+3\nu+\frac{1}{2}\right)}{2^{n+2\nu+\frac{1}{2}} n!\,\Gamma(2\nu+1)} F\left(n+\nu+\frac{1}{2}, n+3\nu+\frac{1}{2}; 2\nu+1; -\frac{1}{2}a\right)$$
$\left[\operatorname{Re} a > -2, \quad \operatorname{Re}(n+\nu) > -\frac{1}{2}, \quad \operatorname{Re}(n+3\nu) > -\frac{1}{2}\right]$ ET II 370(44)

7.421

1. $$\int_0^\infty x e^{-\frac{1}{2}\alpha x^2}\, L_n\left(\frac{1}{2}\beta x^2\right) J_0(xy)\, dx = \frac{(\alpha-\beta)^n}{\alpha^{n+1}} e^{-\frac{1}{2\alpha}y^2}\, L_n\left[\frac{\beta y^2}{2\alpha(\beta-\alpha)}\right]$$
$[y > 0, \quad \operatorname{Re}\alpha > 0]$ ET II 13(4)a

2. $$\int_0^\infty x e^{-x^2}\, L_n\left(x^2\right) J_0(xy)\, dx = \frac{2^{-2n-1}}{n!} y^{2n} e^{-\frac{1}{4}y^2}$$
ET II 13(5)

3. $$\int_0^\infty x^{2n+\nu+1} e^{-\frac{1}{2}x^2} L_n^{\nu+n}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = y^{2n+\nu} e^{-\frac{1}{2}y^2} L_n^{\nu+n}\left(\frac{1}{2}y^2\right)$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ MO 183

4. $$\int_0^\infty x^{\nu+1} e^{-\beta x^2} L_n^\nu\left(\alpha x^2\right) J_\nu(xy)\,dx = 2^{-\nu-1}\beta^{-\nu-n-1}(\beta-\alpha)^n y^\nu e^{-\frac{y^2}{4\beta}} L_n^\nu\left[\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right]$$
ET II 43(5)

5. $$\int_0^\infty e^{-\frac{1}{2q}x^2} x^{\nu+1} L_n^\nu\left[\frac{x^2}{2q(1-q)}\right] J_\nu(xy)\,dx = \frac{q^{n+\nu+1}}{(q-1)^n} e^{-\frac{qy^2}{2}} y^\nu L_n^\nu\left(\frac{y^2}{2}\right)$$
$[\nu > 0]$ MO 183

6.* $$\int_0^\infty x^{\nu+1} e^{-x^2} L_n^\nu\left(x^2\right) J_\nu(xy)\,dx = \frac{1}{2n!}\left(\frac{y}{2}\right)^{2n+\nu} e^{-\frac{1}{4}y^2}$$

7.422

1. $$\int_0^\infty x^{\nu+1} e^{-\beta x^2}\left[L_n^{\frac{1}{2}\nu}\left(\alpha x^2\right)\right]^2 J_\nu(xy)\,dx$$
$$= \frac{y^\nu}{\pi n!}\Gamma\left(n+1+\tfrac{1}{2}\nu\right)(2\beta)^{-\nu-1} e^{-\frac{y^2}{4\beta}} \times \sum_{l=0}^n \frac{(-1)^l\,\Gamma\left(n-l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma\left(l+1+\frac{1}{2}\nu\right)(n-l)!}\left(\frac{2\alpha-\beta}{\beta}\right)^{2l} L_{2l}^\nu\left[\frac{\alpha y^2}{2\beta(2\alpha-\beta)}\right]$$
$[y > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\nu > -1]$ ET II 43(7)

2.[9] $$\int_0^\infty x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}\left(\alpha x^2\right) L_n^\sigma\left(\alpha x^2\right) J_\nu(xy)\,dx$$
$$= (-1)^{m+n}(2\alpha)^{-\nu-1} y^\nu e^{-\frac{y^2}{4\alpha}} L_n^{m-n-\sigma}\left(\frac{y^2}{4\alpha}\right) L_m^{n-m+\sigma-\nu}\left(\frac{y^2}{4\alpha}\right)$$
$[y > 0, \quad \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\nu > -1, \quad n \neq 0, \quad \sigma \neq 0, \quad \alpha \neq 1]$ ET II 43(8)

7.423

1. $$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n+1}\left(\frac{x}{2\sqrt{2}}\right)\sin(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n+1}\left(\frac{y}{2\sqrt{2}}\right)$$
ET II 294(13)a

2. $$\int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n}\left(\frac{x}{2\sqrt{2}}\right)\cos(xy)\,dx = \left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n}\left(\frac{y}{2\sqrt{2}}\right)$$
ET II 294(14)a

7.5 Hypergeometric Functions

7.51 Combinations of hypergeometric functions and powers

7.511 $$\int_0^\infty F(a,b;c;-z) z^{-s-1}\,dx = \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c)\,\Gamma(-s)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+s)}$$
$[c \neq 0, -1, -2, \ldots, \quad \operatorname{Re}s < 0, \quad \operatorname{Re}(a+s) > 0, \quad \operatorname{Re}(b+s) > 0]$ EH I 79(4)

7.512

1. $$\int_0^1 x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1} F(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)\,\Gamma\left(\gamma-\frac{\alpha}{2}-\beta\right)}{\Gamma(1+\alpha)\,\Gamma\left(1+\frac{\alpha}{2}-\beta\right)\Gamma\left(\gamma-\frac{\alpha}{2}\right)}$$
$$\left[\operatorname{Re}\alpha+1>\operatorname{Re}\gamma>\operatorname{Re}\beta, \quad \operatorname{Re}\left(\gamma-\frac{\alpha}{2}-\beta\right)>0\right]$$ ET II 398(1)

2. $$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n} F(-n,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\gamma+1)\,\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\rho)\,\Gamma(\beta-\gamma+\rho+1)}$$
$$[n=0,1,2\ldots; \quad \operatorname{Re}\rho>0, \quad \operatorname{Re}(\beta-\gamma)>n-1]$$ ET II 398(2)

3. $$\int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1} F(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\rho)\,\Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\rho)}$$
$$[\operatorname{Re}\rho>0, \quad \operatorname{Re}(\beta-\rho)>0, \quad \operatorname{Re}(\gamma-\alpha-\rho)>0]$$ ET II 399(3)

4. $$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} F(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}$$
$$[\operatorname{Re}\gamma>0, \quad \operatorname{Re}\rho>0, \quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0]$$ ET II 399(4)

5. $$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1} F(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\,{}_3F_2(\alpha,\beta,\rho;\gamma,\rho+\sigma;1)$$
$$[\operatorname{Re}\rho>0, \quad \operatorname{Re}\sigma>0, \quad \operatorname{Re}(\gamma+\sigma-\alpha-\beta)>0]$$ ET II 399(5)

6.[10] $$\int_0^1 x^{\lambda-1}(1-x)^{\beta-\lambda-1} F\left(\alpha,\beta;\lambda;\frac{zx}{b}\right)dx = \mathrm{B}(\lambda,\beta-\lambda)(1-z/b)^{-\alpha}$$ BU 9

7.[11] $$\int_0^1 x^{\gamma-1}(1-x)^{\delta-\gamma-1} F(\alpha,\beta;\gamma;xz)\,F\left(\delta-\alpha,\delta-\beta;\delta-\gamma;(1-x)\zeta\right)\,dx$$
$$= \frac{\Gamma(\gamma)\,\Gamma(\delta-\gamma)}{\Gamma(\delta)}(1-\zeta)^{\alpha+\beta-\delta} F(\alpha,\beta;\delta;z+\zeta-z\zeta)$$
$$[0<\operatorname{Re}\gamma<\operatorname{Re}\delta, \quad |\arg(1-z)|<\pi, \quad |\arg(1-\zeta)|<\pi]$$ ET II 400(11)

8. $$\int_0^1 x^{\gamma-1}(1-x)^{\epsilon-1}(1-xz)^{-\delta} F(\alpha,\beta;\gamma;xz)\,F\left[\delta,\beta-\gamma;\epsilon;\frac{(1-x)z}{(1-xz)}\right]dx$$
$$= \frac{\Gamma(\gamma)\,\Gamma(\epsilon)}{\Gamma(\gamma+\epsilon)} F(\alpha+\delta,\beta;\gamma+\epsilon;z)$$
$$[\operatorname{Re}\gamma>0, \quad \operatorname{Re}\epsilon>0, \quad |\arg(z-1)|<\pi]$$ ET II 400(12), Eh I 78(3)

9. $$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha,\beta;\gamma;x)\,dx$$
$$= \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}(1-z)^{-\sigma}$$
$$\times\,{}_3F_2\left(\rho,\sigma,\gamma+\rho-\alpha-\beta;\gamma+\rho-\alpha,\gamma+\rho-\beta;\frac{z}{z-1}\right)$$
$$[\operatorname{Re}\gamma>0, \quad \operatorname{Re}\rho>0, \quad \operatorname{Re}(\gamma+\rho-\alpha-\beta)>0, \quad |\arg(1-z)|<\pi]$$ ET II 399(6)

10. $$\int_0^\infty x^{\gamma-1}(x+z)^{-\sigma} F(\alpha,\beta;\gamma;-x)\,dx = \frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+\sigma)\,\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\,\Gamma(\alpha+\beta-\gamma+\sigma)} \times F(\alpha-\gamma+\sigma,\beta-\gamma+\sigma;\alpha+\beta-\gamma+\sigma;1-z)$$

$[\operatorname{Re}\gamma>0,\quad \operatorname{Re}(\alpha-\gamma+\sigma)>0,\quad \operatorname{Re}(\beta-\gamma+\sigma)>0,\quad |\arg z|<\pi]$ ET II 400(10)

11. $$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\,{}_pF_q(a_1,\ldots,a_p;\nu,b_2,\ldots,b_q;ax)\,dx = \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\,{}_pF_q(a_1,\ldots,a_p;\mu+\nu,b_2,\ldots,b_q;a)$$

$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad p\le q+1;\ \text{if } p=q+1, \text{ then } |a|<1]$ ET II 200(94)

12. $$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;ax)\,dx = \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}\,{}_{p+1}F_{q+1}(\nu,a_1,\ldots,a_p;\mu+\nu,b_1,\ldots,b_q;a)$$

$[\operatorname{Re}\mu>0,\quad \operatorname{Re}\nu>0,\quad p\le q+1,\ \text{if } p=q+1, \text{ then } |a|<1]$ ET II 200(95)

7.513 $$\int_0^1 x^{s-1}\left(1-x^2\right)^\nu F\left(-n,a;b;x^2\right)\,dx = \frac{1}{2}\,\mathrm{B}\left(\nu+1,\frac{s}{2}\right)\,{}_3F_2\left(-n,a,\frac{s}{2};b,\nu+1+\frac{s}{2};1\right)$$

$[\operatorname{Re}s>0,\quad \operatorname{Re}\nu>-1]$ ET I 336(4)

7.52 Combinations of hypergeometric functions and exponentials

7.521 $$\int_0^\infty e^{-st}\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q,t)\,dt = \frac{1}{s}\,{}_{p+1}F_q\left(1,a_1,\ldots,a_p;b_1,\ldots,b_q,s^{-1}\right)$$

$[p\le q]$ EH I 192

7.522

1.[11] $$\int_0^\infty e^{-\lambda x}x^{\gamma-1}\,{}_2F_1(\alpha,\beta;\delta;-x)\,dx = \frac{\Gamma(\delta)\lambda^{-\gamma}}{\Gamma(\alpha)\,\Gamma(\beta)}\,E(\alpha,\beta,\gamma:\delta:\lambda)$$

$[\operatorname{Re}\lambda>0,\quad \operatorname{Re}\gamma>0]$ EH I 205(10)

2.[6] $$\int_0^\infty e^{-bx}x^{a-1}\,F\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu;a;-\frac{x}{2}\right)\,dx = 2^a e^b \frac{1}{\sqrt{\pi}}\,\Gamma(a)(2b)^{\frac{1}{2}-a}\,K_\nu(b)$$

$[\operatorname{Re}a>0,\quad \operatorname{Re}b>0]$ ET I 212(1)

3. $$\int_0^\infty e^{-bx}x^{\gamma-1}\,F(2\alpha,2\beta;\gamma;-\lambda x)\,dx = \Gamma(\gamma)b^{-\gamma}\left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}} e^{\frac{b}{2\lambda}}\,W_{\frac{1}{2}-\alpha-\beta,\alpha-\beta}\left(\frac{b}{\lambda}\right)$$

$[\operatorname{Re}b>0,\quad \operatorname{Re}\gamma>0,\quad |\arg\lambda|<\pi]$
BU 78(30), ET I 212(4)

4.[6] $$\int_0^\infty e^{-xt}t^{b-1}\,F(a,a-c+1;b;-t)\,dt = x^{a-b}\,\Gamma(b)\Psi(a,c;x)$$

$[\operatorname{Re}b>0,\quad \operatorname{Re}x>0]$ EH I 273(11)

5. $$\int_0^\infty e^{-x}x^{s-1}\,{}_pF_q(a_1,\ldots,a_p,b_1,\ldots,b_q;ax)\,dx = \Gamma(s)\,{}_{p+1}F_q(s,a_1,\ldots,a_p;b_1,\ldots,b_q;a)$$

$[p<q,\quad \operatorname{Re}s>0]$ ET I 337(11)

6. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\ {}_2F_2(-n,n+1;1,\beta;x)\,dx = \Gamma(\beta)\mu^{-\beta}\,P_n\left(1-\frac{2}{\mu}\right)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0]$ ET I 218(6)

7. $$\int_0^\infty x^{\beta-1}e^{-\mu x}\ {}_2F_2\left(-n,n;\beta,\frac{1}{2};x\right)\,dx = \Gamma(\beta)\mu^{-\beta}\cos\left[2n\arcsin\left(\frac{1}{\sqrt{\mu}}\right)\right]$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta > 0]$ ET I 218(7)

8. $$\int_0^\infty x^{\rho_n-1}e^{-\mu x}\ {}_mF_n\left(a_1,\dots,a_m;\rho_1,\dots,\rho_n;\lambda x\right)\,dx = \Gamma\left(\rho_n\right)\mu^{-\rho_n}\ {}_mF_{n-1}\left(a_1,\dots,a_m;\rho_1,\dots,\rho_{n-1};\frac{\lambda}{\mu}\right)$$
$[m \le n; \quad \operatorname{Re}\rho_n > 0, \quad \operatorname{Re}\mu > 0, \text{ if } m < n; \operatorname{Re}\mu > \operatorname{Re}\lambda, \text{ if } m = n]$ ET I 219(16)a

9. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\ {}_mF_n\left(a_1,\dots,a_m;\rho_1,\dots,\rho_n;\lambda x\right)\,dx = \Gamma(\sigma)\mu^{-\sigma}\ {}_{m+1}F_n\left(a_1,\dots,a_m,\sigma;\rho_1,\dots,\rho_n;\frac{\lambda}{\mu}\right)$$
$[m \le n, \quad \operatorname{Re}\sigma > 0, \quad \operatorname{Re}\mu > 0, \text{ if } m < n; \operatorname{Re}\mu > \operatorname{Re}\lambda, \text{ if } m = n]$ ET I 219(17)

7.523 $$\int_1^\infty (x-1)^{\mu-1}x^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}ax}\ W_{2\mu+\frac{1}{2},\lambda}(ax)\,dx = \Gamma(\mu)e^{-\frac{1}{2}a}\ W_{\mu+\frac{1}{2},\lambda}(a)$$
$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}a > 0]$

7.524

1. $$\int_0^\infty e^{-\lambda x}\,F\left(\alpha,\beta;\frac{1}{2};-x^2\right)\,dx = \lambda^{\alpha+\beta-1}\,S_{1-\alpha-\beta,\alpha-\beta}(\lambda)$$
$[\operatorname{Re}\lambda > 0]$ ET II 401(13)

2. $$\int_0^\infty e^{-st}\ {}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;t^2\right)\,dx = s^{-1}\ {}_{p+2}F_q\left(a_1,\dots,a_p,1,\frac{1}{2};b_1,\dots,b_q;\frac{4}{s^2}\right)$$
$[p < q]$ MO 176

3. $$\int_0^\infty e^{-st}\ {}_0F_q\left(\frac{1}{q},\frac{2}{q},\dots,\frac{q-1}{q},1;\frac{t^q}{q^q}\right)\,dt = s^{-1}\exp\left(s^{-q}\right)$$ MO 176

7.525

1. $$\int_0^\infty x^{\sigma-1}e^{-\mu x}\ {}_mF_n\left(a_1,\dots,a_m;\rho_1,\dots,\rho_n;(\lambda x)^k\right)\,dx = \Gamma(\sigma)\mu^{-\sigma}\ {}_{m+k}F_n\left(a_1,\dots,a_m,\frac{\sigma}{k},\frac{\sigma+1}{k},\dots,\frac{\sigma+k-1}{k};\rho_1,\dots,\rho_n;\left(\frac{k\lambda}{\mu}\right)^k\right)$$
$\left[m+k \le n+1, \quad \operatorname{Re}\sigma > 0; \quad \operatorname{Re}\mu > 0, \text{ if } m+k \le n;\right.$
$\left.\operatorname{Re}\left(\mu + k\lambda e^{\frac{2\pi i}{k}}\right) > 0; \quad r = 0,1,\dots,k-1 \text{ for } m+k = n+1\right]$
ET I 220(19)

2. $$\int_0^\infty xe^{-\lambda x}\, F\left(\alpha, \beta; \tfrac{3}{2}; -x^2\right)\, dx = \lambda^{\alpha+\beta-2}\, S_{1-\alpha-\beta,\alpha-\beta}(\lambda)$$

$[\operatorname{Re}\lambda > 0]$ ET II 401(14)

7.526

1. $$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} s^{-b}\, F\left(a, b; a+b-c+1; 1-\frac{1}{s}\right)\, dx = 2\pi i \frac{\Gamma(a+b-c+1)}{\Gamma(b)\,\Gamma(b-c+1)} t^{b-1}\, \Psi(a; c; t)$$

$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}(b-c) > -1, \quad \gamma > \frac{1}{2}\right]$

EH I 273(12)

2. $$\int_0^\infty e^{-t} t^{\gamma-1} (x+t)^{-\alpha} (y+t)^{-a'}\, F\left[a, a'; \gamma; \frac{t(x+y+t)}{(x+t)(y+t)}\right]\, dt = \Gamma(\gamma)\Psi(a, c; x)\Psi\left(a', c; y\right),$$

$\gamma = a + a' - c + 1 \qquad [\operatorname{Re}\gamma > 0, \quad xy \neq 0]$ EH I 287(21)

3. $$\int_0^\infty x^{\gamma-1} (x+y)^{-\alpha} (x+z)^{-\beta} e^{-x}\, F\left[\alpha, \beta; \gamma; \frac{x(x+y+z)}{(x+y)(x+z)}\right]\, dx$$

$$= \Gamma(\gamma)(zy)^{-\frac{1}{2}-\mu} e^{\frac{y+z}{2}}\, W_{\nu,\mu}(y)\, W_{\lambda,\mu}(z)$$

$2\nu = 1 - \alpha + \beta - \gamma; \quad 2\lambda = 1 + \alpha - \beta - \gamma; \quad 2\mu = \alpha + \beta - \gamma$

$[\operatorname{Re}\gamma > 0, \quad |\arg y| < \pi, \quad |\arg z| < \pi]$

ET II 401(15)

7.527

1. $$\int_0^\infty \left(1 - e^{-x}\right)^{\lambda-1} e^{-\mu x}\, F\left(\alpha, \beta; \gamma; \delta e^{-x}\right)\, dx = \mathrm{B}(\mu, \lambda)\; {}_3F_2(\alpha, \beta, \mu; \gamma, \mu+\lambda; \delta)$$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\mu > 0, \quad |\arg(1-\delta)| < \pi]$ ET I 213(9)

2. $$\int_0^\infty \left(1 - e^{-x}\right)^{\mu} e^{-\alpha x}\, F\left(-n, \mu+\beta+n; \beta; e^{-x}\right)\, dx = \frac{\mathrm{B}(\alpha, \mu+n+1)\,\mathrm{B}(\alpha, \beta+n-\alpha)}{\mathrm{B}(\alpha, \beta-\alpha)}$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\mu > -1]$ ET I 213(10)

3. $$\int_0^\infty \left(1 - e^{-x}\right)^{\gamma-1} e^{-\mu x}\, F\left(\alpha, \beta; \gamma; 1 - e^{-x}\right)\, dx = \frac{\Gamma(\mu)\,\Gamma(\gamma-\alpha-\beta+\mu)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha+\mu)\,\Gamma(\gamma-\beta+\mu)}$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\mu > \operatorname{Re}(\alpha+\beta-\gamma), \quad \operatorname{Re}\gamma > 0]$ ET I 213(11)

4. $$\int_0^\infty \left(1 - e^{-x}\right)^{\gamma-1} e^{-\mu x}\, F\left[\alpha, \beta; \gamma; \delta\left(1 - e^{-x}\right)\right]\, dx = \mathrm{B}(\mu, \gamma)\, F(\alpha, \beta; \mu+\gamma; \delta)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\gamma > 0, \quad |\arg(1-\delta)| < \pi]$ ET I 213(12)

7.53 Hypergeometric and trigonometric functions

7.531

1. $$\int_0^\infty x \sin \mu x \, F\left(\alpha, \beta; \frac{3}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-2} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\,\Gamma(\beta)}$$
$\left[\mu > 0, \quad \operatorname{Re}\alpha > \frac{1}{2}, \quad \operatorname{Re}\beta > \frac{1}{2}\right]$
ET I 115(6)

2. $$\int_0^\infty \cos \mu x \, F\left(\alpha, \beta; \frac{1}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-1} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\,\Gamma(\beta)}$$
$[\mu > 0, \quad \operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad c > 0]$
ET I 61(9)

7.54 Combinations of hypergeometric and Bessel functions

7.541 $$\int_0^\infty x^{\alpha+\beta-2\nu-1} (x+1)^{-\nu} e^{xz} K_\nu[(x+1)z] \, F(\alpha, \beta; \alpha+\beta-2\nu; -x) \, dx$$
$$= \pi^{-\frac{1}{2}} \cos(\nu\pi) \, \Gamma\left(\tfrac{1}{2} - \alpha + \nu\right) \Gamma\left(\tfrac{1}{2} - \beta + \nu\right) \Gamma(\gamma) (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} \, W_{\frac{1}{2}\gamma, \frac{1}{2}(\beta-\alpha)}(2z)$$
$\gamma = \alpha + \beta - 2\nu \qquad \left[\operatorname{Re}(\alpha+\beta-2\nu) > 0, \quad \operatorname{Re}\left(\tfrac{1}{2} - \alpha + \nu\right) > 0, \quad \operatorname{Re}\left(\tfrac{1}{2} - \beta + \nu\right) > 0, \quad |\arg z| < \tfrac{3}{2}\pi\right]$
ET II 401(16)

7.542

1. $$\int_0^\infty x^{\sigma-1} \, {}_pF_{p-1}\left(a_1, \ldots, a_p; b_1, \ldots, b_{p-1}; -\lambda x^2\right) Y_\nu(xy) \, dx$$
$$= \frac{\Gamma(b_1) \ldots \Gamma(b_{p-1})}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \ldots \Gamma(a_p)} G^{p+2,1}_{p+2,p+3}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} b_0^*, \ldots, b_{p+1}^*, \\ h, k, a_1^*, \ldots, a_p^*, l \end{matrix}\right)$$
$$a_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \ldots, p; \quad b_0^* = 1 - \frac{\sigma}{2}; \quad b_j^* = b_j - \frac{\sigma}{2},$$
$$j = 1, \ldots, p-1; h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$$
$\left[|\arg\lambda| < \pi, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|, \quad \operatorname{Re}a_j > \frac{1}{2}\operatorname{Re}\sigma - \frac{3}{4}, \quad y > 0\right]$
ET II 118(53)

2. $$\int_0^\infty x^{\sigma-1} \, {}_pF_p\left(a_1, \ldots, a_p; b_1, \ldots, b_p; -\lambda x^2\right) Y_\nu(xy) \, dx$$
$$= \frac{\Gamma(b_1) \ldots \Gamma(b_p)}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \ldots \Gamma(a_p)} G^{p+2,1}_{p+2,p+3}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} b_0^*, \ldots, b_p^*, l \\ h, k, a_1^*, \ldots, a_p^*, l \end{matrix}\right)$$
$$b_0^* = 1 - \frac{\sigma}{2}; \quad a_j^* = a_j - \frac{\sigma}{2}, \quad b_j* = b_j - \frac{\sigma}{2}; \quad j = 1, \ldots, p; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$$
$\left[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|, \quad \operatorname{Re}a_j > \frac{1}{2}\operatorname{Re}\sigma - \frac{3}{4}, \quad y > 0\right]$
ET II 119(54)

3. $$\int_0^\infty x^{\sigma-1}\, {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-\lambda x^2\right) Y_\nu(xy)\,dx$$
$$= -\pi^{-1}2^{\sigma-1}y^{-\sigma}\cos\left[\frac{\pi}{2}(\sigma-\nu)\right]\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right)$$
$$\times\, {}_{p+2}F_q\left(a_1,\ldots,a_p,\frac{\sigma+\nu}{2},\frac{\sigma-\nu}{2};b_1,\ldots,b_q;-\frac{4\lambda}{y^2}\right)$$
$[y>0, \quad p \le q-1, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|]$ ET II 119(55)

4. $$\int_0^\infty x^{\sigma-1}\, {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-\lambda x^2\right) K_\nu(xy)\,dx$$
$$= 2^{\sigma-2}y^{-\sigma}\,\Gamma\left(\frac{\sigma+\nu}{2}\right)\Gamma\left(\frac{\sigma-\nu}{2}\right)\, {}_{p+2}F_q\left(a_1,\ldots,a_p,\frac{\sigma+\nu}{2},\frac{\sigma-\nu}{2};b_1,\ldots,b_q;\frac{4\lambda}{y^2}\right)$$
$[\operatorname{Re}y>0, \quad p \le q-1, \quad \operatorname{Re}\sigma > |\operatorname{Re}\nu|]$ ET II 153(88)

5. $$\int_0^\infty x^{2\rho}\, {}_pF_p\left(a_1,\ldots,a_p;b_1,\ldots,b_p;-\lambda x^2\right) J_\nu(xy)\,dx$$
$$= \frac{2^{2\rho}\,\Gamma(b_1)\ldots\Gamma(b_p)}{y^{2\rho+1}\,\Gamma(a_1)\ldots\Gamma(a_p)}\, G^{p+1,1}_{p+1,p+2}\left(\frac{y^2}{4\lambda}\middle|\begin{matrix}1, & b_1,\ldots,b_p & \\ h, & a_1,\ldots,a_p, & k\end{matrix}\right)$$
$$h = \tfrac12+\rho+\tfrac12\nu, \qquad k = \tfrac12+\rho-\tfrac12\nu$$
$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -1-\operatorname{Re}\nu < 2\operatorname{Re}\rho < \tfrac12 + 2\operatorname{Re}a_r, \quad r=1,\ldots,p\right]$ ET II 91(18)

6. $$\int_0^\infty x^{2\rho}\, {}_{m+1}F_m\left(a_1,\ldots,a_{m+1};b_1,\ldots,b_m;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$= \frac{2^{2\rho}\,\Gamma(b_1)\ldots\Gamma(b_m)\,y^{-2\rho-1}}{\Gamma(a_1)\ldots\Gamma(a_{m+1})}\, G^{m+2,1}_{m+1,m+3}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1, & b_1,\ldots,b_m & \\ h, & a_1,\ldots,a_{m+1}, & k\end{matrix}\right)$$
$$h = \tfrac12+\rho+\tfrac12\nu, \qquad k = \tfrac12+\rho-\tfrac12\nu,$$
$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}(2\rho+\nu) > -1, \quad \operatorname{Re}(\rho-a_r) < \tfrac14; \quad r=1,\ldots,m+1\right]$ ET II 91(19)

7. $$\int_0^\infty x^{\delta}\, F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$= \frac{2^\delta\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}y^{-\delta-1}\, G^{22}_{24}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1-\alpha, \quad 1-\beta \\ \dfrac{1+\delta+\nu}{2}, \quad 0, \quad 1-\gamma, \quad \dfrac{1+\delta-\nu}{2}\end{matrix}\right)$$
$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -1-\operatorname{Re}\nu-2\min(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) < \operatorname{Re}\delta < -\tfrac12\right]$ ET II 82(9)

8. $$\int_0^\infty x^{\delta}\, F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right) J_\nu(xy)\,dx = \frac{2^\delta y^{-\delta-1}\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\, G^{31}_{24}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1, \quad \gamma \\ \dfrac{1+\delta+\nu}{2},\alpha,\beta,\dfrac{1+\delta-\nu}{2}\end{matrix}\right)$$
$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -\operatorname{Re}\nu-1 < \operatorname{Re}\delta < 2\max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \tfrac12\right]$ ET II 81(6)

9. $$\int_0^\infty x^{\nu+1}\, F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right) J_\nu(xy)\,dx = \frac{2^{\nu+1}\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}y^{-\nu-2}\, G^{30}_{13}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}\gamma \\ \nu+1, \quad \alpha, \quad \beta\end{matrix}\right)$$
$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -1 < \operatorname{Re}\nu < 2\max(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta) - \tfrac32\right]$ ET II 81(5)

10. $$\int_0^\infty x^{\nu+1} F\left(\alpha,\beta;\nu+1;-\lambda^2x^2\right) J_\nu(xy)\,dx = \frac{2^{\nu-\alpha-\beta+2}\,\Gamma(\nu+1)}{\lambda^{\alpha+\beta}\,\Gamma(\alpha)\,\Gamma(\beta)} y^{\alpha+\beta-\nu-2}\,K_{\alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -1<\operatorname{Re}\nu<2\max\left(\operatorname{Re}\alpha, \quad \operatorname{Re}\beta\right)-\tfrac{3}{2}\right]$$ ET II 81(3)

11. $$\int_0^\infty x^{\nu+1} F\left(\alpha,\beta;\nu+1;-\lambda^2x^2\right) K_\nu(xy)\,dx = 2^{\nu+1}\lambda^{-\alpha-\beta}y^{\alpha+\beta-\nu-2}\,\Gamma(\nu+1)\,S_{1-\alpha-\beta,\alpha-\beta}\left(\frac{y}{\lambda}\right)$$
$$[\operatorname{Re}y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\nu>-1]$$ ET II 152(86)

12. $$\int_0^\infty x^{\nu+1}F\left(\alpha,\beta;\frac{\beta+\nu}{2}+1;-\lambda^2x^2\right)J_\nu(xy)\,dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right)y^{\beta-1}\lambda^{-\nu-\beta-1}}{\pi^{\frac{1}{2}}\,\Gamma(\alpha)\,\Gamma(\beta)2^{\beta-1}}\,K_{\frac{1}{2}(\nu-\beta+1)}\left(\frac{y}{2\lambda}\right)^2$$
$$\left[y>0, \quad -1<\operatorname{Re}\nu<\left(2\max\left(\operatorname{Re}\alpha,\operatorname{Re}\beta\right)-\tfrac{3}{2}\right)\right]$$ ET II 81(4)

13. $$\int_0^\infty x^{\sigma+\frac{1}{2}} F\left(\alpha,\beta;\gamma;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{\lambda^{-\sigma-1}y^{-\frac{1}{2}}\,\Gamma(\gamma)}{\sqrt{2}\,\Gamma(\alpha)\,\Gamma(\beta)}\,G^{41}_{35}\left(\frac{y^2}{4\lambda^2}\middle|\begin{matrix}1-p,\gamma-p,l\\ h,\quad k,\alpha-p,\beta-p,l\end{matrix}\right)$$
$$h=\tfrac{1}{4}+\tfrac{1}{2}\nu, \quad k=\tfrac{1}{4}-\tfrac{1}{2}\nu, \quad l=-\tfrac{1}{4}-\tfrac{1}{2}\nu, \quad p=\tfrac{1}{2}+\tfrac{1}{2}\sigma$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\sigma>|\operatorname{Re}\nu|-\tfrac{3}{2}, \quad \operatorname{Re}\sigma<2\operatorname{Re}\alpha, \quad \operatorname{Re}\sigma<2\operatorname{Re}\beta\right]$$ ET II 118(52)

14. $$\int_0^\infty x^{\nu+2} F\left(\frac{1}{2},\frac{1}{2}-\nu;\frac{3}{2};-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{2^\nu y^{-\nu-1}}{\pi^{\frac{1}{2}}\lambda^2\,\Gamma\left(\frac{1}{2}-\nu\right)}\,K_\nu\left(\frac{y}{2\lambda}\right)K_{\nu+1}\left(\frac{y}{2\lambda}\right)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -\tfrac{3}{2}<\operatorname{Re}\nu<-\tfrac{1}{2}\right]$$ ET II 117(49)

15. $$\int_0^\infty x^{\nu+2} F\left(1,2\nu+\frac{3}{2};\nu+2;-\lambda^2x^2\right) Y_\nu(xy)\,dx = \pi^{-\frac{1}{2}}2^{-\nu}\lambda^{-2\nu-3}\frac{\Gamma(\nu+2)}{\Gamma\left(2\nu+\frac{3}{2}\right)}\left[K_\nu\left(\frac{y}{2\lambda}\right)\right]^2$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -\tfrac{1}{2}<\operatorname{Re}\nu<\tfrac{1}{2}\right]$$ ET II 117(50)

16. $$\int_0^\infty x^{\nu+2} F\left(1,\mu+\nu+\frac{3}{2};\frac{3}{2};-\lambda^2x^2\right) Y_\nu(xy)\,dx = \frac{\pi^{\frac{1}{2}}2^{-\mu-\nu-1}\lambda^{-\mu-2\nu-3}y^{\mu+\nu}}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)}\,K_\mu\left(\frac{y}{\lambda}\right)$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad -\tfrac{3}{2}<\operatorname{Re}\nu<\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu)>-\tfrac{3}{2}\right]$$ ET II 118(51)

17. $$\int_0^\infty x^{2\alpha+\nu} F\left(\alpha-\nu-\frac{1}{2},\alpha;2\alpha;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$= \frac{i\,\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\alpha+\nu\right)}{\pi 2^{1-\nu-2\alpha}\lambda^{2\alpha-1}y^{\nu+2}}\,W_{\frac{1}{2}-\alpha,-\frac{1}{2}-\nu}\left(\frac{y}{\lambda}\right)\left[W_{\frac{1}{2}-\alpha,-\frac{1}{2}-\nu}\left(e^{-i\pi}\frac{y}{\lambda}\right)-W_{\frac{1}{2}-\alpha,-\frac{1}{2}-\nu}\left(e^{i\pi}\frac{y}{\lambda}\right)\right]$$
$$\left[y>0, \quad \operatorname{Re}\lambda>0, \quad \operatorname{Re}\nu<-\tfrac{1}{2}, \quad \operatorname{Re}(\alpha+\nu)>-\tfrac{1}{2}\right]$$ ET II 80(1)

18. $$\int_0^\infty x^{2\alpha-\nu} F\left(\nu+\alpha-\frac{1}{2},\alpha;2\alpha;-\lambda^2x^2\right) J_\nu(xy)\,dx$$
$$= \frac{2^{2\alpha-\nu}\,\Gamma\left(\frac{1}{2}+\alpha\right)y^{\nu-2}}{\lambda^{2\alpha-1}\,\Gamma(2\nu)}\,M_{\alpha-\frac{1}{2},\nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right)W_{\frac{1}{2}-\alpha,\nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right)$$ ET II 80(2)

7.543

1. $$\int_0^\infty x^{-2\alpha-1} F\left(\frac{1}{2}+\alpha, 1+\alpha; 1+2\alpha; -\frac{4\lambda^2}{x^2}\right) J_\nu(xy)\,dx = \lambda^{-2\alpha} I_{\frac{1}{2}\nu+\alpha}(\lambda y)\, K_{\frac{1}{2}\nu-\alpha}(\lambda y)$$
$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\alpha > -\frac{1}{2}\right]$ ET II 81(7)

2. $$\int_0^\infty x^{\nu+1-4\alpha} F\left(\alpha, \alpha+\frac{1}{2}; \nu+1; -\frac{\lambda^2}{x^2}\right) J_\nu(xy)\,dx = \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1} I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right)$$
$\left[y > 0, \quad \operatorname{Re}\lambda > 0, \quad \operatorname{Re}\alpha - 1 < \operatorname{Re}\nu < 4\operatorname{Re}\alpha - \frac{3}{2}\right]$ ET II 81(8)

7.544 $$\int_0^\infty x^{\nu+1}(1+x)^{-2\alpha} F\left[\alpha, \nu+\frac{1}{2}; 2\nu+1; \frac{4x}{(1+x)^2}\right] J_\nu(xy)\,dx = \frac{\Gamma(\nu+1)\,\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)} J_\nu(y)$$
$\left[y > 0, \quad -1 < \operatorname{Re}\nu < 2\operatorname{Re}\alpha - \frac{3}{2}\right]$ ET II 82(10)

7.6 Confluent Hypergeometric Functions

7.61 Combinations of confluent hypergeometric functions and powers

7.611

1. $$\int_0^\infty x^{-1}\, W_{k,\mu}(x)\,dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma\left(\frac{3}{4}-\frac{1}{2}k+\frac{1}{2}\mu\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}\mu\right)}$$
$\left[|\operatorname{Re}\mu| < \frac{1}{2}\right]$ ET II 406(22)

2. $$\int_0^\infty x^{-1}\, M_{k,\mu}(x)\, W_{\lambda,\mu}(x)\,dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\,\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\lambda) > 0\right]$ BU 116(11), ET II 409(39)

3. $$\int_0^\infty x^{-1}\, W_{k,\mu}(x)\, W_{\lambda,\mu}(x)\,dx = \frac{1}{(k-\lambda)\sin(2\mu\pi)}\left[\frac{1}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-\lambda-\mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2}-k-\mu\right)\Gamma\left(\frac{1}{2}-\lambda+\mu\right)}\right]$$
$\left[|\operatorname{Re}\mu| < \frac{1}{2}\right]$ BU 116(12), ET II 409(40)

4. $$\int_0^\infty \left\{W_{\kappa,\mu}(z)\right\}^2 \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi\left(\frac{1}{2}+\mu-\kappa\right)-\psi\left(\frac{1}{2}-\mu-\kappa\right)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}$$
$\left[|\operatorname{Re}\mu| < \frac{1}{2}\right]$ BU 117(12a)

5. $$\int_0^\infty \frac{1}{z}\left[W_{\kappa,0}(z)\right]^2\,dx = \frac{\psi'\left(\frac{1}{2}-\kappa\right)}{\left[\Gamma\left(\frac{1}{2}-\kappa\right)\right]^2}$$ BU 117(12b)

6. $$\int_0^\infty x^{\rho-1}\, W_{k,\mu}(x)\, W_{-k,\mu}(x)\, dx = \frac{\Gamma(\rho+1)\,\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}\rho+\frac{1}{2}-\mu\right)}{2\,\Gamma\left(1+\frac{1}{2}\rho+k\right)\Gamma\left(1+\frac{1}{2}\rho-k\right)}$$

$[\operatorname{Re}\rho > 2|\operatorname{Re}\mu| - 1]$ ET II 409(41)

7.[11] $$\int_0^\infty x^{\rho-1}\, W_{k,\mu}(x)\, W_{\lambda,\nu}(x)\, dx$$
$$= \frac{\Gamma(1-\mu+\nu+\rho)\,\Gamma(1+\mu+\nu+\rho)\,\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\lambda-\nu\right)\Gamma\left(\frac{3}{2}-k+\nu+\rho\right)} \times {}_3F_2\left(1-\mu+\nu+\rho, 1+\mu+\nu+\rho, \frac{1}{2}-\lambda+\nu; 1+2\nu, \frac{3}{2}-k+\nu+\rho; 1\right)$$
$$+\frac{\Gamma(1+\mu-\nu+\rho)\,\Gamma(1-\mu-\nu+\rho)\,\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right)\Gamma\left(\frac{3}{2}-k-\nu+\rho\right)} \times {}_3F_2\left(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, \frac{1}{2}-\lambda-\nu; 1-2\nu, \frac{3}{2}-k-\nu+\rho; 1\right)$$

$[|\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\rho + 1]$ ET II 410(42)

7.612

1. $$\int_0^\infty t^{b-1}\, {}_1F_1(a;c;-t)\, dt = \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}$$ $[0 < \operatorname{Re} b < \operatorname{Re} a]$ EH I 285(10)

2. $$\int_0^\infty t^{b-1}\Psi(a,c;t)\, dt = \frac{\Gamma(b)\,\Gamma(a-b)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(a-c+1)}$$ $[0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b + 1]$

EH I 285(11)

7.613

1. $$\int_0^t x^{\gamma-1}(t-x)^{c-\gamma-1}\, {}_1F_1(a;\gamma;x)\, dx = t^{c-1}\frac{\Gamma(\gamma)\,\Gamma(c-\gamma)}{\Gamma(c)}\, {}_1F_1(a;c;t)$$

$[\operatorname{Re} c > \operatorname{Re}\gamma > 0]$

BU 9(16)a, EH I 271(16)

2. $$\int_0^t x^{\beta-1}(t-x)^{\gamma-1}\, {}_1F_1(t;\beta;x)\, dx = \frac{\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma)} t^{\beta+\gamma-1}\, {}_1F_1(t;\beta+\gamma;t)$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$ ET II 401(1)

3. $$\int_0^1 x^{\lambda-1}(1-x)^{2\mu-\lambda}\, {}_1F_1\left(\frac{1}{2}+\mu-\nu;\lambda;xz\right) dx = \mathrm{B}(\lambda, 1+2\mu-\lambda)e^{\frac{1}{2}z}z^{-\frac{1}{2}-\mu}\, M_{\nu,\mu}(z)$$

$[\operatorname{Re}\lambda > 0, \quad \operatorname{Re}(2\mu-\lambda) > -1]$

BU 14(14)

4. $$\int_0^t x^{\beta-1}(t-x)^{\delta-1}\, {}_1F_1(t;\beta;x)\, {}_1F_1(\gamma;\delta;t-x)\, dx = \frac{\Gamma(\beta)\,\Gamma(\delta)}{\Gamma(\beta+\delta)} t^{\beta+\delta-1}\, {}_1F_1(t+\gamma;\beta+\delta;t)$$

$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\delta > 0]$

ET II 402(2), EH I 271(15)

5. $$\int_0^t x^{\mu-\frac{1}{2}}(t-x)^{\nu-\frac{1}{2}}\, M_{k,\mu}(x)\, M_{\lambda,\nu}(t-x)\, dx = \frac{\Gamma(2\mu+1)\,\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} t^{\mu+\nu}\, M_{k+\lambda,\mu+\nu+\frac{1}{2}}(t)$$

$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$

BU 128(14), ET II 402(7)

6. $$\int_0^1 x^{\beta-1}(1-x)^{\sigma-\beta-1}\ {}_1F_1(\alpha;\beta;\lambda x)\ {}_1F_1[\sigma-\alpha;\sigma-\beta;\mu(1-x)]\,dx = \frac{\Gamma(\beta)\,\Gamma(\sigma-\beta)}{\Gamma(\sigma)}e^{\lambda}\ {}_1F_1\left(\alpha;\sigma;\mu-\lambda\right)$$
$[0<\operatorname{Re}\beta<\operatorname{Re}\sigma]$ ET II 402(3)

7.62–7.63 Combinations of confluent hypergeometric functions and exponentials

7.621

1. $$\int_0^\infty e^{-st}t^{\alpha}\,M_{\mu,\nu}(t)\,dt = \frac{\Gamma\left(\alpha+\nu+\frac{3}{2}\right)}{\left(\frac{1}{2}+s\right)^{\alpha+\nu+\frac{3}{2}}}\,F\left(\alpha+\nu+\frac{3}{2},-\mu+\nu+\frac{1}{2};2\nu+1;\frac{2}{2s+1}\right)$$
$\left[\operatorname{Re}\left(\alpha+\mu+\frac{3}{2}\right)>0,\quad \operatorname{Re}s>\frac{1}{2}\right]$
BU 118(1), MO 176a, EH I 270(12)a

2. $$\int_0^\infty e^{-st}t^{\mu-\frac{1}{2}}\,M_{\lambda,\mu}(qt)\,dt = q^{\mu+\frac{1}{2}}\,\Gamma(2\mu+1)\left(s-\tfrac{1}{2}q\right)^{\lambda-\mu-\frac{1}{2}}\left(s+\tfrac{1}{2}q\right)^{-\lambda-\mu-\frac{1}{2}}$$
$\left[\operatorname{Re}\mu>-\frac{1}{2},\quad \operatorname{Re}s>\frac{|\operatorname{Re}q|}{2}\right]$
BU 119(4c), MO 176a, EH I 271(13)a

3. $$\int_0^\infty e^{-st}t^{\alpha}\,W_{\lambda,\mu}(qt)\,dt = \frac{\Gamma\left(\alpha+\mu+\frac{3}{2}\right)\Gamma\left(\alpha-\mu+\frac{3}{2}\right)q^{\mu+\frac{1}{2}}}{\Gamma(\alpha-\lambda+2)}\left(s+\frac{1}{2}q\right)^{-\alpha-\mu-\frac{3}{2}} \times F\left(\alpha+\mu+\frac{3}{2},\mu-\lambda+\frac{1}{2};\alpha-\lambda+2;\frac{2s-q}{2s+q}\right)$$
$\left[\operatorname{Re}\left(\alpha\pm\mu+\frac{3}{2}\right)>0,\quad \operatorname{Re}s>-\frac{q}{2},\quad q>0\right]$ EH I 271(14)a, BU 121(6), MO 176

4. $$\int_0^\infty e^{-st}t^{b-1}\ {}_1F_1(a;c;kt)\,dt = \Gamma(b)s^{-b}\,F\left(a,b;c;ks^{-1}\right)$$ $[|s|>|k|]$
$$= \Gamma(b)(s-k)^{-b}\,F\left(c-a,b;c;\frac{k}{k-s}\right)$$ $[|s-k>|k||]$
$[\operatorname{Re}b>0,\quad \operatorname{Re}s>\max\left(0,\operatorname{Re}k\right)]$ EH I 269(5)

5. $$\int_0^\infty t^{c-1}\ {}_1F_1(a;c;t)e^{-st}\,dt = \Gamma(c)s^{-c}\left(1-s^{-1}\right)^{-a}$$ $[\operatorname{Re}c>0,\quad \operatorname{Re}s>1]$ EH I 270(6)

6. $$\int_0^\infty t^{b-1}\Psi\left(a,c;t\right)e^{-st}\,dt = \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}\,F\left(b,b-c+1;a+b-c+1;1-s\right)$$
$[\operatorname{Re}b>0,\quad \operatorname{Re}c<\operatorname{Re}b+1,\quad |1-s|<1]$
$$= \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}s^{-b}\,F\left(a,b;a+b-c+1;1-s^{-1}\right)$$
$\left[\operatorname{Re}s>\frac{1}{2}\right]$
EH I 270(7)

7. $$\int_0^\infty e^{-\frac{b}{2}x}x^{\nu-1}\,M_{\kappa,\mu}(bx)\,dx = \frac{\Gamma(1+2\mu)\,\Gamma\left(\kappa-\nu\right)\Gamma\left(\frac{1}{2}+\mu+\nu\right)}{\Gamma\left(\frac{1}{2}+\mu+\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\nu\right)}b^{\nu}$$
$\left[\operatorname{Re}\left(\nu+\frac{1}{2}+\mu\right)>0,\quad \operatorname{Re}\left(\kappa-\nu\right)>0\right]$
BU 119(3)a, ET I 215(11)a

8. $$\int_0^\infty e^{-sx}\, M_{\kappa,\mu}(x)\frac{dx}{x} = \frac{2\,\Gamma(1+2\mu)e^{-i\pi\kappa}}{\Gamma\left(\frac{1}{2}+\mu+\kappa\right)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} Q^{\kappa}_{\mu-\frac{1}{2}}(2s)$$

$\left[\operatorname{Re}\left(\frac{1}{2}+\mu\right)>0, \quad \operatorname{Re} s > \frac{1}{2}\right]$ BU 119(4a)

9. $$\int_0^\infty e^{-sx}\, W_{\kappa,\mu}(x)\frac{dx}{x} = \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{\frac{\kappa}{2}} P^{\kappa}_{\mu-\frac{1}{2}}(2s)$$

$\left[\operatorname{Re}\left(\frac{1}{2}\pm\mu\right)>0, \quad \operatorname{Re} s > -\frac{1}{2}\right]$ BU 121(7)

10. $$\int_0^\infty x^{k+2\mu-1}e^{-\frac{3}{2}x}\, W_{k,\mu}(x)\, dx = \frac{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2k+6\mu+5)\right]}{\left(k+3\mu+\frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2\mu-2k+3)\right]}$$

$\left[\operatorname{Re}(k+\mu)>-\frac{1}{2}, \quad \operatorname{Re}(k+3\mu)>-\frac{1}{2}\right]$ BU 122(8a), ET II 406(23)

11. $$\int_0^\infty e^{-\frac{1}{2}x}x^{\nu-1}\, W_{\kappa,\mu}(x)\, dx = \frac{\Gamma\left(\nu+\frac{1}{2}-\mu\right)\Gamma\left(\nu+\frac{1}{2}+\mu\right)}{\Gamma\left(\nu-\kappa+1\right)}$$

$\left[\operatorname{Re}\left(\nu+\frac{1}{2}\pm\mu\right)>0\right]$ BU 122(8b)

12. $$\int_0^\infty e^{\frac{1}{2}x}x^{\nu-1}\, W_{\kappa,\mu}(x)\, dx = \Gamma\left(-\kappa-\mu\right)\frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}$$

$\left[\operatorname{Re}\left(\nu+\frac{1}{2}\pm\mu\right)>0, \quad \operatorname{Re}\left(\kappa+\nu\right)<0\right]$ BU 122(8c)a

7.622

1. $$\int_0^\infty e^{-st}t^{c-1}\ {}_1F_1(a;c;t)\ {}_1F_1\left(\alpha;c;\lambda t\right)\, dt$$
$$= \Gamma(c)(s-1)^{-a}(s-\lambda)^{-\alpha}s^{a+\alpha-c}\, F\left[a,\alpha;c;\lambda(s-1)^{-1}(s-\lambda)^{-1}\right]$$

$[\operatorname{Re} c>0, \quad \operatorname{Re} s>\operatorname{Re}\lambda+1]$ EH I 287(22)

2. $$\int_0^\infty e^{-t}t^{\rho}\ {}_1F_1(a;c;t)\Psi\left(a';c';\lambda t\right)\, dt$$
$$= C\frac{\Gamma(c)\,\Gamma(\beta)}{\Gamma(\gamma)}\lambda^{\sigma}\, F\left(c-a,\beta;\gamma;1-\lambda^{-1}\right),$$

$$\rho=c-1, \quad \sigma=-c, \quad \beta=c-c'+1, \quad \gamma=c-a+a'-c'+1, \quad C=\frac{\Gamma\left(a'-a\right)}{\Gamma\left(a'\right)}, \text{ or}$$

$$\rho=c+c'-2, \quad \sigma=1-c-c', \quad \beta=c+c'-1, \quad \gamma=a'-a+c, \quad C=\frac{\Gamma\left(a'-a-c'+1\right)}{\Gamma\left(a'-c'+1\right)}$$

EH I 287(24)

3. $$\int_0^\infty x^{\nu-1} e^{-bx} M_{\lambda_1,\mu_1-\frac{1}{2}}(a_1 x) \dots M_{\lambda_n,\mu_n-\frac{1}{2}}(a_n x)\, dx$$
$$= a_1^{\mu_1} \dots a_n^{\mu_n} (b+A)^{-\nu-M}\, \Gamma(\nu+M)$$
$$\times F_A\left(\nu+M; \mu_1-\lambda_1, \dots, \mu_n-\lambda_n; 2\mu_1, \dots, 2\mu_n; \frac{a_1}{b+A}, \dots, \frac{a_n}{b+A}\right),$$
$$M = \mu_1 + \dots + \mu_n, \qquad A = \tfrac{1}{2}(a_1 + \dots + a_n)$$
$\left[\operatorname{Re}(\nu+M) > 0, \quad \operatorname{Re}\left(b \pm \frac{1}{2}a_1 \pm \dots + \frac{1}{2}a_n\right) > 0\right]$ ET I 216(14)

7.623

1. $$\int_0^\infty e^{-x} x^{c+n-1} (x+y)^{-1}\, {}_1F_1(a;c;x)\, dx = (-1)^n\, \Gamma(c)\, \Gamma(1-a) y^{c+n-1} \Psi(c-a, c; y)$$
$[-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \quad n = 0, 1, 2, \dots, \quad |\arg y| < \pi]$ EH I 285(16)

2. $$\int_0^t x^{-1} (t-x)^{k-1} e^{\frac{1}{2}(t-x)} M_{k,\mu}(x)\, dx = \frac{\Gamma(k)\, \Gamma(2\mu+1)}{\Gamma\left(k+\mu+\frac{1}{2}\right)} \pi^{\frac{1}{2}} t^{k-\frac{1}{2}} l_\mu\left(\frac{1}{2}t\right)$$
$\left[\operatorname{Re} k > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right]$ ET II 402(5)

3. $$\int_0^t x^{k-1} (t-x)^{\lambda-1} e^{\frac{1}{2}(t-x)} M_{k+\lambda,\mu}(x)\, dx = \frac{\Gamma(\lambda)\, \Gamma\left(k+\mu+\frac{1}{2}\right) t^{k+\lambda-1}}{\Gamma\left(k+\lambda+\mu+\frac{1}{2}\right)} M_{k,\mu}(t)$$
$\left[\operatorname{Re}(k+\mu) > -\frac{1}{2}, \quad \operatorname{Re} \lambda > 0\right]$ ET II 402(6)

4. $$\int_0^t x^{-k-\lambda-1} (t-x)^{\lambda-1} e^{\frac{1}{2}x} W_{k,\mu}(x)\, dx = \frac{\Gamma(\lambda)\, \Gamma\left(\frac{1}{2}-k-\lambda+\mu\right) \Gamma\left(\frac{1}{2}-k-\lambda-\mu\right)}{t^{k+1}\, \Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right)} W_{k+\lambda,\mu}(t)$$
$\left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\lambda) < \frac{1}{2} - |\operatorname{Re} \mu|\right]$ ET II 405(21)

5. $$\int_1^\infty (x-1)^{\mu-1} x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{k,\lambda}(ax)\, dx = \frac{\Gamma(\mu)\, \Gamma\left(\frac{1}{2}-k-\lambda-\mu\right)}{\Gamma\left(\frac{1}{2}-k-\lambda\right)} a^{-\frac{1}{2}\mu} e^{\frac{1}{2}a} W_{k+\frac{1}{2}\mu, \lambda+\frac{1}{2}\mu}(a)$$
$\left[|\arg(a)| < \frac{3}{2}\pi, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re}(k+\lambda)\right]$ ET II 211(72)a

6.[11] $$\int_1^\infty (x-1)^{\mu-1} x^{\mu-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{2\mu+\frac{1}{2},\lambda}(ax)\, dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{\mu+\frac{1}{2},\lambda}(a)$$
$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0]$ ET II 211(74)a

7. $$\int_1^\infty (x-1)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax)\, dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{k-\mu,\lambda}(a)$$
$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0]$ ET II 211(73)a

8. $$\int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax}\, W_{k,\lambda}(ax)\, dx = \Gamma(\mu) e^{-\frac{1}{2}a} \sec[(k-\mu-\lambda)\pi] \times \left\{ \sin(\mu\pi) \frac{\Gamma\left(k-\mu+\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda+1)}\, M_{k-\mu,\lambda}(a) + \cos[(k-\lambda)\pi]\, W_{k-\mu,\lambda}(a) \right\}$$
$\left[0 < \operatorname{Re}\mu < \operatorname{Re}k - |\operatorname{Re}\lambda| + \frac{1}{2}\right]$ ET II 200(93)a

7.624

1. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, M_{k,\mu}(x)\, dx = \frac{-\sigma\,\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+k+\mu\right)}\, G_{34}^{23}\left(a \,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, -\sigma, \sigma, \frac{1}{2}-\mu+\rho \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}(\mu+\rho) > -\frac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > 0\right]$ ET II 403(8)

2. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = -\pi^{-\frac{1}{2}} \sigma a^\sigma\, G_{34}^{32}\left(a \,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1-k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}\right]$ ET II 406(24)

3. $$\int_0^\infty x^{\rho-1} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = -\frac{\sigma \pi^{-\frac{1}{2}} a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)}\, G_{34}^{33}\left(a \,\middle|\, \begin{matrix} \frac{1}{2}, 1, 1+k+\rho \\ \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho, -\sigma, \sigma \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < 0\right]$ ET II 406(25)

4. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, M_{k,\mu}(x)\, dx = \frac{\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}+k+\mu\right)}\, G_{34}^{23}\left(a \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k-\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}(\rho+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k-\rho-\sigma) > -\frac{1}{2}\right]$ ET II 403(9)

5. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = \frac{\pi^{-\frac{1}{2}} a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)}\, G_{34}^{33}\left(a \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}+k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho+\sigma) < \frac{1}{2}\right]$ ET II 406(26)

6. $$\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} \left[x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\right]^{2\sigma} e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, dx = \pi^{-\frac{1}{2}} a^\sigma\, G_{34}^{32}\left(a \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix}\right)$$
$\left[|\arg a| < \pi, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}\right]$ ET II 406(27)

7.625

1. $$\int_0^\infty x^{\rho-1} \exp\left[-\tfrac{1}{2}(\alpha+\beta)x\right] M_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$$
$$= \frac{\Gamma(1+\mu+\nu+\rho)\,\Gamma(1+\mu-\nu+\rho)}{\Gamma\left(\frac{3}{2}-\lambda+\mu+\rho\right)} \alpha^{\mu+\frac{1}{2}} \beta^{-\mu-\rho-\frac{1}{2}}$$
$$\times\, {}_3F_2\left(\frac{1}{2}+k+\mu, 1+\mu+\nu+\rho, 1+\mu-\nu+\rho; 2\mu+1, \frac{3}{2}-\lambda+\mu+\rho; -\frac{\alpha}{\beta}\right)$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\rho+\mu) > |\operatorname{Re}\nu| - 1]$ ET II 410(43)

2. $$\int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$$
$$= \beta^{-\rho} \left[\Gamma\left(\tfrac{1}{2}-k+\mu\right)\Gamma\left(\tfrac{1}{2}-k-\mu\right)\Gamma\left(\tfrac{1}{2}-\lambda+\nu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\nu\right)\right]^{-1}$$
$$\times\, G^{33}_{33}\left(\frac{\beta}{\alpha}\middle| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, -k \end{matrix}\right)$$
$[|\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\rho + 1, \quad \operatorname{Re}(k+\lambda+\rho) < 0]$ ET II 410(44)a

3. $$\int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx = \beta^{-\rho}\, G^{22}_{33}\left(\frac{\beta}{\alpha}\middle| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\nu, 1-\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k \end{matrix}\right)$$
ET II 411(46)

4. $$\int_0^\infty x^{\rho-1} \exp\left[-\frac{1}{2}(\alpha-\beta)x\right] W_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, dx$$
$$= \beta^{-\rho} \left[\Gamma\left(\tfrac{1}{2}-\lambda+\nu\right)\Gamma\left(\tfrac{1}{2}-\lambda-\nu\right)\right]^{-1} G^{23}_{33}\left(\frac{\beta}{\alpha}\middle| \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, k \end{matrix}\right)$$
$[\operatorname{Re}\alpha > 0, \quad |\operatorname{Re}\mu| + |\operatorname{Re}\nu| < \operatorname{Re}\rho + 1]$ ET II 411(45)

7.626

1. $$\int_0^1 \left[\frac{k}{x} - \frac{1}{4}(\xi+\eta) \exp\left[-\frac{1}{2}(\xi+\eta)x\right] x^c\right]\, {}_1F_1(a;c;\xi x)\, {}_1F_1(a;c;\eta x)\, dx$$
$$= 0 \qquad [\xi \neq \eta, \quad \operatorname{Re} c > 0]$$
$$= \frac{a}{\xi} e^{-\xi} \left[{}_1F_1(a+1;c;\xi)\right]^2 \qquad [\xi = \eta, \quad \operatorname{Re} c > 0]$$
[where ξ and η are any two zeros of the function ${}_1F_1(a;c;x)$] EH I 285

2. $$\int_1^\infty \left[\frac{k}{x} - \frac{1}{4}(\xi+\eta)\right] e^{-\frac{1}{2}(\xi+\eta)x} x^c \Psi(a,c;\xi x)\, \Psi(a,c;\eta x)\, dx = 0 \qquad [\xi \neq \eta];$$
$$= -\xi^{-1} e^{-\xi} [\Psi(a-1,c;\xi)]^2 \qquad [\xi = \eta]$$
[where ξ and η are any two zeros of the function $\Psi(a,c;x)$] EH I 286

7.627

1. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx = \frac{\Gamma(2\lambda)\,\Gamma\left(\frac{1}{2}-k+\mu-2\lambda\right)}{\Gamma\left(\frac{1}{2}-k+\mu\right)}a^{\lambda-\mu-\frac{1}{2}}\,W_{k+\lambda,\mu-\lambda}(a)$$
$$\left[|\arg a|<\pi, \quad 0<2\operatorname{Re}\lambda<\frac{1}{2}-\operatorname{Re}(k+\mu)\right]$$ ET II 411(50)

2. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\,M_{k,\mu}^{-\frac{1}{2}x}(a+x)\,dx$$
$$= \frac{\Gamma(2\lambda)\,\Gamma(2\mu+1)\,\Gamma\left(k+\mu-2\lambda+\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(1-2\lambda+2\mu)}a^{\lambda-\mu-\frac{1}{2}}\,M_{k-\lambda,\mu-\lambda}(a)$$
$$\left[\operatorname{Re}\lambda>0, \quad \operatorname{Re}(k+\mu-2\lambda)>-\tfrac{1}{2}\right]$$ ET II 405(20)

3. $$\int_0^\infty x^{2\lambda-1}(a+x)^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx = \Gamma(2\lambda)a^{\lambda-\mu-\frac{1}{2}}\,W_{k-\lambda,\mu-\lambda}(a)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\lambda>0]$$ ET II 411(47)

4. $$\int_0^\infty x^{\lambda-1}(a+x)^{k-\lambda-1}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx = \Gamma(\lambda)a^{k-1}\,W_{k-\lambda,\mu}(a)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\lambda>0]$$ ET II 411(48)

5. $$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{-\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx = \Gamma(\rho)a^{\rho}e^{\frac{1}{2}a}\,G_{23}^{30}\left(a\,\middle|\,\begin{matrix}0, 1-k-\sigma\\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma\end{matrix}\right)$$
$$[|\arg a|<\pi, \quad \operatorname{Re}\rho>0]$$ ET II 411(49)

6. $$\int_0^\infty x^{\rho-1}(a+x)^{-\sigma}e^{\frac{1}{2}x}\,W_{k,\mu}(a+x)\,dx$$
$$= \frac{\Gamma(\rho)a^{\rho}e^{-\frac{1}{2}a}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)}\,G_{23}^{31}\left(a\,\middle|\,\begin{matrix}k-\sigma+1, 0\\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma\end{matrix}\right)$$
$$[|\arg a|<\pi, \quad 0<\operatorname{Re}\rho<\operatorname{Re}(\sigma-k)]$$ ET II 412(51)

7. $$\int_0^\infty e^{-\frac{1}{2}(a+x)}\frac{(a+x)^{2\kappa-1}}{(ax)^{\kappa}}\,W_{\kappa,\mu}(x)\frac{dx}{x} = \frac{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}{a\,\Gamma\left(1-2\kappa\right)}\,W_{\kappa,\mu}(a)$$
$$\left[\operatorname{Re}\left(\tfrac{1}{2}\pm\mu-\kappa\right)>0\right]$$ BU 126(7a)

8. $$\int_0^\infty e^{-\frac{1}{2}x}x^{\gamma+\alpha-1}\,M_{\kappa,\mu}(x)\frac{dx}{(x+a)^{\alpha}}$$
$$= \frac{\Gamma(1+2\mu)\,\Gamma\left(\frac{1}{2}+\mu+\gamma\right)\Gamma\left(\kappa-\gamma\right)}{\Gamma\left(\frac{1}{2}+\mu-\gamma\right)\Gamma\left(\frac{1}{2}+\mu+\kappa\right)}\,{}_2F_2\left(\alpha, \kappa-\gamma; \frac{1}{2}+\mu-\gamma, \frac{1}{2}-\mu-\gamma; a\right)$$
$$+\frac{\Gamma\left(\alpha+\gamma+\frac{1}{2}+\mu\right)\Gamma\left(-\gamma-\frac{1}{2}-\mu\right)}{\Gamma(\alpha)}a^{\gamma+\frac{1}{2}+\mu}$$
$$\times\,{}_2F_2\left(\alpha+\gamma+\mu+\frac{1}{2}, \kappa+\mu+\frac{1}{2}; 1+2\mu, \frac{3}{2}+\mu+\gamma; a\right)$$
$$\left[\operatorname{Re}\left(\gamma+\alpha+\tfrac{1}{2}+\mu\right)>0, \quad \operatorname{Re}\left(\gamma-\kappa\right)<0\right]$$ BU 126(8)a

9. $$\int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}}\, M_{\kappa,\mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a}\, \Gamma(1+2\mu)\, \Gamma\left(\frac{1}{2}-\mu+\kappa\right) W_{-\kappa,\mu}(a)$$
$$\left[n = 0, 1, 2, \ldots, \quad \operatorname{Re}\left(\mu + 1 + \frac{n}{2}\right) > 0, \quad \operatorname{Re}\left(\kappa - \mu - \frac{1}{2}\right) < n, \quad |\arg a| < \pi\right]$$ BU 127(10a)a

7.628

1. $$\int_0^\infty e^{-st} e^{-t^2} t^{2c-2}\, {}_1F_1\left(a; c; t^2\right) dt = 2^{1-2c}\, \Gamma(2c-1) \Psi\left(c - \frac{1}{2}, a + \frac{1}{2}; \frac{1}{4}s^2\right)$$
$$\left[\operatorname{Re} c > \tfrac{1}{2}, \quad \operatorname{Re} s > 0\right]$$ EH I 270(11)

2. $$\int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st}\, M_{-3\nu,\nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}}\, \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8}\, K_{2\nu}\left(\frac{as^2}{8}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\tfrac{1}{4}, \quad \operatorname{Re} s > 0\right]$$ ET I 215(12)

3. $$\int_0^\infty t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st}\, M_{\lambda,\mu}\left(\frac{t^2}{a}\right) dt$$
$$= 2^{-3\mu-\lambda}\, \Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}}\, W_{-\frac{1}{2}(\lambda+3\mu), \frac{1}{2}(\lambda-\mu)}\left(\frac{as^2}{4}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\tfrac{1}{4}, \quad \operatorname{Re} s > 0\right]$$ ET I 215(13)

7.629

1.[8] $$\int_0^\infty t^k \exp\left(\frac{a}{2t}\right) e^{-st}\, W_{k,\mu}\left(\frac{a}{t}\right) dt = 2^{1-2k} \sqrt{a} s^{-k-\frac{1}{2}}\, S_{2k,2\mu}\left(2\sqrt{as}\right)$$
$$\left[|\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\tfrac{1}{2}, \quad \operatorname{Re} s > 0\right]$$ ET I 217(21)

2. $$\int_0^\infty t^{-k} \exp\left(-\frac{a}{2t}\right) e^{-st}\, W_{k,\mu}\left(\frac{a}{t}\right) dt = 2\sqrt{a} s^{k-\frac{1}{2}}\, K_{2\mu}\left(2\sqrt{as}\right)$$
$$\left[\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0\right]$$ ET I 217(22)

7.631

1. $$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\alpha^{-1}x - \beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right) W_{\lambda,\nu}\left(\beta x^{-1}\right) dx$$
$$= \beta^\rho \left[\Gamma\left(\tfrac{1}{2} - k + \mu\right) \Gamma\left(\tfrac{1}{2} - k - \mu\right)\right]^{-1}$$
$$\times G^{41}_{24}\left(\frac{\beta}{\alpha}\middle|\begin{matrix} 1+k, & 1-\lambda-\rho \\ \tfrac{1}{2}+\mu, & \tfrac{1}{2}-\mu, & \tfrac{1}{2}+\nu-\rho, & \tfrac{1}{2}-\nu-\rho \end{matrix}\right)$$
$$\left[|\arg \alpha| < \tfrac{3}{2}\pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(k+\rho) < -|\operatorname{Re} \nu| - \tfrac{1}{2}\right]$$ ET II 412(55)

2. $$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\alpha^{-1}x - \beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right) W_{\lambda,\nu}\left(\beta x^{-1}\right) dx$$
$$= \beta^\rho \left[\Gamma\left(\tfrac{1}{2} - k + \mu\right) \Gamma\left(\tfrac{1}{2} - k - \mu\right) \Gamma\left(\tfrac{1}{2} - \lambda + \nu\right) \Gamma\left(\tfrac{1}{2} - \lambda - \nu\right)\right]^{-1}$$
$$\times G^{42}_{24}\left(\frac{\beta}{\alpha}\middle|\begin{matrix} 1+k, & 1+\lambda-\rho \\ \tfrac{1}{2}+\mu, & \tfrac{1}{2}-\mu, & \tfrac{1}{2}+\nu-\rho, & \tfrac{1}{2}-\nu-\rho \end{matrix}\right)$$
$$\left[|\arg \alpha| < \tfrac{3}{2}\pi, \quad |\arg \beta| < \tfrac{3}{2}\pi, \quad \operatorname{Re}(\lambda-\rho) < \tfrac{1}{2} - |\operatorname{Re} \mu|, \quad \operatorname{Re}(k+\rho) < \tfrac{1}{2} - |\operatorname{Re} \nu|\right]$$ ET II 412(57)

3. $$\int_0^\infty x^{\rho-1} \exp\left[\frac{1}{2}\left(\alpha^{-1}x + \beta x^{-1}\right)\right] W_{k,\mu}\left(\alpha^{-1}x\right) W_{\lambda,\nu}\left(\beta x^{-1}\right) dx = \beta^\rho G^{40}_{24}\left(\frac{\beta}{\alpha}\middle| \begin{matrix} 1-k, & 1-\lambda-\rho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\rho, & \frac{1}{2}-\nu-\rho \end{matrix}\right)$$

$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0]$ ET II 412(54)

7.632 $$\int_0^\infty e^{-st}\left(e^t - 1\right)^{\mu-\frac{1}{2}} \exp\left(-\frac{1}{2}\lambda e^t\right) M_{k,\mu}\left(\lambda e^t - \lambda\right) dt = \frac{\Gamma(2\mu+1)\,\Gamma\left(\frac{1}{2}+k-\mu+s\right)}{\Gamma(s+1)} W_{-k-\frac{1}{2}s,\mu-\frac{1}{2}s}(\lambda)$$

$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}s > \operatorname{Re}(\mu-k) - \frac{1}{2}\right]$ ET I 216(15)

7.64 Combinations of confluent hypergeometric and trigonometric functions

7.641 $$\int_0^\infty \cos(ax)\ {}_1F_1(\nu+1;1;ix)\ {}_1F_1\left(\nu+1;1;-ix\right) dx$$

$$= -a^{-1}\sin(\nu\pi)\, P_\nu\left(2a^{-2}-1\right) \qquad [0 < a < 1];$$

$$= 0 \qquad [1 < a < \infty]$$

$[-1 < \operatorname{Re}\nu < 0]$ ET II 402(4)

7.642[11] $$\int_0^\infty \cos(2xy)\ {}_1F_1\left(a;c;-x^2\right) dx = \frac{1}{2}\pi^{\frac{1}{2}}\frac{\Gamma(c)}{\Gamma(a)}|y|^{2\alpha-1}e^{-y^2}\Psi\left(c-\tfrac{1}{2}, a+\tfrac{1}{2}; y^2\right)$$ EH I 285(12)

7.643

1. $$\int_0^\infty x^{4\nu}e^{-\frac{1}{2}x^2}\sin(bx)\ {}_1F_1\left(\frac{1}{2}-2\nu; 2\nu+1; \frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}}b^{4\nu}c^{-\frac{1}{2}b^2}\ {}_1F_1\left(\frac{1}{2}-2\nu; 1+2\nu; \frac{1}{2}b^2\right)$$

$\left[b > 0, \quad \operatorname{Re}\nu > -\frac{1}{4}\right]$ ET I 115(5)

2. $$\int_0^\infty x^{2\nu-1}e^{-\frac{1}{4}x^2}\sin(bx)\, M_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}}b^{2\nu-1}e^{-\frac{1}{4}b^2}\, M_{3\nu,\nu}\left(\frac{1}{2}b^2\right)$$

$\left[b > 0, \quad \operatorname{Re}\nu > -\frac{1}{4}\right]$ ET I 116(10)

3. $$\int_0^\infty x^{-2\nu-1}e^{\frac{1}{4}x^2}\cos(bx)\, W_{3\nu,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}}b^{-2\nu-1}e^{\frac{1}{4}b^2}\, W_{3\nu,\nu}\left(\frac{1}{2}b^2\right)$$

$\left[\operatorname{Re}\nu < \frac{1}{4}, \quad b > 0\right]$ ET I 61(7)

4. $$\int_0^\infty x^{-2\nu}e^{\frac{1}{4}x^2}\sin(bx)\, W_{3\nu-1,\nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}}b^{-2\nu}e^{\frac{1}{4}b^2}\, W_{3\nu-1,\nu}\left(\frac{1}{2}b^2\right)$$

$\left[\operatorname{Re}\nu < \frac{1}{2}, \quad b > 0\right]$ ET I 116(9)

7.644

1.[11] $$\int_0^\infty x^{-\mu-\frac{1}{2}}e^{-\frac{1}{2}x}\sin\left(2ax^{\frac{1}{2}}\right) M_{k,\mu}(x)\, dx = \pi^{\frac{1}{2}}a^{k+\mu-1}\frac{\Gamma(3-2\mu)}{\Gamma\left(\frac{1}{2}+k+\mu\right)}\exp\left(-\frac{a^2}{2}\right) W_{\rho,\sigma}\left(a^2\right),$$

$2\rho = k - 3\mu + 1, \qquad 2\sigma = k + \mu - 1 \qquad [a > 0, \quad \operatorname{Re}(k+\mu) > 0]$ ET II 403(10)

2. $$\int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\,dx = \frac{c\,\Gamma(1+\mu+\rho)\,\Gamma(1-\mu+\rho)}{\Gamma\left(\frac{3}{2}-k+\rho\right)} \times {}_2F_2\left(1+\mu+\rho, 1-\mu+\rho; \frac{3}{2}, \frac{3}{2}-k+\rho; -\frac{c^2}{4}\right)$$
$[\operatorname{Re}\rho > |\operatorname{Re}\mu| - 1]$ ET II 407(28)

3. $$\int_0^\infty x^{\rho-1} \sin\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x}\, W_{k,\mu}(x)\,dx = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G^{22}_{23}\left(\frac{c^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho\\ \frac{1}{2}, -k-\rho, 0\end{matrix}\right)$$
$\left[c > 0, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - 1, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}\right]$ ET II 407(29)

4. $$\int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\,dx = \frac{\Gamma\left(\frac{1}{2}+\mu+\rho\right)\Gamma\left(\frac{1}{2}-\mu+\rho\right)}{\Gamma(1-k+\rho)} \times {}_2F_2\left(\frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho; \frac{1}{2}, 1-k+\rho; -\frac{c^2}{4}\right)$$
$\left[\operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}\right]$ ET II 407(30)

5. $$\int_0^\infty x^{\rho-1} \cos\left(cx^{\frac{1}{2}}\right) e^{\frac{1}{2}x}\, W_{k,\mu}(x)\,dx = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G^{22}_{23}\left(\frac{c^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho\\ 0, -k-\rho, \frac{1}{2}\end{matrix}\right)$$
$\left[c > 0, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\rho) < \frac{1}{2}\right]$ ET II 407(31)

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.651

1. $$\int_0^\infty J_\nu(xy)\, M_{-\frac{1}{2}\mu,\frac{1}{2}\nu}(ax)\, W_{\frac{1}{2}\mu,\frac{1}{2}\nu}(ax)\,dx = ay^{-\mu-1}\frac{\Gamma(\nu+1)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu\right)}\left[a+\left(a^2+y^2\right)^{\frac{1}{2}}\right]^\mu \left(a^2+y^2\right)^{-\frac{1}{2}}$$
$\left[y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu < \frac{1}{2}, \quad \operatorname{Re} a > 0\right]$ ET II 85(19)

2. $$\int_0^\infty M_{k,\frac{1}{2}\nu}(-iax)\, M_{-k,\frac{1}{2}\nu}(-iax)\, J_\nu(xy)\,dx = \frac{ae^{-\frac{1}{2}(\nu+1)\pi i}\left[\Gamma(1+\nu)\right]^2}{\Gamma\left(\frac{1}{2}+k+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-k+\frac{1}{2}\nu\right)} y^{-1-2k} \times \left(a^2-y^2\right)^{-\frac{1}{2}}\left\{\left[a+\left(a^2-y^2\right)^{\frac{1}{2}}\right]^{2k} + \left[a-\left(a^2-y^2\right)^{\frac{1}{2}}\right]^{2k}\right\} \quad [0 < y < a];$$
$$= 0 \quad [a < y < \infty]$$
$\left[a > 0, \quad \operatorname{Re}\nu > -1, \quad |\operatorname{Re} k| < \frac{1}{4}\right]$ ET II 85(18)

7.652 $$\int_0^\infty M_{-\mu,\frac{1}{2}\nu}\left\{a\left[\left(b^2+x^2\right)^{\frac{1}{2}}-b\right]\right\} W_{\mu,\frac{1}{2}\nu}\left\{a\left[\left(b^2+x^2\right)^{\frac{1}{2}}+b\right]\right\} J_\nu(xy)\,dx = \frac{ay^{-2\mu-1}\,\Gamma(1+\nu)\left[\left(a^2+y^2\right)^{\frac{1}{2}}+a\right]^{2\mu}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\mu\right)\left(A^2+Y^2\right)^{\frac{1}{2}}}\exp\left[-b\left(a^2+y^2\right)^{\frac{1}{2}}\right]$$
$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu<\frac{1}{4}, \quad \operatorname{Re}a>0, \quad \operatorname{Re}b>0\right]$ ET II 87(29)

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

1. $$\int_0^\infty x^{-1}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_0(xy)\,dx = e^{-ik\pi}\frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu+k\right)}\,P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]Q^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$\left[y>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}k<\frac{3}{4}\right]$ ET II 18(44)

2. $$\int_0^\infty x^{-1}\,W_{k,\mu}(ax)\,W_{-k,\mu}(ax)\,J_0(xy)\,dx = \frac{1}{2}\pi\cos(\mu\pi)\,P^k_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]P^{-k}_{\mu-\frac{1}{2}}\left[\left(1+\frac{y^2}{a^2}\right)^{\frac{1}{2}}\right]$$
$\left[y>0, \quad \operatorname{Re}a>0, \quad |\operatorname{Re}\mu|<\frac{1}{2}\right]$
ET II 18(45)

3. $$\int_0^\infty x^{2\mu-\nu}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_\nu(xy)\,dx = 2^{2\mu-\nu+2k}a^{2k}y^{\nu-2\mu-2k-1}\frac{\Gamma(2\mu+1)}{\Gamma\left(\nu-k-\mu+\frac{1}{2}\right)} \times {}_3F_2\left(\frac{1}{2}-k, 1-k, \frac{1}{2}-k+\mu; 1-2k, \frac{1}{2}-k-\mu+\nu; -\frac{y^2}{a^2}\right)$$
$\left[y>0, \quad \operatorname{Re}\mu>-\frac{1}{2}, \quad \operatorname{Re}a>0, \quad \operatorname{Re}(2\mu+2k-\nu)<\frac{1}{2}\right]$ ET II 85(20)

4. $$\int_0^\infty x^{2\rho-\nu}\,W_{k,\mu}(iax)\,W_{k,\mu}(-iax)\,J_\nu(xy)\,dx = 2^{2\rho-\nu}y^{\nu-2\rho-1}\pi^{-\frac{1}{2}}\left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1}G^{24}_{44}\left(\frac{y^2}{a^2}\middle|\begin{matrix}\frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu\\ \rho+\frac{1}{2}, -k, k, \rho-\nu+\frac{1}{2}\end{matrix}\right)$$
$\left[y>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\rho>|\operatorname{Re}\mu|-1, \quad \operatorname{Re}(2\rho+2k-\nu)<\frac{1}{2}\right]$ ET II 86(23)a

5. $$\int_0^\infty x^{2\rho-\nu}\,W_{k,\mu}(ax)\,M_{-k,\mu}(ax)\,J_\nu(xy)\,dx = \frac{2^{2\rho-\nu}\,\Gamma(2\mu+1)}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}-k+\mu\right)}y^{\nu-2\rho-1}\,G^{23}_{44}\left(\frac{y^2}{a^2}\middle|\begin{matrix}\frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu\\ \rho+\frac{1}{2}, -k, k, \rho-\nu+\frac{1}{2}\end{matrix}\right)$$
$\left[y>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\rho>-1, \quad \operatorname{Re}(\rho+\mu)>-1, \quad \operatorname{Re}(2e+2k+\nu)<\frac{1}{2}\right]$ ET II 86(21)a

6. $$\int_0^\infty x^{2\rho-\nu}\, W_{k,\mu}(ax)\, W_{-k,\mu}(ax)\, J_\nu(xy)\, dx = \frac{\Gamma(\rho+1+\mu)\,\Gamma(\rho+1-\mu)\,\Gamma(2\rho+2)}{\Gamma\left(\frac{3}{2}+k+\rho\right)\Gamma\left(\frac{3}{2}-k+\rho\right)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\rho-1} \times {}_4F_3\left(\rho+1, \rho+\frac{3}{2}, \rho+1+\mu, \rho+1-\mu; \frac{3}{2}+k+\rho, \frac{3}{2}-k+\rho, 1+\nu; -\frac{y^2}{a^2}\right)$$
$[y > 0, \quad \operatorname{Re}\rho > |\operatorname{Re}\mu| - 1, \quad \operatorname{Re} a > 0]$ ET II 86(22)a

7.662

1. $$\int_0^\infty x^{-1}\, M_{-\mu,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) W_{\mu,\frac{1}{4}\nu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\nu-\mu\right)} I_{\frac{1}{4}\nu-\mu}\left(\frac{1}{4}y^2\right) K_{\frac{1}{4}\nu+\mu}\left(\frac{1}{4}y^2\right)$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 86(24)

2. $$\int_0^\infty x^{-1}\, M_{\alpha-\beta,\frac{1}{4}\nu-\gamma}\left(\frac{1}{2}x^2\right) W_{\alpha+\beta,\frac{1}{4}\nu+\gamma}\left(\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{\Gamma\left(1+\frac{1}{2}\nu-2\gamma\right)}{\Gamma\left(1+\frac{1}{2}\nu-2\beta\right)} y^{-2}\, M_{\alpha-\gamma,\frac{1}{4}\nu-\beta}\left(\frac{1}{2}y^2\right) W_{\alpha+\gamma,\frac{1}{4}\nu+\beta}\left(\frac{1}{2}y^2\right)$$
$\left[y > 0, \quad \operatorname{Re}\beta < \frac{1}{8}, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu-4\gamma) > -2\right]$ ET II 86(25)

3. $$\int_0^\infty x^{-1}\, M_{k,0}\left(iax^2\right) M_{k,0}\left(-iax^2\right) K_0(xy)\, dx = \frac{\pi}{16}\left\{\left[J_k\left(\frac{y^2}{8a}\right)\right]^2 + \left[Y_k\left(\frac{y^2}{8a}\right)\right]^2\right\}$$
$[a > 0]$ ET II 152(83)

4. $$\int_0^\infty x^{-1}\, M_{k,\mu}\left(iax^2\right) M_{k,\mu}\left(-iax^2\right) K_0(xy)\, dx = ay^{-2}\left[\Gamma(2\mu+1)\right]^2 W_{-\mu,k}\left(\frac{iy^2}{4a}\right) W_{-\mu,k}\left(-\frac{iy^2}{4a}\right)$$
$\left[a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ ET II 152(84)

7.663

1. $$\int_0^\infty x^{2\rho}\, {}_1F_1\left(a; b; -\lambda x^2\right) J_\nu(xy)\, dx = \frac{2^{2\rho}\,\Gamma(b)}{\Gamma(a) y^{2\rho+1}} G_{23}^{21}\left(\frac{y^2}{4\lambda}\,\middle|\, \begin{matrix} 1, b \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, a, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix}\right)$$
$\left[y > 0, \quad -1-\operatorname{Re}\nu < 2\operatorname{Re}\rho < \frac{1}{2}+2\operatorname{Re} a, \quad \operatorname{Re}\lambda > 0\right]$ ET II 88(6)

2. $$\int_0^\infty x^{\nu+1}\, {}_1F_1\left(2a-\nu; a+1; -\frac{1}{2}x^2\right) J_\nu(xy)\, dx = \frac{2^{\nu-a+\frac{1}{2}}\,\Gamma(a+1)}{\pi^{\frac{1}{2}}\,\Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} K_{a-\nu-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$\left[y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(4a-3\nu) > \frac{1}{2}\right]$ ET II 87(1)

3. $$\int_0^\infty x^a\, {}_1F_1\left(a; \frac{1+a+\nu}{2}; -\frac{1}{2}x^2\right) J_\nu(xy)\, dx = y^{a-1}\, {}_1F_1\left(a; \frac{1+a+\nu}{2}; -\frac{y^2}{2}\right)$$
$\left[y > 0, \quad \operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a+\nu) > -1\right]$ ET II 87(2)

4. $$\int_0^\infty x^{\nu+1-2a}\,{}_1F_1\left(a;1+\nu-a;-\frac{1}{2}x^2\right)J_\nu(xy)\,dx$$
$$=\frac{\pi^{\frac{1}{2}}\,\Gamma(1+\nu-a)}{\Gamma(a)}2^{-2a+\nu+\frac{1}{2}}y^{2a-\nu-1}e^{-\frac{1}{4}y^2}\,I_{a-\frac{1}{2}}\left(\frac{1}{4}y^2\right)$$
$\left[y>0, \quad \operatorname{Re} a-1<\operatorname{Re}\nu<4\operatorname{Re} a-\frac{1}{2}\right]$ ET II 87(3)

5. $$\int_0^\infty x\,{}_1F_1\left(\lambda;1;-x^2\right)J_0(xy)\,dx=\left[2^{2\lambda-1}\,\Gamma(\lambda)\right]^{-1}y^{2\lambda-2}e^{-\frac{1}{4}y^2}$$
$[y>0, \quad \operatorname{Re}\lambda>0]$ ET II 18(46)

6. $$\int_0^\infty x^{\nu+1}\,{}_1F_1\left(a;b;-\lambda x^2\right)J_\nu(xy)\,dx$$
$$=\frac{2^{1-a}\,\Gamma(b)}{\Gamma(a)\lambda^{\frac{1}{2}a+\frac{1}{2}\nu}}y^{a-2}e^{-\frac{y^2}{8\lambda}}\,W_{k,\mu}\left(\frac{y^2}{4\lambda}\right), \qquad 2k=a-2b+\nu+2, \qquad 2\mu=a-\nu-1$$
$\left[y>0, \quad -1<\operatorname{Re}\nu<2\operatorname{Re} a-\frac{1}{2}, \quad \operatorname{Re}\lambda>0\right]$ ET II 88(4)

7. $$\int_0^\infty x^{2b-\nu-1}\,{}_1F_1\left(a;b;-\lambda x^2\right)J_\nu(xy)\,dx=\frac{2^{2b-2a-\nu-1}\,\Gamma(b)}{\Gamma(a-b+\nu+1)}\lambda^{-a}y^{2a-2b+\nu}$$
$$\times\,{}_1F_1\left(a;1+a-b+\nu;-\frac{y^2}{4\lambda}\right)$$
$\left[y>0, \quad 0<\operatorname{Re} b<\frac{3}{4}+\operatorname{Re}\left(a+\frac{1}{2}\nu\right), \quad \operatorname{Re}\lambda>0\right]$ ET II 88(5)

7.664

1. $$\int_0^\infty x\,W_{\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right)W_{-\frac{1}{2}\nu,\mu}\left(\frac{a}{x}\right)K_\nu(xy)\,dx=2ay^{-1}\,K_{2\mu}\left[(2ay)^{\frac{1}{2}}e^{\frac{1}{4}i\pi}\right]K_{2\mu}\left[(2ay)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}\right]$$
$[\operatorname{Re} y>0, \quad \operatorname{Re} a>0]$ ET II 152(85)

2. $$\int_0^\infty x\,W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)J_\nu(xy)\,dx$$
$$=-4y^{-1}\left\{\sin\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]J_{2\mu}\left(2y^{\frac{1}{2}}\right)+\cos\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]Y_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}K_{2\mu}\left(2y^{\frac{1}{2}}\right)$$
$[y>0, \quad \operatorname{Re}(\nu\pm 2\mu)>-1]$ ET II 87(27)

3. $$\int_0^\infty x\,W_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)Y_\nu(xy)\,dx$$
$$=4y^{-1}\left\{\left\{\cos\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]J_{2\mu}\left(2y^{\frac{1}{2}}\right)-\sin\left[\left(\mu-\tfrac{1}{2}\nu\right)\pi\right]Y_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}K_{2\mu}\left(2y^{\frac{1}{2}}\right)\right\}$$
$\left[y>0, \quad |\operatorname{Re}\mu|<\frac{1}{4}\right]$ ET II 117(48)

4. $$\int_0^\infty x\,W_{-\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)M_{\frac{1}{2}\nu,\mu}\left(\frac{2}{x}\right)J_\nu(xy)\,dx=\frac{4\,\Gamma(1+2\mu)y^{-1}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\mu\right)}J_{2\mu}\left(2y^{\frac{1}{2}}\right)K_{2\mu}\left(2y^{\frac{1}{2}}\right)$$
$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}\mu>-\frac{1}{4}\right]$
ET II 86(26)

5. $$\int_0^\infty x\, W_{-\frac{1}{2}\nu,\mu}\left(\frac{ia}{x}\right) W_{-\frac{1}{2}\nu,\mu}\left(-\frac{ia}{x}\right) J_\nu(xy)\,dx$$
$$= 4ay^{-1}\left[\Gamma\left(\tfrac{1}{2}+\mu+\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}-\mu+\tfrac{1}{2}\nu\right)\right]^{-1} K_\mu\left[(2iay)^{\frac{1}{2}}\right] K_\mu\left[(-2iay)^{\frac{1}{2}}\right]$$
$$\left[y>0, \quad \operatorname{Re} a>0, \quad |\operatorname{Re}\mu|<\tfrac{1}{2}, \quad \operatorname{Re}\nu>-1\right]$$ ET II 87(28)

7.665

1. $$\int_0^\infty x^{-\frac{1}{2}} J_\nu\left(ax^{\frac{1}{2}}\right) K_{\frac{1}{2}\nu-\mu}\left(\frac{1}{2}x\right) M_{k,\mu}(x)\,dx$$
$$= \frac{\Gamma(2\mu+1)}{a\,\Gamma\left(k+\frac{1}{2}\nu+1\right)}\, W_{\frac{1}{2}(k-\mu),\frac{1}{2}k-\frac{1}{4}\nu}\left(\frac{a^2}{2}\right) M_{\frac{1}{2}(k+\mu),\frac{1}{2}k+\frac{1}{4}\nu}\left(\frac{a^2}{2}\right)$$
$$\left[a>0, \quad \operatorname{Re} k>-\tfrac{1}{4}, \quad \operatorname{Re}\mu>-\tfrac{1}{2}, \quad \operatorname{Re}\nu>-1\right]$$ ET II 405(18)

2. $$\int_0^\infty x^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi(a,c;x)\ {}_1F_1\left(a';c';-x\right) J_{c+c'-2}\left[2(xy)^{\frac{1}{2}}\right]\,dx$$
$$= \frac{\Gamma\left(c'\right)}{\Gamma\left(a+a'\right)} y^{\frac{1}{2}c+\frac{1}{2}c'-1}\Psi\left(c'-a',c+c'-a-a';y\right)\ {}_1F_1\left(a';a+a';-y\right)$$
$$\left[\operatorname{Re} c'>0, \quad 1<\operatorname{Re}\left(c+c'\right)<2\operatorname{Re}\left(a+a'\right)+\tfrac{1}{2}\right]$$ EH I 287(23)

7.666 $$\int_0^\infty x^{\frac{1}{2}c-\frac{1}{2}}\ {}_1F_1\left(a;c;-2x^{\frac{1}{2}}\right)\Psi\left(a,c;2x^{\frac{1}{2}}\right) J_{c-1}\left[2(xy)^{\frac{1}{2}}\right]\,dx$$
$$= 2^{-c}\frac{\Gamma(c)}{\Gamma(a)} y^{a-\frac{1}{2}c-\frac{1}{2}}\left[1+(1+y)^{\frac{1}{2}}\right]^{c-2a}(1+y)^{-\frac{1}{2}}$$
$$\left[\operatorname{Re} c>2, \quad \operatorname{Re}(c-2a)<\tfrac{1}{2}\right]$$ EH I 285(13)

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

1. $$\int_0^\infty x^{k-\frac{3}{2}}\exp\left[-\frac{1}{2}(a+1)x\right] K_\nu\left(\frac{1}{2}ax\right) M_{k,\nu}(x)\,dx$$
$$= \frac{\pi^{\frac{1}{2}}\,\Gamma(k)\,\Gamma(k+2\nu)}{a^{k+\nu}\,\Gamma\left(k+\nu+\frac{1}{2}\right)}\ {}_2F_1\left(k,k+2\nu;2\nu+1;-a^{-1}\right)$$
$$[\operatorname{Re} a>0, \quad \operatorname{Re} k>0, \quad \operatorname{Re}(k+2\nu)>0]$$ ET II 405(17)

2. $$\int_0^\infty x^{-k-\frac{3}{2}}\exp\left[-\frac{1}{2}(a-1)x\right] K_\mu\left(\frac{1}{2}ax\right) W_{k,\mu}(x)\,dx$$
$$= \frac{\pi\,\Gamma(-k)\,\Gamma(2\mu-k)\,\Gamma(-2\mu-k)}{\Gamma\left(\frac{1}{2}-k\right)\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} 2^{2k+1}a^{k-\nu}\ {}_2F_1\left(-k,2\mu-k;-2k;1-a^{-1}\right)$$
$$[\operatorname{Re} a>0, \quad \operatorname{Re} k<2\operatorname{Re}\mu<-\operatorname{Re} k]$$ ET II 408(36)

7.672

1. $$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} M_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} 2^{2\rho} y^{-2\rho-1} G^{21}_{23}\left(\frac{y^2}{4a}\middle|\begin{matrix}\frac{1}{2}-\mu, \frac{1}{2}+\mu\\ \frac{1}{2}+\rho+\frac{1}{2}\nu, k, \frac{1}{2}+\rho-\frac{1}{2}\nu\end{matrix}\right)$$

$\left[y>0, \quad -1-\operatorname{Re}\left(\frac{1}{2}\nu+\mu\right)<\operatorname{Re}\rho<\operatorname{Re}k-\frac{1}{4}, \quad \operatorname{Re}a>0\right]$ ET II 83(10)

2. $$\int_0^\infty x^{2\rho} e^{-\frac{1}{2}ax^2} W_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{\Gamma\left(1+\mu+\frac{1}{2}\nu+\rho\right)\Gamma\left(1-\mu+\frac{1}{2}\nu+\rho\right)2^{-\nu-1}}{\Gamma(\nu+1)\,\Gamma\left(\frac{3}{2}-k+\frac{1}{2}\nu+\rho\right)} a^{-\frac{1}{2}\nu-\rho-\frac{1}{2}} y^\nu \times {}_2F_2\left(\lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-k+\lambda; -\frac{y^2}{4a}\right),$$

$\lambda = 1+\frac{1}{2}\nu+\rho \qquad \left[y>0, \quad \operatorname{Re}a>0, \quad \operatorname{Re}\left(\rho\pm\mu+\frac{1}{2}\nu\right)>-1\right]$ ET II 85(16)

3. $$\int_0^\infty x^{2\rho} e^{\frac{1}{2}ax^2} W_{k,\mu}\left(ax^2\right) J_\nu(xy)\,dx = \frac{2^{2\rho}y^{-2\rho-1}}{\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} \times G^{22}_{23}\left(\frac{y^2}{4a}\middle|\begin{matrix}\frac{1}{2}-\mu, & \frac{1}{2}+\mu & \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, & -k, & \frac{1}{2}+\rho-\frac{1}{2}\nu\end{matrix}\right)$$

$\left[y>0, \quad |\arg a|<\pi, \quad -1-\operatorname{Re}\left(\frac{1}{2}\nu\pm\mu\right)<\operatorname{Re}\rho<-\frac{1}{4}-\operatorname{Re}k\right]$ ET II 85(17)

4. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx = \frac{2^\lambda y^{-1/2}\,\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} G^{31}_{34}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & & \\ h, & \kappa, & -\lambda-\frac{1}{2}, & l\end{matrix}\right)$$

$h=\frac{1}{4}+\frac{1}{2}\nu, \quad \kappa=\frac{1}{4}-\frac{1}{2}\nu, \quad l=-\frac{1}{4}-\frac{1}{2}\nu$

$\left[y>0, \quad \operatorname{Re}(k-\lambda)>0, \quad \operatorname{Re}\left(2\lambda+2\mu\pm\nu\right)>-\frac{5}{2}\right]$ ET II 116(45)

5. $$\int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx = 2^\lambda\left[\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1} G^{32}_{34}\left(\frac{y^2}{2}\middle|\begin{matrix}-\mu-\lambda, & \mu-\lambda, & l & \\ h, & \kappa, & -\frac{1}{2}-k-\lambda, & l\end{matrix}\right) y^{-1/2},$$

$h=\frac{1}{4}+\frac{1}{2}\nu, \quad \kappa=\frac{1}{4}-\frac{1}{2}\nu, \quad l=-\frac{1}{4}-\frac{1}{2}\nu$

$\left[y>0, \quad \operatorname{Re}(k+\lambda)<0, \quad \operatorname{Re}\left(2\lambda\pm2\mu\pm\nu\right)>-\frac{5}{2}\right]$ ET II 117(47)

6. $$\int_0^\infty x^{-1/2} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}\left(x^2\right) J_\nu(xy)\,dx = (2\nu+1)2^{-\nu}y^{\nu-1}\left[1-\Phi\left(\frac{1}{2}y\right)\right]$$

$\left[y>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right]$ ET II 82(1)

7. $$\int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}\left(x^2\right) J_\nu(xy)dx = \frac{\Gamma(\nu+2)y^\nu}{\Gamma\left(\nu+\frac{3}{2}\right)2^\nu}\left[1-\Phi\left(\frac{1}{2}y\right)\right]$$

$[y>0, \operatorname{Re}\nu>-1]$ ET II 82(2)

8. $$\int_0^\infty e^{-\frac{1}{4}x^2} M_{k,\frac{1}{2}\nu}\left(\frac{1}{2}\right) x^2 J_\nu(xy)\,dx = \frac{2^{-k}\,\Gamma(\nu+1)}{\Gamma\left(k+\frac{1}{2}\nu+\frac{1}{2}\right)} y^{2k-1} e^{-\frac{1}{2}y^2}$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}k<\tfrac{1}{2}\right]$$
ET II 83(7)

9. $$\int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}\right) x^2 J_\nu(xy)\,dx$$
$$= 2^{\frac{1}{2}\left(\frac{1}{2}-k-3\mu+\nu\right)} \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} W_{\alpha,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = k-3\mu+\nu+\tfrac{1}{2}, \qquad 2\beta = k+\mu-\nu-\tfrac{1}{2}$$
$$\left[y>0, \quad -1<\operatorname{Re}\nu<2\operatorname{Re}(k+\mu)-\tfrac{1}{2}\right]$$
ET II 83(9)

10. $$\int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2} W_{k,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx = \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\beta}\left(\frac{1}{2}y^2\right)$$
$$2\alpha = \tfrac{1}{2}+k+\nu-3\mu, \quad 2\beta = \tfrac{1}{2}-k+\nu-\mu$$
$$[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-2\mu)>-1]$$
ET II 84(14)

11. $$\int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} W_{k,\pm\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx$$
$$= \frac{\Gamma(1+\nu-2\mu)}{\Gamma\left(\frac{1}{2}+\mu-k\right)} 2^{\frac{1}{2}\left(\frac{1}{2}+k-3\mu+\nu\right)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2} W_{\alpha,\beta}\left(\frac{1}{2}y^2\right),$$
$$2\alpha = k+3\mu-\nu-\tfrac{1}{2}, \qquad 2\beta = k-\mu+\nu+\tfrac{1}{2}$$
$$\left[y>0, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(\nu-2\mu)>-1, \quad \operatorname{Re}\left(k-\mu+\tfrac{1}{2}\nu\right)<-\tfrac{1}{4}\right]$$
ET II 84(15)

12. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) J_\nu(xy)\,dx$$
$$= \frac{\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k-\mu+\nu\right)} 2^{\frac{1}{2}\left(\frac{1}{2}-k+3\mu-\nu\right)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha,\beta}\left(\frac{1}{2}y^2\right)$$
$$2\alpha = \tfrac{1}{2}+k+3\mu-\nu, \qquad 2\beta = -\tfrac{1}{2}+k-\mu+\nu$$
$$\left[y>0, \quad -\tfrac{1}{2}<\operatorname{Re}\mu<\operatorname{Re}\left(k+\tfrac{1}{2}\nu\right)-\tfrac{1}{4}\right]$$
ET II 83(8)

13. $$\int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx$$
$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2}\,\Gamma(2\mu+1)$$
$$\times \Gamma\left(\tfrac{1}{2}-k-\mu\right) \left\{ \cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)} M_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right) \right.$$
$$\left. -\ \sin[(\nu+k-\mu)\pi]\, W_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right) \right\}$$
$$2\alpha = 3\mu-\nu+k+\tfrac{1}{2}, \qquad 2\beta = \mu-\nu-k+\tfrac{1}{2}$$
$$\left[y>0, \quad -1<2\operatorname{Re}\mu<\operatorname{Re}(2k+\nu)+\tfrac{1}{2}, \quad \operatorname{Re}(2\mu-\nu)>-1\right]$$
ET II 116(44)

14. $$\int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\mu}\left(\frac{1}{2}x^2\right) Y_\nu(xy)\,dx$$
$$= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}}\,\Gamma(2\mu+1)$$
$$\times \Gamma\left(\tfrac{1}{2}-\mu-k\right) e^{-\frac{1}{4}y^2} \left\{\cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma\left(\mu+\nu-k+\frac{3}{2}\right)} M_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right) + \sin[(\mu-k)\pi]\, W_{\alpha,\beta}\left(\tfrac{1}{2}y^2\right)\right\}$$
$$2\alpha = 3\mu+\nu+k+\tfrac{1}{2}, \qquad 2\beta = \mu+\nu-k+\tfrac{1}{2}$$
$$\left[y>0, \quad -1 < 2\operatorname{Re}\mu < \operatorname{Re}(2k-\nu)+\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1\right]$$ ET II 116(43)

15. $$\int_0^\infty x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\mu}\left(ax^2\right) K_\nu(xy)\,dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}}$$
$$\times \Gamma(2\mu+1)\,\Gamma(2\mu+\nu+1) \exp\left(\frac{y^2}{8a}\right) W_{\kappa,m}\left(\frac{y^2}{4a}\right),$$
$$2\kappa = -3\mu-\nu-k-\tfrac{1}{2}, \qquad 2m = \mu+\nu-k+\tfrac{1}{2}$$
$$\left[\operatorname{Re} y>0, \quad \operatorname{Re} a>0, \quad \operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1\right]$$ ET II 152(82)

7.673

1.[10] $$\int_0^\infty e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\kappa,\frac{1}{2}\mu}(ax)\, J_\nu\left(2\sqrt{bx}\right)\,dx$$
$$= \left(\frac{b}{a}\right)^{\frac{\kappa-1}{2}-\frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)}\,\Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma\left(1+\dfrac{\kappa+\nu}{2}-\dfrac{1+\mu}{4}\right)}$$
$$\times M_{\frac{1}{2}(\kappa-\nu-1)+\frac{3}{4}(1+\mu),\frac{\kappa+\nu}{2}-\frac{1+\mu}{4}}\left(\frac{b}{a}\right)$$
$$\left[\operatorname{Re}(1+\mu)>0, \quad \operatorname{Re}\left(\kappa+\frac{\nu-\mu}{2}\right) > -\frac{3}{4}, \quad \operatorname{Im} b = 0\right]$$ BU 128(12)a

2. $$\int_0^\infty e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\kappa,\frac{1}{2}\mu}(ax)\, J_\nu\left(2\sqrt{bx}\right)\,dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu)\, e^{\frac{b}{2a}}}{\Gamma\left(\frac{1\pm\mu}{2}-\kappa\right)} \left(\frac{a}{b}\right)^{\frac{1}{2}(\kappa+1)+\frac{1}{4}(1\mp\nu)}$$
$$\times W_{\frac{1}{2}(\kappa+1-\nu)-\frac{3}{4}(1\mp\mu),\frac{1}{2}(\kappa+\nu)+\frac{1}{4}(1\mp\mu)}\left(\frac{b}{a}\right)$$
$$\left[\operatorname{Re}\left(\frac{\nu\mp\mu}{2}+\kappa\right) < \frac{3}{4}, \quad \operatorname{Re}\nu > -1\right]$$ BU 128(13)

7.674

1. $$\int_0^\infty x^{\rho-1}e^{-\frac{1}{2}\kappa}\,J_{\lambda+\nu}\left(ax^{1/2}\right)J_{\lambda-\nu}\left(ax^{1/2}\right)W_{k,\mu}(x)\,dx$$
$$=\frac{\left(\frac{1}{2}a\right)^{2\lambda}\Gamma\left(\frac{1}{2}+\lambda+\mu+\rho\right)\Gamma\left(\frac{1}{2}+\lambda-\mu+\rho\right)}{\Gamma\left(1+\lambda+\nu\right)\Gamma(1+\lambda-\nu)\Gamma(1+\lambda-k+\rho)}$$
$$\times\ {}_4F_4\left(1+\lambda,\tfrac{1}{2}+\lambda,\tfrac{1}{2}+\lambda+\mu+\rho,\tfrac{1}{2}+\lambda-\mu+\rho;1+\lambda+\nu,\right.$$
$$\left.1+\lambda-\nu,1+2\lambda,1+\lambda-k+\rho;-a^2\right)$$
$\left[|\operatorname{Re}\mu|<\operatorname{Re}(\lambda+\rho)+\frac{1}{2}\right]$ ET II 409(37)

2. $$\int_0^\infty x^{\rho-1}e^{-\frac{1}{2}\kappa}\,I_{\lambda+\nu}\left(ax^{1/2}\right)K_{\lambda-\nu}\left(ax^{1/2}\right)W_{k,\mu}(x)\,dx$$
$$=\frac{\pi^{-1/2}}{2}\,G^{24}_{45}\left(a^2\left|\begin{matrix}0,\frac{1}{2},\frac{1}{2}+\mu-\rho,\frac{1}{2}-\mu-\rho\\ \lambda,\nu,-\lambda,-\nu,k-\rho\end{matrix}\right.\right)$$
$\left[|\operatorname{Re}\mu|<\operatorname{Re}(\lambda+\rho)+\frac{1}{2},\quad |\operatorname{Re}\mu|<\operatorname{Re}(\nu+\rho)+\frac{1}{2}\right]$ ET II 409(38)

Combinations of Struve functions and confluent hypergeometric functions

7.675

1. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{-\frac{1}{4}x^2}\,M_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx=\frac{2^{-\lambda}\,\Gamma(2\mu+1)}{y^{1/2}\,\Gamma\left(\frac{1}{2}+k+\mu\right)}\,G^{22}_{34}\left(\frac{y^2}{2}\left|\begin{matrix}l,-\mu-\lambda,mu-\lambda\\ l,k-\lambda-\frac{1}{2},h,\kappa\end{matrix}\right.\right)$$
$h=\frac{1}{4}+\frac{1}{2}\nu,\quad \kappa=\frac{1}{4}-\frac{1}{2}\nu,\quad l=\frac{3}{4}+\frac{1}{2}\nu$
$\left[\operatorname{Re}(2\lambda+2\mu+\nu)>-\frac{7}{2},\quad \operatorname{Re}(k-\lambda)>0,\quad y>0,\quad \operatorname{Re}(2\lambda-2k+\nu)<-\frac{1}{2}\right]$
ET II 171(42)

2. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{-\frac{1}{4}x^2}\,W_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx$$
$$=2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu}\pi^{-1/2}y^{\nu+1}\frac{\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda+\mu\right)\Gamma\left(\frac{7}{4}+\frac{1}{2}\nu+\lambda-\mu\right)}{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{9}{4}+\lambda-k-\frac{1}{2}\nu\right)}$$
$$\times\ {}_3F_3\left(1,\frac{7}{4}+\frac{\nu}{2}+\lambda+\mu,\frac{7}{4}+\frac{\nu}{2}+\lambda-\mu;\frac{3}{2},\nu+\frac{3}{2},\frac{9}{4}+\lambda-k+\frac{\nu}{2};-\frac{y^2}{2}\right)$$
$\left[\operatorname{Re}(2\lambda+\nu)>2|\operatorname{Re}\mu|-\frac{7}{4},\quad y>0\right]$ ET II 171(43)

3. $$\int_0^\infty x^{2\lambda+\frac{1}{2}}e^{\frac{1}{4}x^2}\,W_{k,\mu}\left(\frac{1}{2}x^2\right)\mathbf{H}_\nu(xy)\,dx$$
$$=\left[2^\lambda\,\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)\right]^{-1}y^{-1/2}\,G^{23}_{34}\left(\frac{y^2}{2}\left|\begin{matrix}l,-\mu-\lambda,\mu-\lambda\\ l,-k-\lambda-\frac{1}{2},h,\kappa\end{matrix}\right.\right)$$
$h=\frac{1}{4}+\frac{1}{2}\nu,\quad \kappa=\frac{1}{4}-\frac{1}{2}\nu,\quad l=\frac{3}{4}+\frac{1}{2}\nu$
$\left[y>0,\quad \operatorname{Re}(2\lambda+\nu)>2|\operatorname{Re}\mu|-\frac{7}{2},\quad \operatorname{Re}\left(2k+2\lambda+\nu\right)<-\frac{1}{2},\quad \operatorname{Re}(k+\lambda)<0\right]$ ET II 172(46)a

4. $$\int_0^\infty e^{\frac{1}{2}x^2}\,W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}\left(x^2\right)\mathbf{H}_\nu(xy)\,dx=2^{-\nu-1}y^\nu\pi e^{\frac{1}{4}y^2}\left[1-\Phi\left(\frac{y}{2}\right)\right]$$
$\left[y>0,\quad \operatorname{Re}\nu>-1\right]$ ET II 171(44)

7.68 Combinations of confluent hypergeometric functions and other special functions

Combinations of confluent hypergeometric functions and associated Legendre functions

7.681

1. $$\int_0^\infty x^{-1/2}(a+x)^{\mu}e^{-\frac{1}{2}x}\,P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right)M_{k,\mu}(x)\,dx = -\frac{\sin(\nu\pi)}{\pi\,\Gamma(k)}\Gamma(2\mu+1)\,\Gamma\left(k-\mu+\nu+\tfrac{1}{2}\right)\Gamma\left(k-\mu-\nu-\tfrac{1}{2}\right)e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a),$$
$$\rho=\tfrac{1}{2}-k+\mu,\qquad \sigma=\tfrac{1}{2}+\nu$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(k-\mu)>\left|\operatorname{Re}\nu+\tfrac{1}{2}\right|\right]$$ ET II 403(11)

2. $$\int_0^\infty x^{-1/2}(a+x)^{-\mu}e^{-\frac{1}{2}x}\,P_\nu^{-2\mu}\left(1+2\frac{x}{a}\right)M_{k,\mu}(x)\,dx = \frac{\Gamma(2\mu+1)\,\Gamma\left(k+\mu+\nu+\frac{1}{2}\right)\Gamma\left(k+\mu-\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(2\mu+\nu+1)\,\Gamma(2\mu-\nu)}\,W_{\frac{1}{2}-k-\mu,\frac{1}{2}+\nu}(a)$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(k+\mu)>\left|\operatorname{Re}\nu+\tfrac{1}{2}\right|\right]$$ ET II 403(12)

3. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{\frac{1}{2}\mu}e^{-\frac{1}{2}x}\,P^{\mu}_{k+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right)W_{k,\nu}(x)\,dx = \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}+2\mu+\nu-k,\quad 2\sigma=k+3\nu-\tfrac{3}{2}$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+2\nu)<1]$$ ET II 407(32)

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-\frac{1}{2}\mu}e^{-\frac{1}{2}x}\,P^{\mu}_{k+\mu+\nu-\frac{3}{2}}\left(1+2\frac{x}{a}\right)W_{k,\nu}(x)\,dx = \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)}a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}-k+\nu,\qquad 2\sigma=k+2\mu+3\nu-\tfrac{3}{2}$$
$$[|\arg a|<\pi,\quad \operatorname{Re}\mu<1,\quad \operatorname{Re}(\mu+2\nu)<1]$$ ET II 408(33)

5. $$\int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}}(a+x)^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}\,Q^{\nu}_{\mu-k+\frac{3}{2}}\left(1+2\frac{x}{a}\right)M_{k,\nu}(x)\,dx = \frac{e^{\nu\pi i}\,\Gamma(1+2\mu-\nu)\,\Gamma(1+2\mu)\,\Gamma\left(\frac{5}{2}-k+\mu+\nu\right)}{2\,\Gamma\left(\frac{1}{2}+k+\mu\right)}a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)}e^{\frac{1}{2}a}\,W_{\rho,\sigma}(a)$$
$$2\rho=\tfrac{1}{2}-k-\mu+2\nu,\qquad 2\sigma=k-3\mu-\tfrac{3}{2}$$
$$\left[|\arg a|<\pi,\quad \operatorname{Re}\mu>-\tfrac{1}{2},\quad \operatorname{Re}(2\mu-\nu)>-1\right]$$ ET II 404(14)

7.682

1. $$\int_0^\infty x^{-1/2} e^{-\frac{1}{2}x} P_\nu^{-2\mu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right] M_{k,\mu}(x)\,dx = \frac{\Gamma(2\mu+1)\,\Gamma\left(k+\frac{1}{2}\nu\right)\Gamma\left(k-\frac{1}{2}\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{2^{2\mu}a^{1/4}\,\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{2}\nu\right)} W_{\frac{3}{4}-k,\frac{1}{4}+\frac{1}{2}\nu}(a)$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re} k > \tfrac{1}{2}\operatorname{Re}\nu - \tfrac{1}{2}, \quad \operatorname{Re} k > -\tfrac{1}{2}\operatorname{Re}\nu\right]$$ ET II 404(13)

2. $$\int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1}(a+x)^{-1/2} e^{-\frac{1}{2}x} Q_{k-\mu-\nu-1}^{1-k+\mu-\nu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right] M_{k,\mu}(x)\,dx = e^{(1-k+\mu-\nu)\pi i} 2^{\mu-k-\nu} a^{\frac{1}{2}(k+\mu-1)} \frac{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+2\mu)\,\Gamma(k+\mu+\nu)}{\Gamma\left(k+\mu+\frac{1}{2}\right)} e^{\frac{1}{2}a} W_{\rho,\sigma}(a),$$
$$\rho = \tfrac{1}{2}-k-\tfrac{1}{2}\nu, \qquad \sigma = \mu+\tfrac{1}{2}\nu$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}(k+\mu+\nu)>0\right]$$ ET II 404(15)

3. $$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x} Q_{2k-2\nu-3}^{2\mu-2\nu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right] M_{k,\mu}(x)\,dx = e^{2(\mu-\nu)\pi i} 2^{2\mu-2\nu-1} a^{\frac{1}{2}(k+\mu-1)} e^{\frac{1}{2}a} \frac{\Gamma(2\mu+1)\,\Gamma(\nu+1)\,\Gamma\left(k+\mu-2\nu-\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)} W_{\rho,\sigma}(a),$$
$$2\rho = 1-k+\mu-2\nu, \qquad 2\sigma = k-\mu-2\nu-2$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>-\tfrac{1}{2}, \quad \operatorname{Re}\nu>-1, \quad \operatorname{Re}(k+\mu-2\nu)>\tfrac{1}{2}\right]$$ ET II 404(16)

4. $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-3}^{\mu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right] W_{k,\nu}(x)\,dx = \frac{2^\mu\,\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a),$$
$$2\rho = 1-k+\mu+\nu, \qquad 2\sigma = k+\mu+3\nu-2$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu<1, \quad \operatorname{Re}(\mu+2\nu)<1\right]$$ ET II 408(34)

5.[8] $$\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu}(a+x)^{-1/2} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-2}^{\mu}\left[\left(1+\frac{x}{a}\right)^{1/2}\right] W_{k,\nu}(x)\,dx = \frac{2^\mu\,\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a), \quad 2\rho = \mu+\nu-k, \qquad 2\sigma = k+\mu+3\nu-1$$
$$\left[|\arg a|<\pi, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}\nu>0\right]$$ ET II 408(35)

A combination of confluent hypergeometric functions and orthogonal polynomials

7.683[8] $$\int_0^1 e^{-\frac{1}{2}ax} x^\alpha (1-x)^{\frac{\mu-\alpha}{2}-1} L_n^\alpha(ax)\, M_{\alpha-\frac{1+\alpha}{2},\frac{\mu-\alpha-1}{1}}[a(1-x)]\,dx = \frac{\Gamma(\mu-\alpha)}{\Gamma(1+\mu)} \frac{\Gamma(1+n+\alpha)}{n!} a^{-\frac{1+\alpha}{2}} M_{\alpha+n,\frac{\mu}{2}}(a)$$
$$\left[\operatorname{Re} a > -1, \quad \operatorname{Re}(\mu-\alpha)>0, \quad n = 0,1,2,\ldots\right]$$ BU 129(14b)

A combination of hypergeometric and confluent hypergeometric functions

7.684 $$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} M_{\gamma+\rho,\beta+\rho+\frac{1}{2}}(x)\, {}_2F_1\left(\alpha,\beta;\gamma;-\frac{\lambda}{x}\right) dx = \frac{\Gamma(\alpha+\beta+2\rho)\,\Gamma(2\beta+2\rho)\,\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta+\gamma+2\rho)} \lambda^{\frac{1}{2}\beta+\rho-\frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k,\mu}(\lambda);$$
$$k = \tfrac{1}{2} - \alpha - \tfrac{1}{2}\beta - \rho, \qquad \mu = \tfrac{1}{2}\beta + \rho$$
$$[|\arg\lambda| < \pi, \quad \mathrm{Re}(\beta+\rho) > 0, \quad \mathrm{Re}\,(\alpha+\beta+2\rho) > 0, \quad \mathrm{Re}\,\gamma > 0]$$
ET II 405(19)

7.69 Integration of confluent hypergeometric functions with respect to the index

7.691 $$\int_{-\infty}^\infty \mathrm{sech}(\pi x)\, W_{ix,0}(\alpha)\, W_{-ix,0}(\beta)\, dx = 2\frac{(a\beta)^{1/2}}{\alpha+\beta} \exp\left[-\frac{1}{2}(\alpha+\beta)\right]$$ ET II 414(61)

7.692 $$\int_{-i\infty}^{i\infty} \Gamma(-a)\,\Gamma(c-a)\Psi(a,c;x)\Psi(c-a,c;y)\, da = 2\pi i\,\Gamma(c)\Psi(c,2c;x+y)$$ EH I 285(15)

7.693

1. $$\int_{-\infty}^\infty \Gamma(ix)\,\Gamma(2k+ix)\, W_{k+ix,k-\frac{1}{2}}(\alpha)\, W_{-k-ix,k-\frac{1}{2}}(\beta)\, dx = 2\pi^{1/2}\,\Gamma(2k)(a\beta)^k(\alpha+\beta)^{\frac{1}{2}-2k} K_{2k-\frac{1}{2}}\left(\frac{a+\beta}{2}\right)$$
ET II 414(62)

2. $$\int_{-i\infty}^{i\infty} \Gamma\left(\tfrac{1}{2}+\nu+\mu+x\right)\Gamma\left(\tfrac{1}{2}+\nu+\mu-x\right)\Gamma\left(\tfrac{1}{2}+\nu-\mu+x\right)\Gamma\left(\tfrac{1}{2}+\nu-\mu-x\right) \times M_{\mu+ix,\nu}(\alpha)\, M_{\mu-ix,\nu}(\beta)\, dx = \frac{2\pi(a\beta)^{\nu+\frac{1}{2}}\left[\Gamma(2\nu+1)\right]^2\Gamma(2\nu+2\mu+1)\,\Gamma(2\nu-2\mu+1)}{(\alpha+\beta)^{2\nu+1}\,\Gamma(4\nu+2)} M_{2\mu,2\nu+\frac{1}{2}}(\alpha+\beta)$$
$$\left[\mathrm{Re}\,\nu > |\mathrm{Re}\,\mu| - \tfrac{1}{2}\right]$$ ET II 413(59)

7.694[11] $$\int_{-\infty}^\infty e^{-2\rho x i}\,\Gamma\left(\tfrac{1}{2}+\nu+ix\right)\Gamma\left(\tfrac{1}{2}+\nu-ix\right) M_{ix,\nu}(\alpha)\, M_{ix,\nu}(\beta)\, dx = \pi\sqrt{\alpha\beta}\left[\Gamma(2\nu+1)\right]^2 \mathrm{sech}\,\rho \exp\left[-\frac{1}{2}(\alpha+\beta)\tanh\rho\right] J_{2\nu}\left(\sqrt{\alpha\beta}\,\mathrm{sech}\,\rho\right)$$
$$\left[|\mathrm{Im}\,\rho| < \tfrac{1}{2}\pi, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}\right]$$

7.7 Parabolic Cylinder Functions

7.71 Parabolic cylinder functions

7.711

1. $$\int_{-\infty}^\infty D_n(x)\, D_m(x)\, dx = 0 \qquad [m \neq n]$$
$$= n!(2\pi)^{1/2} \qquad [m = n]$$
WH

2. $$\int_0^\infty D_\mu(\pm t)\, D_\nu(t)\, dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu-\nu}\left[\frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu\right)\Gamma\left(-\frac{1}{2}\nu\right)} \mp \frac{1}{\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu\right)}\right]$$

[when the lower sign is taken, $\operatorname{Re}\mu > \operatorname{Re}\nu$] BU 11 117(13a), EH II 122(21)

3. $$\int_0^\infty [D_\nu(t)]^2\, dt = \pi^{1/2} 2^{-3/2} \frac{\psi\left(\frac{1}{2}-\frac{1}{2}\nu\right) - \psi\left(-\frac{1}{2}\nu\right)}{\Gamma(-\nu)}$$ BU 117(13b)a, EH II 122(22)a

7.72 Combinations of parabolic cylinder functions, powers, and exponentials

7.721

1. $$\int_{-\infty}^\infty e^{-\frac{1}{4}x^2}(x-z)^{-1}\, D_n(x)\, dx = \pm i e^{\mp n\pi i}(2\pi)^{1/2} n! e^{-\frac{1}{4}z^2}\, D_{-n-1}(\mp iz)$$

[The upper or lower sign is taken accordingly as the imaginary part of z is positive or negative.] WH

2. $$\int_1^\infty x^\nu (x-1)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp\left[-\frac{(x-1)^2 a^2}{4}\right] D_\mu(ax)\, dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2}-\frac{\nu}{2}-1}\, \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$$

[$\operatorname{Re}(\mu-\nu) > 0$] ET II 395(4)a

7.722

1. $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu\, D_{\nu+1}(x)\, dx = 2^{-\frac{1}{2}-\frac{1}{2}\nu}\, \Gamma(\nu+1) \sin\frac{1}{4}(1-\nu)\pi$$

[$\operatorname{Re}\nu > -1$] WH

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\mu-1}\, D_{-\nu}(x)\, dx = \frac{\pi^{1/2} 2^{-\frac{1}{2}\mu-\frac{1}{2}\nu}\, \Gamma(\mu)}{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}$$ [$\operatorname{Re}\mu > 0$] EH II 122(20)

3.[11] $$\int_0^\infty e^{-\frac{3}{4}x^2} x^\nu\, D_{\nu-1}(x)\, dx = 2^{-\frac{1}{2}\nu}\, \Gamma(\nu) \sin\left(\frac{1}{4}\pi\nu\right)$$ [$\operatorname{Re}\nu > -1$] ET II 395(2)

7.723

1. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^\nu \left(x^2+y^2\right)^{-1} D_\nu(x)\, dx = \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2}\, D_{-\nu-1}(y)$$

[$\operatorname{Re} y > 0, \quad \operatorname{Re}\nu > -1$]
EH II 121(18)a, ET II 396(6)a

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} x^{\nu-1} \left(x^2+y^2\right)^{-1/2} D_\nu(x)\, dx = y^{\nu-1}\, \Gamma(\nu) e^{\frac{1}{4}y^2}\, D_{-\nu}(y)$$

[$\operatorname{Re} y > 0, \quad \operatorname{Re}\nu > 0$] ET II 396(7)

3. $$\int_0^1 x^{2\nu-1} \left(1-x^2\right)^{\lambda-1} e^{\frac{a^2x^2}{4}}\, D_{-2\lambda-2\nu}(ax)\, dx = \frac{\Gamma(\lambda)\,\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} e^{\frac{a^2}{4}}\, D_{-2\nu}(a)$$

[$\operatorname{Re}\lambda > 0, \quad \operatorname{Re}\nu > 0$] ET II 395(3)a

7.724 $$\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2}\, D_\nu(x)\, dx = (2\pi\mu)^{1/2}(1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}}\, D_\nu\left[y(1-\mu)^{-1/2}\right]$$ [$0 < \operatorname{Re}\mu < 1$]

EH II 121(15)

7.725

1. $$\int_0^\infty e^{-pt}(2t)^{\frac{\nu-1}{2}}e^{-\frac{t}{2}}\,D_{-\nu-2}\left(\sqrt{2t}\right)\,dt = \left(\frac{\pi}{2}\right)^{1/2}\frac{\left(\sqrt{p+1}-1\right)^{\nu+1}}{(\nu+1)p^{\nu+1}}$$
$[\operatorname{Re}\nu > -1]$ MO 175

2. $$\int_0^\infty e^{-pt}(2t)^{\frac{\nu-1}{2}}e^{-\frac{t}{2}}\,D_{-\nu}\left(\sqrt{2t}\right)\,dt = \left(\frac{\pi}{2}\right)^{1/2}\frac{\left(\sqrt{p+1}-1\right)^{\nu}}{p^{\nu}\sqrt{p+1}}$$
$[\operatorname{Re}\nu > -1]$ MO 175

3. $$\int_0^\infty e^{-bx}\,D_{2n+1}\left(\sqrt{2x}\right)\,dx = (-2)^n\,\Gamma\left(n+\tfrac{3}{2}\right)\left(b-\tfrac{1}{2}\right)^n\left(b+\tfrac{1}{2}\right)^{-n-\frac{3}{2}}$$
$\left[\operatorname{Re}b > -\tfrac{1}{2}\right]$ ET I 210(3)

4. $$\int_0^\infty \left(\sqrt{x}\right)^{-1}e^{-bx}\,D_{2n}\left(\sqrt{2x}\right)\,dx = (-2)^n\,\Gamma\left(n+\frac{1}{2}\right)\left(b-\frac{1}{2}\right)^n\left(b+\frac{1}{2}\right)^{-n-\frac{1}{2}}$$
$\left[\operatorname{Re}b > -\tfrac{1}{2}\right]$ ET I 210(5)

5. $$\int_0^\infty x^{-\frac{1}{2}(\nu+1)}e^{-sx}\,D_\nu\left(\sqrt{x}\right)\,dx = \sqrt{\pi}\left(1+\sqrt{\tfrac{1}{2}+2s}\right)^\nu\frac{1}{\sqrt{\tfrac{1}{4}+s}}$$
$\left[\operatorname{Re}s > -\tfrac{1}{4},\quad \operatorname{Re}\nu < 1\right]$ ET I 210(7)

6. $$\int_0^\infty e^{-zt}t^{-1+\frac{\beta}{2}}\,D_{-\nu}\left[2(kt)^{1/2}\right]\,dt = \frac{2^{1-\beta-\frac{\nu}{2}}\pi^{1/2}\,\Gamma(\beta)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}\right)}(z+k)^{-\frac{\beta}{2}}\,F\left(\frac{\nu}{2},\frac{\beta}{2};\frac{\nu+\beta+1}{2};\frac{z-k}{z+k}\right)$$
$\left[\operatorname{Re}(z+k) > 0,\quad \operatorname{Re}\frac{z}{k} > 0\right]$
EH II 121(11)

7.726 $$\int_{-\infty}^\infty e^{ixy-\frac{(1+\lambda)x^2}{4}}\,D_\nu\left[x(1-\lambda)^{1/2}\right]\,dx = (2\pi)^{1/2}\lambda^{\frac{1}{2}\nu}e^{-\frac{(1+\lambda)y^2}{4\lambda}}\,D_\nu\left[i\left(\lambda^{-1}-1\right)^{1/2}y\right]$$
$[\operatorname{Re}\lambda > 0]$
EH II 121(16)

7.727 $$\int_0^\infty \frac{e^{\frac{1}{2}x}e^{-bx}}{(e^x-1)^{\mu+\frac{1}{2}}}\exp\left(-\frac{a}{1-e^{-x}}\right)D_{2\mu}\left(\frac{2\sqrt{a}}{\sqrt{1-e^{-x}}}\right)\,dx = e^{-a}2^{b+\mu}\,\Gamma(b+\mu)\,D_{-2b}\left(2\sqrt{a}\right)$$
$[\operatorname{Re}a > 0,\quad \operatorname{Re}b > -\operatorname{Re}\mu]$
ET I 211(13)

7.728 $$\int_0^\infty (2t)^{-\frac{\nu}{2}}e^{-pt}e^{-\frac{q^2}{8t}}\,D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right)\,dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}p^{\frac{1}{2}\nu-1}e^{-q\sqrt{p}}$$
MO 175

7.73 Combinations of parabolic cylinder and hyperbolic functions

7.731

1. $$\int_0^\infty \cosh(2\mu x)\exp\left[-\left(a\sinh x\right)^2\right]D_{2k}\left(2a\cosh x\right)\,dx = 2^{k-\frac{3}{2}}\pi^{1/2}a^{-1}\,W_{k,\mu}\left(2a^2\right)$$
$\left[\operatorname{Re}^2 a > 0\right]$ ET II 398(20)

2. $$\int_0^\infty \cosh(2\mu x) \exp\left[(a \sinh x)^2\right] D_{2k}(2a \cosh x)\, dx = \frac{\Gamma(\mu - k)\,\Gamma(-\mu - k)}{2^{k+\frac{5}{2}} a\, \Gamma(-2k)}\, W_{k+\frac{1}{2},\mu}\left(2a^2\right)$$
$$\left[|\arg a| < \frac{3\pi}{4}, \quad \operatorname{Re} k + |\operatorname{Re} \mu| < 0\right]$$
ET II 398(21)

7.74 Combinations of parabolic cylinder and trigonometric functions

7.741

1. $$\int_0^\infty \sin(bx) \left\{[D_{-n-1}(ix)]^2 - [D_{-n-1}(-ix)]^2\right\} dx = (-1)^{n+1} \frac{i}{n!} \pi \sqrt{2\pi} e^{-\frac{1}{2}b^2} L_n\left(b^2\right)$$
$$[b > 0]$$
ET I 115(3)

2. $$\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx)\, D_{2n+1}(x)\, dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2}$$
$$[b > 0]$$
ET I 115(1)

3. $$\int_0^\infty e^{-\frac{1}{4}x^2} \cos(bx)\, D_{2n}(x)\, dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} \qquad [b > 0]$$
ET I 60(2)

4. $$\int_0^\infty e^{-\frac{1}{4}x^2} \sin(bx) \left[D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x)\right] dx = \sqrt{2\pi} \sin\left[\left(\nu - \tfrac{1}{4}\right)\pi\right] b^{2\nu-\frac{1}{2}} e^{-\frac{1}{2}b^2}$$
$$\left[\operatorname{Re} \nu > \tfrac{1}{4}, \quad b > 0\right]$$
ET I 115(2)

5. $$\int_0^\infty e^{-\frac{1}{2}x^2} \cos(bx) \left[D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x)\right] dx = \frac{2^{\frac{1}{4}-2\nu} \sqrt{\pi} b^{2\nu-\frac{1}{2}} e^{-\frac{1}{4}b^2}}{\operatorname{cosec}\left[\left(\nu + \frac{1}{4}\right)\pi\right]}$$
$$\left[\operatorname{Re} \nu > \tfrac{1}{4}, \quad b > 0\right]$$
ET I 61(4)

7.742

1. $$\int_0^\infty x^{2\rho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x)\, dx = 2^{\nu-\rho-\frac{1}{2}} \pi^{1/2} a \frac{\Gamma(2\rho + 1)}{\Gamma(\rho - \nu + 1)}$$
$$\times\ {}_2F_2\left(\rho + \frac{1}{2}, \rho + 1; \frac{3}{2}, \rho - \nu + 1; -\frac{a^2}{2}\right)$$
$$\left[\operatorname{Re} \rho > -\tfrac{1}{2}\right]$$
ET II 396(8)

2. $$\int_0^\infty x^{2\rho-1} \sin(ax) e^{\frac{x^2}{4}} D_{2\nu}(x)\, dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G^{22}_{23}\left(\frac{a^2}{2} \middle| \begin{matrix} \frac{1}{2} - \rho, 1 - \rho \\ -\rho - \nu, \frac{1}{2}, 0 \end{matrix}\right)$$
$$\left[a > 0, \quad \operatorname{Re} \rho > -\tfrac{1}{2}, \quad \operatorname{Re}(\rho + \nu) < \tfrac{1}{2}\right]$$
ET II 396(9)

3. $$\int_0^\infty x^{2\rho-1} \cos(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x)\, dx = \frac{2^{\nu-\rho}\,\Gamma(2\rho)\pi^{1/2}}{\Gamma\left(\rho - \nu + \frac{1}{2}\right)}\ {}_2F_2\left(\rho, \rho + \frac{1}{2}; \frac{1}{2}, \rho - \nu + \frac{1}{2}; -\frac{a^2}{2}\right)$$
$$[\operatorname{Re} \rho > 0]$$
ET II 396(10)a

4. $$\int_0^\infty x^{2\rho-1}\cos(ax)e^{\frac{x^2}{4}}\,D_{2\nu}(x)\,dx = \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)}\,G^{22}_{23}\left(\frac{a^2}{2}\,\middle|\,\begin{matrix}\frac{1}{2}-\rho, 1-\rho\\ -\rho-\nu, 0, \frac{1}{2}\end{matrix}\right)$$

$\left[a>0, \quad \operatorname{Re}\rho>0, \quad \operatorname{Re}(\rho+\nu)<\frac{1}{2}\right]$ ET II 396(11)

7.743 $$\int_0^{\pi/2}(\cos x)^{-\mu-2}(\sin x)^{-\nu}\,D_\nu\left(a\sin x\right)D_\mu\left(a\cos x\right)\,dx = -\left(\tfrac{1}{2}\pi\right)^{1/2}(1+\mu)^{-1}\,D_{\mu+\nu+1}(a)$$

$[\operatorname{Re}\nu<1, \quad \operatorname{Re}\mu<-1]$ ET II 397(19)

7.744

1. $$\int_0^\infty \sin(bx)\left[D_{-\nu-\frac{1}{2}}\left(\sqrt{2x}\right)-D_{-\nu-\frac{1}{2}}\left(-\sqrt{2x}\right)\right]D_{\nu-\frac{1}{2}}\left(\sqrt{2x}\right)\,dx$$
$$= -\sqrt{2\pi}\sin\left[\left(\tfrac{1}{4}+\tfrac{1}{2}\nu\right)\pi\right]b^{-\nu-\frac{1}{2}}\frac{\left(1+\sqrt{1+b^2}\right)^\nu}{\sqrt{1+b^2}}$$

$[b>0]$ ET I 115(4)

2. $$\int_0^\infty \cos(bx)\left[D_{-2\nu-\frac{1}{2}}\left(\sqrt{2x}\right)+D_{-2\nu-\frac{1}{2}}\left(-\sqrt{2x}\right)\right]D_{2\nu-\frac{1}{2}}\left(\sqrt{2x}\right)\,dx$$
$$= -\frac{\sqrt{\pi}\sin\left[\left(\nu-\frac{1}{4}\right)\pi\right]\left(1+\sqrt{1+b^2}\right)^{2\nu}}{\sqrt{1+b^2}b^{2\nu+\frac{1}{2}}}$$

$[b>0]$ ET I 60(3)

7.75 Combinations of parabolic cylinder and Bessel functions

7.751

1. $$\int_0^\infty [D_n(ax)]^2\,J_1(xy)\,dx = (-1)^{n-1}y^{-1}\left[D_n\left(\frac{y}{a}\right)\right]^2$$ $[y>0]$ ET II 20(24)

2. $$\int_0^\infty J_0(xy)\,D_n(ax)\,D_{n+1}(ax)\,dx = (-1)^n y^{-1}\,D_n\left(\frac{y}{a}\right)D_{n+1}\left(\frac{y}{a}\right)$$

$\left[y>0, \quad |\arg a|<\frac{1}{4}\pi\right]$ ET II 17(42)

3. $$\int_0^\infty J_0(xy)\,D_\nu(x)\,D_{\nu+1}(x)\,dx = 2^{-1}y^{-1}\left[D_\nu(-y)\,D_{\nu+1}(y)-D_{\nu+1}(-y)\,D_\nu(y)\right]$$ ET II 397(17)a

7.752

1. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2}\,D_{2\nu-1}(x)\,J_\nu(xy)\,dx = -\frac{1}{2}\sec(\nu\pi)y^{\nu-1}e^{-\frac{1}{4}y^2}\left[D_{2\nu-1}(y)-D_{2\nu-1}(-y)\right]$$

$\left[y>0, \quad \operatorname{Re}\nu>-\frac{1}{2}\right]$

ET II 76(1), MO 183

2. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2}\,D_{2\nu-1}(x)\,J_\nu(xy)\,dx = 2^{\frac{1}{2}-\nu}\pi\sin(\nu\pi)y^{-\nu}\,\Gamma(2\nu)e^{\frac{1}{4}y^2}\,K_\nu\left(\tfrac{1}{4}y^2\right)$$

$\left[y>0, \quad -\frac{1}{2}<\operatorname{Re}\nu<\frac{1}{2}\right]$ ET II 77(4)

3. $$\int_0^\infty x^{\nu+1}e^{-\frac{1}{4}x^2}\,D_{2\nu}(x)\,J_\nu(xy)\,dx = \frac{1}{2}\sec(\nu\pi)y^{\nu-1}e^{-\frac{1}{4}y^2}\left[D_{2\nu+1}(y)-D_{2\nu+1}(-y)\right]$$

$[y>0, \quad \operatorname{Re}\nu>-1]$ ET II 78(13)

4. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{2\nu+1}(x)\, J_\nu(xy)\, dx = \frac{1}{2}\sec(\nu\pi) e^{-\frac{1}{4}y^2} y^\nu \left[D_{2\nu}(y) + D_{2\nu}(-y)\right]$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 77(5)

5. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu+2}(x)\, J_\nu(xy)\, dx = -\tfrac{1}{2}\sec(\nu\pi) y^\nu e^{-\frac{1}{4}y^2} \left[D_{2\nu+2}(y) + D_{2\nu+2}(-y)\right]$$
$[\operatorname{Re}\nu > -1, \quad y > 0]$ ET II 78(16)

6. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{2\nu+2}(x)\, J_\nu(xy)\, dx = \pi^{-1}\sin(\nu\pi)\,\Gamma(2\nu+3) y^{-\nu-2} e^{\frac{1}{4}y^2} K_{\nu+1}\left(\tfrac{1}{4}y^2\right)$$
$\left[y > 0, \quad -1 < \operatorname{Re}\nu < -\frac{5}{6}\right]$
ET II 78(19)

7. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} D_{-2\nu}(x)\, J_\nu(xy)\, dx = 2^{-1/2}\pi^{1/2} y^{-\nu} e^{-\frac{1}{4}y^2} I_\nu\left(\tfrac{1}{4}y^2\right)$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 77(8)

8. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu}(x)\, J_\nu(xy)\, dx = y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu}(y)$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad y > 0\right]$
ET II 77(9), EH II 121(17)

9. $$\int_0^\infty x^\nu e^{\frac{1}{4}x^2} D_{-2\nu-2}(x)\, J_\nu(xy)\, dx = (2\nu+1)^{-1} y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-1}(y)$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 77(10)

10. $$\int_0^\infty x^\nu e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{2^{\mu-\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right) y^\nu}{\Gamma(\nu-\mu+1) a^{1+2\nu}}\ {}_1F_1\left(\nu+\frac{1}{2}; \nu-\mu+1; -\frac{y^2}{2a^2}\right)$$
$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$
ET II 77(11)

11. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{\Gamma\left(\frac{1}{2}+\nu\right) a^{2k} 2^{m+\mu}}{\Gamma\left(\frac{1}{2}-\mu\right) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}}\ W_{k,m}\left(\frac{y^2}{4a^2}\right)$$
$2k = \frac{1}{2}+\mu-\nu, \quad 2m = \frac{1}{2}+\mu+\nu$
$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re}\nu < \operatorname{Re}\left(\frac{1}{2}-2\mu\right)\right]$
ET II 78(12)

12. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{2^\mu\,\Gamma\left(\nu+\frac{3}{2}\right) y^\nu}{\Gamma\left(\nu-\mu+\frac{3}{2}\right) a^{2\nu+2}}\ {}_1F_1\left(\nu+\frac{3}{2}; \nu-\mu+\frac{3}{2}; -\frac{y^2}{2a^2}\right)$$
$\left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re}\nu > -1\right]$
ET II 79(23)

13. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax)\, J_\nu(xy)\, dx = \frac{\Gamma\left(\frac{3}{2}+\nu\right) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu) y^{\mu+2}} e^{\frac{y^2}{4a^2}}\ W_{k,m}\left(\frac{y^2}{2a^2}\right)$$
$2k = \mu-\nu-1, \quad 2m = \mu+\nu+1$
$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -1 < \operatorname{Re}\nu < -\frac{1}{2} - 2\operatorname{Re}\mu\right]$
ET II 79(24)

14. $$\int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2x^2} D_\mu(ax)\, J_\nu(xy)\, dx = \frac{2^{\lambda-\frac{1}{2}\mu}\pi^{-\frac{1}{2}}}{\Gamma(-\mu)y^{\lambda+\frac{3}{2}}} G^{22}_{23}\left(\frac{y^2}{2a^2}\middle|\begin{matrix}\frac{1}{2}, 1\\ \frac{3}{4}+\frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2}\end{matrix}\right)$$
$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad \operatorname{Re}\mu < -\operatorname{Re}\lambda < \operatorname{Re}\nu + \frac{3}{2}\right]$ ET II 80(26)

15. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-1}(x)\, J_\nu(xy)\, dx = (2\nu+1)y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu-2}(y)$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 79(20)

16. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{-2\nu-3}(x)\, J_\nu(xy)\, dx = 2^{-1/2}\pi^{1/2} y^{-\nu-2} e^{-\frac{1}{4}y^2} I_{\nu+1}\left(\tfrac{1}{4}y^2\right)$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 79(21)

17. $$\int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x)\, J_\nu(xy)\, dx = y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-3}(y)$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 79(22)

18. $$\int_0^\infty x^\nu e^{\frac{1}{4}a^2x^2} D_{\frac{1}{2}\nu-\frac{1}{2}}(ax)\, Y_\nu(xy)\, dx = -\pi^{-1} 2^{\frac{3}{4}\nu+\frac{3}{4}} a^{-\nu} y^{-1}\, \Gamma(\nu+1) e^{\frac{y^2}{4a^2}}\, W_{-\frac{1}{2}\nu-\frac{1}{2},\frac{1}{2}\nu}\left(\frac{y^2}{2a^2}\right)$$
$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re}\nu < \frac{2}{3}\right]$ ET II 115(39)

7.753

1. $$\int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2} I_{\nu-\frac{1}{2}}(2ax)\, D_\nu(2x)\, dx = \frac{1}{2}\pi^{-1/2}\, \Gamma(\nu) a^{\nu-\frac{1}{2}} D_{-\nu}(2a)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 0]$ ET II 397(12)

2. $$\int_0^\infty x^{\nu-\frac{3}{2}} e^{-(x+a)^2} I_{\nu-\frac{3}{2}}(2ax)\, D_\nu(2x)\, dx = \frac{1}{2}\pi^{-1/2}\, \Gamma(\nu) a^{\nu-\frac{3}{2}} D_{-\nu}(2a)$$
$[\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > 1]$ ET II 397(13)

7.754

1. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu-1}(x) - D_{2\nu-1}(-x)\}\, J_\nu(xy)\, dx$$
$$= \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu-1}(y) - D_{2\nu-1}(-y)\}$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 76(2, 3)

2. $$\int_0^\infty x^\nu e^{-\frac{1}{4}x^2} \{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu+1}(x) - D_{2\nu+1}(-x)\}\, J_\nu(xy)\, dx$$
$$= \mp y^\nu e^{-\frac{1}{4}y^2} \{[1 \mp 2\cos(\nu\pi)]\, D_{2\nu}(y) + D_{2\nu}(-y)\}$$
$\left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}\right]$ ET II 77(6, 7)

3. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \{[1 \pm 2\cos(\nu\pi)]\, D_{2\nu}(x) + D_{2\nu}(-x)\}\, J_\nu(xy)\, dx$$
$$= \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \pm 2\cos(\nu\pi)]\, D_{2\nu+1}(y) - D_{2\nu+1}(-y)\}$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 78(14, 15)

4. $$\int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} \left\{\left[1 \mp 2\cos(\nu\pi)\right] D_{2\nu+2}(x) + D_{2\nu+2}(-x)\right\} J_\nu(xy)\, dx$$
$$= \pm y^\nu e^{-\frac{1}{4}y^2} \left\{\left[1 \mp 2\cos(\nu\pi)\right] D_{2\nu+2}(y) + D_{2\nu+2}(-y)\right\}$$
$[y > 0, \quad \operatorname{Re}\nu > -1]$ ET II 78(17, 18)

7.755

1. $$\int_0^\infty x^{-1/2} D_\nu\left(\sqrt{ax}\right) D_{-\nu-1}\left(\sqrt{ax}\right) J_0(xy)\, dx$$
$$= 2^{-3/2}\pi a^{-1/2} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}} \left[\left(1+\frac{4y^2}{a^2}\right)^{1/2}\right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}} \left[\left(1+\frac{4y^2}{a^2}\right)^{1/2}\right]$$
$[y > 0, \operatorname{Re} a > 0]$ ET II 17(43)

2. $$\int_0^\infty x^{1/2} D_{-\frac{1}{2}-\nu}\left(ae^{\frac{1}{4}\pi i} x^{1/2}\right) D_{-\frac{1}{2}-\nu}\left(ae^{-\frac{1}{4}\pi i} x^{1/2}\right) J_\nu(xy)\, dx$$
$$= 2^{-\nu}\pi^{1/2} y^{-\nu-1} \left(a^2+2y\right)^{-1/2} \left[\Gamma\left(\nu+\tfrac{1}{2}\right)\right]^{-1} \left[\left(a^2+2y\right)^{1/2} - a\right]^{2\nu}$$
$\left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 80(27)

3. $$a\int_0^\infty D_{-\frac{1}{2}-\nu}\left(ae^{\frac{1}{4}\pi i} x^{-1/2}\right) D_{-\frac{1}{2}-\nu}\left(ae^{-\frac{1}{4}\pi i} x^{-1/2}\right) J_\nu(xy)\, dx$$
$$= 2^{1/2}\pi^{1/2} y^{-1} \left[\Gamma\left(\nu+\tfrac{1}{2}\right)\right]^{-1} \exp\left[-a(2y)^{1/2}\right]$$
$\left[y > 0, \quad \operatorname{Re} a > 0, \quad e\operatorname{Re}\nu > -\tfrac{1}{2}\right]$ ET II 80(28)a

4. $$\int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}\left(ax^{-1/2}\right) D_{-\nu-\frac{1}{2}}\left(ax^{-1/2}\right) Y_\nu(xy)\, dx$$
$$= y^{-3/2} \exp\left(-ay^{1/2}\right) \sin\left[ay^{1/2} - \tfrac{1}{2}\left(\nu - \tfrac{1}{2}\right)\pi\right]$$
$\left[y > 0, \quad |\arg a| < \tfrac{1}{4}\pi\right]$ ET II 115(40)

5. $$\int_0^\infty x^{1/2} D_{\nu-\frac{1}{2}}\left(ax^{-1/2}\right) D_{-\nu-\frac{1}{2}}\left(ax^{-1/2}\right) K_\nu(xy)\, dx = 2^{-1} y^{-3/2}\pi \exp\left[-a(2y)^{1/2}\right]$$
$\left[\operatorname{Re} y > 0, \quad |\arg a| < \tfrac{1}{4}\pi\right]$ ET II 151(81)

Combinations of parabolic cylinder and Struve functions

7.756 $$\int_0^\infty x^{-\nu} e^{-\frac{1}{4}x^2} \left[D_\mu(x) - D_\mu(-x)\right] \mathbf{H}_\nu(xy)\, dx$$
$$= \frac{2^{3/2}\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu+\nu+1\right)} y^{\mu+\nu} \sin\left(\frac{1}{2}\mu\pi\right)\ {}_1F_1\left(\frac{1}{2}\mu+\frac{1}{2}; \frac{1}{2}\mu+\nu+1; -\frac{1}{2}y^2\right)$$
$\left[y > 0, \quad \operatorname{Re}(\mu+\nu) > -\tfrac{3}{2}, \quad \operatorname{Re}\mu > -1\right]$ ET II 171(41)

7.76 Combinations of parabolic cylinder functions and confluent hypergeometric functions

7.761

1. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1} D_{-\nu}(t)\ {}_1F_1\left(a;c;-\frac{1}{2}pt^2\right) dt = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c)\,\Gamma\left(\frac{1}{2}\nu-c+a\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}+\frac{1}{2}\nu\right)} F\left(a, c+\frac{1}{2}; a+\frac{1}{2}+\frac{1}{2}\nu; 1-p\right)$$
$$[|1-p|<1, \quad \operatorname{Re} c>0, \quad \operatorname{Re}\nu > 2\operatorname{Re}(c-a)]$$ EH II 121(12)

2. $$\int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2} D_{-\nu}(t)\ {}_1F_1\left(a;c;-\frac{1}{2}pt^2\right) dt = \frac{\pi^{1/2}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1)\,\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}-c+a\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)\Gamma\left(a+\frac{1}{2}\nu\right)} F\left(a, c-\frac{1}{2}; a+\frac{1}{2}\nu; 1-p\right)$$
$$\left[|1-p|<1, \quad \operatorname{Re} c>\tfrac{1}{2}, \quad \operatorname{Re}\nu > 2\operatorname{Re}(c-a)-1\right]$$ EH II 121(13)

7.77 Integration of a parabolic cylinder function with respect to the index

7.771 $$\int_0^\infty \cos(ax)\, D_{x-\frac{1}{2}}(\beta)\, D_{-x-\frac{1}{2}}(\beta)\, dx = \frac{1}{2}\left(\frac{\pi}{\cos a}\right)^{1/2} \exp\left(-\frac{\beta^2\cos a}{2}\right) \qquad \left[|a|<\tfrac{1}{2}\pi\right]$$
$$= 0 \qquad \left[|a|>\tfrac{1}{2}\pi\right]$$
ET II 298(22)

7.772

1. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left[\frac{\left(\tan\frac{1}{2}\varphi\right)^\nu}{\cos\frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\xi\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) + \frac{\left(\cot\frac{1}{2}\varphi\right)^\nu}{\sin\frac{1}{2}\varphi} D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\xi\right) D_\nu\left(-e^{\frac{1}{4}i\pi}\eta\right)\right] \frac{d\nu}{\sin\nu\pi} = -2i(2\pi)^{1/2}\exp\left[-\tfrac{1}{4}i\left(\xi^2-\eta^2\right)\cos\varphi - \tfrac{1}{2}i\xi\eta\sin\varphi\right]$$
EH II 125(7)

2. $$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\left(\tan\frac{1}{2}\varphi\right)^\nu}{\cos\frac{1}{2}\varphi} D_\nu\left(-e^{\frac{1}{4}i\pi}\zeta\right) D_{-\nu-1}\left(e^{\frac{1}{4}i\pi}\eta\right) \frac{d\nu}{\sin\nu\pi} = -2i\, D_0\left[e^{\frac{1}{4}i\pi}\left(\zeta\cos\tfrac{1}{2}\varphi+\eta\sin\tfrac{1}{2}\varphi\right)\right] D_{-1}\left[e^{\frac{1}{4}i\pi}\left(\eta\cos\tfrac{1}{2}\varphi-\zeta\sin\tfrac{1}{2}\varphi\right)\right]$$
EH II 125(8)

7.773

1. $$\int_{c-i\infty}^{c+i\infty} D_\nu(z) t^\nu\, \Gamma(-\nu)\, d\nu = 2\pi i e^{-\frac{1}{4}z^2-zt-\frac{1}{2}t^2} \qquad \left[c<0, \quad |\arg t|<\frac{\pi}{4}\right]$$ EH II 126(10)

2. $$\int_{c-i\infty}^{c+i\infty} \left[D_\nu(x)\, D_{-\nu-1}(iy) + D_\nu(-x)\, D_{-\nu-1}(iy)\right] \frac{t^{-\nu-1}\, d\nu}{\sin(-\nu\pi)} = \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{1/2}} \left(1+t^2\right)^{-\frac{1}{2}} \exp\left[\frac{1}{4}\frac{1-t^2}{1+t^2}\left(x^2+y^2\right) + i\frac{txy}{1+t^2}\right]$$

$$\left[-1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi\right]$$ EH II 126(11)

7.774 $$\int_{c-i\infty}^{c+i\infty} D_\nu\left[k^{\frac{1}{2}}(1+i)\xi\right] D_{-\nu-1}\left[k^{\frac{1}{2}}(1+i)\eta\right] \Gamma\left(-\tfrac{1}{2}\nu\right)\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu\right) d\nu = 2^{1/2}\pi^2\, H_0^{(2)}\left[\tfrac{1}{2}k\left(\xi^2+\eta^2\right)\right]$$

$$[-1 < c < 0, \quad \operatorname{Re} ik \geq 0]$$ EH II 125(9)

7.8 Meijer's and MacRobert's Functions (G and E)

7.81 Combinations of the functions G and E and the elementary functions

7.811

1. $$\int_0^\infty G_{p,q}^{m,n}\left(\eta x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) G_{\sigma,\tau}^{\mu,\nu}\left(\omega x \,\middle|\, \begin{matrix} c_1, \ldots, c_\sigma \\ d, \ldots, d_\tau \end{matrix}\right) dx = \frac{1}{\eta} G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}\left(\frac{\omega}{\eta} \,\middle|\, \begin{matrix} -b_1, \ldots, -b_m, c_1, \ldots, c_\sigma, -b_{m+1}, \ldots, -b_q \\ -a_1, \ldots, -a_n, d_1, \ldots, d_\tau, -a_{n+1}, \ldots, -a_p \end{matrix}\right)$$

subject to the following constraints

- $m, n, p, q, \mu, \nu, \sigma, \tau$ are integers;
- $1 \leq n \leq p < q < p + \tau - \sigma$
- $\frac{1}{2}p + \frac{1}{2}q - n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma + \frac{1}{2}\tau - \nu < \mu \leq \tau$
- $\operatorname{Re}(b_j + d_k) > -1 \qquad (j = 1, \ldots, m; k = 1, \ldots, \mu)$
- $\operatorname{Re}(a_j + c_k) < 1 \qquad (j = 1, \ldots, n; k = 1, \ldots, \tau)$
- $\omega \neq 0, \quad \eta \neq 0, \quad |\arg\eta| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi, \quad |\arg\omega| < \left(\mu + \nu - \frac{1}{2}\sigma - \frac{1}{2}\tau\right)\pi$
- The following must not be integers:

$$\begin{aligned} &b_j - b_k \quad (j = 1, \ldots, m; k = 1, \ldots, m; j \neq k), \\ &a_j - a_k \quad (j = 1, \ldots, n; k = 1, \ldots, n; j \neq k), \\ &d_j - d_k \quad (j = 1, \ldots, \mu; k = 1, \ldots, \mu; j \neq k), \\ &a_j + d_k \quad (j = 1, \ldots, n; k = 1, \ldots, n); \end{aligned}$$

- The following must not be positive integers:

$$\begin{aligned} &a_j - b_k \quad (j = 1, \ldots, n; k = 1, \ldots, m) \\ &c_j - d_k \quad (j = 1, \ldots, \nu; k = 1, \ldots, \mu) \end{aligned}$$

Formula **7.811** 1 also holds for four sets of restrictions. See C. S. Meijer, Neue Integraldarstellungen für Whittakersche Funktionen, Nederl. Akad. Wetensch. Proc. **44** (1941), 82–92.
ET II 422(14)

Hereafter, $G_{p,q}^{m,n}$ will be written as G_{pq}^{mn}, and commas will only be inserted in entries like $G_{p+1,q+1}^{m,n+1}$, where their omission could cause ambiguity.

2. $$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\, G^{mn}_{pq}\left(\alpha x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \Gamma(\sigma)\, G^{m,n+1}_{p+1,q+1}\left(\alpha \middle| \begin{matrix} 1-\rho, a_1,\dots,a_p \\ b_1,\dots,b_q, 1-\rho-\sigma \end{matrix}\right)$$
where

- $(p+q) < 2(m+n)$
- $|\arg a| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $\operatorname{Re}(\rho+b_j) > 0,\ j=1,\dots,m$
- $\operatorname{Re}\sigma > 0$
- either

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}\rho-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\rho+b_j) > 0; \quad j=1,\dots,m; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q)\left(\rho-\frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$p < q \quad (\text{or } p \le q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(p+b_j) > 0; \quad j=1,\dots,m; \quad \operatorname{Re}\sigma > 0$$

ET II 417(1)

3. $$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\, G^{mn}_{pq}\left(\alpha x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \Gamma(\sigma)\, G^{m+1,n}_{p+1,q+1}\left(\alpha \middle| \begin{matrix} a_1,\dots,a_p,\rho \\ \rho-\sigma, b_1,\dots,b_q \end{matrix}\right)$$
where

- $p+q < 2(m+n)$
- $|\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $\operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\dots,n$
- $\operatorname{Re}\sigma > 0$
- either

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\dots,n; \quad \operatorname{Re}\sigma > 0,$$
$$\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)\left(\rho-\sigma+\frac{1}{2}\right)\right] > -\frac{1}{2},$$

or

$$q < p \quad (\text{or } q \le p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\rho-\sigma-a_j) > -1; \quad j=1,\dots,n; \quad \operatorname{Re}\sigma > 0$$

ET II 417(2)

4. $$\int_0^\infty x^{\rho-1}\, G^{mn}_{pq}\left(\alpha x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \frac{\prod_{j=1}^m \Gamma(b_j+\rho) \prod_{j=1}^n \Gamma(1-a_j-\rho)}{\prod_{j=m+1}^q \Gamma(1-b_j-\rho) \prod_{j=n+1}^p \Gamma(a_j+\rho)} \alpha^{-\rho}$$
$p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad -\min_{1\le j\le m} \operatorname{Re} b_j < \operatorname{Re}\rho < 1 - \max_{1\le j\le n} \operatorname{Re} a_j$

ET II 418(3)a, ET I 337(14)

5. $$\int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma}\, G_{pq}^{mn}\left(\alpha x \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)}\, G_{p+1,q+1}^{m+1,n+1}\left(\alpha\beta \,\middle|\, \begin{matrix} 1-\rho, a_1,\dots,a_p \\ \sigma-\rho, b_1,\dots,b_q \end{matrix}\right)$$

where

- $p+q < 2(m+n)$
- $|\arg\alpha| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$
- $|\arg\beta| < \pi$
- $\operatorname{Re}(\rho+b_j) > 0, \quad j=1,\dots,m$
- $\operatorname{Re}(\rho-\sigma+a_j) < 1, \quad j=1,\dots,n$
- either

$$p \le q, \quad p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad |\arg\beta| < \pi$$
$$\operatorname{Re}(\rho=b_j) > 0, \quad j=1,\dots,m, \quad \operatorname{Re}(\rho-\sigma+a_j) < 1, \quad j=1,\dots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j - (q-p)\left(\rho-\sigma-\frac{1}{2}\right)\right] > 1,$$

or

$$p \ge q, \quad p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \quad |\arg\beta| < \pi,$$
$$\operatorname{Re}(\rho+b_j) > 0, \quad j=1,\dots,m, \quad \operatorname{Re}(\rho-\sigma+a_j) < 1, \quad j=1,\dots,n,$$
$$\operatorname{Re}\left[\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j + (p-q)\left(\rho-\frac{1}{2}\right)\right] > 1$$

ET II 418(4)

7.812

1. $$\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q; \frac{z}{x^m}\right) dx$$
$$= \Gamma(\gamma-\beta) m^{\beta-\gamma}\, E\left(a_1,\dots,a_{p+m} : \rho_1,\dots,\rho_{q+m} : z\right)$$
$$a_{p+k} = \frac{\beta+k-1}{m}, \quad \rho_{q+k} = \frac{\gamma+k-1}{m}, \qquad k=1,\dots,m$$
$[\operatorname{Re}\gamma > \operatorname{Re}\beta > 0, \quad m=1,2,\dots]$ ET II 414(2)

2. $$\int_0^\infty x^{\rho-1}(1+x)^{-\sigma}\, E\left[a_1,\dots,a_p : \rho_1,\dots,\rho_q : (1+x)z\right] dx$$
$$= \Gamma(\rho)\, E\left(a_1,\dots,a_p, \sigma-\rho; \rho_1,\dots,\rho_q, \sigma; z\right)$$
$[\operatorname{Re}\sigma > \operatorname{Re}\rho > 0]$ ET II 415(3)

3. $$\int_0^\infty (1+x)^{-\beta} x^{s-1}\, G^{mn}_{pq}\left(\frac{ax}{1+x}\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \Gamma(\beta-s)\, G^{m,n+1}_{p+1,q+1}\left(a\,\middle|\,\begin{matrix}1-s,a_1,\dots,a_p\\ b_1,\dots,b_q,1-\beta\end{matrix}\right)$$

$$\left[-\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re}\beta, \quad 1 \le k \le m; \quad (p+q) < 2(m+n), \quad |\arg a| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi\right]$$

ET I 338(19)

7.813

1. $$\int_0^\infty x^{-\rho} e^{-\beta x}\, G^{mn}_{pq}\left(\alpha x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \beta^{\rho-1}\, G^{m,n+1}_{p+1,q}\left(\frac{\alpha}{\beta}\,\middle|\,\begin{matrix}\rho,a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$

$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad |\arg\beta| < \tfrac{1}{2}\pi, \quad \operatorname{Re}(b_j-\rho) > -1, \quad j=1,\dots,m\right]$$

ET II 419(5)

2. $$\int_0^\infty e^{-\beta x}\, G^{mn}_{pq}\left(\alpha x^2\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \pi^{-1/2}\beta^{-1}\, G^{m,n+2}_{p+2,q}\left(\frac{4\alpha}{\beta^2}\,\middle|\,\begin{matrix}0,\frac{1}{2},a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$

$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad |\arg\beta| < \tfrac{1}{2}\pi, \quad \operatorname{Re} b_j > -\tfrac{1}{2}; \quad j=1,\dots,m\right]$$

ET II 419(6)

7.814

1. $$\int_0^\infty x^{\beta-1} e^{-x}\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q : xz\right) dx$$
$$= \pi \operatorname{cosec}(\beta\pi)\left[E\left(a_1,\dots,a_p : 1-\beta,\rho_1,\dots,\rho_q : e^{\pm i\pi}z\right) - z^{-\beta}\, E\left(a_1+\beta,\dots,a_p+\beta : 1+\beta,\rho_1+\beta,\dots,\rho_l+\beta : e^{\pm i\pi}z\right)\right]$$

[$p \ge q+1$, $\operatorname{Re}(a_r+\beta) > 0$, $r=1,\dots,p$, $|\arg z| < \pi$. The formula holds also for $p < q+1$, provided the integral converges.] ET II 415(4)

2. $$\int_0^\infty x^{\beta-1} e^{-x}\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q : x^{-m}z\right) dx$$
$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}}\, E\left(a_1,\dots,a_{p+m} : \rho_1,\dots,\rho_q : m^{-m}z\right)$$

$$\left[\operatorname{Re}\beta > 0, \quad a_{p+k} = \frac{\beta+k-1}{m}, \quad k=1,\dots,m; \quad m=1,2,\dots\right]$$ ET II 415(5)

7.815

1. $$\int_0^\infty \sin(cx)\, G^{mn}_{pq}\left(\alpha x^2\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) dx = \sqrt{\pi}c^{-1}\, G^{m,n+1}_{p+2,q}\left(\frac{4\alpha}{c^2}\,\middle|\,\begin{matrix}0,a_1,\dots,a_p,\frac{1}{2}\\ b_1,\dots,b_q\end{matrix}\right)$$

$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad c > 0, \quad \operatorname{Re} b_j > -1, \quad j=1,2,\dots,m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j=1,\dots,n\right]$$

ET II 420(7)

2. $$\int_0^\infty \cos(cx)\, G^{mn}_{pq}\left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right) dx = \pi^{1/2} c^{-1}\, G^{m,n+1}_{p+2,q}\left(\frac{4\alpha}{c^2} \left| \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right.\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad c > 0, \quad \operatorname{Re} b_j > -\tfrac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \tfrac{1}{2}, \quad j = 1, \dots, n\right]$$

ET II 420(8)

7.82 Combinations of the functions G and E and Bessel functions

7.821

1. $$\int_0^\infty x^{-\rho} J_\nu\left(2\sqrt{x}\right) G^{mn}_{pq}\left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right) dx = G^{m,n+1}_{p+2,q}\left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix} \right.\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi \quad -\tfrac{3}{4} + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < 1 + \tfrac{1}{2}\operatorname{Re}\nu + \min_{1\le j\le m} \operatorname{Re} b_j\right]$$

ET II 420(9)

2. $$\int_0^\infty x^{-\rho} Y_\nu\left(2\sqrt{x}\right) G^{mn}_{pq}\left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right) dx$$
$$= G^{m,n+2}_{p+3,q+1}\left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \rho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix} \right.\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad -\tfrac{3}{4} + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < \min_{1\le j\le m} \operatorname{Re} b_j + \tfrac{1}{2}|\operatorname{Re}\nu| + 1\right]$$

ET II 420(10)

3. $$\int_0^\infty x^{-\rho} K_\nu\left(2\sqrt{x}\right) G^{mn}_{pq}\left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right) dx = \frac{1}{2} G^{m,n+2}_{p+2,q}\left(\alpha \left| \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right)$$
$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad \operatorname{Re}\rho < 1 - \tfrac{1}{2}|\operatorname{Re}\nu| + \min_{1\le j\le m} \operatorname{Re} b_j\right]$$

ET II 421(11)

7.822

1. $$\int_0^\infty x^{2\rho} J_\nu(xy)\, G^{mn}_{pq}\left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right.\right) dx = \frac{2^{2\rho}}{y^{2\rho+1}} G^{m,n+1}_{p+2,q}\left(\frac{4\lambda}{y^2} \left| \begin{matrix} h, a_1, \dots, a_p, k \\ b_1, \dots, b_q \end{matrix} \right.\right)$$
$$h = \tfrac{1}{2} - \rho - \tfrac{1}{2}\nu, \quad k = \tfrac{1}{2} - \rho + \tfrac{1}{2}\nu$$
$$\left[p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad \operatorname{Re}\left(b_j + \rho + \tfrac{1}{2}\nu\right) > -\tfrac{1}{2}, \quad j = 1, 2, \dots, m, \quad \operatorname{Re}(a_j + \rho) < \tfrac{3}{4}, \quad j = 1, \dots, n, \quad y > 0\right]$$

ET II 91(20)

2. $$\int_0^\infty x^{1/2}\, Y_\nu(xy)\, G_{pq}^{mn}\left(\lambda x^2 \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx$$
$$= (2\lambda)^{-1/2} y^{-1/2}\, G_{q+1,p+3}^{n+2,m}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2}-b_1,\dots,\frac{1}{2}-b_q, l \\ h, k, \frac{1}{2}-a_1,\dots,\frac{1}{2}-a_p, l \end{matrix}\right)$$
$$h = \tfrac{1}{4} + \tfrac{1}{2}\nu, \quad k = \tfrac{1}{4} - \tfrac{1}{2}\nu, \quad l = -\tfrac{1}{4} - \tfrac{1}{2}\nu$$
$$\left[p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi, \quad y > 0,\right.$$
$$\left.\operatorname{Re} a_j < 1, \quad j = 1,\dots,n, \quad \operatorname{Re}\left(b_j \pm \tfrac{1}{2}\nu\right) > -\frac{3}{4}, \quad j = 1,\dots,m\right]$$
ET II 119(56)

3. $$\int_0^\infty x^{1/2}\, K_\nu(xy)\, G_{pq}^{mn}\left(\lambda x^2 \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx$$
$$= 2^{-3/2}\lambda^{-1/2} y^{-1/2}\, G_{q,p+2}^{n+2,m}\left(\frac{y^2}{4\lambda} \middle| \begin{matrix} \frac{1}{2}-b_1,\dots,\frac{1}{2}-b_q \\ h, k, \frac{1}{2}-a_1,\dots,\frac{1}{2}-a_p \end{matrix}\right)$$
$$h = \tfrac{1}{4} + \tfrac{1}{2}\nu, \quad k = \tfrac{1}{4} - \tfrac{1}{2}\nu$$
$$\left[\operatorname{Re} y > 0, \quad p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi,\right.$$
$$\left.\operatorname{Re} b_j > \tfrac{1}{2}|\operatorname{Re}\nu| - \tfrac{3}{4}, \quad j = 1,\dots,m\right]$$
ET II 153(90)

7.823

1. $$\int_0^\infty x^{\beta-1}\, J_\nu(x)\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q : x^{-2m} z\right) dx$$
$$= (2\pi)^{-m}(2m)^{\beta-1}\left\{\exp\left[\tfrac{1}{2}\pi(\beta-\nu-1)i\right] E\left[a_1,\dots,a_{p+2m} : \rho_1,\dots,\rho_q : (2m)^{-2m} z e^{-m\pi i}\right]\right.$$
$$\left. + \exp\left[-\tfrac{1}{2}\pi(\beta-\nu-1)i\right] E\left[a_1,\dots,a_{p+2m} : \rho_1,\dots,\rho_q : (2m)^{-2m} z e^{m\pi i}\right]\right\},$$
$$a_{p+k} = \frac{\beta+\nu+2k-2}{2m}, \quad a_{p+m+k} = \frac{\beta-\nu+2k-2}{2m}, \qquad m = 1,2,\dots; \quad k = 1,\dots,m$$
$$\left[\operatorname{Re}(\beta+\nu) > 0, \quad \operatorname{Re}(2a_r m - \beta) > -\tfrac{3}{2}, \quad r = 1,\dots,p\right]$$ ET II 415(7)

2. $$\int_0^\infty x^{\beta-1}\, K_\nu(x)\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q : x^{-2m} z\right) dx$$
$$= (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1}$$
$$\times E\left[a_1,\dots,a_{p+2m} : \rho_1,\dots,\rho_q : (2m)^{-2m} z\right],$$
$$a_{p+k} = \frac{\beta+\nu+2k-2}{2m}, \quad a_{p+m+k} = \frac{\beta-\nu+2k-2}{2m}, \qquad k = 1,2,\dots,m$$
$$\left[\operatorname{Re}\beta > |\operatorname{Re}\nu|, \quad m = 1,2,\dots\right]$$
ET II 416(8)

7.824

1. $$\int_0^\infty x^{1/2}\,\mathbf{H}_\nu(xy)\,G^{\,mn}_{pq}\left(\lambda x^2 \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = (2\lambda y)^{-1/2}\,G^{\,n+1,m+1}_{q+1,p+3}\left(\frac{y^2}{4\lambda} \,\middle|\, \begin{matrix} l, \frac12 - b_1,\dots,\frac12 - b_q \\ l, \frac12 - a_1,\dots,\frac12 - a_p, h, k \end{matrix}\right)$$

$$h = \frac14 + \frac{\nu}{2}, \quad k = \frac14 - \frac{\nu}{2}, \quad l = \frac34 + \frac{\nu}{2}$$

$$\left[p+q < 2(m+n), \quad |\arg\lambda| < \left(m+n-\tfrac12 p - \tfrac12 q\right)\pi, \quad y > 0, \right.$$
$$\left. \operatorname{Re} a_j < \min\left(1, \tfrac34 - \tfrac12\nu\right), \quad j = 1,\dots,n, \quad \operatorname{Re}(2b_j+\nu) > -\tfrac52, \quad j = 1,\dots,m\right]$$

ET II 172(47)

2. $$\int_0^\infty x^{-\rho}\,\mathbf{H}_\nu\left(2\sqrt{x}\right) G^{\,mn}_{pq}\left(\alpha x \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = G^{\,m+1,n+1}_{p+3,q+1}\left(\alpha \,\middle|\, \begin{matrix} \rho - \frac12 - \frac12\nu, a_1,\dots,a_p, \rho + \frac12\nu, \rho - \frac12\nu \\ \rho - \frac12 - \frac12\nu, b_1,\dots,b_q \end{matrix}\right)$$

$$\left[p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac12 p - \tfrac12 q\right)\pi\ , \right.$$
$$\left. \max\left(-\frac34, \operatorname{Re}\frac{\nu-1}{2}\right) + \max_{1\le j\le n} \operatorname{Re} a_j < \operatorname{Re}\rho < \min_{1\le j\le m} \operatorname{Re} b_j + \tfrac12 \operatorname{Re}\nu + \tfrac32\right]$$

ET II 421(12)

7.83 Combinations of the functions G and E and other special functions

7.831 $$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\,F(k+\sigma-\rho, \lambda+\sigma-\rho; \sigma; 1-x)\,G^{\,mn}_{pq}\left(\alpha x \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dx = \Gamma(\sigma)\,G^{\,m+2,n}_{p+2,q+2}\left(\alpha \,\middle|\, \begin{matrix} a_1,\dots,a_p, k+\lambda+\sigma-\rho, \rho \\ k, \lambda, b_1,\dots,b_q \end{matrix}\right)$$

where

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- $\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)\left(k+\frac12\right)\right] > -\frac12$
- $\operatorname{Re}\left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)\left(\lambda+\frac12\right)\right] > -\frac12$
- either

$$p+q < 2(m+n), \quad |\arg\alpha| < \left(m+n-\tfrac12 p - \tfrac12 q\right)\pi,$$
$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re} k \ge \operatorname{Re}\lambda > \operatorname{Re} a_j - 1, \quad j = 1,\dots,n,$$

or

$$p+q \le 2(m+n), \quad |\arg\alpha| \le \left(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q\right)\pi,$$
$$\operatorname{Re}\sigma > 0, \quad \operatorname{Re}k \ge \operatorname{Re}\lambda > \operatorname{Re}a_j - 1, \quad j=1,\dots,n,$$

7.832
$$\int_0^\infty x^{\beta-1}e^{-\frac{1}{2}x}\, W_{\kappa,\mu}(x)\, E\left(a_1,\dots,a_p : \rho_1,\dots,\rho_q : x^{-m}z\right)\, dx$$
$$= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta+\kappa-\frac{1}{2}}\, E\left(a_1,\dots,a_{p+2m} : \rho_1,\dots,\rho_{q+m} : m^{-m}z\right),$$
$$a_{p+k} = \frac{\beta+k+\mu-\frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta-\mu+k-\frac{1}{2}}{m}, \quad \rho_{q+k} = \frac{\beta-\kappa+k}{m}, \qquad k=1,\dots,m$$
$$\left[\operatorname{Re}\beta > |\operatorname{Re}\mu| - \tfrac{1}{2}, \quad m=1,2,\dots\right]$$ ET II 416(10)

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8–9 Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.110

1. Every integral of the form $\int R\left(x, \sqrt{P(x)}\right)\,dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}, \qquad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx, \qquad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)\,(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form.* The results of this reduction for the more frequently encountered integrals are given in formulas **3.13–3.17**. The number k is called the *modulus** of these integrals; the number $k' = \sqrt{1-k^2}$ is called the complementary modulus, and the number n is called the parameter of the integral of the third kind. BY (110.04)

2. By means of the substitution $x = \sin\varphi$, elliptic integrals can be reduced to the normal trigonometric forms

$$\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad \int \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \qquad \int \frac{d\varphi}{\left(1-n\sin^2\varphi\right)\sqrt{1-k^2\sin^2\varphi}}. \qquad \text{BY (110.04)}$$

The results of reducing integrals of trigonometric functions to normal form are given in **2.58–2.62**.

3.[11] Elliptic integrals from 0 to 1 in the **8.110 1** formulation (or from 0 to $\frac{\pi}{2}$ in the **8.110 2** formulation) are called *complete elliptic integrals.*

4.* Take note that in mathematical software, and elsewhere, the notation for elliptic integrals is often modified by replacing the parameter k^2 that is used here with k.

8.111

Notations:

1. $\Delta\varphi = \sqrt{1-k^2\sin^2\varphi}; \quad k' = \sqrt{1-k^2}; \quad k^2 < 1$

*The quantity k is sometimes called the *module* of the functions.

2. The elliptic integral of the first kind:

$$F(\varphi,k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}$$

3. The elliptic integral of the second kind:

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\alpha}\, d\alpha = \int_0^{\sin\varphi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx$$

FI II 135

4.[11] The elliptic integral of the third kind:

$$\Pi(\varphi,n,k) = \int_0^{\varphi} \frac{d\alpha}{\left(1-n\sin^2\alpha\right)\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{\left(1-nx^2\right)\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}$$

BY (110.04)

5. $$D(\varphi,k) = \frac{F(\varphi,k)-E(\varphi,k)}{k^2} = \int_0^{\varphi} \frac{\sin^2\alpha\, d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{x^2\, dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}$$

6.* $$\int_0^{\pi/2} \frac{dx}{\sqrt{a^2+\sin^2 x}} \arctan\left(\frac{b}{\sqrt{a^2+\sin^2 x}}\right) = \frac{\pi}{2|a|} F\left(\arcsin\left(\frac{b}{\sqrt{a^2+b^2+1}}\right), \frac{i}{a}\right)$$

[a and b are real]

8.112 Complete elliptic integrals

1. $\boldsymbol{K}(k) = F\left(\frac{\pi}{2}, k\right) = \boldsymbol{K}'(k')$

2. $\boldsymbol{E}(k) = E\left(\frac{\pi}{2}, k\right) = \boldsymbol{E}'(k')$

3. $\boldsymbol{K}'(k) = F\left(\frac{\pi}{2}, k'\right) = \boldsymbol{K}(k')$

4. $\boldsymbol{E}'(k) = E\left(\frac{\pi}{2}, k'\right) = \boldsymbol{E}(k')$

5. $\boldsymbol{D} = D\left(\frac{\pi}{2}, k\right) = \frac{\boldsymbol{K}-\boldsymbol{E}}{k^2}$

In writing complete elliptic integrals, the modulus k, which acts as an independent variable, is often omitted, and we write

$$\boldsymbol{K}\,(\equiv \boldsymbol{K}(k)), \quad \boldsymbol{K}'\left(\equiv \boldsymbol{K}'(k)\right), \quad \boldsymbol{E}\,(\equiv \boldsymbol{E}(k)), \quad \boldsymbol{E}'\left(\equiv \boldsymbol{E}'(k)\right).$$

Series representations

8.113

1. $$\boldsymbol{K} = \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \dots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2 k^{2n} + \dots\right\} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

FI II 487, WH 499

2. $$\boldsymbol{K} = \frac{\pi}{1+k'}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left(\frac{(2n-1)!!}{2^n n!}\right)^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \dots\right\}$$ DW

3. $$\boldsymbol{K} = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2}\right)k'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)k'^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{2}{5\cdot 6}\right)k'^6 + \dots$$ DW

See also **8.197** 1 and **8.197** 2.

8.114

1.[6] $$\boldsymbol{E} = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2\cdot 3}{2^2\cdot 4^2}k^4 - \dots - \left(\frac{(2n-1)!!}{2^n n!}\right)^2\frac{k^{2n}}{2n-1} - \dots\right\} = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right)$$ WH 518, FI II 487

2. $$\boldsymbol{E} = \frac{(1+k')\,\pi}{4}\left\{1 + \frac{1}{2^2}\left(\frac{1-k'}{1+k'}\right)^2 + \frac{1^2}{2^2\cdot 4^2}\left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left(\frac{(2n-3)!!}{2^n n!}\right)^2\left(\frac{1-k'}{1+k'}\right)^{2n} + \dots\right\}$$ DW

3. $$\boldsymbol{E} = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{1\cdot 2}\right)k'^2 + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)k'^4 + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)k'^6 + \dots$$ DW

8.115 $$\boldsymbol{D} = \pi\left\{\frac{1}{1}\left(\frac{1}{2}\right)^2 + \frac{2}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \dots + \frac{n}{2n-1}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2(n-1)} + \dots\right\}$$ ZH 43(158)

8.116 $$\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-k^2\sin^2\varphi}}{1-n^2\sin^2\varphi}\,d\varphi = \sqrt{n'^2-k'^2}\left(\frac{\arccos\frac{1}{n'}}{n'\sqrt{n'^2-1}} + \mathbf{R}\right),$$ where ZH 44(163)

$$\mathbf{R} = \frac{k'^2}{2}\left(p+\frac{1}{2}\right)\frac{1}{n'^3} + \frac{k'^4}{16}\left[-1 + \left(p+\frac{1}{4}\right)\frac{1}{n'^3}\left(1+\frac{6}{n'^2}\right)\right] + \frac{k'^6}{16}\left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p+\frac{1}{6}\right)\frac{1}{n'^3}\left(\frac{3}{8}+\frac{1}{n'^2}+\frac{5}{n'^4}\right)\right] + \frac{15k'^8}{256}\left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p+\frac{1}{8}\right)\frac{1}{n'^3}\left(\frac{5}{24}+\frac{9}{20n'^2}+\frac{1}{n'^4}+\frac{14}{3n'^6}\right)\right] + \dots,$$

$$p = \ln\frac{4}{k'},\quad k' = 4e^{-p},\quad k'^2 = 1-k^2,\quad n'^2 = 1-n^2$$ ZH 44(163)

Trigonometric series

8.117 For *small* values of k and φ, we may use the series

1. $$F(\varphi,k) = \frac{2}{\pi}\boldsymbol{K}\varphi - \sin\varphi\cos\varphi\left(a_0 + \frac{2}{3}a_1\sin^2\varphi + \frac{2\cdot 4}{3\cdot 5}a_2\sin^4\varphi + \dots\right), \qquad \text{where}$$

$$a_0 = \frac{2}{\pi}\boldsymbol{K} - 1; \quad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}$$ ZH 10(19)

2. $$E(\varphi,k) = \frac{2}{\pi}\boldsymbol{E}\varphi + \sin\varphi\cos\varphi\left(b_0 + \frac{2}{3}b_1\sin^2\varphi + \frac{2\cdot 4}{3\cdot 5}b_2\sin^4\varphi + \dots\right), \qquad \text{where}$$

$$b_0 = 1 - \frac{2}{\pi}\boldsymbol{E}, \quad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1}$$ ZH 27(86)

8.118 For k close to 1, we may use the series

1. $$F(\varphi,k) = \frac{2}{\pi}\boldsymbol{K}'\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right) - \frac{\tan\varphi}{\cos\varphi}\left(a_0' - \frac{2}{3}a_1'\tan^2\varphi + \frac{2\cdot 4}{3\cdot 5}a_2'\tan^4\varphi - \dots\right), \qquad \text{where}$$

$$a_0' = \frac{2}{\pi}\boldsymbol{K}' - 1; \quad a_n' = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k'^{2n}$$ ZH 10(23)

2. $$E(\varphi,k) = \frac{2}{\pi}\left(\boldsymbol{K}' - \boldsymbol{E}'\right)\ln\tan\left(\frac{\varphi}{2}+\frac{\pi}{2}\right) + \frac{\tan\varphi}{\cos\varphi}\left(b_1' - \frac{2}{3}b_2'\tan^2\varphi + \frac{2\cdot 4}{3\cdot 5}b_3'\tan^4\varphi - \dots\right) + \frac{1}{\sin\varphi}\left[1 - \cos\varphi\sqrt{1-k^2\sin\varphi}\right],$$

where

$$b_0' = \frac{2}{\pi}\left(\boldsymbol{K}' - \boldsymbol{E}'\right), b_n' = b_{n-1}' - \left[\frac{(2n-3)!!}{2^{n-1}(n-1)!}\right]^2\left(\frac{2n-1}{2n}\right)k'^{2n}$$ ZH 27(90)

For the expansion of complete elliptic integrals in Legendre polynomials, see **8.928**.

8.119 Representation in the form of an infinite product:

1. $$\boldsymbol{K}(k) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 + k_n\right), \qquad \text{where}$$

$$k_n = \frac{1 - \sqrt{1-k_{n-1}^2}}{1 + \sqrt{1-k_{n-1}^2}}; \qquad k_0 = k$$ FI II 166

See also **8.197**.

8.12 Functional relations between elliptic integrals

8.121

1. $F(-\varphi,k) = -F(\varphi,k)$ JA

2. $E(-\varphi,k) = -E(\varphi,k)$ JA

3. $F(n\pi \pm \varphi, k) = 2n\,\boldsymbol{K}(k) \pm F(\varphi,k)$ JA

4. $E(n\pi \pm \varphi, k) = 2n\,\boldsymbol{E}(k) \pm E(\varphi,k)$ JA

8.122 $\boldsymbol{E}(k)\,\boldsymbol{K}'(k) + \boldsymbol{E}'(k)\,\boldsymbol{K}(k) - \boldsymbol{K}(k)\,\boldsymbol{K}'(k) = \frac{\pi}{2}$ FI II 691, 791

8.123

1. $$\frac{\partial F}{\partial k} = \frac{1}{k'^2}\left(\frac{E - k'^2 F}{k} - \frac{k\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right)$$ MO 138, BY (710.07)

2. $$\frac{d\,\boldsymbol{K}(k)}{dk} = \frac{\boldsymbol{E}(k)}{kk'^2} - \frac{\boldsymbol{K}(k)}{k}$$ FI II 691

3. $$\frac{\partial E}{\partial k} = \frac{E-F}{k}$$ MO 138

4. $$\frac{d\,\boldsymbol{E}(k)}{dk} = \frac{\boldsymbol{E}(k) - \boldsymbol{K}(k)}{k}$$ FI II 690

8.124

1. The functions $\boldsymbol{K}$ and $\boldsymbol{K}'$ satisfy the equation

$$\frac{d}{dk}\left\{kk'^2\frac{du}{dk}\right\} - ku = 0.$$ WH 499, WH 502

2. The functions $\boldsymbol{E}$ and $\boldsymbol{E}' - \boldsymbol{K}'$ satisfy the equation

$$k'^2\frac{d}{dk}\left(k\frac{du}{dk}\right) + ku = 0.$$ WH

8.125

1. $$F\left(\psi, \frac{1-k'}{1+k'}\right) = (1+k')\,F(\varphi,k) \qquad [\tan(\psi-\varphi) = k'\tan\varphi]$$ MO 130

2. $$E\left(\psi, \frac{1-k'}{1+k'}\right) = \frac{2}{1+k'}\left[E(\varphi,k) + k'\,F(\varphi,k)\right] - \frac{1-k'}{1+k'}\sin\psi$$
$$[\tan(\psi-\varphi) = k'\tan\varphi]$$ MO 131

3. $$F\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = (1+k)\,F(\varphi,k) \qquad \left[\sin\psi = \frac{(1+k)\sin\varphi}{1+k\sin^2\varphi}\right]$$

4. $$E\left(\psi, \frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}\left[2E(\varphi,k) - k'^2\,F(\varphi,k) + 2k\frac{\sin\varphi\cos\varphi}{1+k\sin^2\varphi}\sqrt{1-k^2\sin^2\varphi}\right]$$
$$\left[\sin\psi = \frac{(1+k)\sin\varphi}{1+k\sin^2\varphi}\right]$$ MO 131

8.126 In particular,

1. $$\boldsymbol{K}\left(\frac{1-k'}{1+k'}\right)=\frac{1+k'}{2}\,\boldsymbol{K}(k)$$ MO 130

2. $$\boldsymbol{E}\left(\frac{1-k'}{1+k'}\right)=\frac{1}{1+k'}\left[\boldsymbol{E}(k)+k'\,\boldsymbol{K}(k)\right]$$ MO 130

3. $$\boldsymbol{K}\left(\frac{2\sqrt{k}}{1+k}\right)=(1+k)\,\boldsymbol{K}(k)$$ MO 130

4. $$\boldsymbol{E}\left(\frac{2\sqrt{k}}{1+k}\right)=\frac{1}{1+k}\left[2\,\boldsymbol{E}(k)-k'^2\,\boldsymbol{K}(k)\right]$$ MO 130

8.127[11]

k_1	$\sin\varphi_1$	$\cos\varphi_1$	$F(\varphi_1,k_1)$	$E(\varphi_1,k_1)$
$i\frac{k}{k'}$	$k'\frac{\sin\varphi}{\Delta\varphi}$	$\frac{\cos\varphi}{\Delta\varphi}$	$k'\,F(\varphi,k)$	$\frac{1}{k'}\left[E(\varphi,k)-\frac{k^2\sin\varphi\cos\varphi}{\Delta\varphi}\right]$
k'	$-i\tan\varphi$	$\sec\varphi$	$-i\,F(\varphi,k)$	$ia\left[E(\varphi,k)-F(\varphi,k)-\Delta\varphi\tan\varphi\right]$
$\frac{1}{k}$	$k\sin\varphi$	$\Delta\varphi$	$k\,F(\varphi,k)$	$\frac{1}{k}\left[E(\varphi,k)-k'^2\,F(\varphi,k)\right]$
$\frac{1}{k'}$	$-ik'\tan\varphi$	$\frac{\Delta\varphi}{\cos\varphi}$	$-ik'\,F(\varphi,k)$	$\frac{i}{k'}\left[E(\varphi,k)-k'^2\,F(\varphi,k)-\Delta\varphi\tan\varphi\right]$
$\frac{k'}{ik}$	$\frac{-ik\sin\varphi}{\Delta\varphi}$	$\frac{1}{\Delta\varphi}$	$-ik\,F(\varphi,k)$	$\frac{i}{k}\left[E(\varphi,k)-F(\varphi,k)-\frac{k^2\sin\varphi\cos\varphi}{\Delta\varphi}\right]$

(see **8.111** 1) MO 131

8.128 In particular,

1. $$\boldsymbol{K}\left(i\frac{k}{k'}\right)=k'\,\boldsymbol{K}(k)\qquad[\operatorname{Im}(k)<0]$$ MO 130

2. $$\boldsymbol{K}\left(i\frac{k}{k'}\right)=k'\left[\boldsymbol{K}'(k')-i\,\boldsymbol{K}(k)\right]\qquad[\operatorname{Im}(k)<0]$$ MO 130

3. $$\boldsymbol{K}\left(\frac{1}{k}\right)=k\left[\boldsymbol{K}(k)+i\,\boldsymbol{K}'(k)\right]\qquad[\operatorname{Im}(k)<0]$$ MO 130

For integrals of elliptic integrals, see **6.11–6.15**. For indefinite integrals of complete elliptic integrals, see **5.11**.

8.129 Special values:

1. $$\boldsymbol{K}\left(\sin\frac{\pi}{4}\right)=\boldsymbol{K}\left(\frac{\sqrt{2}}{2}\right)=\boldsymbol{K}'\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^4}}=\frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$ MO 130

2. $$\boldsymbol{K}'\left(\sqrt{2}-1\right)=\sqrt{2}\,\boldsymbol{K}\left(\sqrt{2}-1\right)$$ MO 130

3. $$\boldsymbol{K}'\left(\sin\frac{\pi}{12}\right) = \sqrt{3}\,\boldsymbol{K}\left(\sin\frac{\pi}{12}\right)$$ MO 130

4. $$\boldsymbol{K}'\left(\tan^2\frac{\pi}{8}\right) = \boldsymbol{K}'\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2\,\boldsymbol{K}\left(\tan^2\frac{\pi}{8}\right)$$ MO 130

5.* $$\boldsymbol{K}\left(\sin\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

6.* $$\boldsymbol{E} = \frac{\pi\sqrt{3}}{12\boldsymbol{K}} + \sqrt{\frac{2}{3}}k'\boldsymbol{K}$$

7.* $$\boldsymbol{E}' = \frac{\pi\sqrt{3}}{4\boldsymbol{K}'} + \sqrt{\frac{2}{3}}k\boldsymbol{K}'$$

8.13 Elliptic functions

8.130 Definition and general properties.

1. A single-valued function $f(z)$ of a complex variable, which is not a constant, is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$, that is

$$f\left(z + 2m\omega_1 + 2n\omega_2\right) = f(z) \qquad [m, n \text{ integers}]\,.$$

The ratio of the periods of an analytic function cannot be a real number. For an elliptic function $f(z)$, the z-plane can be partitioned into parallelograms—the period parallelograms—the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$. At corresponding points of these parallelograms, the function $f(z)$ has the same value. ZH 117, SI 299

2. Suppose that α is the angle between the sides a and b of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b}e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi\sin\alpha}\left[\cos\left(\frac{a}{b}\pi\cos\alpha\right) + i\sin\left(\frac{a}{b}\pi\cos\alpha\right)\right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods. SM III 598

4. A non-constant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points $a_1, a_2, \ldots, a_n$ and that their orders are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Suppose that the zeros of an analytic function that occur in a single parallelogram are $b_1, b_2, \ldots, b_m$ and that the orders of the zeros are $\beta_1, \beta_2, \ldots, \beta_m$, respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_m.$$ ZH 118

The number γ representing this sum is called the *order* of the elliptic function.

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.

6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.

7. Every two elliptic functions with the same periods are related by an algebraic relationship. GO II 151

8.[7] A non-constant single-valued function which is not constant cannot have more than two periods. GO II 147

9. An elliptic function of order γ assumes *an arbitrary value* γ times in a period parallelogram. SM 601, SI 301

8.14 Jacobian elliptic functions

8.141 Consider the upper limit φ of the integral

$$u = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of u. Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude*. The quantity u is called the *argument*, and its dependence on φ is written

$$u = \arg \varphi.$$

8.142 The amplitude is an *infinitely many-valued* function of u and has a period of $4\boldsymbol{K}i$. The *branch points* of the amplitude correspond to the values of the argument

$$u = 2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i,$$ ZH 67–69

where m and n are arbitrary integers (see also **8.151**).

8.143 The first two of the following functions

$$\operatorname{sn} u = \sin\varphi = \sin \operatorname{am} u, \qquad \operatorname{cn} u = \cos\varphi = \cos \operatorname{am} u,$$
$$\operatorname{dn} u = \Delta\varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the *delta amplitude*. All these elliptic functions were exhibited by Jacobi and they bear his name. SI 16

The Jacobian elliptic functions are *doubly periodic* functions and have *two simple poles* in a period parallelogram. ZH 69

8.144

1. $$u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$ SI 21(23)

2. $$u = \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$$ SI 21(23)

3. $$u = \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}$$ SI 21(23)

8.145 Power series representations:

1.[11] $$\operatorname{sn} u = u - \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \frac{1+135k^2+135k^4+k^6}{7!}u^7 + \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}u^9 - \dots$$

$[|u| < |\boldsymbol{K}'|]$ ZH 81(97)

2. $$\operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \frac{1+408k^2+912k^4+64k^6}{8!}u^8 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ ZH 81(98)

3. $$\operatorname{dn} u = 1 - \frac{k^2}{2!}u^2 + \frac{k^2\left(4+k^2\right)}{4!}u^4 - \frac{k^2\left(16+44k^2+k^4\right)}{6!}u^6 + \frac{k^2\left(64+912k^2+408k^4+k^6\right)}{8!}u^8 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ ZH 81(99)

4. $$\operatorname{am} u = u - \frac{k^2}{3!}u^3 + \frac{k^2\left(4+k^2\right)}{5!}u^5 - \frac{k^2\left(16+44k^2+k^4\right)}{7!}u^7 + \frac{k^2\left(64+912k^2+408k^4+k^6\right)}{9!}u^9 - \dots$$
$[|u| < |\boldsymbol{K}'|]$ LA 380(4)

8.146 Representation as a trigonometric series or a product $\left(q = e^{-\frac{\pi K'}{K}} = e^{\pi i \tau}\right)^*$

1.[11] $$\operatorname{sn} u = \frac{2\pi}{k\boldsymbol{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ WH 511a, ZH 84(108)

2.[11] $$\operatorname{cn} u = \frac{2\pi}{k\boldsymbol{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ WH 511a, ZH 84(109)

3. $$\operatorname{dn} u = \frac{\pi}{2\boldsymbol{K}} + \frac{2\pi}{\boldsymbol{K}} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos\frac{n\pi u}{\boldsymbol{K}}$$ WH 511a, ZH 84(110)

4.[11] $$\operatorname{am} u = \frac{\pi u}{2\boldsymbol{K}} + 2\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin\frac{n\pi u}{\boldsymbol{K}}$$ WH 511a

5. $$\frac{1}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} + 4\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} \sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(3)

6. $$\frac{1}{\operatorname{cn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[\frac{1}{\cos\frac{\pi u}{2\boldsymbol{K}}} + 4\sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1+q^{2n-1}} \cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(3)

7. $$\frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[1 + 4\sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(3)

8. $$\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'\boldsymbol{K}}\left[\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(4)

9.[11] $$\frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'\boldsymbol{K}} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(4)

10. $$\frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin\frac{\pi n u}{\boldsymbol{K}}\right]$$ LA 369(5)

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*The expansions 1–22 are valid in every strip of the form $\left|\operatorname{Im}\frac{\pi u}{2\boldsymbol{K}}\right| < \frac{1}{2}\pi \operatorname{Im}\tau$. The expansions 23–25 are valid in an arbitrary bounded portion of u.

11. $$\frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{k\boldsymbol{K}}\sum_{n=1}^{\infty}(-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(5)

12. $$\frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{2n-1}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(6)

13. $$\frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\frac{1}{\cos\frac{\pi u}{2\boldsymbol{K}}} - 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n-1}}{1-q^{2n-1}}\cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}\right]$$ LA 369(6)

14. $$\frac{\operatorname{cn} u\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(7)

15. $$\frac{\operatorname{sn} u\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2\boldsymbol{K}}\left\{\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}\frac{q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right\}$$ LA 369(7)

16. $$\frac{\operatorname{sn} u\operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2\boldsymbol{K}}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}$$ LA 369(7)

17. $$\frac{\operatorname{sn} u}{\operatorname{cn} u\operatorname{dn} u} = \frac{\pi}{2\left(1-k^2\right)\boldsymbol{K}}\left[\tan\frac{\pi u}{2\boldsymbol{K}} + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1-q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

18. $$\frac{\operatorname{cn} u}{\operatorname{sn} u\operatorname{dn} u} = \frac{\pi}{2\boldsymbol{K}}\left[\cot\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+(-1)^n q^n}\sin\frac{n\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

19. $$\frac{\operatorname{dn} u}{\operatorname{sn} u\operatorname{cn} u} = \frac{\pi}{\boldsymbol{K}}\left[\frac{1}{\sin\frac{\pi u}{\boldsymbol{K}}} + 4\sum_{n=1}^{\infty}\frac{q^{2(2n-1)}}{1-q^{2(2n-1)}}\sin(2n-1)\frac{\pi u}{\boldsymbol{K}}\right]$$ LA 369(8)

20.[11] $$\ln\operatorname{sn} u = \ln\frac{2\boldsymbol{K}}{\pi} + \ln\sin\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}$$ LA 369(2)

21. $$\ln\operatorname{cn} u = \ln\cos\frac{\pi u}{2\boldsymbol{K}} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n}\sin^2\frac{n\pi u}{2\boldsymbol{K}}$$ LA 369(2)

22. $$\ln\operatorname{dn} u = -8\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{q^{2n-1}}{1-q^{2(2n-1)}}\sin^2(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ LA 369(2)

23.[11] $$\operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}}\sin\frac{\pi u}{2\boldsymbol{K}}\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(145)

24. $$\operatorname{cn} u = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}}\cos\frac{\pi u}{2\boldsymbol{K}}\prod_{n=1}^{\infty}\frac{1+2q^{2n}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(146)

25. $$\operatorname{dn} u = \sqrt{k'}\prod_{n=1}^{\infty}\frac{1+2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}{1-2q^{2n-1}\cos\frac{\pi u}{\boldsymbol{K}}+q^{4n-2}}$$ WH 508a, ZH 86(147)

26. $$\operatorname{sn}^3 u = \sum_{n=0}^{\infty}\left[\frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3}\frac{\pi^2}{4\boldsymbol{K}^2}\right]\frac{2\pi q^{n+\frac{1}{2}}\sin(2n+1)\frac{\pi u}{2\boldsymbol{K}}}{\boldsymbol{K}\left(1-q^{2n+1}\right)}$$

$$\left[\left|\operatorname{Im}\frac{u}{2\boldsymbol{K}}\right| < \operatorname{Im}\tau\right]$$ MO 147

27. $$\frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4\boldsymbol{K}^2} \operatorname{cosec}^2 \frac{\pi u}{2\boldsymbol{K}} + \frac{\boldsymbol{K} - \boldsymbol{E}}{\boldsymbol{K}} - \frac{2\pi^2}{\boldsymbol{K}^2} \sum_{n=1}^{\infty} \frac{nq^{2n} \cos \frac{n\pi u}{\boldsymbol{K}}}{1 - q^{2n}}$$

$$\left[\left|\operatorname{Im} \frac{u}{2\boldsymbol{K}}\right| < \frac{1}{2} \operatorname{Im} \tau\right]$$ MO 148

8.147

1. $$\operatorname{sn} u = \frac{\pi}{2k\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{1}{\sin \frac{\pi}{2\boldsymbol{K}} [u - (2n-1)i\boldsymbol{K}']}$$ MO 149

2. $$\operatorname{cn} u = \frac{\pi i}{2k\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin \frac{\pi}{2\boldsymbol{K}} [u - (2n-1)i\boldsymbol{K}']}$$ MO 150

3. $$\operatorname{dn} u = \frac{\pi i}{2\boldsymbol{K}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\tan \frac{\pi}{2\boldsymbol{K}} [u - (2n-1)i\boldsymbol{K}']}$$ MO150

8.148 The Weierstrass expansions of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$:

$$\operatorname{sn} u = \frac{B}{A}, \qquad \operatorname{cn} u = \frac{C}{A}, \qquad \operatorname{dn} u = \frac{D}{A},$$ ZH 82–83(105,106,107)

where

$$A = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!} \qquad B = \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!}$$

$$C = \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!} \qquad D = \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!}$$

and

$$a_2 = 2k^2, \quad a_3 = 8\left(k^2 + k^4\right), \quad a_4 = 32\left(k^2 + k^6\right) + 68k^4, \quad a_5 = 128\left(k^2 + k^8\right) + 480\left(k^4 + k^6\right),$$
$$a_6 = 512\left(k^2 + k^{10}\right) + 3008\left(k^4 + k^8\right) + 5400k^6, \quad \ldots$$
$$b_0 = 1, \quad b_1 = 1 + k^2, \quad b_2 = 1 + k^4 + 4k^2, \quad b_3 = 1 + k^6 + 9\left(k^2 + k^4\right),$$
$$b_4 = 1 + k^8 + 16\left(k^2 + k^6\right) - 6k^4, \quad b_5 = 1 + k^{10} + 25\left(k^2 + k^8\right) - 494\left(k^4 + k^6\right),$$
$$b_6 = 1 + k^{12} + 36\left(k^2 + k^{10}\right) - 5781\left(k^4 + k^8\right) - 12184k^6, \quad \ldots$$
$$c_0 = 1, \quad c_1 = 1, \quad c_2 = 1 + 2k^2, \quad c_3 = 1 + 6k^2 + 8k^4, \quad c_4 = 1 + 12k^2 + 60k^4 + 32k^6,$$
$$c_5 = 1 + 20k^2 + 348k^4 + 448k^6 + 128k^8, \quad c_6 = 1 + 30k^2 + 2372k^4 + 4600k^6 + 2880k^8 + 512k^{10}, \quad \ldots$$
$$d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2 + k^4, \quad d_3 = 8k^2 + 6k^4 + k^6, \quad d_4 = 32k^2 + 60k^4 + 12k^4 + k^8,$$
$$d_5 = 128k^2 + 448k^4 + 348k^6 + 20k^8 + k^{10},$$
$$d_6 = 512k^2 + 2880k^4 + 4600k^6 + 2372k^8 + 30k^{10} + k^{12}, \quad \ldots$$

8.15 Properties of Jacobian elliptic functions and functional relationships between them

8.151 The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
$\operatorname{sn} u$	$4m\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$(-1)^m \frac{1}{k}$
$\operatorname{cn} u$	$4m\boldsymbol{K} + 2n(\boldsymbol{K} + \boldsymbol{K}'i)$	$(2m+1)\boldsymbol{K} + 2n\boldsymbol{K}'i$	$2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$(-1)^{m-1}\frac{i}{k}$
$\operatorname{dn} u$	$2m\boldsymbol{K} + 4n\boldsymbol{K}'i$	$(2m+1)\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$2m\boldsymbol{K} + (2n+1)\boldsymbol{K}'i$	$(-1)^{n-1}i$

SM 630, ZH 69–72

2.

$u^* = u + \boldsymbol{K}$	$u + i\boldsymbol{K}$	$u + \boldsymbol{K} + i\boldsymbol{K}'$	$u + 2\boldsymbol{K}$	$u + 2i\boldsymbol{K}'$	$u + 2\boldsymbol{K} + 2i\boldsymbol{K}'$
$\operatorname{sn} u^* = \frac{\operatorname{cn} u}{\operatorname{dn} u}$	$\frac{1}{k \operatorname{sn} u}$	$\frac{1}{k}\frac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\operatorname{sn} u$	$\operatorname{sn} u$	$-\operatorname{sn} u$
$\operatorname{cn} u^* = -k'\frac{\operatorname{sn} u}{\operatorname{dn} u}$	$-\frac{i}{k}\frac{\operatorname{dn} u}{\operatorname{sn} u}$	$-\frac{ik'}{k \operatorname{cn} u}$	$-\operatorname{cn} u$	$-\operatorname{cn} u$	$\operatorname{cn} u$
$\operatorname{dn} u^* = k'\frac{1}{\operatorname{dn} u}$	$-i\frac{\operatorname{cn} u}{\operatorname{sn} u}$	$ik'\frac{\operatorname{sn} u}{\operatorname{cn} u}$	$\operatorname{dn} u$	$-\operatorname{dn} u$	$-\operatorname{dn} u$

SM 630

3.

$u^* = 0$	$-u$	$\frac{1}{2}\boldsymbol{K}$	$\frac{1}{2}(\boldsymbol{K} + i\boldsymbol{K}')$	$\frac{1}{2}i\boldsymbol{K}'$	$u + 2m\boldsymbol{K} + 2n\boldsymbol{K}'i$
$\operatorname{sn} u^* = 0$	$-\operatorname{sn} u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k} + i\sqrt{1-k}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m \operatorname{sn} u$
$\operatorname{cn} u^* = 1$	$\operatorname{cn} u$	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n} \operatorname{cn} u$
$\operatorname{dn} u^* = 1$	$\operatorname{dn} u$	$\sqrt{k'}$	$\frac{\sqrt{k'}\left(\sqrt{1+k'} - i\sqrt{1-k'}\right)}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n \operatorname{dn} u$

SI 19, SI 18(13), WH, WH WH WH

8.152 Transformation formulas

u_1	l_1	$sn\,(u_1, k_1)$	$\operatorname{cn}\,(u_1, k_1)$	$\operatorname{dn}\,(u_1, k_1)$
ku	$\frac{1}{k}$	$k \operatorname{sn}(u,k)$	$\operatorname{dn}(u,k)$	$\operatorname{cn}(u,k)$
iu	k'	$i\frac{\operatorname{sn}(u,k)}{\operatorname{cn}(u,k)}$	$\frac{1}{\operatorname{cn}(u,k)}$	$\frac{\operatorname{dn}(u,k)}{\operatorname{cn}(u,k)}$
$k'u$	$i\frac{k}{k'}$	$k'\frac{\operatorname{sn}(u,k)}{\operatorname{dn}(u,k)}$	$\frac{\operatorname{cn}(u,k)}{\operatorname{dn}(u,k)}$	$\frac{1}{\operatorname{dn}(u,k)}$
iku	$i\frac{k'}{k}$	$ik\frac{\operatorname{sn}(u,k)}{\operatorname{dn}(u,k)}$	$\frac{1}{\operatorname{dn}(u,k)}$	$\frac{\operatorname{cn}(u,k)}{\operatorname{dn}(u,k)}$
$ik'u$	$\frac{1}{k'}$	$ik'\frac{\operatorname{sn}(u,k)}{\operatorname{cn}(u,k)}$	$\frac{\operatorname{dn}(u,k)}{\operatorname{cn}(u,k)}$	$\frac{1}{\operatorname{cn}(u,k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$\frac{(1+k)\operatorname{sn}(u,k)}{1+k\operatorname{sn}^2(u,k)}$	$\frac{\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{1+k\operatorname{sn}^2(u,k)}$	$\frac{1-k\operatorname{sn}^2(u,k)}{1+k\operatorname{sn}^2(u,k)}$
$(1+k')\,u$	$\frac{1-k'}{1+k'}$	$(1+k')\,\frac{\operatorname{sn}(u,k)\operatorname{cn}(u,k)}{\operatorname{dn}(u,k)}$	$\frac{1-(1+k')\operatorname{sn}^2(u,k)}{\operatorname{dn}(u,k)}$	$\frac{1-(1-k')\operatorname{sn}^2(u,k)}{\operatorname{dn}(u,k)}$
$\frac{\left(1+\sqrt{k'}\right)^2}{2}u$	$\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)^2$	$\frac{k^2\operatorname{sn}(u,k) d\operatorname{cn}(u,k)}{\sqrt{k_1}\,[1+\operatorname{dn}(u,k)]\,[k'+\operatorname{dn}(u,k)]}$	$\frac{\operatorname{dn}(u,k)-\sqrt{k'}}{1-\sqrt{k'}} \times\sqrt{\frac{2(1+k')}{[1+\operatorname{dn}(u,k)][k'+\operatorname{dn}(u,k)]}}$	$\frac{\sqrt{1+k_1}\left(\operatorname{dn}(u,k)+\sqrt{k'}\right)}{\sqrt{[1+\operatorname{dn}(u,k)]\,[k'+\operatorname{dn}(u,k)]}}$

8.153

1. $\operatorname{sn}(iu,k) = i\dfrac{\operatorname{sn}(u,k')}{\operatorname{cn}(u,k')}$ SI 50(64)

2. $\operatorname{cn}(iu,k) = \dfrac{1}{\operatorname{cn}(u,k')}$ SI 50(65)

3. $\operatorname{dn}(iu,k) = \dfrac{\operatorname{dn}(u,k')}{\operatorname{cn}(u,k')}$ SI 50(65)

4. $\operatorname{sn}(u,k) = k^{-1}\operatorname{sn}\left(ku,k^{-1}\right)$

5. $\operatorname{cn}(u,k) = \operatorname{dn}\left(ku,k^{-1}\right)$

6. $\operatorname{dn}(u,k) = \operatorname{cn}\left(ku,k^{-1}\right)$

7.[11] $\operatorname{sn}(u,ik) = \dfrac{1}{\sqrt{1+k^2}}\dfrac{\operatorname{sn}\left(u\sqrt{1+k^2}, k\left(1+k^2\right)^{-1/2}\right)}{\operatorname{dn}\left(u\sqrt{1+k^2}, k\left(1+k^2\right)^{-1/2}\right)}$

8.[11] $\operatorname{cn}(u,ik) = \dfrac{\operatorname{sn}\left(u\left(1+k^2\right)^{1/2}, k\left(1+k^2\right)^{-1/2}\right)}{\operatorname{dn}\left(u\left(1+k^2\right)^{1/2}, k\left(1+k^2\right)^{-1/2}\right)}$

9.[11] $\operatorname{dn}(u,ik) = \dfrac{1}{\operatorname{dn}\left(u\left(1+k^2\right)^{1/2}, k\left(1+k^2\right)^{-1/2}\right)}$

Functional relations

8.154

1. $\operatorname{sn}^2 u = \dfrac{1-\operatorname{cn} 2u}{1+\operatorname{dn} 2u}$ MO 146

2. $\operatorname{cn}^2 u = \dfrac{\operatorname{cn} 2u+\operatorname{dn} 2u}{1+\operatorname{dn} 2u}$ MO 146

3. $\operatorname{dn}^2 u = \dfrac{\operatorname{dn} 2u+k^2\operatorname{cn} 2u+k'^2}{1+\operatorname{dn} 2u}$ MO 146

4. $\operatorname{sn}^2 u+\operatorname{cn}^2 u = 1$ SI 16(9)

5. $\operatorname{dn}^2 u+k^2\operatorname{sn}^2 u = 1$ SI 16(9)

8.155

1. $\dfrac{1-\operatorname{dn} 2u}{1+\operatorname{dn} 2u} = k^2\dfrac{\operatorname{sn}^2 u\operatorname{cn}^2 u}{\operatorname{dn}^2 u}$ MO 146

2. $\dfrac{1-\operatorname{cn} 2u}{1+\operatorname{cn} 2u} = \dfrac{\operatorname{sn}^2 u\operatorname{dn}^2 u}{\operatorname{cn}^2 u}$ MO 146

8.156

1. $\operatorname{sn}(u\pm v) = \dfrac{\operatorname{sn} u\operatorname{cn} v\operatorname{dn} v\pm\operatorname{sn} v\operatorname{cn} u\operatorname{dn} u}{1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 v}$ SI 46(56)

2. $$\operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$ SI 46(57)

3. $$\operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$ SI 46(58)

8.157

1. $$\operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}$$ SI 47(61), SU 67(15)

2. $$\operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}}$$ SI 48(62), SI 67(16)

3. $$\operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}}$$ SI 48(63), SI 67(17)

8.158

1. $$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u$$ SI 21(21)

2. $$\frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$$ SI 21(21)

3.[8] $$\frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{dn} u \operatorname{cn} u$$ SI 21(21)

8.159 Jacobian elliptic functions are solutions of the following differential equations:

1. $$\frac{d}{du} \operatorname{sn} u = \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)}$$ SI 21(22)

2. $$\frac{d}{du} \operatorname{cn} u = -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)},$$ SI 21(22)

3. $$\frac{d}{du} \operatorname{dn} u = -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}$$ SI 21(22)

For the indefinite integrals of Jacobi's elliptic functions, see **5.13**.

8.16 The Weierstrass function $\wp(u)$

8.160 The Weierstrass elliptic function $\wp(u)$ is defined by

1. $$\wp(u) = \frac{1}{u^2} + \sum_{m,n}{}' \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},$$ SI 307(6)

where the symbol $\sum'$ means that the summation is made over all combinations of integers m and n except for the combination $m = n = 0$; $2\omega_1$ and $2\omega_2$ are the periods of the function $\wp(u)$. Obviously,

2. $$\wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \text{ and } \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0,$$

3. $$\frac{d}{du}\wp(u) = -2\sum_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3},$$

where the summation is made over all integral values of m and n.

The series **8.160** 1 and **8.160** 3 converge everywhere except at the poles, that is, at the points $2m\omega_1 + 2n\omega_2$ (where m and n are integers).

4. The function $\wp(u)$ is a *doubly periodic function* and has *one second-order pole* in a period parallelogram. SI 306

8.161 The function $\wp(u)$ satisfies the differential equation

1. $$\left[\frac{d\,\wp(u)}{du}\right]^2 = 4\,\wp^3(u) - g_2\,\wp(u) - g_3,$$ SI 142, 310, WH

where

2. $$g_2 = 60\sum_{m,n}{}' (m\omega_1 + n\omega_2)^{-4}; \qquad g_3 = 140\sum_{m,n}{}' (m\omega_1 + n\omega_2)^{-6}$$ WH, SI 310

The functions g_2 and g_3 are called the *invariants* of the function $\wp(u)$.

8.162 $$u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4\,(z - e_1)\,(z - e_2)\,(z - e_3)}},$$

where e_1, e_2, and e_3 are the roots of the equation $4z^3 - g_2 z - g_3 = 0$; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4}, \quad e_1 e_2 e_3 = \frac{g_3}{4}$$ SI 142, 143, 144

8.163 $\wp(\omega_1) = e_1$, $\wp(\omega_1) + \omega_2 = e_2$, $\wp(\omega_2) = e_3$. Here, it is assumed that if e_1, e_2, and e_3 lie on a straight line in the complex plane, e_2 lies between e_1 and e_3.

8.164 The number $\Delta = g_2^3 - 27g_3^2$ is called the *discriminant* of the function $\wp(u)$. If $\Delta > 0$, all roots e_1, e_2, and e_3 of the equation $4z^3 - g_2 z - g_3 = 0$ (where g_2 and g_3 are real numbers) are *real*. In this case, the roots e_1, e_2, and e_3 are numbered in such a way that $e_1 > e_2 > e_3$.

1. If $\Delta > 0$, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \qquad \omega_2 = i\int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2 z - 4z^3}},$$

where ω_1 is real and ω_2 is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that ω_1 and $\frac{\omega_2}{i}$ will be positive.

2. If $\Delta < 0$, the root e_2 of the equation $4z^3 - g_2 z - g_3 = 0$ is *real*, and the remaining two roots (e_1 and e_3) are *complex conjugates*. Suppose that $e_1 = \alpha + i\beta$, and $e_3 = \alpha - i\beta$. In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} \quad \text{and} \quad \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

as basic semiperiods.

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower half-plane. SI 151(21, 22)

8.165 Series representation:

1. $$\wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4\cdot 5} + \frac{g_3 u^4}{4\cdot 7} + \frac{g_2^2 u^6}{2^4\cdot 3\cdot 5^2} + \frac{3 g_2 g_3 u^8}{2^4\cdot 5\cdot 7\cdot 11} + \dots$$ WH

8.166 Functional relations

1. $$\wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u)$$

2. $$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$ SI 163(32)

8.167 $$\wp\left(u; g_2, g_3\right) = \mu^2\,\wp\left(\mu u; \frac{g_2}{\mu^4}, \quad \frac{g_3}{\mu^6}\right)$$ (the formula for homogeneity)

SI 149(13)

The special case: $\mu = i$.

1. $$\wp\left(u; g_2, g_3\right) = -\wp\left(iu; g_2, -g_3\right)$$

8.168 An arbitrary elliptic function can be expressed in terms of the elliptic function $\wp(u)$ having the same periods as the original function and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

8.169 A connection with the Jacobian elliptic functions. For $\Delta > 0$ (see **8.164** 1).

1. $$\begin{aligned}\wp\left(\frac{u}{\sqrt{e_1-e_2}}\right) &= e_1 + (e_1-e_3)\frac{\operatorname{cn}^2(u;k)}{\operatorname{sn}^2(u;k)}\\ &= e_2 + (e_1-e_3)\frac{\operatorname{dn}^2(u;k)}{\operatorname{sn}^2(u;k)}\\ &= e_3 + (e_1-e_3)\frac{1}{\operatorname{sn}^2(u;k)}\end{aligned}$$

SI 145(5), ZH 120(197–199)a

2. $$\omega_1 = \frac{\boldsymbol{K}}{\sqrt{e_1-e_3}}, \qquad \omega_2 = \frac{i\boldsymbol{K}'}{\sqrt{e_1-e_3}},$$ SI 154(29)

where

3. $$k = \sqrt{\frac{e_2-e_3}{e_1-e_3}}, \qquad k' = \sqrt{\frac{e_1-e_2}{e_1-e_3}}$$ SI 145(7)

For $\Delta < 0$ (see **8.164** 2)

4. $$\wp\left(\frac{u}{\sqrt[4]{9\alpha^2+\beta^2}}\right) = e_2 + \sqrt{9\alpha^2+\beta^2}\,\frac{1+\operatorname{cn}(2u;k)}{1-\operatorname{cn}(2u;k)};$$ SI 147(12)

5. $$\omega' = \frac{\boldsymbol{K} - i\boldsymbol{K}'}{2\sqrt{9\alpha^2+\beta^2}}, \qquad \omega'' = \frac{\boldsymbol{K} + i\boldsymbol{K}'}{\sqrt[4]{9\alpha^2+\beta^2}},$$ SI 153(28)

where

6.[11] $$k = \sqrt{\frac{1}{2} - \frac{3e_2}{4\sqrt{9\alpha^2+\beta^2}}}; \qquad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{4\sqrt{9\alpha^2+\beta^2}}}$$ SI 147

For $\Delta = 0$, all the roots e_1, e_2, and e_3 are real, and if $g_2 g_3 \neq 0$, two of them are equal to each other. If $e_1 = e_2 \neq e_3$, then

7. $\wp(u) = \frac{3g_3}{g_2} - \frac{9g_3}{2g_2}\coth^2\left(u\sqrt{-\frac{9g_3}{2g_2}}\right)$ SI 148

If $e_1 \neq e_2 = e_3$, then

8. $\wp(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2}\frac{1}{\sin^2\left(u\sqrt{\frac{9g_3}{2g_2}}\right)}$ SI 149

If $g_2 = g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and

9. $\wp(u) = \frac{1}{u^2}$ SI 149

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.171 Definitions:

1. $\zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2}\right) dz$ SI 181(45)

2. $\sigma(u) = u\exp\left\{\int_0^u \left(\wp(z) - \frac{1}{z^2}\right) dz\right\}$ SI 181(46)

8.172 Series and infinite-product representation

1. $\zeta(u) = \frac{1}{u} + {\sum_{m,n}}' \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{\left(2m\omega_1 - 2n\omega_2\right)^2}\right)$ SI 307(8)

2. $\sigma(u) = u {\prod_{mn,}}' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2}\right)\exp\left\{\frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2\left(2m\omega_1 + 2n\omega_2\right)^2}\right\}$ SI 308(9)

8.173

1. $\zeta(u) = u - \frac{g_2 u^3}{2^2\cdot 3\cdot 5} - \frac{g_3 u^5}{2^2\cdot 5\cdot 7} - \frac{g_2^2 u^7}{2^4\cdot 3\cdot 5^2\cdot 7} - \frac{3g_2 g_3 u^9}{2^4\cdot 5\cdot 7\cdot 9\cdot 11} - \cdots$ SI 181(49)

2. $\sigma(u) = u - \frac{g_2 u^5}{2^4\cdot 3\cdot 5} - \frac{g_3 u^7}{2^3\cdot 3\cdot 5\cdot 7} - \frac{g_2^2 u^9}{2^9\cdot 3^2\cdot 5\cdot 7} - \frac{3g_2 g_3 u^{11}}{2^7\cdot 3^2\cdot 5^2\cdot 7\cdot 11} - \cdots$ SI 181(49)

8.174 $\zeta(u) = \frac{\zeta\left(\omega_1\right)}{\omega_1}u + \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{\pi}{2\omega_1}\sum_{n=1}^{\infty}\left\{\cot\left(\frac{\pi u}{2\omega_1} + n\pi\frac{\omega_2}{\omega_1}\right)\right.$

$\left. + \cot\left(\frac{\pi u}{2\omega_1} - n\pi\frac{\omega_2}{\omega_1}\right)\right\}$ MO 154

$= \frac{\zeta\left(\omega_1\right)}{\omega_1}u + \frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1 - q^{2n}}\sin\frac{\pi n u}{\omega_1}$ MO 155

Functional relations and properties

8.175 $\zeta(u) = -\zeta(-u), \quad \sigma(u) = -\sigma(-u)$ SI 181

8.176

1. $\zeta\left(u + 2\omega_1\right) = \zeta(u) + 2\zeta\left(\omega_1\right)$ SI 184(57)

2. $\zeta\left(u+2\omega_2\right)=\zeta(u)+2\zeta\left(\omega_2\right)$ SI 184(57)

3. $\sigma\left(u+2\omega_1\right)=-\sigma(u)\exp\left\{2\left(u+\omega_1\right)\zeta\left(\omega_1\right)\right\}.$ SI 185(60)

4. $\sigma\left(u+2\omega_2\right)=-\sigma(u)\exp\left\{2\left(u+\omega_2\right)\zeta\left(\omega_2\right)\right\}.$ SI 185(60)

5. $\omega_2\,\zeta\left(\omega_1\right)-\omega_1\,\zeta\left(\omega_2\right)=\frac{\pi}{2}i$ SI 186(62)

8.177

1. $\zeta(u+v)-\zeta(u)-\zeta(v)=\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$ SI 182(53)

2. $\wp(u)-\wp(v)=-\frac{\sigma(u-v)\,\sigma\left(u+v\right)}{\sigma^2(u)\,\sigma^2(v)}$ SI 183(54)

3. $\zeta(u-v)+\zeta(u+v)-2\zeta(u)=\frac{\wp'(u)}{\wp(u)-\wp(v)}$ SI 182(51)

8.178

1. $\zeta\left(u;\omega_1,\omega_2\right)=t\,\zeta\left(tu;t\omega_1,t\omega_2\right)$ MO 154

2.[8] $\sigma\left(u;\omega_1,\omega_2\right)=t^{-1}\,\sigma\left(tu;t\omega_1,t\omega_2\right)$ MO 156

For the indefinite integrals of Weierstrass elliptic functions, see **5.14**.

8.18–8.19 Theta functions

8.180 *Theta functions* are defined as the sums (for $|q|<1$) of the following series:

1. $\vartheta_4(u)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2nui}=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nu$ WH

2. $\vartheta_1(u)=\frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)ui}=2\sum_{n=1}^{\infty}(-1)^{n+1}q^{\left(n-\frac{1}{2}\right)^2}\sin(2n-1)u$ WH

3.[11] $\vartheta_2(u)=\sum_{n=-\infty}^{\infty}q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)ui}=2\sum_{n=1}^{\infty}q^{\left(n-\frac{1}{2}\right)^2}\cos(2n-1)u$ WH

4. $\vartheta_3(u)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2nui}=1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nu$ WH

The notations $\vartheta(u,q)$ and $\vartheta\left(u\mid\tau\right)$, where τ and q are related by $q=e^{i\pi\tau}$, are also used. Here, q is called the *nome* of the theta function and τ its *parameter*.

8.181 Representation of theta functions in terms of infinite products

1. $\vartheta_4(u)=\prod_{n=1}^{\infty}\left(1-2q^{2n-1}\cos 2u+q^{2(2n-1)}\right)\left(1-q^{2n}\right)$ SI 200(9), ZH 90(9)

2. $\vartheta_3(u)=\prod_{n=1}^{\infty}\left(1+2q^{2n-1}\cos 2u+q^{2(2n-1)}\right)\left(1-q^{2n}\right)$ SI 200(9), ZH 90(9)

3. $$\vartheta_1(u) = 2\sqrt[4]{q}\sin u \prod_{n=1}^{\infty}\left(1 - 2q^{2n}\cos 2u + q^{4n}\right)\left(1 - q^{2n}\right)$$ SI 200(9), ZH 90(9)

4.[8] $$\vartheta_2(u) = 2\sqrt[4]{q}\cos u \prod_{n=1}^{\infty}\left(1 + 2q^{2n}\cos 2u + q^{4n}\right)\left(1 - q^{2n}\right)$$ SI 200(0), ZH 90(9)

Functional relations and properties

8.182 Quasiperiodicity. Suppose that $q = e^{\pi\tau i}\,(\operatorname{Im}\tau > 0)$. Then, theta functions that are periodic functions of u are called *quasiperiodic functions* of τ and u. This property follows from the equations

1. $\vartheta_4(u+\pi) = \vartheta_4(u)$ SI 200(10)
2. $\vartheta_4(u+\tau\pi) = -\frac{1}{q}e^{-2iu}\,\vartheta_4(u)$ SI 200(10)
3. $\vartheta_1(u+\pi) = -\vartheta_1(u)$ SI 200(10)
4. $\vartheta_1(u+\tau\pi) = -\frac{1}{q}e^{-2iu}\,\vartheta_1(u)$ SI 200(10)
5. $\vartheta_2(u+\pi) = -\vartheta_2(u)$ SI 200(10)
6. $\vartheta_2(u+\tau\pi) = \frac{1}{q}e^{-2iu}\,\vartheta_2(u)$ SI 200(10)
7. $\vartheta_3(u+\pi) = \vartheta_3(u)$ SI 200(10)
8. $\vartheta_3(u+\tau\pi) = \frac{1}{q}e^{-2iu}\,\vartheta_3(u)$ SI 200(10)

8.183

1. $\vartheta_4\left(u+\frac{1}{2}\pi\right) = \vartheta_3(u)$ WH
2. $\vartheta_1\left(u+\frac{1}{2}\pi\right) = \vartheta_2(u)$ WH
3. $\vartheta_2\left(u+\frac{1}{2}\pi\right) = -\vartheta_1(u)$ WH
4. $\vartheta_3\left(u+\frac{1}{2}\pi\right) = \vartheta_4(u)$ WH
5. $\vartheta_4\left(u+\frac{1}{2}\pi\tau\right) = iq^{-1/4}e^{-iu}\,\vartheta_1(u)$ WH
6. $\vartheta_1\left(u+\frac{1}{2}\pi\tau\right) = iq^{-1/4}e^{-iu}\,\vartheta_4(u)$ WH
7. $\vartheta_2\left(u+\frac{1}{2}\pi\tau\right) = q^{-1/4}e^{-iu}\,\vartheta_3(u)$ WH
8. $\vartheta_3\left(u+\frac{1}{2}\pi\tau\right) = q^{-1/4}e^{-iu}\,\vartheta_2(u)$ WH

8.184 Even and odd theta functions

1. $\vartheta_1(-u) = -\vartheta_1(u)$ WH
2. $\vartheta_2(-u) = \vartheta_2(u)$ WH
3. $\vartheta_3(-u) = \vartheta_3(u)$ WH
4. $\vartheta_4(-u) = \vartheta_4(u)$ WH

8.185 $\vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u)$ WH

8.186[7] Considering the theta functions as functions of two independent variables u and τ, we have

$$\pi i \frac{\partial^2 \vartheta_k(u \mid \tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u \mid \tau)}{\partial \tau} = 0 \qquad [k = 1, 2, 3, 4]$$ WH

8.187 We denote the partial derivatives of the theta functions with respect to u by a prime and consider them as functions of the single argument u. Then,

1. $\vartheta_1'(0) = \vartheta_2(0)\,\vartheta_3(0)\,\vartheta_4(0)$ WH

2. $$\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}$$ WH

8.188 $\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)\,\vartheta_4(0) = \frac{1}{2}\vartheta_1(2u)\,\vartheta_2(0)\,\vartheta_3(0)\,\vartheta_4(0)$ WH

8.189 The zeros of the theta functions:

1.[8] $\vartheta_4(u) = 0$ for $u = 2m\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2}$ SI 201

2.[10] $\vartheta_1(u) = 0$ for $u = 2m\frac{\pi}{2} + 2n\frac{\pi\tau}{2}$ SI 201

3. $\vartheta_2(u) = 0$ for $u = (2m-1)\frac{\pi}{2} + 2n\frac{\pi\tau}{2}$ SI 201

4. $\vartheta_3(u) = 0$ for $u = (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2}$ [m and n are integers or zero] SI 201

For integrals of theta functions, see **6.16**.

8.191 Connections with the Jacobian elliptic functions:

For $\tau = i\frac{\boldsymbol{K}'}{\boldsymbol{K}}$, i.e. for $q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right)$,

1. $$\operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}$$ SI 206(22), SI 209(35)

2. $$\operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}$$ SI 207(23), SI 209(35)

3. $$\operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi u}{2\boldsymbol{K}}\right)}{\vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right)} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}$$ SI 207(24), SI 209(35)

8.192 Series representation of the functions H, H_1, Θ, Θ_1.

In these formulas, $q = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right)$.

1. $$\Theta(u) = \vartheta_4\left(\frac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos\frac{n\pi u}{\boldsymbol{K}}$$ SI 207(25), SI 212(42)

2. $$H(u) = \vartheta_1\left(\frac{\pi u}{2\boldsymbol{K}}\right) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[4]{q^{(2n+1)^2}}\sin(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ SI 207(25), SI 212(43)

3. $$\Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2\boldsymbol{K}}\right) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos\frac{n\pi u}{\boldsymbol{K}}$$ SI 207(25), SI 212(45)

4. $$H_1(u) = \vartheta_2\left(\frac{\pi u}{2\boldsymbol{K}}\right) = 2\sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1)\frac{\pi u}{2\boldsymbol{K}}$$ SI 207(25), SI 212(44)

8.193 Connections with the Weierstrass elliptic functions

1. $$\wp(u) = e_1 + \left[\frac{H_1\left(u\sqrt{\lambda}\right) H'(0)}{H_1(0)\, H\left(u\sqrt{\lambda}\right)}\right]^2 \lambda = e_2 + \left[\frac{\Theta_1\left(u\sqrt{\lambda}\right) H'(0)}{\Theta_1(0)\, H'\left(u\sqrt{\lambda}\right)}\right]^2 \lambda = e_3 + \left[\frac{\Theta\left(u\sqrt{\lambda}\right) H'(0)}{\Theta(0)\, H'\left(u\sqrt{\lambda}\right)}\right]^2 \lambda$$ SI 235(77,78)

2. $$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda}\frac{H'\left(u\sqrt{\lambda}\right)}{H\left(u\sqrt{\lambda}\right)}$$ SI 234(73)

3. $$\sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H\left(u\sqrt{\lambda}\right)}{H'(0)}$$ SI 234(72)

where

$$\lambda = e_1 - e_3; \qquad \eta_1 = \zeta\left(\omega_1\right) = -\frac{\omega_1 \lambda}{3} \frac{H'''(0)}{H'(0)}$$ SI 236

8.194 The connection with elliptic integrals:

1. $$E(u,k) = u - u\frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)}$$ SI 228(65)

2.[11] $$\Pi\left(u, -k^2 \sin^2 a, k\right) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2 a \operatorname{sn}^2\varphi} = u + \frac{\operatorname{sn} a}{\operatorname{cn} a \operatorname{dn} a}\left[\frac{\Theta'(a)}{\Theta(a)}u + \frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}\right]$$ SI 228(65)

q-series and products, $q = \exp\left(-\pi \frac{\mathsf{K}'}{\mathsf{K}}\right)$

8.195 $$\frac{\pi}{2}\left[1 + 2\sum_{n=1}^{\infty} q^{n^2}\right]^2 = \boldsymbol{K} = \frac{\pi}{2}\,\Theta^2\left(\boldsymbol{K}\right)$$ (cf. **8.197** 1) SI 219

8.196 $$\boldsymbol{E} = \boldsymbol{K} - \boldsymbol{K}\frac{\Theta''(0)}{\Theta(0)} = \boldsymbol{K} - \frac{2\pi^2}{\boldsymbol{K}} \frac{\sum_{n=1}^{\infty}(-1)^{n+1} n^2 q^{n^2}}{1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}}$$ SI 230(67)

8.197

1. $$1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2\boldsymbol{K}}{\pi}} = \vartheta_3(0)$$ (cf. **8.195**) WH

2. $$\sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{k\boldsymbol{K}}{2\pi}} = \frac{1}{2}\,\vartheta_2(0)$$ WH

3. $$4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4 = k$$ SI 206(17, 18)

4. $$\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4 = k'$$ SI 206(19, 20)

5. $$2\sqrt[4]{q}\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right)^2 = 2\sqrt{k}\frac{\boldsymbol{K}}{\pi}$$ WH

6. $$\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2 = 2\sqrt{k'}\frac{\boldsymbol{K}}{\pi}$$ WH

8.198

1. $$\lambda = \frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1+2\sum_{n=1}^{\infty} q^{4n^2}}$$ [for $0 < k < 1$, we have $0 < \lambda < \frac{1}{2}$] WH

The series

2. $$q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \dots$$ WH

is used to determine q from the given modulus k.

8.199[10] Identities involving products of theta functions

1. $$\vartheta_1(x,q)\,\vartheta_1(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right) - \vartheta_2\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right)$$ LW 7(1.4.7)

2. $$\vartheta_1(x,q)\,\vartheta_2(y,q) = \vartheta_1\left(x+y,q^2\right)\vartheta_4\left(x-y,q^2\right) + \vartheta_4\left(x+y,q^2\right)\vartheta_1\left(x-y,q^2\right)$$ LW 8(1.4.8)

3. $$\vartheta_2(x,q)\,\vartheta_2(y,q) = \vartheta_2\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) + \vartheta_3\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.9)

4. $$\vartheta_3(x,q)\,\vartheta_3(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) + \vartheta_2\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.10)

5. $$\vartheta_3(x,q)\,\vartheta_4(y,q) = \vartheta_4\left(x+y,q^2\right)\vartheta_4\left(x-y,q^2\right) - \vartheta_1\left(x+y,q^2\right)\vartheta_1\left(x-y,q^2\right)$$ LW 8(1.4.11)

6. $$\vartheta_4(x,q)\,\vartheta_4(y,q) = \vartheta_3\left(x+y,q^2\right)\vartheta_3\left(x-y,q^2\right) - \vartheta_2\left(x+y,q^2\right)\vartheta_2\left(x-y,q^2\right)$$ LW 8(1.4.12)

7. $$\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_4^2(0) = \vartheta_3^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_3^2(y) = \vartheta_1^2(x)\,\vartheta_4^2(y) - \vartheta_4^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.16)

8. $$\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_3^2(y) = \vartheta_2^2(x)\,\vartheta_4^2(y) - \vartheta_3^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.17)

9. $$\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_3^2(y) - \vartheta_1^2(x)\,\vartheta_2^2(y) = \vartheta_3^2(x)\,\vartheta_4^2(y) - \vartheta_2^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.18)

10. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_4^2(0) = \vartheta_4^2(x)\,\vartheta_4^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y)$$ LW 8(1.4.15)

11. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_4^2(0) = \vartheta_3^2(x)\,\vartheta_3^2(y) - \vartheta_2^2(x)\,\vartheta_2^2(y) = \vartheta_4^2(x)\,\vartheta_4^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y)$$ LW 9(1.4.19)

12. $$\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_3^2(y) - \vartheta_3^2(x)\,\vartheta_1^2(y) = \vartheta_4^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.23)

13. $$\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_3^2(0) = \vartheta_2^2(x)\,\vartheta_3^2(y) - \vartheta_4^2(x)\,\vartheta_1^2(y) = \vartheta_3^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.24)

14. $$\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_1^2(y) + \vartheta_3^2(x)\,\vartheta_3^2(y) = \vartheta_2^2(x)\,\vartheta_2^2(y) + \vartheta_4^2(x)\,\vartheta_4^2(y)$$ LW 9(1.4.25)

15. $$\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(y) + \vartheta_3^2(x)\,\vartheta_4^2(y) = \vartheta_2^2(x)\,\vartheta_1^2(y) + \vartheta_4^2(x)\,\vartheta_3^2(y)$$ LW 9(1.4.26)

16. $\vartheta_1(x+y)\,\vartheta_1(x-y)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_1^2(y) = \vartheta_4^2(x)\,\vartheta_3^2(y) - \vartheta_3^2(x)\,\vartheta_4^2(y)$ LW 9(1.4.30)

17. $\vartheta_2(x+y)\,\vartheta_2(x-y)\,\vartheta_2^2(0) = \vartheta_2^2(x)\,\vartheta_2^2(y) - \vartheta_1^2(x)\,\vartheta_1^2(y) = \vartheta_3^2(x)\,\vartheta_3^2(y) - \vartheta_4^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.31)

18. $\vartheta_3(x+y)\,\vartheta_3(x-y)\,\vartheta_2^2(0) = \vartheta_3^2(x)\,\vartheta_2^2(y) + \vartheta_4^2(x)\,\vartheta_1^2(y) = \vartheta_2^2(x)\,\vartheta_3^2(y) + \vartheta_1^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.32)

19. $\vartheta_4(x+y)\,\vartheta_4(x-y)\,\vartheta_2^2(0) = \vartheta_4^2(x)\,\vartheta_2^2(y) + \vartheta_3^2(x)\,\vartheta_1^2(y) = \vartheta_1^2(x)\,\vartheta_3^2(y) + \vartheta_2^2(x)\,\vartheta_4^2(y)$ LW 10(1.4.33)

20. $\vartheta_3^2(x)\,\vartheta_3^2(0) = \vartheta_4^2(x)\,\vartheta_4^2(0) + \vartheta_2^2(x)\,\vartheta_2^2(0)$ LW 11(1.4.49)

21. $\vartheta_4^2(x)\,\vartheta_3^2(0) = \vartheta_1^2(x)\,\vartheta_2^2(0) + \vartheta_3^2(x)\,\vartheta_4^2(0)$ LW 11(1.4.50)

22. $\vartheta_4^2(x)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_3^2(0) + \vartheta_2^2(x)\,\vartheta_4^2(0)$ LW 11(1.4.51)

23. $\vartheta_3^2(x)\,\vartheta_2^2(0) = \vartheta_1^2(x)\,\vartheta_4^2(0) + \vartheta_2^2(x)\,\vartheta_3^2(0)$ LW 11(1.4.52)

24.[8] $\vartheta_3^4(x) = \vartheta_2^4(0) + \vartheta_4^4(0)$ LW 11(1.4.53)

8.199(2)[10] Derivatives of ratios of theta functions

1. $\frac{d}{dx}\,(\vartheta_1\,/\,\vartheta_4) = \vartheta_4^2(0)\,\vartheta_2(x)\,\vartheta_3(x)/\,\vartheta_4^2(x)$ LW 19(1.9.3)

2. $\frac{d}{dx}\,(\vartheta_2\,/\,\vartheta_4) = -\,\vartheta_3^2(0)\,\vartheta_1(x)\,\vartheta_3(x)/\,\vartheta_4^2(x)$ LW 19(1.9.6)

3. $\frac{d}{dx}\,(\vartheta_3\,/\,\vartheta_4) = -\,\vartheta_2^2(0)\,\vartheta_1(x)\,\vartheta_2(x)/\,\vartheta_4^2(x)$ LW 19(1.9.7)

4. $\frac{d}{dx}\,(\vartheta_1\,/\,\vartheta_3) = \vartheta_3^2(0)\,\vartheta_2(x)\,\vartheta_4(x)/\,\vartheta_3^2(x)$ LW 19(1.9.8)

5. $\frac{d}{dx}\,(\vartheta_2\,/\,\vartheta_3) = -\,\vartheta_4^2(0)\,\vartheta_1(x)\,\vartheta_4(x)/\,\vartheta_3^2(x)$ LW 19(1.9.9)

6. $\frac{d}{dx}\,(\vartheta_1\,/\,\vartheta_2) = \vartheta_2^2(0)\,\vartheta_3(x)\,\vartheta_4(x)/\,\vartheta_2^2(x)$ LW 19(1.9.10)

7. $\frac{d}{dx}\,(\vartheta_4\,/\,\vartheta_1) = -\,\vartheta_4^2(0)\,\vartheta_2(x)\,\vartheta_3(x)/\,\vartheta_1^2(x)$ LW 19(1.9.11)

8. $\frac{d}{dx}\,(\vartheta_4\,/\,\vartheta_2) = \vartheta_3^2(0)\,\vartheta_1(x)\,\vartheta_3(x)/\,\vartheta_2^2(x)$ LW 20(1.9.12)

9. $\frac{d}{dx}\,(\vartheta_4\,/\,\vartheta_3) = \vartheta_2^2(0)\,\vartheta_1(x)\,\vartheta_2(x)/\,\vartheta_3^2(x)$ LW 20(1.9.13)

10. $\frac{d}{dx}\,(\vartheta_3\,/\,\vartheta_1) = -\,\vartheta_3^2(0)\,\vartheta_2(x)\,\vartheta_4(x)/\,\vartheta_1^2(x)$ LW 20(1.9.14)

11. $\frac{d}{dx}\,(\vartheta_3\,/\,\vartheta_2) = \vartheta_4^2(0)\,\vartheta_1(x)\,\vartheta_4(x)/\,\vartheta_2^2(x)$ LW 20(1.9.15)

12. $\frac{d}{dx}\,(\vartheta_2\,/\,\vartheta_1) = -\,\vartheta_2^2(0)\,\vartheta_3(x)\,\vartheta_4(x)/\,\vartheta_1^2(x)$ LW 20(1.9.16)

8.199(3)[10] Derivatives of theta functions

1. $$\frac{d}{du}\ln\vartheta_1(u) = \cot u + 4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n}}{1-2q^{2n}\cos 2u+q^{4n}}$$

2. $$\frac{d}{du}\ln\vartheta_2(u) = -\tan u - 4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n}}{1+2q^{2n}\cos 2u+q^{4n}}$$

3. $$\frac{d}{du}\ln\vartheta_3(u) = -4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+2q^{2n}\cos 2u+q^{4n-2}}$$

4. $$\frac{d}{du}\ln\vartheta_4(u) = 4\sin 2u\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-2q^{2n}\cos 2u+q^{4n-2}}$$

5. $$\frac{d^2}{du^2}\ln\vartheta_2(u) = -\sum_{n=-\infty}^{\infty}\operatorname{sech}^2\{i(u+n\pi\tau)\}$$

8.2 The Exponential Integral Function and Functions Generated by It

8.21 The exponential integral function Ei(x)

8.211

1. $$\operatorname{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}\,dt = \int_{-\infty}^{x}\frac{e^{t}}{t}\,dt = \operatorname{li}(e^x) \qquad [x<0]$$

2.[11] $$\operatorname{Ei}(x) = -\lim_{\varepsilon\to 0+}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}\,dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}\,dt\right] = \operatorname{PV}\int_{-\infty}^{x}\frac{e^{t}}{t}\,dt \qquad [x>0]$$

3.[7] $$\operatorname{Ei}(x) = \tfrac{1}{2}\{\operatorname{Ei}(x+i0)+\operatorname{Ei}(x-i0)\} \qquad [x>0]$$ ET I 386

8.212

1.[8] $$\operatorname{Ei}(-x) = \boldsymbol{C} + \ln x + \int_0^x \frac{e^{-t}-1}{t}\,dt \qquad [x>0]$$ NT 11(1)

$$= \boldsymbol{C} + e^{-x}\ln x + \int_0^x e^{-t}\ln t\,dt \qquad [x>0]$$ NT 11(10)

2.[7] $$\operatorname{Ei}(x) = e^x\left[\frac{1}{x} + \int_0^{\infty}\frac{e^{-t}\,dt}{(x-t)^2}\right] \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$

3. $$\operatorname{Ei}(-x) = e^{-x}\left[-\frac{1}{x} + \int_0^{\infty}\frac{e^{-t}\,dt}{(x+t)^2}\right] \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$ LA 281(28)

4. $$\operatorname{Ei}(\pm x) = \pm e^{\pm x}\int_0^1 \frac{dt}{x\pm\ln t} \qquad [x>0] \qquad (\text{cf. } \mathbf{8.211}\ 1)$$

5. $$\operatorname{Ei}(\pm xy) = \pm e^{\pm xy}\int_0^{\infty}\frac{e^{-xt}}{y\mp t}\,dt \qquad [\operatorname{Re} y>0, \quad x>0]$$ NT 19(11)

6. $$\operatorname{Ei}(\pm x) = -e^{\pm x}\int_0^{\infty}\frac{e^{-it}}{t\pm ix}\,dt \qquad [x>0]$$ NT 23(2, 3)

7.[8] $$\operatorname{Ei}(xy) = e^{xy}\int_0^1 \frac{t^{y-1}}{x+\ln t}\,dt$$ LA 282(44)a

8. $$\mathrm{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t}\, dt$$ LA 282(45)a

$$= x^{-1} e^{-xy} \left[\int_0^1 \frac{t^{x-1}}{(y - \ln t)^2}\, dt - y^{-1} \right] \qquad [x > 0, \quad y > 0]$$ LA 283(47)a

9. $$\mathrm{Ei}(x) = e^x \int_1^\infty \frac{1}{x - \ln t} \frac{dt}{t^2} \qquad [x > 0]$$ LA 283(48)

10. $$\mathrm{Ei}(-x) = -e^{-x} \int_1^\infty \frac{1}{x + \ln t} \frac{dt}{t^2} \qquad [x > 0]$$ LA 283(48)

11. $$\mathrm{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t + x \sin t}{t^2 + x^2}\, dt \qquad [x > 0]$$ NT 23(6)

12. $$\mathrm{Ei}(-x) = -e^{-x} \int_0^\infty \frac{t \cos t - x \sin t}{t^2 + x^2}\, dt \qquad [x < 0]$$ NT 23(6)

13. $$\mathrm{Ei}(-x) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \arctan \frac{t}{x}\, dt \qquad [\operatorname{Re} x > 0]$$ NT 25(13)

14. $$\mathrm{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^\infty \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t\, dt \qquad [x > 0]$$ NT 26(7)

15. $$\mathrm{Ei}(x) = 2 \ln x - \frac{2e^x}{\pi} \int_0^\infty \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t\, dt \qquad [x > 0]$$ NT 27(8)

16. $$\mathrm{Ei}(-x) = -x \int_1^\infty e^{-tx} \ln t\, dt \qquad [x > 0]$$ NT 32(12)

See also **3.327**, **3.881** 8, **3.916** 2 and 3, **4.326** 1, **4.326** 2, **4.331** 2, **4.351** 3, **4.425** 3, **4.581**. For integrals of the exponential integral function, see **6.22–6.23**, **6.78**.

Series and asymptotic representations

8.213

1. $$\mathrm{li}(x) = \boldsymbol{C} + \ln(-\ln x) + \sum_{k=1}^\infty \frac{(\ln x)^k}{k \cdot k!} \qquad [0 < x < 1]$$ NT 3(9)

2. $$\mathrm{li}(x) = \boldsymbol{C} + \ln \ln x + \sum_{k=1}^\infty \frac{(\ln x)^k}{k \cdot k!} \qquad [x > 1]$$ NT 3(10)

8.214

1. $$\mathrm{Ei}(x) = \boldsymbol{C} + \ln(-x) + \sum_{k=1}^\infty \frac{x^k}{k \cdot k!} \qquad [x < 0]$$

2. $$\mathrm{Ei}(x) = \boldsymbol{C} + \ln x + \sum_{k=1}^\infty \frac{x^k}{k \cdot k!} \qquad [x > 0]$$

3. $$\mathrm{Ei}(x) - \mathrm{Ei}(-x) = 2x \sum_{k=0}^\infty \frac{x^{2k}}{(2k+1)(2k+1)!} \qquad [x > 0]$$ NT 39(13)

8.215[7] $$\mathrm{Ei}(z) = \frac{e^z}{z}\left[\sum_{k=0}^{n}\frac{k!}{z^k} + R_n(z)\right] \qquad |R_n(z)| = O\left(|z|^{-n-1}\right)$$

$[z \to \infty, \quad |\arg(-z)| \le \pi - \delta; \quad \delta > 0 \text{ small}], \qquad |R_n(z)| \le (n+1)!|z|^{-n-1} \quad [\mathrm{Re}\, z \le 0]$

8.216[7] $$\mathrm{Ei}(nx) - \mathrm{Ei}(-nx) = e^{nx'}\left(\frac{1}{nx} + \frac{1}{n^2x^2} + \frac{k_n}{n^3x^3}\right),$$

where $x' = x\,\mathrm{sign}\,\mathrm{Re}(x), \quad k_n = O(1),$ and $n \to \infty$ NT 39(15)

8.217 Functional relations:

1. $$e^{x'}\,\mathrm{Ei}\,(-x') - e^{-x'}\,\mathrm{Ei}\,(x') = -2\int_0^\infty \frac{x'\sin t}{t^2+x^2}\,dt$$ NT 24(11)

$$= \frac{4}{\pi}\int_0^\infty \frac{x'\cos t}{t^2+x^2}\ln t\,dt - 2e^{-x'}\ln x' \qquad [x' = x\,\mathrm{sign}\,\mathrm{Re}\,x]$$ NT 27(9)

2. $$e^{x'}\,\mathrm{Ei}\,(-x') + e^{-x'}\,\mathrm{Ei}\,(x') = -2\int_0^\infty \frac{t\cos t}{t^2+x^2}\,dt = 2e^{-x'}\ln x' - \frac{4}{\pi}\int_0^\infty \frac{t\sin t}{t^2+x^2}\ln t\,dt$$

$[x' = x\,\mathrm{sign}\,\mathrm{Re}\,x]$ NT 24(10), NT 27(10)

3. $$\mathrm{Ei}(-x) - \mathrm{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi}\int_0^\infty \frac{\cos t}{t}\arctan\frac{t\left(x-\frac{1}{x}\right)}{1+t^2}\,dt$$

$[\mathrm{Re}\, x > 0]$ NT 25(14)

4. $$\mathrm{Ei}(-\alpha x)\,\mathrm{Ei}(-\beta x) - \ln(\alpha\beta)\,\mathrm{Ei}[-(\alpha+\beta)x] = e^{-(\alpha+\beta)x}\int_0^\infty \frac{e^{-tx}\ln[(\alpha+t)(\beta+t)]}{t+\alpha+\beta}\,dt$$ NT 32(9)

See also **3.723** 1 and 5, **3.742** 2 and 4, **3.824** 4, **4.573** 2.

- For a connection with a confluent hypergeometric function, see **9.237**.
- For integrals of the exponential integral function, see **5.21**, **5.22**, **5.23**, **6.22**, and **6.23**.

8.218 Two numerical values:

1. $\mathrm{Ei}(-1) = -0.219\ 383\ 934\ 395\ 520\ 273\ 665\ldots$ NT 89
2. $\mathrm{Ei}(1) = 1.895\ 117\ 816\ 355\ 936\ 755\ 478\ldots$ NT 89

8.219* Definite integrals of exponential functions

1.* $$\int_0^\infty \mathrm{Ei}^2(x)e^{-2x}\,dx = \frac{\pi^2}{4}$$

2.* $$\int_0^\infty \mathrm{Ei}^2(-x)e^{2x}\,dx = \frac{\pi^2}{4}$$

3.* $$\int_0^\infty \mathrm{Ei}(x)\,\mathrm{Ei}(-x)\,dx = 0$$

8.22 The hyperbolic sine integral $\operatorname{shi} x$ and the hyperbolic cosine integral $\operatorname{chi} x$

8.221

1. $$\operatorname{shi} x = \int_0^x \frac{\sinh t}{t}\,dt = -i\left[\frac{\pi}{2} + \operatorname{si}(ix)\right]$$ (see **8.230** 1) EH II 146(17)

2.[11] $$\operatorname{chi} x = \boldsymbol{C} + \ln x + \int_0^x \frac{\cosh t - 1}{t}\,dt$$ EH II 146(18)

8.23 The sine integral and the cosine integral: $\operatorname{si} x$ and $\operatorname{ci} x$

8.230

1.[10] $$\operatorname{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,dt = -\frac{\pi}{2} + \operatorname{Si}(x), \text{ where } \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$ NT 11(3)

2.[10] $$\operatorname{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = \boldsymbol{C} + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt$$ [ci(x) is also written Ci(x)] NT 11(2)

8.231

1. $$\operatorname{si}(xy) = -\int_x^\infty \frac{\sin ty}{t}\,dt$$ NT 18(7)

2. $$\operatorname{ci}(xy) = -\int_x^\infty \frac{\cos ty}{t}\,dt$$ NT 18(6)

3. $$\operatorname{si}(x) = -\int_0^{\pi/2} e^{-x\cos t}\cos(x\sin t)\,dt$$ NT 13(26)

8.232

1. $$\operatorname{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^\infty \frac{(-1)^{k+1}x^{2k-1}}{(2k-1)(2k-1)!}$$ NT 7(4)

2.[7] $$\operatorname{ci}(x) = \boldsymbol{C} + \ln(x) + \sum_{k=1}^\infty (-1)^k \frac{x^{2k}}{2k(2k)!}$$ NT 7(3)

8.233

1. $$\operatorname{ci}(x) \pm i\operatorname{si}(x) = \operatorname{Ei}(\pm ix)$$ NT 6a

2. $$\operatorname{ci}(x) - \operatorname{ci}\left(xe^{\pm\pi i}\right) = \mp\pi i$$ NT 7(5)

3. $$\operatorname{si}(x) + \operatorname{si}(-x) = -\pi$$ NT 7(7)

8.234

1.[7] $$\operatorname{Ei}(-x) - \operatorname{ci}(x) = \int_0^{\pi/2} e^{-x\cos\varphi}\sin(s\sin\varphi)\,d\varphi$$ NT 13(27)

2. $$[\operatorname{ci}(x)]^2 + [\operatorname{si}(x)]^2 = -2\int_0^{\pi/2} \frac{\exp(-x\tan\varphi)\ln\cos\varphi}{\sin\varphi\cos\varphi}\,d\varphi$$ [$\operatorname{Re} x > 0$] (see also **4.366**) NT 32(11)

See also **3.341**, **3.351** 1 and 2, **3.354** 1 and 2, **3.721** 2 and 3, **3.722** 1, 3, 5 and 7, **3.723** 8 and 11, **4.338** 1, **4.366** 1.

8.235

1. $$\lim_{x\to+\infty} \left(x^{\varrho} \operatorname{si}(x)\right) = 0, \qquad \lim_{x\to+\infty} \left(x^{\varrho} \operatorname{ci}(x)\right) = 0 \qquad [\varrho < 1]$$ NT 38(5)

2. $$\lim_{x\to-\infty} \operatorname{si}(x) = -\pi, \qquad \lim_{x\to-\infty} \operatorname{ci}(x) = \pm\pi i$$ NT 38(6)

- For integrals of the sine integral and cosine integral, see **6.24–6.26**, **6.781**, **6.782**, and **6.783**.
- For indefinite integrals of the sine integral and cosine integral, see **5.3**.

8.24 The logarithm integral li(x)

8.240

1. $$\operatorname{li}(x) = \int_0^x \frac{dt}{\ln t} = \operatorname{Ei}(\ln x) \qquad [x < 1]$$ JA

2. $$\operatorname{li}(x) = \lim_{\varepsilon\to 0}\left[\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t}\right] = \operatorname{Ei}(\ln x) \qquad [x > 1]$$ JA

3. $$\operatorname{li}\left\{\exp\left(-xe^{\pm\pi i}\right)\right\} = \operatorname{Ei}\left(-xe^{\pm i\pi}\right) = \operatorname{Ei}(x \mp i0) = \operatorname{Ei}(x) \pm i\pi = \operatorname{li}(e^x) \pm i\pi \qquad [x > 0]$$ JA, NT 2(6)

Integral representations

8.241

1. $$\operatorname{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t}\, dt = x \ln\ln\frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t\, dt \qquad [x < 1]$$ LA 281(33)

2. $$\operatorname{li}(x) = x\int_0^1 \frac{dt}{\ln x + \ln t}$$ LA 280(22)
$$= \frac{x}{\ln x} + x\int_0^1 \frac{dt}{(\ln x + \ln t)^2}$$ LA 280(29)
$$= x\int_1^{\infty} \frac{1}{\ln x - \ln t}\frac{dt}{t^2} \qquad [x < 1]$$ LA 280(30)

3. $$\operatorname{li}(a^x) = \frac{1}{\ln a}\int_{-\infty}^{x} \frac{a^t}{t}\, dt \qquad [x > 0]$$

For integrals of the logarithm integral, see **6.21**

8.25 The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$ and $C(x)$, the error function $\operatorname{erf}(x)$, and the complementary error function $\operatorname{erfc}(x)$

8.250 Definition:

1.[11] $$\Phi(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$ (called the error function)

2. $$S(x) = \frac{2}{\sqrt{2\pi}}\int_0^x \sin t^2\, dt$$

3. $$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 \, dt$$

4.[11] $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ (called the complementary error function)

5.* $$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \sin\left(a\sqrt{x}\right) \, dx = -\sinh\left(a\sqrt{p}\right) + \frac{1}{2} e^{-a\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} - \sqrt{py}\right) + \frac{1}{2} e^{a\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right)$$

6.* $$\int_0^\infty \frac{e^{-(p+x)y}}{\pi(p+x)} \cos\left(a\sqrt{x}\right) \, dx = \frac{1}{\sqrt{\pi y}} \exp\left(-\frac{a^2}{4y} - py\right) - \frac{\sqrt{p}}{2} e^{-a\sqrt{p}} \, \Phi\left(\frac{a}{a\sqrt{y}} - \sqrt{py}\right) + \frac{\sqrt{p}}{2} e^{\sqrt{p}} \, \Phi\left(\frac{a}{2\sqrt{y}} + \sqrt{py}\right) - \sqrt{p} \cosh\left(a\sqrt{p}\right)$$
[$\operatorname{Re} p > 0$, a, b are real]

7.* $$\int_0^p \exp\left(-x^2\right) \Phi(p-x) \, dx = \int_0^p \exp\left(-x^2\right) \operatorname{erf}(p-x) \, dx = \frac{\sqrt{\pi}}{2} \left[\Phi\left(\frac{p}{\sqrt{2}}\right)\right]^2$$

8.* $$\int_0^p x^2 \exp\left(-x^2\right) \Phi(p-x) \, dx = \int_0^p x^2 \exp\left(-x^2\right) \operatorname{erf}(p-x) \, dx = \frac{\sqrt{\pi}}{4} \left[\Phi\left(\frac{p}{\sqrt{2}}\right)\right]^2 - \frac{p}{2\sqrt{2}} \, \Phi\left(-\frac{x^2}{2}\right) \operatorname{erf}\left(\frac{p}{\sqrt{2}}\right)$$

9.* $$\int_{(b-a)/\sqrt{2}}^{(b+a)/\sqrt{2}} \exp\left(-x^2\right) \Phi\left(b\sqrt{2} - x\right) \, dx + \int_{(a-b)/\sqrt{2}}^{(a+b)/\sqrt{2}} \exp\left(-x^2\right) \Phi\left(a\sqrt{2} - x\right) \, dx = \sqrt{\pi} \, \Phi(a) \, \Phi(b)$$

Integral representations

8.251

1. $$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} \, dt$$
(see also **3.361** 1)

2. $$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} \, dt$$

3. $$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} \, dt$$

8.252

1. $$\Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2 y^2} \, dt$$
$\left[\operatorname{Re} y^2 > 0\right]$

2. $$S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin\left(t^2 y^2\right) \, dt$$

3. $$C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos\left(t^2 y^2\right) \, dt$$

4. $$\Phi(xy) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2}\, ty\, dt}{\sqrt{t^2+x^2}} \qquad \left[\operatorname{Re} y^2 > 0\right]$$ NT 19(11)a

$$= 1 - \frac{2x}{\pi} e^{-x^2y^2} \int_0^\infty \frac{e^{-t^2y^2}\, dt}{t^2+x^2} \qquad \left[\operatorname{Re} y^2 > 0\right]$$ NT 19(13)a

5.[7] $$\Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2y^2} \sin(ty)\, dt \qquad \left[\operatorname{Re} x^2 > 0\right]$$ NT 28(3)a

6.[8] $$\Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} x e^{-\frac{y^2}{4}} \int_0^\infty e^{-t^2x^2-ty}\, dt \qquad \left[\operatorname{Re} x^2 > 0\right]$$ NT 27(1)a

See also **3.322**, **3.362** 2, **3.363**, **3.468**, **3.897**, **6.511** 4 and 5.

8.253[8] Series representations:

1.[11] $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} x\, F_1\left(1; \frac{3}{2}; x^2\right) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^\infty (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!}$$ NT 7(9)a

$$= \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^\infty \frac{2^k x^{2k+1}}{(2k+1)!!}$$ NT 10(11)a

2. $$S(x) = \frac{2}{\sqrt{2\pi}} \left(x \sin x^2\, F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right) - \frac{2}{3} x^3 \cos x^2\, F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right)\right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)}$$ NT 8(14)a

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\}$$ NT 10(13)a

3. $$C(x) = \frac{2}{\sqrt{2\pi}} \left(\frac{2}{3} x^3 \sin x^2\, F\left(1; \frac{7}{4}, \frac{5}{4}; -\frac{1}{4}x^2\right) - x \cos x^2\, F\left(1; \frac{5}{4}, \frac{3}{4}; -\frac{1}{4}x^2\right)\right)$$

$$= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)}$$ NT 8(13)a

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin^2 x \sum_{k=0}^\infty \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^\infty \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\}$$ NT 10(12)a

For the expansions in Bessel functions, see **8.515** 2, **8.515** 3.

Asymptotic representations

8.254[8] $$\Phi(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left[\sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2z^2)^k} + O\left(|z|^{-2n-z}\right) \right],$$

$$[z \to \infty, \quad |\arg(-z)| \le \pi - \delta; \quad \delta > 0 \text{ small}]$$

where

$$|R_n| < \frac{\Gamma\left(n+\frac{1}{2}\right)}{|x|^{n+\frac{1}{2}}} \cos\frac{\varphi}{2}, \quad x = |x| e^{i\varphi} \text{ and } \varphi^2 < \pi^2$$ NT 37(10)a

8.255

1. $$S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi x}} \cos x^2 + O\left(\frac{1}{x^2}\right) \qquad [x \to \infty]$$ MO 127a

2. $$C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}x} \sin x^2 + O\left(\frac{1}{x^2}\right)$$ $[x \to \infty]$ MO 127a

8.256 Functional relations:

1. $$C(z) + i\,S(z) = \sqrt{\frac{i}{2}}\,\Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{it^2}\,dt$$

2. $$C(z) - i\,S(z) = \frac{1}{\sqrt{2i}}\,\Phi\left(z\sqrt{i}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-it^2}\,dt$$

3. $$\left[\cos^2 u\,C(u) + \sin u^2\,S(u)\right] = \frac{1}{2}\left[\cos^2 u + \sin u^2\right] + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2\,dt$$ $[\operatorname{Re} u \geq 0]$ NT 28(6)a

4. $$\left[\cos^2 u\,S(u) - \sin u^2\,C(u)\right] = \frac{1}{2}\left[\cos^2 u - \sin u^2\right] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2\,dt$$ $[\operatorname{Re} u \geq 0]$ NT 28(5)a

5.[11] $$\left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 = \frac{2}{\pi} \int_0^{\pi/2} \frac{\exp\left(-x^2 \tan\varphi\right) \sin\frac{\varphi}{2} \sqrt{\cos\varphi}}{\sin 2\varphi}\,d\varphi$$ (see also **6.322**) NT 33(18)a

- For a connection with a confluent hypergeometric function, see **9.236**.
- For a connection with a parabolic cylinder function, see **9.254**.

8.257

1. $$\lim_{x\to+\infty} \left(x^\varrho \left[S(x) - \tfrac{1}{2}\right]\right) = 0$$ $[\varrho < 1]$ NT 38(11)

2. $$\lim_{x\to+\infty} \left(x^\varrho \left[C(x) - \tfrac{1}{2}\right]\right) = 0$$ $[\varrho < 1]$ NT 38(11)

3. $$\lim_{x\to+\infty} S(x) = \frac{1}{2}$$ NT 38(12)a

4. $$\lim_{x\to+\infty} C(x) = \frac{1}{2}$$ NT 38(12)a

- For integrals of the probability integral, see **6.28–6.31**.
- For integrals of Fresnel's sine integral and cosine integral, see **6.32**.

8.258[10] Integrals involving the complementary error function

1. $$\int_0^\infty \operatorname{erfc}^2(x) e^{-\beta x^2}\,dx = \frac{1}{\sqrt{\beta\pi}} \left(-\arccos\left(\frac{1}{1+\beta}\right) + 2\arctan\left(\sqrt{\beta}\right)\right)$$ $[\beta > 0]$

2. $$\int_0^\infty x \operatorname{erfc}^2(x) e^{-\beta x^2}\,dx = \frac{1}{2\beta} \left(1 - \frac{4}{\pi} \frac{\arctan\left(\sqrt{1+\beta}\right)}{\sqrt{1+\beta}}\right)$$ $[\beta > 0]$

3. $$\int_0^\infty x^3 \operatorname{erfc}^2(x) e^{-\beta x^2}\, dx = \frac{1}{2\beta^2}\left(1 - \frac{4}{\pi}\frac{\arctan\left(\sqrt{1+\beta}\right)}{\sqrt{1+\beta}}\right) + \frac{1}{\beta\pi}\left(\frac{1}{(1+\beta)\left(\beta^2+2\beta+2\right)} - \frac{\arctan\left(\sqrt{1+\beta}\right)}{(1+\beta)^{\frac{3}{2}}}\right)$$ $[\beta > 0]$

4. $$\int_0^\infty x \operatorname{erfc}\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{1}{\beta^2}\left[1 - \frac{1+\frac{3}{2}\beta}{(1+\beta)^{\frac{3}{2}}}\right]$$ $[\beta > 0]$

5.[11] $$\int_0^\infty \sqrt{x} \operatorname{erfc}\left(\sqrt{x}\right) e^{-\beta x}\, dx = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\frac{\arctan\left(\sqrt{\beta}\right)}{\beta^{\frac{3}{2}}} - \frac{1}{2\beta(1+\beta)}\right)$$ $[\beta > 0]$

8.259* Integrals involving the error function and an exponential function

1. $$\int_{-\infty}^\infty e^{-px^2}\, \Phi(a+bx)\, dx = \sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right)$$ $[\operatorname{Re} p > 0]$, a, b real

2. $$\int_{-\infty}^\infty x^2 e^{-px^2}\, \Phi(a+bx)\, dx = \frac{1}{2p}\sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right) - \frac{ab^2}{p\left(b^2+p\right)^{3/2}} \exp\left(-\frac{a^2 p}{b^2+p}\right)$$ $[\operatorname{Re} p > 0, \quad a, b \text{ are real}]$

3. $$\int_{-\infty}^\infty x^{2n} e^{-px^2}\, \Phi(a+bx)\, dx = (-1)^n \frac{\partial^n}{\partial p^n}\left[\sqrt{\frac{\pi}{p}}\, \Phi\left(\frac{a\sqrt{p}}{\sqrt{b^2+p}}\right)\right]$$ $[n = 0, 1, \ldots, \quad \operatorname{Re} p > 0, \quad a, b \text{ are real}]$

8.26 Lobachevskiy's function $L(x)$

8.260 Definition:

$$L(x) = -\int_0^x \ln\cos t\, dt$$ LO III 184(10)

For integral representations of the function $L(x)$, see also **3.531** 8, **3.532** 2, **3.533**, and **4.224**.

8.261 Representation in the form of a series:

$$L(x) = x \ln 2 - \frac{1}{2}\sum_{k=1}^\infty (-1)^{k-1} \frac{\sin 2kx}{k^2}$$ LO III 185(11)

8.262 Functional relationships:

1. $$L(-x) = -L(x)$$ $\left[-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right]$ LO III 185(13)

2. $$L(\pi - x) = \pi \ln 2 - L(x)$$ LO III 286

3. $$L(\pi + x) = \pi \ln 2 + L(x)$$ LO III 286

4. $$L(x) - L\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right)\ln 2 - \frac{1}{2} L\left(\frac{\pi}{2} - 2x\right)$$ $\left[0 \le x < \frac{\pi}{4}\right]$ LO III 186(14)

8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

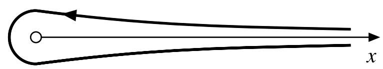
8.310 Definition:

1. $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad [\operatorname{Re} z > 0] \qquad \text{(Euler)}$$ FI II 777(6)

Generalization:

2. $$\Gamma(z) = -\frac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t}\, dt$$

for z not an integer. The contour C is shown in the drawing: WH



$\Gamma(z)$ is an analytic function z with simple poles at the points $z = -l$ (for $l = 0, 1, 2, \ldots$) to which correspond to residues $\frac{(-1)^l}{l!}$. $\Gamma(z)$ satisfies the relation $\Gamma(1) = 1$. WH, MO 1

Integral representations

8.311 $$\Gamma(z) = \frac{1}{e^{2\pi i z} - 1} \int_\infty^{(0+)} e^{-t} t^{z-1}\, dt$$ MO 2

8.312

1. $$\Gamma(z) = \int_0^1 \left(\ln \frac{1}{t}\right)^{z-1} dt \qquad [\operatorname{Re} z > 0]$$ FI II 778

2. $$\Gamma(z) = x^z \int_0^\infty e^{-xt} t^{z-1}\, dt \qquad [\operatorname{Re} z > 0, \quad \operatorname{Re} x > 0]$$ FI II 779(8)

3. $$\Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^\infty e^{-at^2} \left(1 + t^2\right)^{z - \frac{1}{2}} \cos\left[2at + (2z-1)\arctan t\right] dt$$
$[a > 0]$ WH

4. $$\Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^\infty e^{-t^2} t^{z-1} \left(1 + t^2\right)^{\frac{z}{2}} \left\{3 \sin\left[t + z \operatorname{arccot}(-t)\right] + \sin\left[t + (z-2) \operatorname{arccot}(-t)\right]\right\} dt$$
[arccot denotes an obtuse angle] WH

5. $$\Gamma(y) = x^y e^{-i\beta y} \int_0^\infty t^{y-1} \exp\left(-xte^{-i\beta}\right) dt$$
$\left[x, y, \beta \text{ real}, \quad x > 0, \quad y > 0, \quad |\beta| < \frac{\pi}{2}\right]$ MO 8

6. $$\Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^\infty e^{bti} (it)^{z-1}\, dt \qquad [b > 0, \quad 0 < \operatorname{Re} z < 1]$$ NH 154(3)

7. $$\Gamma(z) = \frac{\left(\sqrt{a^2+b^2}\right)^z}{\cos\left(z \arctan \frac{b}{a}\right)} \int_0^\infty e^{-at} \cos(bt) t^{z-1}\, dt$$ NH 152(1)a

$$= \frac{\left(\sqrt{a^2+b^2}\right)^z}{\sin\left(z \arctan \frac{b}{a}\right)} \int_0^\infty e^{-at} \sin(bt) t^{z-1}\, dt$$ NH 152(2)

$[a > 0, \quad b \geq 0, \quad \operatorname{Re} z > 0]$

8. $$\Gamma(z) = \frac{b^z}{\cos\frac{\pi z}{2}} \int_0^\infty \cos(bt) t^{z-1}\, dt$$

$$= \frac{b^z}{\sin\frac{\pi z}{2}} \int_0^\infty \sin(bt) t^{z-1}\, dt$$

$[b > 0, \quad 0 < \operatorname{Re} z < 1]$ NH 152(5)

9. $$\Gamma(z) = \int_0^\infty e^{-t}(t-z) t^{z-1} \ln t\, dt$$

$[\operatorname{Re} z > 0]$ NH 173(7)

10. $$\Gamma(z) = \int_{-\infty}^\infty \exp\left(zt - e^t\right)\, dt$$

$[\operatorname{Re} z > 0]$ NH 145(14)

11.[11] $$\Gamma(x) \cos \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \cos\left(\lambda t \sin\alpha\right)\, dt$$

$\left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right]$ WH

12. $$\Gamma(x) \sin \alpha x = \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos\alpha} \sin\left(\lambda t \sin\alpha\right)\, dt$$

$\left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right]$ WH

13. $$\Gamma(-z) = \int_0^\infty \left[\frac{e^{-t} - \sum_{k=0}^{n} (-1)^k \frac{t^k}{k!}}{t^{z+1}}\right] dt$$

$[n = \lfloor \operatorname{Re} z \rfloor]$ MO 2

8.313 $$\Gamma\left(\frac{z+1}{v}\right) = v u^{\frac{z+1}{v}} \int_0^\infty \exp\left(-u t^v\right) t^z\, dt$$

$[\operatorname{Re} u > 0, \quad \operatorname{Re} v > 0, \quad \operatorname{Re} z > -1]$

JA, MO 7a

8.314* $$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1}\, dt + \sum_{n=0}^\infty \frac{(-1)^k}{k!(z+k)}$$

$[z \to 0, \text{ in } |\arg z| < \pi]$

8.315

1.[11] $$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t}\, dt$$

[for the contour C, see **8.310** 2]

2.[8] $$\int_{-\infty}^\infty \frac{e^{bti}}{(a+it)^z}\, dt = \frac{2\pi e^{-ab} b^{z-1}}{\Gamma(z)}$$

$$\int_{-\infty}^\infty \frac{e^{-bti}}{(a+it)^z}\, dt = 0$$

$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} z > 0, \quad |\arg(a+it)| < \frac{1}{2}\pi\right]$

3. $$\frac{1}{\Gamma(z)} = a^{1-z}\frac{e^a}{\pi}\int_0^{\pi/2} \cos\left(a\tan\theta - z\theta\right)\cos^{z-2}\theta\,d\theta \qquad [\operatorname{Re} z > 1]$$ NH 157(14)

See also **3.324** 2, **3.326**, **3.328**, **3.381** 4, **3.382** 2, **3.389** 2, **3.433**, **3.434**, **3.478** 1, **3.551** 1, 2, **3.827** 1, **4.267** 7, **4.272**, **4.353** 1, **4.369** 1, **6.214**, **6.223**, **6.246**, **6.281**.

8.32 Representation of the gamma function as series and products

8.321 Representation in the form of a series:

1.[6] $$\Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$
$$\left[c_0 = 1, \quad c_{n+1} = \frac{\sum_{k=0}^{n}(-1)^{k+1}s_{k+1}c_{n-k}}{n+1}; \quad s_1 = \boldsymbol{C}, \quad s_n = \zeta(n) \text{ for } n \geq 2, \quad |z| < 1\right]$$
NH 40(1, 3)

2.[11] $$\frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$
$$\left[d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^{n}(-1)^{k}s_{k+1}d_{n-k}}{n+1}; \quad s_1 = \boldsymbol{C}, \quad s_n = \zeta(n) \text{ for } n \geq 2\right]$$ NH 41(4, 6)

Infinite-product representation

8.322[11] $$\Gamma(z) = e^{-\boldsymbol{C}z}\frac{1}{z}\prod_{k=1}^{\infty}\frac{e^{z/k}}{1+\frac{z}{k}} \qquad [\operatorname{Re} z > 0]$$ SM 269

$$= \frac{1}{z}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}} \qquad [\operatorname{Re} z > 0]$$ WH

$$= \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k} \qquad [\operatorname{Re} z > 0]$$ SM 267(130)

8.323[7] $$\Gamma(z) = 2z^z e^{-z}\prod_{k=1}^{\infty}\sqrt[2^k]{\mathrm{B}\left(2^{k-1}z, \tfrac{1}{2}\right)}$$ NH 98(12)

8.324[7] $$\Gamma(1+z) = 4^z\prod_{k=1}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{z}{2^k}\right)}{\sqrt{\pi}}$$ MO 3

8.325

1. $$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\gamma)\,\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{\gamma}{\alpha+k}\right)\left(1-\frac{\gamma}{\beta+k}\right)\right]$$ NH 62(2)

2.[11] $$\frac{e^{\boldsymbol{C}x}\,\Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty}\left[\left(1-\frac{x}{z+k}\right)e^{x/k}\right] \qquad [z \neq 0, -1, -2, \ldots; \quad \operatorname{Re} z > 0, \quad \operatorname{Re}(z-x) > 0]$$

3.[7] $$\frac{\sqrt{\pi}}{\Gamma\left(1+\frac{z}{2}\right)\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)} = \prod_{k=1}^{\infty}\left(1-\frac{z}{2k-1}\right)\left(1+\frac{z}{2k}\right)$$ MO 2

8.326

1. $$\frac{\frac{[\Gamma(x)]^2}{\Gamma(2x)}}{\mathrm{B}(x+iy, x-iy)} = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2 = \prod_{k=0}^{\infty}\left(1+\frac{y^2}{(x+k)^2}\right)$$

[x, y are real, $x \neq 0, -1, -2, \dots$]

LO V, NH 63(4)

2.[11] $$\frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{xe^{-iCy}}{x+iy}\prod_{n=1}^{\infty}\frac{\exp\left(\frac{iy}{n}\right)}{1+\frac{iy}{x+n}}$$ [x, y are real, $x \neq 0, -1, -2, \dots$]

MO 2

8.327 Asymptotic representation for large arguments:

1.* $$\Gamma(z) \sim z^{z-\frac{1}{2}}e^{-z}\sqrt{2\pi}\left\{1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+O\left(z^{-5}\right)\right\}$$

[$|\arg z| < \pi$] WH

For z real and positive, the remainder of the series is less than the last term that is retained.

2.* $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{or equivalently} \quad \Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

[Stirling's asymptotic formula for $n \gg 0$] AS 6.1.38

3.* $$\ln\Gamma(z) \sim \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \dots$$

[$z \to \infty$, $|\arg z| < \pi$] AS 6.1.38

8.328

1. $$\lim_{|y|\to\infty} |\Gamma(x+iy)|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi}$$ [x and y are real] MO 6

2. $$\lim_{|z|\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\ln z} = 1$$ MO 6

8.33 Functional relations involving the gamma function

8.331

1. $$\Gamma(x+1) = x\,\Gamma(x)$$

2.* $$\begin{aligned}\Gamma(x+a) &= (x+a-1)\,\Gamma(x+a-1)\\ &= \frac{\Gamma(x+a+1)}{(x+a)}\end{aligned}$$

3.* $$\begin{aligned}\Gamma(x-a) &= (x-a-1)\,\Gamma(x-a-1)\\ &= \frac{\Gamma(x-a+1)}{(x-a)}\end{aligned}$$

8.332

1. $|\Gamma(iy)|^2 = \dfrac{\pi}{y \sinh \pi y}$ [y is real] MO 3

2. $\left|\Gamma\left(\frac{1}{2} + iy\right)\right|^2 = \dfrac{\pi}{\cosh \pi y}$ [y is real]

3. $\Gamma(1 + ix)\,\Gamma(1 - ix) = \dfrac{\pi x}{\sinh x\pi}$ [x is real] LO V

4. $\Gamma(1 + x + iy)\,\Gamma(1 - x + iy)\,\Gamma(1 + x - iy)\,\Gamma(1 - x - iy) = \dfrac{2\pi^2\left(x^2 + y^2\right)}{\cosh 2y\pi - \cos 2x\pi}$
[x and y are real] LO V

8.333 $[\Gamma(n+1)]^n = G(n+1) \prod_{k=1}^{n} k^k,$

where n is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left[-\frac{z(z+1)}{2} - \frac{\boldsymbol{C}}{2} z^2\right] \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n \exp\left(-z + \frac{z^2}{2n}\right) \right\}$$ WH

8.334

1. $\displaystyle\prod_{k=1}^{n} \frac{1}{\Gamma\left(-z \exp \frac{2\pi k i}{n}\right)} = -z^n \prod_{k=1}^{\infty} \left[1 - \left(\frac{z}{k}\right)^n\right]$ [$n = 2, 3, 3 \ldots$] MO 2

2. $\Gamma\left(\frac{1}{2} + x\right)\Gamma\left(\frac{1}{2} - x\right) = \dfrac{\pi}{\cos \pi x}$

3. $\Gamma(1 - x)\,\Gamma(x) = \dfrac{\pi}{\sin \pi x}$ FI II 430

Special cases

8.335[7] $\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx - \frac{1}{2}} \displaystyle\prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right)$ [product theorem] FI II 782a, WH

1. $\Gamma(2x) = \dfrac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\,\Gamma\left(x + \frac{1}{2}\right)$ [doubling formula]

2. $\Gamma(3x) = \dfrac{3^{3x - \frac{1}{2}}}{2\pi} \Gamma(x)\,\Gamma\left(x + \frac{1}{3}\right)\Gamma\left(x + \frac{2}{3}\right)$

3. $\displaystyle\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(1 - \frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}$ WH

4.[10] $\displaystyle\sum_{n=0}^{\infty} \frac{\Gamma^2\left(n - \frac{1}{2}\right)}{4\,(n!)^2\,\Gamma^2\left(-\frac{1}{2}\right)} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{1024} + \frac{25}{65536} + \cdots = \frac{1}{\pi}$

8.336 $\Gamma\left(-\dfrac{yz + xi}{2y}\right)\Gamma(1 - z) = (2i)^{z+1} y\,\Gamma\left(1 + \dfrac{yz - xi}{2y}\right) \displaystyle\int_0^\infty e^{-tx} \sin^z(ty)\,dt$
[$\operatorname{Re}(yi) > 0, \quad \operatorname{Re}(x - yzi) > 0$]
NH 133(10)

- For a connection with the psi function, see **8.361** 1.
- For a connection with the beta function, see **8.384** 1.
- For integrals of the gamma function, see **8.412** 4, **8.414**, **9.223**, **9.242** 3, **9.242** 4.

8.337

1. $\left[\Gamma'(x)\right]^2 < \Gamma(x)\,\Gamma''(x)$ $\quad [x > 0]$ MO 1
2. For $x > 0$, $\min \Gamma(1+x) = 0.88560\ldots$ is attained when $x = 0.46163\ldots$ JA

Particular values

8.338

1. $\Gamma(1) = \Gamma(2) = 1$
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
4. $$\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2\left[(4k+1)^2-1\right]}{\left[(4k-1)^2-1\right](4k+1)^2}$$ MO 1a
5. $$\prod_{k=1}^{8} \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3$$ WH

8.339 For n a natural number

1. $\Gamma(n) = (n-1)!$
2. $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!$
3. $\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \frac{2^n\sqrt{\pi}}{(2n-1)!!}$
4. $$\frac{\Gamma\left(p+n+\frac{1}{2}\right)}{\Gamma\left(p-n+\frac{1}{2}\right)} = \frac{\left(4p^2-1^2\right)\left(4p^2-3^2\right)\ldots\left[4p^2-(2n-1)^2\right]}{2^{2n}}$$ WA 221

5.* $$\Gamma(n+k) = (n+k-1)! = \frac{\Gamma(n+k+1)}{(n+k)} \qquad [n+k \geq 0, 1, \ldots]$$

6.* $$\Gamma(n-k) = (n-k-1)! = \frac{\Gamma(n-k+1)}{(n-k)} \qquad [n-k \geq 0, 1, \ldots]$$

8.34 The logarithm of the gamma function

8.341 Integral representation:

1. $$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}\,dt$$ $[\operatorname{Re} z > 0]$ WH

2.[11] $$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + 2\int_0^\infty \frac{\arctan\frac{t}{z}}{e^{2\pi t}-1}\,dt$$ $\left[\operatorname{Re} z > 0 \text{ and } \arctan w = \int_0^w \frac{du}{1+u^2} \text{ is taken over a rectangular path in the } w\text{-plane}\right]$ WH

3. $$\ln\Gamma(z) = \int_0^\infty \left\{\frac{e^{-zt}-e^{-t}}{1-e^{-t}} + (z-1)e^{-t}\right\}\frac{dt}{t}$$ $[\operatorname{Re} z > 0]$ WH

4. $$\ln\Gamma(z) = \int_0^\infty \left\{(z-1)e^{-t} + \frac{(1+t)^{-z}-(1+t)^{-1}}{\ln(1+t)}\right\}\frac{dt}{t}$$ $[\operatorname{Re} z > 0]$ WH

5. $$\ln\Gamma(x) = \frac{\ln\pi - \ln\sin\pi x}{2} + \frac{1}{2}\int_0^\infty \left\{\frac{\sinh\left(\frac{1}{2}-x\right)t}{\sinh\frac{t}{2}} - (1-2x)e^{-t}\right\}\frac{dt}{t}$$ $[0 < x < 1]$ WH

6. $$\ln\Gamma(z) = \int_0^1 \left\{\frac{t^z-t}{t-1} - t(z-1)\right\}\frac{dt}{t\ln t}$$ $[\operatorname{Re} z > 0]$ WH

7. $$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz}-e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}$$ $[\operatorname{Re} z > 0]$ NH 187(7)

See also **3.427** 9, **3.554** 5.

8.342 Series representations:

1.[11] $\ln\Gamma(z+1)$

$$= \frac{1}{2}\left[\ln\left(\frac{\pi z}{\sin\pi z}\right) - \ln\frac{1+z}{1-z}\right] + (1-\boldsymbol{C})\,z + \sum_{k=1}^\infty \frac{1-\zeta(2k+1)}{2k+1}z^{2k+1}$$
$$= -\boldsymbol{C}z + \sum_{k=2}^\infty (-1)^k\frac{z^k}{k}\,\zeta(k)$$ $[|z| < 1]$ NH 38(16, 12)

2. $$\ln\Gamma(1+x) = \frac{1}{2}\ln\frac{\pi x}{\sin\pi x} - \boldsymbol{C}x - \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1}\,\zeta(2n+1)$$ $[|x| < 1]$ NH 38(14)

8.343

1. $$\ln\Gamma(x) = \ln\sqrt{2\pi} + \sum_{n=1}^\infty \left\{\frac{1}{2n}\cos 2n\pi x + \frac{1}{n\pi}\left(\boldsymbol{C}+\ln 2n\pi\right)\sin 2n\pi x\right\}$$ $[0 < x < 1]$ FI III 558

2. $$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)(m+2)}\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}}$$
$[|\arg z| < \pi]$ MO 9

8.344[7] Asymptotic expansion for large values of $|z|$:

$$\ln\Gamma(z) = z\ln z - z - \frac{1}{2}\ln z + \ln\sqrt{2\pi} + \sum_{k=1}^{n-1}\frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1}\cos^{2n-1}\left(\frac{1}{2}\arg z\right)}$$ MO5

For integrals of $\ln\Gamma(x)$, see **6.44**.

8.35 The incomplete gamma function

8.350 Definition:

1. $$\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}\,dt$$ $[\operatorname{Re}\alpha > 0]$ EH II 133(1), NH 1(1)

2.[11] $$\Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}\,dt$$ EH II 133(2), NH 2(2), LE 339

3.* $\Gamma(z, 0) = \Gamma(z)$

4.* $\Gamma(a, \infty) = 0$

5.* $\gamma(a, 0) = 0$

8.351

1. $\gamma^*(\alpha, x) = \dfrac{x^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha, x)$ is an analytic function with respect to α and x EH II 133(5)

2. Another definition of $\Gamma(\alpha, x)$ that is also suitable for the case $\operatorname{Re}\alpha \le 0$:

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha}e^{-x}\,\Phi\left(1, 1+\alpha; x\right) = \frac{x^\alpha}{\alpha}\,\Phi(a, 1+a; -x)$$ EH II 133(3)

3. For fixed x, $\Gamma(\alpha, x)$ is an entire function of α. For non-integral α, $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$.

4. A second definition of $\Gamma(\alpha, x)$:

$$\Gamma(\alpha, x) = x^\alpha e^{-x}\Psi(1, 1+\alpha; x) = e^{-x}\Psi(1-\alpha, 1-\alpha; x)$$ EH II 133(4)

8.352 Special cases:

1. $$\gamma(1+n, x) = n!\left[1 - e^{-x}\left(\sum_{m=0}^{n}\frac{x^m}{m!}\right)\right]$$ $[n = 0, 1, \ldots]$

EH II 136(17, 16), NH 6(11)

2. $$\Gamma(1+n, x) = n!e^{-x}\sum_{m=0}^{n}\frac{x^m}{m!}$$ $[n = 0, 1, \ldots]$ EH II 136(16, 18)

3.[11] $$\Gamma(-n,x) = \frac{(-1)^n}{n!}\left[\operatorname{Ei}(-z) - \frac{1}{2}\ln(-z) + \frac{1}{2}\ln\left(-\frac{1}{z}\right) - \ln z\right] - e^{-z}\sum_{k=1}^{n}\frac{z^{k-n-1}}{(-n)_k}$$ $[n = 1, 2, \ldots]$

4.* $$\Gamma(n,x) = (n-1)!e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}$$

5.* $$\Gamma(-n+1,x) = \frac{(-1)^{n+1}}{(n-1)!}\left[\Gamma(0,x) - e^{-z}\sum_{m=0}^{n-2}(-1)^m\frac{m!}{x^{m+1}}\right]$$ $[n = 2, 3, \ldots]$

6.* $$\gamma(n,x) = (n-1)!\left[1 - e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}\right]$$ $[n = 1, 2, \ldots]$

7.* $$\Gamma(n,x) = (n-1)!e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}$$ $[n = 1, 2, \ldots]$

8.* $$\Gamma(-n+k,x) = \frac{(-1)^{n-k}}{(n-k)!}\left[\Gamma(0,x) - e^{-x}\sum_{m=0}^{n-k-1}(-1)^m\frac{m!}{x^{m+1}}\right]$$ $[n - k \geq 1, \quad k = 0, 1, \ldots]$

8.353 Integral representations:

1. $$\gamma(\alpha,x) = x^\alpha \operatorname{cosec}\pi\alpha\int_0^\pi e^x\cos\theta\cos(\alpha\theta + x\sin\theta)\,d\theta$$ $[x \neq 0, \quad \operatorname{Re}\alpha > 0, \quad \alpha \neq 1, 2, \ldots]$ EH II 137(2)

2. $$\gamma(\alpha,x) = x^{\frac{1}{2}\alpha}\int_0^\infty e^{-t}t^{\frac{1}{2}\alpha-1}\,J_\alpha\left(2\sqrt{xt}\right)\,dt$$ $[\operatorname{Re}\alpha > 0]$ EH II 138(4)

3. $$\Gamma(\alpha,x) = \frac{\rho^{-x}x^\alpha}{\Gamma(1-\alpha)}\int_0^\infty\frac{e^{-t}t^{-\alpha}}{x+t}\,dt$$ $[\operatorname{Re}\alpha < 1, \quad x > 0]$ EH II 137(3), NH 19(12)

4. $$\Gamma(\alpha,x) = \frac{2x^{\frac{1}{2}\alpha}e^{-x}}{\Gamma(1-\alpha)}\int_0^\infty e^{-t}t^{-\frac{1}{2}\alpha}\,K_\alpha\left[2\sqrt{xt}\right]\,dt$$ $[\operatorname{Re}\alpha < 1]$ EH II 138(5)

5. $$\Gamma(\alpha,xy) = y^\alpha e^{-xy}\int_0^\infty e^{-ty}(t+x)^{\alpha-1}\,dt$$ $[\operatorname{Re}y > 0, \quad x > 0, \quad \operatorname{Re}\alpha > 1]$ (See also **3.936** 5, **3.944** 1–4) NH 19(10)

For integrals of the gamma function, see **6.45**.

8.354 Series representations:

1. $$\gamma(\alpha,x) = \sum_{n=0}^\infty\frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}$$ EH II 135(4)

2. $$\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \qquad [\alpha \neq 0, -1, -2, \ldots]$$

EH II 135(5), LE 340(2)

3. $$\Gamma(\alpha, x) - \Gamma(\alpha, x+y) = \gamma(\alpha, x+y) - \gamma(\alpha, x)$$
$$= e^{-x} x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k \left[1 - e^{-y} e_k(y)\right] \Gamma(1-\alpha+k)}{x^k\, \Gamma(1-\alpha)}$$
$$e_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!} \qquad [|y| < |x|]$$ EH II 139(2)

4. $$\gamma(\alpha, x) = \Gamma(\alpha) e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} x^{\frac{1}{2}n} I_{n+\alpha}\left(2\sqrt{x}\right) \sum_{m=0}^{n} \frac{(-1)^m}{m!} \qquad [x \neq 0, \quad \alpha \neq 0, \quad -1, -2, \ldots]$$

EH II 139(3)

5. $$\Gamma(\alpha, x) = e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \qquad [x > 0]$$ EH II 140(5)

8.355 $$\Gamma(\alpha, x)\, \gamma(\alpha, y) = e^{-x-y} (xy)^{\alpha} \sum_{n=0}^{\infty} \frac{n!\, \Gamma(\alpha)}{(n+1)\, \Gamma(\alpha+n+1)} L_n^{\alpha}(x)\, L_n^{\alpha}(y)$$
$$[y > 0, \quad x \geq y, \quad \alpha \neq 0, -1, \ldots]$$

EH II 139(4)

8.356 Functional relations:

1.[11] $$\gamma(\alpha+1, x) = \alpha\, \gamma(\alpha, x) - x^{\alpha} e^{-x}$$ EH II 134(2)

2. $$\Gamma(\alpha+1, x) = \alpha\, \Gamma(\alpha, x) + x^{\alpha} e^{-x}$$ EH II 134(3)

3. $$\Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha)$$ EH II 134(1)

4. $$\frac{d\, \gamma(\alpha, x)}{dx} = -\frac{d\, \Gamma(\alpha, x)}{dx} = x^{\alpha-1} e^{-x}$$ EH II 135(8)

5. $$\frac{\Gamma(\alpha+n, x)}{\Gamma(\alpha+n)} = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} + e^{-x} \sum_{s=0}^{n-1} \frac{x^{\alpha+s}}{\Gamma(\alpha+s+1)}$$ NH 4(3)

6.[11] $$\Gamma(\alpha)\, \Gamma(\alpha+n, x) - \Gamma(\alpha+n)\, \Gamma(\alpha, x) = \Gamma(\alpha+n)\, \gamma(\alpha, x) - \Gamma(\alpha)\, \gamma(\alpha+n, x)$$ NH 5

7.* $$\Gamma(a+k, x) = (a+k-1)\, \Gamma(a+k-1, x) + x^{a+k-1} e^{-x}$$
$$= \frac{1}{a+k} \left[\Gamma(a+k+1, x) - x^{a+k} e^{-x}\right]$$

8.* $$\Gamma(a-k, x) = (a-k-1)\, \Gamma(a-k-1, x) + x^{a-k-1} e^{-x}$$
$$= \frac{1}{a-k} \left[\Gamma(a-k+1, x) - x^{a-k} e^{-x}\right]$$

9.* $$\gamma(a+k, x) = (a+k-1)\, \gamma(a+k-1, x) - x^{a+k-1} e^{-x}$$
$$= \frac{1}{a+k} \left[\Gamma(a+k+1, x) + x^{a+k} e^{-x}\right]$$

10.* $$\gamma(a-k,x) = (a-k-1)\,\gamma(a-k-1,x) - x^{a-k-1}e^{-x} = \frac{1}{a-k}\left[\gamma(a-k+1,x) + x^{a-k}e^{-x}\right]$$

8.357 Asymptotic representation for large values of $|x|$:

1. $$\Gamma(\alpha,x) = x^{\alpha-1}e^{-x}\left[\sum_{m=0}^{M-1}\frac{(-1)^m\,\Gamma(1-\alpha+m)}{x^m\,\Gamma(1-\alpha)} + O\left(|x|^{-M}\right)\right]$$
$$\left[|x|\to\infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, \quad M = 1,2,\ldots\right]$$ EH II 135(6), NH 37(7), LE 340(3)

8.358 Representation as a continued fraction:

$$\Gamma(\alpha,x) = \cfrac{e^{-x}x^{\alpha}}{x + \cfrac{1-\alpha}{1 + \cfrac{1}{x + \cfrac{2-\alpha}{1 + \cfrac{2}{x + \cfrac{3-\alpha}{1+\ldots}}}}}}$$ EH II 136(13), NH 42(9)

8.359 Relationships with other functions:

1. $\Gamma(0,x) = -\operatorname{Ei}(-x)$ EH II 143(1)
2. $\Gamma\left(0,\ln\frac{1}{x}\right) = -\operatorname{li}(x)$ EH II 143(2)
3. $\Gamma\left(\frac{1}{2},x^2\right) = \sqrt{\pi} - \sqrt{\pi}\,\Phi(x)$ EH II 147(2)
4.[11] $\gamma\left(\frac{1}{2},x^2\right) = \sqrt{\pi}\,\Phi(x)$ EH II 147(1)

8.36 The psi function $\psi(x)$

8.360 Definition:

1. $$\psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

8.361 Integral representations:

1.[8] $$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt \qquad [\operatorname{Re} z > 0]$$ NH 183(1), WH

2. $$\psi(z) = \int_0^\infty\left\{e^{-t} - \frac{1}{(1+t)^z}\right\}\frac{dt}{t} \qquad [\operatorname{Re} z > 0]$$ NH 184(7), WH

3. $$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty\frac{t\,dt}{(t^2+z^2)\left(e^{2\pi t}-1\right)} \qquad [\operatorname{Re} z > 0]$$ WH

4. $$\psi(z) = \int_0^1\left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t}\right)dt \qquad [\operatorname{Re} z > 0]$$ WH

5. $$\psi(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\, dt - \boldsymbol{C},$$ WH

6. $$\psi(z) = \int_0^\infty \left\{(1+t)^{-1} - (1+t)^{-z}\right\} \frac{dt}{t} - \boldsymbol{C}, \qquad [\operatorname{Re} z > 0]$$ WH

7. $$\psi(z) = \int_0^1 \frac{t^{z-1} - 1}{t - 1}\, dt - \boldsymbol{C}$$ FI II 796, WH

8. $$\psi(z) = \ln z + \int_0^\infty e^{-tz} \left[\frac{1}{t} - \frac{1}{1 - e^{-t}}\right] dt \qquad [\operatorname{Re} z > 0]$$ MO 4

See also **3.244** 3, **3.311** 6, **3.317** 1, **3.457**, **3.458** 2, **3.471** 14, **4.253** 1 and 6, **4.275** 2, **4.281** 4, **4.482** 5.
For integrals of the psi function, see **6.46**, **6.47**.

Series representation

8.362

1. $$\psi(x) = -\boldsymbol{C} - \sum_{k=0}^\infty \left(\frac{1}{x+k} - \frac{1}{k+1}\right)$$ FI II 799(26), KU 26(1)

$$= -\boldsymbol{C} - \frac{1}{x} + x \sum_{k=1}^\infty \frac{1}{k(x+k)}$$ FI II 495

2. $$\psi(x) = \ln x - \sum_{k=0}^\infty \left[\frac{1}{x+k} - \ln\left(1 + \frac{1}{x+k}\right)\right]$$ MO 4

3. $$\psi(x) = -\boldsymbol{C} + \frac{\pi^2}{6}(x-1) - (x-1)\sum_{k=1}^\infty \left(\frac{1}{k+1} - \frac{1}{x+k}\right) \sum_{n=0}^{k-1} \frac{1}{x+n}$$ NH 54(12)

8.363

1. $$\psi(x+1) = -\boldsymbol{C} + \sum_{k=2}^\infty (-1)^k \zeta(k) x^{k-1}$$ NH 37(5)

2. $$\psi(x+1) = \frac{1}{2x} - \frac{\pi}{2}\cot \pi x - \frac{x^2}{1-x^2} - \boldsymbol{C} + \sum_{k=1}^\infty \left[1 - \zeta(2k+1)\right] x^{2k}$$ NH 38(10)

3. $$\psi(x) - \psi(y) = \sum_{k=0}^\infty \left(\frac{1}{y+k} - \frac{1}{x+k}\right)$$
(see also **3.219**, **3.231** 5, **3.311** 7, **3.688** 20, **4.253** 1, **4.295** 37) NH 99(3)

4. $$\psi(x+iy) - \psi(x-iy) = \sum_{k=0}^\infty \frac{2yi}{y^2 + (x+k)^2}$$

5. $$\psi\left(\frac{p}{q}\right) = -\boldsymbol{C} + \sum_{k=0}^\infty \left(\frac{1}{k+1} - \frac{q}{p+kq}\right)$$ (see also **3.244** 3) NH 29(1)

6.[8] $$\psi\left(\frac{p}{q}\right) = -\boldsymbol{C} - \ln(2q) - \frac{\pi}{2}\cot\frac{p\pi}{q} + 2\sum_{k=1}^{\left[\frac{q+1}{2}\right]-1}\left[\cos\frac{2kp\pi}{q}\ln\sin\frac{k\pi}{q}\right]$$
$[q = 2, 3, \ldots, p = 1, 2, \ldots, q-1]$
MO 4, EH I 19(29)

7. $$\psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(p+kq)^n - 1}$$ NH 59(3)

8. $$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n!\, \zeta(n+1, x)$$ NH 37(1)

Infinite-product representation

8.364

1. $$e^{\psi(x)} = x\prod_{k=0}^{\infty}\left(1 + \frac{1}{x+k}\right)e^{-\frac{1}{x+k}}$$ NH 65(12)

2. $$e^{y\,\psi(x)} = \frac{\Gamma(x+y)}{\Gamma(x)}\prod_{k=0}^{\infty}\left(1 + \frac{y}{x+k}\right)e^{-\frac{y}{x+k}}$$ NH 65(11)

See also **8.37**.

- For a connection with Riemann's zeta function, see **9.533** 2.
- For a connection with the gamma function, see **4.325** 12 and **4.352** 1.
- For a connection with the beta function, see **4.253** 1.
- For series of psi functions, see **8.403** 2, **8.446**, and **8.447** 3 (Bessel functions), **8.761** (derivatives of associated Legendre functions with respect to the degree), **9.153**, **9.154** (hypergeometric function), **9.237** (confluent hypergeometric function).
- For integrals containing psi functions, see **6.46–6.47**.

8.365 Functional relations:

1. $$\psi(x+1) = \psi(x) + \frac{1}{x}$$ JA

2. $$\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\,\beta(x)$$ (cf. **8.37** 0)

3. $$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1}\frac{1}{x+k}$$ GA 154(64)a

4. $$\psi(n+1) = -\boldsymbol{C} + \sum_{k=1}^{n}\frac{1}{k}$$ MO 4

5. $$\lim_{n\to\infty}\left[\psi(z+n) - \ln n\right] = 0$$ MO 3

6. $$\psi(nz) = \frac{1}{n}\sum_{k=0}^{n-1}\psi\left(z + \frac{k}{n}\right) + \ln n$$ $[n = 2, 3, 4, \ldots]$ MO 3

7. $\psi(x-n) = \psi(x) - \sum_{k=1}^{n} \frac{1}{x-k}$

8. $\psi(1-z) = \psi(z) + \pi \cot \pi z$ GA 155(68)a

9. $\psi\left(\frac{1}{2}+z\right) = \psi\left(\frac{1}{2}-z\right) + \pi \tan \pi z$ JA

10. $\psi\left(\frac{3}{4}-n\right) = \psi\left(\frac{1}{4}+n\right) + \pi$ $[n = 0, \quad \pm 1, \quad \pm 2, \ldots]$

8.366 Particular values

1. $\psi(1) = -\boldsymbol{C}$ (cf. **8.367** 1)

2. $\psi\left(\frac{1}{2}\right) = -\boldsymbol{C} - 2\ln 2 = -1.963\,510\,026\ldots$ GA 155a

3. $\psi\left(\frac{1}{2} \pm n\right) = -\boldsymbol{C} + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right]$ JA

4. $\psi\left(\frac{1}{4}\right) = -\boldsymbol{C} - \frac{\pi}{2} - 3\ln 2$ GA 157a

5. $\psi\left(\frac{3}{4}\right) = -\boldsymbol{C} + \frac{\pi}{2} - 3\ln 2$ GA 157a

6. $\psi\left(\frac{1}{3}\right) = -\boldsymbol{C} - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3$ GA 157a

7. $\psi\left(\frac{2}{3}\right) = -\boldsymbol{C} + \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3$ GA 157a

8. $\psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\ldots$ JA

9. $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\ldots$ JA

10. $\psi'(-n) = \infty$ [n is a natural number] JA

11. $\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$ [n is a natural number] JA

12. $\psi'\left(\frac{1}{2}+n\right) = \frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$ [n is a natural number] JA

13. $\psi'\left(\frac{1}{2}-n\right) = \frac{\pi^2}{2} + 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}$ [n is a natural number] JA

8.367 Euler's constant (also denoted by γ):

1. $\boldsymbol{C} = -\psi(1) = 0.577\ 215\ 664\ 90\ldots$ FI II 319, 795

2. $\boldsymbol{C} = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n\right]$ FI II 801a

3. $\boldsymbol{C} = \lim_{x\to 1+0}\left[\zeta(x) - \frac{1}{x-1}\right]$ FI II 804

Integral representations:

4. $$\boldsymbol{C} = -\int_0^\infty e^{-t} \ln t \, dt$$ FI II 807

5. $$\boldsymbol{C} = -\int_0^1 \ln\left(\ln \frac{1}{t}\right) dt$$ FI II 807

6. $$\boldsymbol{C} = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t}\right] dt$$ DW

7. $$\boldsymbol{C} = -\int_0^\infty \left[\cos t - \frac{1}{1+t}\right] \frac{dt}{t}$$ MO 10

8. $$\boldsymbol{C} = 1 - \int_0^\infty \left[\frac{\sin t}{t} - \frac{1}{1+t}\right] \frac{dt}{t}$$ MO 10

9. $$\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t}\right] \frac{dt}{t}$$ FI II 795, 802

10. $$\boldsymbol{C} = -\int_0^\infty \left[e^{-t} - \frac{1}{1+t^2}\right] \frac{dt}{t}$$ DW, MO 10

11. $$\boldsymbol{C} = \int_0^\infty \left[\frac{1}{e^t - 1} - \frac{1}{te^t}\right] dt$$ DW

12. $$\boldsymbol{C} = \int_0^1 \left(1 - e^{-t}\right) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} \, dt$$ FI II 802

See also **8.361** 5–**8.361** 7, **3.311** 6, **3.435** 3 and 4, **3.476** 2, **3.481** 1 and 2, **3.951** 10, **4.283** 9, **4.331** 1, **4.421** 1, **4.424** 1, **4.553**, **4.572**, **6.234**, **6.264** 1, **6.468**.

13. Asymptotic expansions

$$\boldsymbol{C} = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$
$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2\,(r+1)} \frac{\theta}{n^{2r+2}}$$
$[0 < \theta < 1]$ FI II 827

8.37 The function $\beta(x)$

8.370 Definition:

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$$ NH 16(13)

8.371 Integral representations:

1.[3] $$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} \, dt$$ $[\operatorname{Re} x > 0]$ WH

2. $$\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} \, dt$$ $[\operatorname{Re} x > 0]$ MO 4

3. $$\beta\left(\frac{x+1}{2}\right) = \int_0^\infty \frac{e^{-xt}}{\cosh t} \, dt$$ $[\operatorname{Re} x > -1]$

See also **3.241** 1, **3.251** 7, **3.522** 2 and 4, **3.623** 2 and 3, **4.282** 2, **4.389** 3, **4.532** 1 and 3.

Series representation

8.372

1.[7] $$\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} \qquad [-x \notin \mathbb{N}]$$ NH 37, 101(1)

2.[7] $$\beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)} \qquad [-x \notin \mathbb{N}]$$ NH 101(2)

3.[8] $$\beta(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k} \qquad [-x \notin \mathbb{N}]$$

[β has simple poles at $x = -n$ with residue $(-1)^n$] NH 246(7)

8.373

1.[6] $$\beta(x+1) = \ln 2 + \sum_{k=1}^{\infty} (-1)^k \left(1 - 2^{-k}\right) \zeta(k+1) x^k \qquad [|x| < 1]$$ NH 37(5)

2.[6] $$\beta(x+1) = \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=1}^{\infty} \left[1 - \left(1 - 2^{-2k}\right) \zeta(2k+1)\right] x^{2k}$$

$[0 < |x| < 2; \quad x \neq \pm 1]$ NH 38(11)

8.374 $$\frac{d^n}{dx^n} \beta(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}} \qquad [-x \in \mathbb{N}]$$ NH 37(2)

8.375 Representation in the form of a finite sum:

1.[6] $$\beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{\left\lfloor \frac{q-1}{2} \right\rfloor} \cos \frac{p(2k+1)\pi}{q} \ln \sin \frac{(2k+1)\pi}{2q}$$

$[q = 2, 3, \dots, p = 1, 2, 3, \dots, q-1]$ (see also **8.362** 5–7) NH 23(9)

2. $$\beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}$$

Functional relations

8.376 $$\sum_{k=0}^{2n} (-1)^k \beta\left(\frac{x+k}{2n+1}\right) = (2n+1)\,\beta(x)$$ NH 19

8.377 $$\sum_{k=1}^{n} \beta\left(2^k x\right) = \psi\left(2^n x\right) - \psi(x) - n \ln 2$$ NH 20(10)

8.38 The beta function (Euler's integral of the first kind): $\mathrm{B}(x, y)$

Integral representation

8.380

1. $$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt^{*}$$
$$= 2\int_0^1 t^{2x-1}\left(1-t^2\right)^{y-1}\,dt \qquad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ FI II 774(1)

2. $$\mathrm{B}(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\varphi\cos^{2y-1}\varphi\,d\varphi \qquad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ KU 10

3. $$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = 2\int_0^\infty \frac{t^{2x-1}}{\left(1+t^2\right)^{x+y}}\,dt \qquad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ FI II 775

4. $$\mathrm{B}(x,y) = 2^{2-y-x}\int_{-1}^1 \frac{(1+t)^{2x-1}(1-t)^{2y-1}}{\left(1+t^2\right)^{x+y}}\,dt \qquad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ MO 7

5. $$\mathrm{B}(x,y) = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt = \int_1^\infty \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt \qquad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ BI (1)(15)

6. $$\mathrm{B}(x,y) = \frac{1}{2^{x+y-1}}\int_0^1 \left[(1+t)^{x-1}(1-t)^{y-1} + (1+t)^{y-1}(1-t)^{x-1}\right]dt$$
$$[\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0]$$ BI (1)(15)

7. $$\mathrm{B}(x,y) = z^y(1+z)^x\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}}\,dt$$
$$[\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad 0 > z > -1, \quad \operatorname{Re}(x+y) < 1]$$ NH 163(8)

8. $$\mathrm{B}(x,y) = z^y(1+z)^x\int_0^{\pi/2} \frac{\cos^{2x-1}\varphi\sin^{2y-1}\varphi}{\left(z+\cos^2\varphi\right)^{x+y}}\,d\varphi$$
$$[\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad 0 > z > -1, \quad \operatorname{Re}(x+y) < 1]$$ NH 163(8)

See also **3.196** 3, **3.198**, **3.199**, **3.215**, **3.238** 3, **3.251** 1–3, 11, **3.253**, **3.312** 1, **3.512** 1 and 2, **3.541** 1, **3.542** 1, **3.621** 5, **3.623** 1, **3.631** 1, 8, 9, **3.632** 2, **3.633** 1, 4, **3.634** 1, 2, **3.637**, **3.642** 1, **3.667** 8, **3.681** 2.

9. $$\mathrm{B}(x,x) = \frac{1}{2^{2x-2}}\int_0^1 \left(1-t^2\right)^{x-1}\,dt = \frac{1}{2^{2x-1}}\int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}}\,dt$$

See **8.384** 4, **8.382** 3, and also **3.621** 1, **3.642** 2, **3.665** 1, **3.821** 6, **3.839** 6.

10. $$\mathrm{B}(x+y, x-y) = 4^{1-x}\int_0^\infty \frac{\cosh 2yt}{\cosh^{2x} t}\,dt \qquad [\operatorname{Re} x > |\operatorname{Re} y|, \quad \operatorname{Re} x > 0]$$ MO 9

11. $$\mathrm{B}\left(x, \frac{y}{z}\right) = z\int_0^1 \left(1-t^z\right)^{x-1} t^{y-1}\,dt \qquad \left[\operatorname{Re} z > 0, \quad \operatorname{Re}\frac{y}{z} > 0, \quad \operatorname{Re} x > 0\right]$$ FI II 787a

*This equation is used as the definition of the function $\mathrm{B}(x, y)$.

8.381

1. $$\int_{-\infty}^{\infty} \frac{dt}{(a+it)^x(b-it)^y} = \frac{2\pi\,(a+b)^{1-x-y}}{(x+y-1)\,\mathrm{B}(x,y)}$$
$[a>0, \quad b>0; \quad x \text{ and } y \text{ are real}, \quad x+y>1]$ MO 7

2. $$\int_{-\infty}^{\infty} \frac{dt}{(a-it)^x(b-it)^y} = 0$$
$[a>0, \quad b>0; \quad x \text{ and } y \text{ are real}, \quad x+y>1]$ MO 7

3. $$\mathrm{B}(x+iy, x-iy) = 2^{1-2x}\alpha e^{-2i\gamma y} \int_{-\infty}^{\infty} \frac{e^{2i\alpha y t}\,dt}{\cosh^{2x}(\alpha t-\gamma)}$$
$[y, \alpha, \gamma \text{ are real}, \quad \alpha>0; \quad \operatorname{Re} x>0]$
MI 8a

For an integral representation of $\ln \mathrm{B}(x,y)$, see **3.428** 7.

4. $$\frac{1}{\mathrm{B}\,(x,y)} = \frac{2^{x+y-1}\,(x+y-1)}{\pi}\int_0^{\pi/2} \cos[(x-y)t]\cos^{x+y-2} t\,dt$$ NH 158(5)a
$$= \frac{2^{x+y-2}(x+y-1)}{\pi\cos\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi} \cos[(x-y)t]\sin^{x+y-2} t\,dt$$ NH 159(8)a
$$= \frac{2^{x+y-2}(x+y-1)}{\pi\sin\left[(x-y)\frac{\pi}{2}\right]}\int_0^{\pi} \sin[(x-y)t]\sin^{x+y-2} t\,dt$$ NH 159(9)a

Series representation

8.382

1. $$\mathrm{B}(x,y) = \frac{1}{y}\sum_{n=0}^{\infty}(-1)^n y \frac{(y-1)\dots(y-n)}{n!(x+n)}$$ $[y>0]$ WH

2. $$\ln \mathrm{B}\left(\frac{1+x}{2}, \frac{1}{2}\right) \ln\sqrt{2\pi} + \frac{1}{2}\left[\ln\left(\frac{\tan\frac{\pi x}{2}}{x}\right) - \ln\left(\frac{1+x}{1-x}\right)\right] + \sum_{k=0}^{\infty}\frac{1-\left(1-2^{-2k}\right)\zeta(2k+1)}{2k+1}x^{2k+1}$$
$[|x|<2]$ NH 39(17)

3. $$\mathrm{B}\left(z, \frac{1}{2}\right) = \sum_{k=1}^{\infty}\frac{(2k-1)!!}{2^k k!}\frac{1}{z+k} + \frac{1}{z}$$ (see also **8.384** and **8.380** 9) WH

8.383 Infinite-product representation:
$$(x+y+1)\,\mathrm{B}(x+1,y+1) = \prod_{k=1}^{\infty}\frac{k(x+y+k)}{(x+k)(y+k)}$$ $[x, \quad y \neq -1, \quad -2, \dots]$ MO 2

8.384 Functional relations involving the beta function:

1. $$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \mathrm{B}(y,x)$$ FI II 779

2. $$\mathrm{B}(x,y)\,\mathrm{B}(x+y,z) = \mathrm{B}(y,z)\,\mathrm{B}(y+z,x)$$ MO 6

3. $$\sum_{k=0}^{\infty} \mathrm{B}(x, y+k) = \mathrm{B}(x-1, y)$$ WH

4. $$\mathrm{B}(x,x) = 2^{1-2x}\,\mathrm{B}\left(\tfrac{1}{2}, x\right)$$ (see also **8.380** 9 and **8.382** 3) FI II 784

5. $$\mathrm{B}(x,x)\,\mathrm{B}\left(x+\tfrac{1}{2}, x+\tfrac{1}{2}\right) = \frac{\pi}{2^{4x-1}x}$$ WH

6. $$\frac{1}{\mathrm{B}(n,m)} = m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1}$$ [m and n are natural numbers]

For a connection with the psi function, see **4.253** 1.

8.39 The incomplete beta function $\mathrm{B}_x(p,q)$

8.391[7] $$\mathrm{B}_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{x^p}{p}\,{}_2F_1\left(p, 1-q; p+1; x\right)$$ ET I 373

8.392 $$I_x(p,q) = \frac{\mathrm{B}_x(p,q)}{\mathrm{B}(p,q)}$$ ET II 429

8.4–8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.401 Bessel functions $Z_\nu(z)$ are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{d Z_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0$$ KU 37(1)

Special types of Bessel functions are what are called Bessel functions of the first kind $J_\nu(z)$, Bessel functions of the second kind $Y_\nu(z)$ (also called Neumann functions and often written $N_\nu(z)$), and Bessel functions of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (also called Hankel's functions).

8.402 $$J_\nu(z) = \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(\nu+k+1)}$$ [$|\arg z| < \pi$] KU 55(1)

8.403

1. $$Y_\nu(z) = \frac{1}{\sin\nu\pi}\left[\cos\nu\pi\, J_\nu(z) - J_{-\nu}(z)\right]$$ [for non-integer ν, $|\arg z| < \pi$] KU 41(3)

2. $$\pi\, Y_n(z) = 2\, J_n(z) \ln\frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$

$$- \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)]$$

KU 43(10)

$$= 2\, J_n(z) \left(\ln\frac{z}{2} + \boldsymbol{C}\right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n}$$

$$- \left(\frac{z}{2}\right)^n \frac{1}{n!} \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k!\,(k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m}\right]$$

[$n+1$ a natural number, $|\arg z| < \pi$]

KU 44, WA 75(3)a

8.404

1. $Y_{-n}(z) = (-1)^n\, Y_n(z)$ [n is a natural number] KU 41(2)
2. $J_{-n}(z) = (-1)^n\, J_n(z)$ [n is a natural number] KU 41(2)

8.405[7]

1. $H_\nu^{(1)}(z) = J_\nu(z) + i\, Y_\nu(z)$ KU 44(1)
2. $H_\nu^{(2)}(z) = J_\nu(z) - i\, Y_\nu(z)$ KU 44(1)

In all relationships that hold for an arbitrary Bessel function $Z_\nu(z)$, that is, for the functions $J_\nu(z)$, $Y_\nu(z)$, and linear combinations of them, for example, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, we shall write simply the letter Z instead of the letters J, Y, $H^{(1)}$, and $H^{(2)}$.

Modified Bessel functions of imaginary argument $I_\nu(z)$ and $K_\nu(z)$

8.406

1. $I_\nu(z) = e^{-\frac{\pi}{2}\nu i}\, J_\nu\left(e^{\frac{\pi}{2} i} z\right)$ $\left[-\pi < \arg z \le \frac{\pi}{2}\right]$ WA 92
2. $I_\nu(z) = e^{\frac{3}{2}\pi\nu i}\, J_\nu\left(e^{-\frac{3}{2}\pi i} z\right)$ $\left[\frac{\pi}{2} < \arg z \le \pi\right]$ WA 92

For integer ν,

3. $I_n(z) = i^{-n}\, J_n(iz)$ KU 46(1)

8.407

1.[8] $K_\nu(z) = \dfrac{\pi i}{2} e^{\frac{\pi}{2}\nu i}\, H_\nu^{(1)}\left(z e^{\frac{1}{2}\pi i}\right)$ $\left[-\pi < \arg z \le \frac{1}{2}\pi\right]$

2.[8] $K_\nu(z) = \dfrac{-\pi i}{2} e^{-\frac{\pi}{2}\nu i}\, H_{-\nu}^{(2)}\left(z e^{-\frac{1}{2}\pi i}\right)$ $\left[-\frac{1}{2}\pi < \arg z \le \pi\right]$ WA 92(8)

For the differential equation defining these functions, see **8.494**.

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

1.[11] $$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos\left(n\theta - z\sin\theta\right)\,d\theta \qquad [n = 0, 1, 2, \ldots]$$ WH

2. $$J_{2n}(z) = \frac{1}{\pi}\int_0^{\pi} \cos 2n\theta \cos\left(z\sin\theta\right)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos 2n\theta \cos\left(z\sin\theta\right)\,d\theta$$ $[n \text{ an integer}]$ WA 30(7)

3.[11] $$J_{2n+1}(z) = \frac{1}{\pi}\int_0^{\pi} \sin(2n+1)\theta \sin\left(z\sin\theta\right)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2} \sin(2n+1)\theta \sin\left(z\sin\theta\right)\,d\theta \qquad [n \text{ an integer}]$$ WA 30(6)

4. $$J_\nu(z) = 2\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi/2} \sin^{2\nu}\theta \cos\left(z\cos\theta\right)\,d\theta$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WH

5. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} \sin^{2\nu}\theta \cos\left(z\cos\theta\right)\,d\theta \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$

6. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2} \cos\left(z\sin\theta\right)\cos^{2\nu}\theta\,d\theta$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ KU 65(5), WA 35(4)a

7. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi} e^{\pm iz\cos\varphi}\sin^{2\nu}\varphi\,d\varphi \qquad \left[\operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right]$$ WH

8. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} \left(1-t^2\right)^{\nu-\frac{1}{2}}\cos zt\,dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$ KU 65(6), WH

9. $$J_\nu(x) = 2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^{\infty} \frac{\sin xt}{\left(t^2-1\right)^{\nu+\frac{1}{2}}}\,dt \qquad \left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x > 0\right]$$ MO 37

10. $$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{izt}\left(1-t^2\right)^{\nu-\frac{1}{2}}\,dt \qquad \left[\operatorname{Re}\nu > -\frac{1}{2}\right]$$ WA 34(3)

11. $$J_\nu(x) = \frac{2}{\pi}\int_0^{\infty} \sin\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh\nu t\,dt$$ WA 199(12)

12. $$J_\nu(z) = \frac{2^{\nu+1}z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\pi/2} \frac{\left(\cos^{\nu-\frac{1}{2}}\theta\right)\sin\left(z-\nu\theta+\frac{1}{2}\theta\right)}{\sin^{2\nu+1}\theta}e^{-2z\cot\theta}\,d\theta$$ $\left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right]$ WH

13.[10] $$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos\left(\nu\theta - z\sin\theta\right)\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu\theta - z\sinh\theta}\,d\theta$$

$[\operatorname{Re} z > 0]$ WA 195(4)

14. $$J_\nu(z) = \frac{e^{\pm\nu\pi i}}{\pi}\left[\int_0^\pi \cos\left(\nu\theta + z\sin\theta\right)\,d\theta - \sin\nu\pi\int_0^\infty e^{-\nu\theta + z\sinh\theta}\,d\theta\right]$$

$\left[\text{for } \frac{\pi}{2} < |\arg z| < \pi, \text{ with the upper sign taken for } |\arg z| > \frac{\pi}{2}\right.$

$\left.\text{and the lower sign taken for } |\arg z| < -\frac{\pi}{2}\right]$

WH

8.412

1. $$J_\nu(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]\,dt$$ $\left[|\arg z| < \frac{\pi}{2}\right]$ WH, WA 195(2)

2. $$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i}\int_{-\infty}^{(0+)} t^{-\nu-1}\exp\left(t - \frac{z^2}{4t}\right)\,dt$$ WA 195(1)

3.[8] $$J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i}\sum_{k=1}^\infty \frac{(-1)^k z^{2k}}{2^{2k}k!}\int_{-\infty}^{(0+)} e^t t^{-\nu-k-1}\,dt$$ WA 195(1)

4. $$J_\nu(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-t)}{\Gamma(\nu+t+1)}\left(\frac{x}{2}\right)^{\nu+2t}\,dt$$ $[\operatorname{Re}\nu > 0, \quad x > 0]$ WA 214(7)

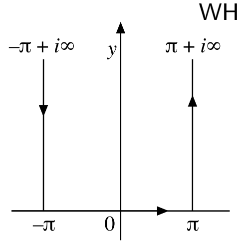
5.[7] $$J_\nu(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{2\pi i\,\Gamma\left(\frac{1}{2}\right)}\int_A^{(1+,-1-)}\left(t^2-1\right)^{\nu-\frac{1}{2}}\cos(zt)\,dt$$

$\left[\nu \neq \frac{1}{2}, \frac{3}{2}, \ldots;\ \text{The point} A \text{ falls to the right of the point } t = 1,\right.$

$\left.\text{and } \arg(t-1) = \arg(t+1) = 0 \text{ at the point } A\right]$

WH

6.[8] $$J_\nu(z) = \frac{1}{2\pi}\int_{-\pi+\infty i}^{\pi+\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta$$ $[\operatorname{Re} z > 0]$



The path of integration being taken around the semi-infinite strip $y \geq 0, -\pi \leq x \leq \pi$.

8.413[8] $$\frac{J_\nu\left(\sqrt{z^2+\zeta^2}\right)}{\left(z^2-\zeta^2\right)^{\frac{\nu}{2}}} = \frac{1}{\pi(z+\zeta)^\nu}\left\{\int_0^\infty e^{\zeta\cos t}\cos\left(z\sin t - \nu t\right)\,dt - \sin\nu\pi\int_0^\infty \exp\left(-z\sinh t - \zeta\cosh t - \nu t\right)\,dt\right\}$$

$[\operatorname{Re}(z+\zeta) > 0]$ MO 40

8.414 $$\int_{2x}^\infty \frac{J_0(t)}{t}\,dt = \frac{1}{4\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{\Gamma(-t)}{t\,\Gamma(1+t)}x^{2t}\,dt$$ $[x > 0]$ MO 41

See **3.715** 2, 9, 10, 13, 14, 19–21, **3.865** 1, 2, 4, **3.996** 4.

- For an integral representation of $J_0(z)$, see **3.714** 2, **3.753** 2, 3, and **4.124**.
- For an integral representation of $J_1(z)$, see **3.697**, **3.711**, **3.752** 2, and **3.753** 5.

8.415

1. $$Y_0(x) = \frac{4}{\pi^2}\int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}}\sin(xt)\,dt - \frac{4}{\pi^2}\int_1^\infty \frac{\ln\left(t+\sqrt{t^2-1}\right)}{\sqrt{t^2-1}}\sin(xt)\,dt$$ $[x > 0]$ MO 37

2. $$Y_\nu(x) = -2\frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)}\int_1^\infty \frac{\cos xt}{\left(t^2-1\right)^{\nu+\frac{1}{2}}}\,dt$$ $\left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x > 0\right]$ KU 89(28)a, MO 38

3. $$Y_\nu(x) = -\frac{2}{\pi}\int_0^\infty \cos\left(x\cosh t - \frac{\nu\pi}{2}\right)\cosh\nu t\,dt$$ $[-1 < \operatorname{Re}\nu < 1, \quad x > 0]$ WA 199(13)

4.[8] $$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin\left(z\sin\theta - \nu\theta\right)\,d\theta - \frac{1}{\pi}\int_0^\infty \left(e^{\nu t} + e^{-\nu t}\cos\nu\pi\right)e^{-z\sinh t}\,dt$$ $[\operatorname{Re} z > 0]$ WA 197(1)

5. $$Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left[\int_0^{\pi/2}\sin\left(z\sin\theta\right)\cos^{2\nu}\theta\,d\theta - \int_0^\infty e^{-z\sinh\theta}\cosh^{2\nu}\theta\,d\theta\right]$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0\right]$ WA 181(5)a

6. $$Y_\nu(z) = -\frac{2^{\nu+1}z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}\theta}\cos\left(z-\nu\theta+\frac{1}{2}\theta\right)}{\sin^{2\nu+1}\theta}e^{-2z\cot\theta}\,d\theta$$ $\left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0\right]$ WA 186(8)

For an integral representation of $Y_0(z)$, see **3.714** 3, **3.753** 4, **3.864**. See also **3.865** 3.

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

8.421

1. $$H_\nu^{(1)}(x) = \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^\infty e^{ix\cosh t - \nu t}\,dt = \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i}\int_0^\infty e^{ix\cosh t}\cosh\nu t\,dt$$ $[-1 < \operatorname{Re}\nu < 1, \quad x > 0]$ WA 199(10)

2. $$H_\nu^{(2)}(x) = -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i}\int_{-\infty}^\infty e^{-ix\cosh t - \nu t}\,dt = -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i}\int_0^\infty e^{-ix\cosh t}\cosh\nu t\,dt$$ $[-1 < \operatorname{Re}\nu < 1, \quad x > 0]$ WA 199(11)

3. $$H_\nu^{(1)}(z) = -\frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t \, e^{i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t} \exp\left(-2z\cot t\right) dt$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0\right]$ WA 186(5)

4. $$H_\nu^{(2)}(z) = \frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi/2} \frac{\cos^{\nu-\frac{1}{2}} t e^{-i\left(z-\nu t+\frac{t}{2}\right)}}{\sin^{2\nu+1} t} \exp\left(-2z\cot t\right) dt$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0\right]$ WA 186(6)

5. $$H_\nu^{(1)}(x) = -\frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{e^{ixt}}{\left(t^2-1\right)^{\nu+\frac{1}{2}}} dt$$
$\left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x > 0\right]$ WA 87(1)

6. $$H_\nu^{(2)}(x) = \frac{2i\left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)} \int_1^\infty \frac{e^{-ixt}}{\left(t^2-1\right)^{\nu+\frac{1}{2}}} dt$$
$\left[-\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}, \quad x > 0\right]$ WA 187(2)

7. $$H_\nu^{(1)}(z) = -\frac{i}{\pi} e^{-\frac{1}{2} i\nu\pi} \int_0^\infty \exp\left[\frac{1}{2} iz\left(t+\frac{1}{t}\right)\right] t^{-\nu-1} dt$$
$[0 < \arg z < \pi; \text{ or } \arg z = 0 \text{ and } -1 < \operatorname{Re}\nu < 1]$ MO 38

8. $$H_\nu^{(1)}(xz) = -\frac{i}{\pi} e^{-\frac{1}{2} i\nu\pi} z^\nu \int_0^\infty \exp\left[\frac{1}{2} ix\left(t+\frac{z^2}{t}\right)\right] t^{-\nu-1} dt$$
$\left[0 < \arg z < \frac{\pi}{2}, \quad x > 0, \quad \operatorname{Re}\nu > -1; \text{ or } \arg z = \frac{\pi}{2}, \quad x > 0 \text{ and } -1 < \operatorname{Re}\nu < 1\right]$ MO 38

9. $$H_\nu^{(1)}(xz) = \sqrt{\frac{2}{\pi z}} \frac{x^\nu \exp\left[i\left(xz-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\right]}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty \left(1+\frac{it}{2z}\right)^{\nu-\frac{1}{2}} t^{\nu-\frac{1}{2}} e^{-xt} dt$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad -\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi, \quad x > 0\right]$ MO 39

10. $$H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi}\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\infty e^{iz\cosh t} \sinh^{2\nu} t \, dt$$
$\left[0 < \arg z < \pi, \quad \operatorname{Re}\nu > -\frac{1}{2} \text{ or } \arg z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right]$ MO 38

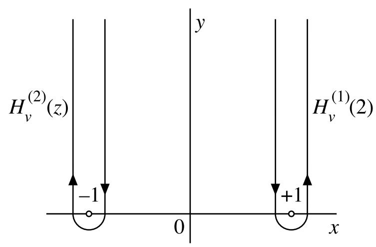
11. $$H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^\infty \frac{\exp\left(i\sqrt{x^2+t^2}\right)}{\sqrt{x^2+t^2}} dt$$
$[x > 0]$ MO 38

8.422

1. $$H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i\,\Gamma\left(\frac{1}{2}\right)} \int_{1+\infty i}^{(1+)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt$$
$[-\pi < \arg z < 2\pi]$ WA 183(4)

2. $$H_\nu^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\left(\frac{z}{2}\right)^\nu}{\pi i\,\Gamma\left(\frac{1}{2}\right)} \int_{-1+\infty i}^{(-1-)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}} dt$$
$[-2\pi < \arg z < \pi]$

The paths of integration are shown in the drawing.

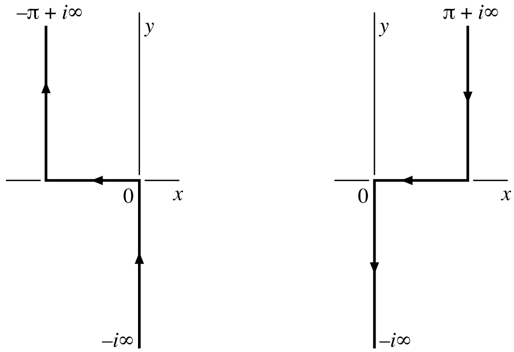


8.423

1. $$H_\nu^{(1)}(z) = -\frac{1}{\pi}\int_{-\infty i}^{-\pi+\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0]$$ WA 197(2)a

2. $$H_\nu^{(2)}(z) = -\frac{1}{\pi}\int_{\pi+\infty i}^{-\infty i} e^{-iz\sin\theta + i\nu\theta}\,d\theta \qquad [\operatorname{Re} z > 0]$$ WA 197(3)a

The path of integration for **8.423** 1 is shown in the left-hand drawing and for **8.423** 2 in the right-hand drawing.



8.424

1. $$H_\nu^{(1)}(z)\,J_\nu(\zeta) = \frac{1}{\pi i}\int_0^{\gamma+i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t}$$
$[\gamma > 0, \quad \operatorname{Re}\nu > -1, \quad |\zeta| < |z|]$ MO 45

2. $$H_\nu^{(2)}(z)\,J_\nu(\zeta) = \frac{i}{\pi}\int_0^{\gamma-i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2+\zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right)\frac{dt}{t}$$
$[\gamma > 0, \quad \operatorname{Re}\nu > -1, \quad |\zeta| < |z|]$ MO 45

8.43 Integral representations of the functions $I_\nu(z)$ and $K_\nu(z)$

The function $I_\nu(z)$

8.431

1. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}} e^{\pm zt}\,dt \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

2. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}} \cosh zt\,dt \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

3. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

4. $$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi \cosh\left(z\cos\theta\right)\sin^{2\nu}\theta\,d\theta \qquad \left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0\right]$$ WA 94(9)

5. $$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\nu\theta\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}\,dt$$
$$\left[|\arg z| \le \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right]$$ WA 201(4)

See also **3.383** 2, **3.387** 1, **3.471** 6, **3.714** 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see **3.366** 1, **3.534** **3.856** 6.

The function $K_\nu(z)$

8.432

1. $$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt$$
$$\left[|\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$$ MO 39

2. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t\,dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\tfrac{1}{2} < \operatorname{Re}\nu < \tfrac{1}{2}\right]$$ WA 190(5), WH

3. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\,dt$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$$ WA 190(4)

4. $$K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty \cos(x\sinh t)\cosh\nu t\,dt$$
$$[x > 0, \quad -1 < \operatorname{Re}\nu < 1]$$ WA 202(13)

5. $$K_\nu(xz) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu}{x^\nu\,\Gamma\left(\frac{1}{2}\right)}\int_0^\infty \frac{\cos xt\,dt}{\left(t^2+z^2\right)^{\nu+\frac{1}{2}}}$$
$$\left[\operatorname{Re}\left(\nu+\tfrac{1}{2}\right) \ge 0, \quad x > 0, \quad |\arg z| < \frac{\pi}{2}\right]$$ WA 191(1)

6.[11] $$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty \frac{e^{-t-z^2/4t}\,dt}{t^{\nu+1}}$$
$$\left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0\right]$$ WA 203(15)

7.[7] $$K_\nu(xz) = \frac{z^\nu}{2}\int_0^\infty \exp\left[-\frac{x}{2}\left(t+\frac{z^2}{t}\right)\right]t^{-\nu-1}\,dt$$
$$\left[|\arg z| < \frac{\pi}{4} \text{ or } |\arg z| = \frac{\pi}{4} \text{ and } \operatorname{Re}\nu < 1\right]$$ MO 39

8. $$K_\nu(xz) = \sqrt{\frac{\pi}{2z}}\frac{x^\nu e^{-xz}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-xt}t^{\nu-\frac{1}{2}}\left(1+\frac{t}{2z}\right)^{\nu-\frac{1}{2}}\,dt$$
$$\left[|\arg z| < \pi, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, x > 0\right]$$ MO 39

9. $$K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2z}\right)^\nu\int_0^\infty \frac{\exp\left(-x\sqrt{t^2+z^2}\right)}{\sqrt{t^2+z^2}}t^{2\nu}\,dt$$
$$\left[\operatorname{Re}\nu > -\tfrac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re}\sqrt{t^2+z^2} > 0, \quad x > 0\right]$$ MO 39

See also **3.383** 3, **3.387** 3, 6, **3.388** 2, **3.389** 4, **3.391**, **3.395** 1, **3.471** 9, **3.483**, **3.547** 2, **3.856**, **3.871** 3, 4, **7.141** 5.

8.433 $K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}}\int_0^\infty \cos\left(t^3 + xt\right)\,dt$ KU 98(31), WA 211(2)

For an integral representation of $K_0(z)$, see **3.754** 2, **3.864**, **4.343**, **4.356**, **4.367**.

8.44 Series representation

The function $J_\nu(z)$

8.440 $$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k} \qquad [|\arg z| < \pi]$$ WH 358 a

8.441 Special cases:

1. $$J_0(z) = \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}\,(k!)^2}$$

2. $$J_1(z) = -\,J_0'(z) = \frac{z}{2}\sum_{k=0}^\infty \frac{(-1)^k z^{2k}}{2^{2k} k!(k+1)!}$$

3. $$J_{\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{4}{3}\right)}\sqrt[3]{\frac{z}{2}}\sum_{k=0}^\infty (-1)^k \frac{\left(z\sqrt{3}\right)^{2k}}{2^{2k}k!\cdot 1\cdot 4\cdot 7\cdot\dots\cdot(3k+1)}$$

4. $$J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\sqrt[3]{\frac{2}{z}}\left\{1 + \sum_{k=1}^\infty (-1)^k \frac{\left(z\sqrt{3}\right)^{2k}}{2^{2k}k!\cdot 2\cdot 5\cdot 8\cdot\dots\cdot(3k-1)}\right\}$$

For the expansion of $J_\nu(z)$ in Laguerre polynomials, see **8.975** 3.

8.442

1.[7] $$J_\nu(z)\,J_\mu(z) = \sum_{m=0}^\infty \frac{(-1)^m \left(\frac{1}{2}z\right)^{\mu+\nu+2m}(\mu+\nu+m+1)_m}{m!\,\Gamma(\mu+m+1)\,\Gamma(\nu+m+1)}$$

2.[8] $$J_\nu(az)\,J_\mu(bz) = \frac{\left(\frac{az}{2}\right)^\nu\left(\frac{bz}{2}\right)^\mu}{\Gamma(\mu+1)}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu-1; \frac{b^2}{a^2}\right)}{k!\,\Gamma(\nu+k+1)}$$ MO 28

The function $Y_\nu(z)$

8.443[11] $$Y_\nu(z) = \frac{1}{\sin\nu\pi}\left\{\cos\nu\pi\left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(\nu+k+1)} - \left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k}k!\,\Gamma(k-\nu+1)}\right\}$$

[$\nu \neq$ an integer] (cf. **8.403** 1)

For $\nu+1$ a natural number, see **8.403** 2.; for ν a negative integer, see **8.404** 1

8.444 Special cases,

1. $$\pi\, Y_0(z) = 2\, J_0(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^{k} \frac{1}{m}$$ KU 44

2.[11] $$\pi\, Y_1(z) = 2\, J_1(z) \left(\ln \frac{z}{2} + \boldsymbol{C}\right) - \frac{2}{z} - \frac{z}{2} - \sum_{k=2}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2}\right)^{2k-1}}{k!(k-1)!} \left(\frac{1}{k} + 2 \sum_{m=1}^{k-1} \frac{1}{m}\right)$$

The functions $I_\nu(z)$ and $K_n(z)$

8.445 $$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\, \Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$ WH 372a

8.446[8] $$K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k! \left(\frac{z}{2}\right)^{n-2k}}$$

$$+(-1)^{n+1} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!} \left[\ln \frac{z}{2} - \frac{1}{2}\, \psi(k+1) - \frac{1}{2}\, \psi(n+k+1)\right]$$

WA 95(15)

$$= (-1)^{n+1}\, I_n(z) \left(\ln \frac{1}{2} z + \boldsymbol{C}\right) + \frac{1}{2}(-1)^n \sum_{l=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2l}}{l!(n+l)!} \left(\sum_{k=1}^{l} \frac{1}{k} + \sum_{k=1}^{n+l} \frac{1}{k}\right)$$

$$+\frac{1}{2} \sum_{l=0}^{n-1} \frac{(-1)^l (n-l-1)!}{l!} \left(\frac{z}{2}\right)^{2l-n}$$

[$n+1$ is a natural number]

MO 29

8.447 Special cases:

1. $$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}$$

2. $$I_1(z) = I_0'(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!}$$

3. $$K_0(z) = -\ln \frac{z}{2}\, I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\, (k!)^2}\, \psi(k+1)$$ WA 95(14)

8.45 Asymptotic expansions of Bessel functions

8.451 For large values of $|z|$ *

1. $$J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}}\left\{\cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma\left(\nu+2k+\frac{1}{2}\right)}{(2k)!\,\Gamma\left(\nu-2k+\frac{1}{2}\right)} + R_1\right]\right.$$
$$\left. - \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma\left(\nu+2k+\frac{3}{2}\right)}{(2k+1)!\,\Gamma\left(\nu-2k-\frac{1}{2}\right)} + R_2\right]\right\}$$
$[|\arg z| < \pi]$ (see **8.339** 4) WA 222(1, 3)

2.[11] $$Y_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}}\left\{\sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k}}\frac{\Gamma\left(\nu+2k+\frac{1}{2}\right)}{(2k)!\,\Gamma\left(\nu-2k+\frac{1}{2}\right)} + R_1\right]\right.$$
$$\left. + \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2z)^{2k+1}}\frac{\Gamma\left(\nu+2k+\frac{3}{2}\right)}{(2k+1)!\,\Gamma\left(\nu-2k-\frac{1}{2}\right)} + R_2\right]\right\}$$
$[|\arg z| < \pi]$ (see **8.339** 4) WA 222(2, 4, 5)

3.[11] $$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{(-1)^k}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)} + \theta_1\frac{(-1)^n}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad |\arg z| < \pi\right]$ (see **8.339** 4) WA 221(5)

4.[11] $$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}e^{-i\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}\left[\sum_{k=0}^{n-1}\frac{1}{(2iz)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)} + \theta_2\frac{1}{(2iz)^n}\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
$\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad |\arg z| < \pi\right]$ (see **8.339** 4) WA 221(6)

For indices of the form $\nu = \frac{2n-1}{2}$ (where n is a natural number), the series **8.451** terminate. In this case, the closed formulas **8.46** are valid for all values.

5. $$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}$$
$$+\frac{\exp\left[-z \pm \left(\nu+\frac{1}{2}\right)\pi i\right]}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{1}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)}$$

[The + sign is taken for $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, the − sign for $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$]* (see **8.339** 4)]
WA 226(2,3)

6.[11] $$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left[\sum_{k=0}^{n-1}\frac{1}{(2z)^k}\frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k!\,\Gamma\left(\nu-k+\frac{1}{2}\right)} + \theta_3\frac{\Gamma\left(\nu+n+\frac{1}{2}\right)}{(2z)^n n!\,\Gamma\left(\nu-n+\frac{1}{2}\right)}\right]$$
(see **8.339** 4) WA 231, 245(9)

An estimate of the remainders of the asymptotic series in formulas **8.451**:

*An estimate of the remainders in formulas **8.451** is given in **8.451** 7 and **8.451** 8.

*The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

7. $$|R_1| < \left|\frac{\Gamma\left(\nu+2n+\frac{1}{2}\right)}{(2z)^{2n}(2n)!\,\Gamma\left(\nu-2n+\frac{1}{2}\right)}\right| \qquad \left[n > \frac{\nu}{2}-\frac{1}{4}\right]$$ WA 231

8. $$|R_2| < \left|\frac{\Gamma\left(\nu+2n+\frac{3}{2}\right)}{(2z)^{2n+1}(2n+1)!\,\Gamma\left(\nu-2n-\frac{1}{2}\right)}\right| \qquad \left[n \ge \frac{\nu}{2}-\frac{3}{4}\right]$$ WA 231

For $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, ν real, and $n+\frac{1}{2} > |\nu|$ WA 245

$$|\theta_1| < \begin{cases} 1, & \text{if } \operatorname{Im} z \ge 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \le 0 \end{cases}$$

For $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$, ν real, and $n+\frac{1}{2} > |\nu|$ WA 246

$$|\theta_2| < \begin{cases} 1, & \text{if } \operatorname{Im} z \le 0 \\ |\sec(\arg z)|, & \text{if } \operatorname{Im} z \ge 0 \end{cases}$$

For ν real, WA 245

$$|\theta_3| < \begin{cases} 1 & \text{if } \operatorname{Re} z \ge 0 \\ |\operatorname{cosec}(\arg z)|, & \text{if } \operatorname{Re} z < 0 \end{cases}$$
$$\operatorname{Re}\theta_3 \ge 0, \qquad \text{if } \operatorname{Re} z \ge 0$$

For ν and z real and $n \ge \nu - \frac{1}{2}$, WA 231

$$0 \le |\theta_3| \le 1$$

In particular, it follows from **8.451** 7 and **8.451** 8 that for real positive values of z and ν, the errors $|R_1|$ and $|R_2|$ are less than the absolute value of the first discarded term. For values of $|\arg z|$ close to π, the series **8.451** 1 and **8.451** 2 may not be suitable for calculations. In particular, the error for $|\arg z| > \pi$ can be greater in absolute value than the first discarded term.

"Approximation by tangents"

8.452[11] For large values of the index (where the argument is less than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cosh\alpha$. Then, for large values of ν, the following expansions are valid:

1. $$J_\nu\left(\frac{\nu}{\cosh\alpha}\right) \sim \frac{\exp(\nu\tanh\alpha - \nu\alpha)}{\sqrt{2\nu\pi\tanh\alpha}}\left\{1+\frac{1}{\nu}\left(\frac{1}{8}\coth\alpha - \frac{5}{24}\coth^3\alpha\right) + \frac{1}{\nu^2}\left(\frac{9}{128}\coth^2\alpha - \frac{231}{576}\coth^4\alpha + \frac{1155}{3456}\coth^6\alpha\right) + \dots\right\}$$

WA 269(3)

2. $$Y_\nu\left(\frac{\nu}{\cosh\alpha}\right) \sim -\frac{\exp\left(\nu\alpha - \nu\tanh\alpha\right)}{\sqrt{\frac{\pi}{2}\nu\tanh\alpha}}\left\{1 - \frac{1}{\nu}\left(\frac{1}{8}\coth\alpha - \frac{5}{24}\coth^3\alpha\right) + \frac{1}{\nu^2}\left(\frac{9}{128}\coth^2\alpha - \frac{231}{576}\coth^4\alpha + \frac{1155}{3456}\coth^6\alpha\right) + \ldots\right\}$$

WA 270(5)

8.453 For large values of the index (where the argument is greater than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cos\beta$. Then, for large values of ν, the following expansions are valid:

1. $$J_\nu\left(\nu\sec\beta\right) \sim \sqrt{\frac{2}{\nu\pi\tan\beta}}\left\{\left[1 - \frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta + \frac{231}{576}\cot^4\beta + \frac{1155}{3456}\cot^6\beta\right) + \ldots\right]\cos\left(\nu\tan\beta - \nu\beta - \frac{\pi}{4}\right) + \left[\frac{1}{\nu}\left(\frac{1}{8}\cot\beta + \frac{5}{24}\cot^3\beta\right) - \ldots\right]\sin\left(\nu\tan\beta - \nu\beta - \frac{\pi}{4}\right)\right]$$

WA 271(4)

2. $$Y_\nu\left(\nu\sec\beta\right) \sim \sqrt{\frac{2}{\nu\pi\tan\beta}}\left\{\left[1 - \frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta + \frac{231}{576}\cot^4\beta + \frac{1155}{3456}\cot^6\beta\right) + \ldots\right]\sin\left(\nu\tan\beta - \nu\beta - \frac{\pi}{4}\right) - \frac{1}{\nu}\left(\frac{1}{8}\cot\beta + \frac{5}{24}\cot^3\beta\right) - \ldots\right]\cos\left(\nu\tan\beta - \nu\beta - \frac{\pi}{4}\right)\right]$$

WA 271(5)

3. $$H_\nu^{(1)}\left(\nu\sec\beta\right) \sim \frac{\exp\left[\nu i\left(\tan\beta - \beta\right) - \frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\tan\beta}}\left\{1 - \frac{i}{\nu}\left(\frac{1}{8}\cot\beta + \frac{5}{24}\cot^3\beta\right) - \frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta + \frac{231}{576}\cot^4\beta + \frac{1155}{3456}\cot^6\beta\right) + \ldots\right\}$$

WA 271(1)

4. $$H_\nu^{(2)}\left(\nu\sec\beta\right) \sim \frac{\exp\left[-\nu i\left(\tan\beta - \beta\right) + \frac{\pi}{4}i\right]}{\sqrt{\frac{\pi}{2}\nu\tan\beta}}\left\{1 + \frac{i}{\nu}\left(\frac{1}{8}\cot\beta + \frac{5}{24}\cot^3\beta\right) - \frac{1}{\nu^2}\left(\frac{9}{128}\cot^2\beta + \frac{231}{576}\cot^4\beta + \frac{1155}{3456}\cot^6\beta\right) + \ldots\right\}$$

WA 271(2)

Formulas **8.453** are not valid when $|x - \nu|$ is of a size comparable to $x^{\frac{1}{3}}$. For arbitrary small (and also large) values of $|x - \nu|$, we may use the following formulas:

8.454 Suppose that $x > 0$ and $\nu > 0$, we set

$$w = \sqrt{\frac{x^2}{\nu^2} - 1};$$

Then,

1. $$H_\nu^{(1)}(x) = \frac{w}{\sqrt{3}} \exp\left\{\left[\frac{\pi}{6} + \nu\left(w - \frac{w^3}{3} - \arctan w\right)\right] i\right\} H_{\frac{1}{3}}^{(1)}\left(\frac{\nu}{3} w^3\right) + O\left(\frac{1}{|\nu|}\right)$$

2. $$H_\nu^{(2)}(x) = \frac{w}{\sqrt{3}} \exp\left\{\left[-\frac{\pi}{6} - \nu\left(w - \frac{w^3}{3} - \arctan w\right)\right] i\right\} H_{\frac{1}{3}}^{(2)}\left(\frac{\nu}{3} w^3\right) + O\left(\frac{1}{|\nu|}\right)$$ MO 34

The absolute value of the error $O\left(\frac{1}{|\nu|}\right)$ is then less than $24\sqrt{2}\left|\frac{1}{\nu}\right|$.

8.455 For x real and ν a natural number ($\nu = n$), if $n \gg 1$, the following approximations are valid:

1.[7] $$J_n(x) \approx \frac{1}{\pi}\sqrt{\frac{2(n-x)}{3x}}\, K_{\frac{1}{3}}\left\{\frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}}\right\}$$

$[n > x]$ (see also **8.433**) WA 276(1)

$$\approx \frac{1}{2} e^{\frac{2}{3}\pi i} \sqrt{\frac{2\,(n-x)}{3x}}\, H_{\frac{1}{3}}^{(1)}\left\{\frac{i}{3}\frac{[2(n-x)]^{\frac{3}{2}}}{\sqrt{x}}\right\}$$

$[n > x]$ MO 34

$$\approx \frac{1}{\sqrt{3}}\sqrt{\frac{2(x-n)}{3x}}\left\{J_{\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right] + J_{-\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]\right\}$$

(see also **8.441** 3, **8.441** 4) WA 276(2)

2. $$Y_n(x) \approx \sqrt{\frac{2(x-n)}{3x}}\left\{J_{-\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right] - J_{\frac{1}{3}}\left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}}\right]\right\}$$

$[x > n]$ WA 276(3)

An estimate of the error in formulas **8.455** has not yet been achieved.

8.456[11] $$J_\nu^2(z) + Y_\nu^2(z) \approx \frac{2}{\pi z}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{2^k z^{2k}}\frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k!\,\Gamma\left(\nu - k + \frac{1}{2}\right)}$$ $[|\arg z| < \pi]$ (see also **8.479** 1)

WA 250(5)

8.457 $$J_\nu^2(x) + J_{\nu+1}^2(x) \approx \frac{2}{\pi x}$$ $[x \gg |\nu|]$ WA 223

8.46 Bessel functions of order equal to an integer plus one-half

The function $J_\nu(z)$

8.461

1.[11]
$$J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!} (2z)^{-2k} + \cos\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!} (2z)^{-(2k+1)} \right\}$$
$[n+1$ is a natural number$]$ (cf. **8.451** 1) KU 59(6), WA 66(2)

2.
$$J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\,(n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} - \sin\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \right\}$$
$[n+1$ is a natural number$]$ (cf. **8.451** 1) KU 58(7), WA 67(5)

8.462

1.
$$J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^{n} \frac{i^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^{n} \frac{(-i)^{-n+k-1}(n+k)!}{k!(n-k)!(2z)^k} \right\}$$
$[n+1$ is a natural number$]$
KU 59(6), WA 66(1)

2.
$$J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^{n} \frac{i^{n+k}(n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^{n} \frac{(-i)^{n+k}(n+k)!}{k!(n-k)!(2z)^k} \right\}$$
$[n+1$ is a natural number$]$
KU 59(7), WA 67(4)

8.463

1.
$$J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z\,dz)^n} \left(\frac{\sin z}{z} \right)$$
KU 58(4)

2.
$$J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(z\,dz)^n} \left(\frac{\cos z}{z} \right)$$
KU 58(5)

8.464 Special cases:

1.
$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$$
DW

2.
$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z$$
DW

3. $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right)$ DW

4. $J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\sin z - \frac{\cos z}{z} \right)$ DW

5.[8] $J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left(\frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}$ DW

6. $J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left(\frac{3}{z^2} - 1 \right) \cos z \right\}$ DW

The function $Y_{n+\frac{1}{2}}(z)$

8.465

1. $Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z)$ JA

2. $Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$ JA

The functions $H^{(1,2)}_{n+\frac{1}{2}}(z)$, $I_{n+\frac{1}{2}}(z)$, $K_{n+\frac{1}{2}}(z)$

8.466

1. $H^{(1)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$

(cf. **8.451** 3)

2. $H^{(2)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} i^{n} e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k}$ (cf. **8.451** 4)

8.467 $I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[e^z \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k} \right]$

(cf. **8.451** 5) KU 60a

8.468 $K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!(2z)^k}$ (cf. **8.451** 6) KU 60

8.469 Special cases:

1. $Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z$

2. $Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$

3. $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$ WA 95(13)

4. $H^{(1)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}$ MO 27

5. $$H^{(2)}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\frac{e^{-iz}}{-i}$$ MO 27

6. $$H^{(1)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}e^{iz}$$ MO 27

7. $$H^{(2)}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}e^{-iz}$$ MO 27

8.47–8.48 Functional relations

8.471[8] Recursion formulas:

1. $$z\, Z_{\nu-1}(z) + z\, Z_{\nu+1}(z) = 2\nu\, Z_\nu(z)$$ KU 56(13), WA 56(1), WA 79(1), WA 88(3)

2. $$Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2\frac{d}{dz}\, Z_\nu(z)$$ KU 56(12), WA 56(2), WA 79(2), We 88(4)

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of z that satisfy the recursion relations **8.471**. Z denotes J, N, $H^{(1)}$, $H^{(2)}$ or any linear combination of these functions, the coefficients of which are independent of z and ν.

8.472 Consequences of the recursion formulas for Z defined as above:

1. $$z\frac{d}{dz}\, Z_\nu(z) + \nu\, Z_\nu(z) = z\, Z_{\nu-1}(z)$$ KU 56(11), WA 56(3), WA 79(3), WA 88(5)

2. $$z\frac{d}{dz}\, Z_\nu(z) - \nu\, Z_\nu(z) = -z\, Z_{\nu+1}(z)$$ KU 56(10), WA 56(4), WA 79(4), WA 88(6)

3. $$\left(\frac{d}{z\, dz}\right)^m (z^\nu\, Z_\nu(z)) = z^{\nu-m}\, Z_{\nu-m}(z)$$ KU 56(8), WA 57(5), WA 89(9)

4. $$\left(\frac{d}{z\, dz}\right)^m \left(z^{-\nu}\, Z_\nu(z)\right) = (-1)^m z^{-\nu-m}\, Z_{\nu+m}(z)$$ WA 89(10), Ku 55(5), WA 57(6)

5. $$Z_{-n}(z) = (-1)^n\, Z_n(z)$$ [n is a natural number] (cf. **8.404**)

8.473 Special cases:

1. $$J_2(z) = \frac{2}{z}\, J_1(z) - J_0(z)$$

2. $$Y_2(z) = \frac{2}{z}\, Y_1(z) - Y_0(z)$$

3. $$H^{(1,2)}_2(z) = \frac{2}{z}\, H^{(1,2)}_1(z) - H^{(1,2)}_0(z)$$

4. $$\frac{d}{dz}\, J_0(z) = -\, J_1(z)$$

5. $$\frac{d}{dz}\, Y_0(z) = -\, Y_1(z)$$

6. $$\frac{d}{dz}\, H^{(1,2)}_0(z) = -\, H^{(1,2)}_1(z)$$

8.474[8] Each of the pairs of functions $J_\nu(z)$ and $J_{-\nu}(z)$ (for $\nu \neq 0, \pm 1, \pm 2, \ldots$), $J_\nu(z)$ and $Y_\nu(z)$, and $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$, which are solutions of equation **8.401**, and also the pair $I_\nu(z)$ and $K_\nu(z)$ is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z}\sin\nu\pi, \quad \frac{2}{\pi z}, \quad -\frac{4i}{\pi z}, \quad -\frac{1}{z}$$ KU 52(10, 11, 12), WA 90(1, 4)

8.475[6] The functions $J_\nu(z)$, and $Y_\nu(z)$, $H_\nu^{(1,2)}(z)$, $I_\nu(z)$, $K_\nu(z)$, with the exception of $J_n(z)$ and $I_n(z)$, for n an integer are *non-single-valued*: $z = 0$ is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut $(-\infty, 0)$ are connected by the relations

8.476

1. $J_\nu\left(e^{m\pi i}z\right) = e^{m\nu\pi i}\, J_\nu(z)$ WA 90(1)

2. $Y_\nu\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, Y_\nu(z) + 2i\sin m\nu\pi\cot\nu\pi\, J_\nu(z)$ WA 90(3)

3. $Y_{-\nu}\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, Y_{-\nu}(z) + 2i\sin m\nu\pi\operatorname{cosec}\nu\pi\, J_\nu(z)$ WA 90(4)

4. $I_\nu\left(e^{m\pi i}z\right) = e^{m\nu\pi i}\, I_\nu(z)$ WA 95(17)

5. $K_\nu\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, K_\nu(z) - i\pi\frac{\sin m\nu\pi}{\sin\nu\pi}\, I_\nu(z)$ [ν not an integer] WA 95(18)

6. $$H_\nu^{(1)}\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, H_\nu^{(1)}(z) - 2e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, J_\nu(z) = \frac{\sin(1-m)\nu\pi}{\sin\nu\pi}\, H_\nu^{(1)}(z) - e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, H_\nu^{(2)}(z)$$ WA 95(5)

7. $$H_\nu^{(2)}\left(e^{m\pi i}z\right) = e^{-m\nu\pi i}\, H_\nu^{(2)}(z) + 2e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, J_\nu(z) = \frac{\sin(1+m)\nu\pi}{\sin\nu\pi}\, H_\nu^{(2)}(z) + e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}\, H_\nu^{(1)}(z)$$ [m an integer] WA 90(6)

8. $H_\nu^{(1)}\left(e^{i\pi}z\right) = -H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu}\, H_\nu^{(2)}(z)$ MO 26

9. $H_\nu^{(2)}\left(e^{-i\pi}z\right) = -H_{-\nu}^{(1)}(z) = -e^{i\pi\nu}\, H_\nu^{(1)}(z)$ MO 26

10.[8] $\overline{H}_\nu^{(2)}(z) = H_{\overline{\nu}}^{(1)}\left(\overline{z}\right)$ MO 26

8.477

1. $J_\nu(z)\, Y_{\nu+1}(z) - J_{\nu+1}(z)\, Y_\nu(z) = -\frac{2}{\pi z}$ WA 91(12)

2. $I_\nu(z)\, K_{\nu+1}(z) + I_{\nu+1}(z)\, K_\nu(z) = \frac{1}{z}$ WA 95(20)

See also **3.863**.

- For a connection with Legendre functions, see **8.722**.
- For a connection with the polynomials $C_n^\lambda(t)$, see **8.936** 4.
- For a connection with a confluent hypergeometric function, see **9.235**.

8.478 For $\nu > 0$ and $x > 0$, the product

$$x\left[J_\nu^2(x) + Y_\nu^2(x)\right],$$

considered as a function of x, decreases monotonically, if $\nu > \frac{1}{2}$ and increases monotonically if $0 < \nu < \frac{1}{2}$.
MO 35

8.479

1.[11] $$\frac{1}{\sqrt{x^2-\nu^2}} > \frac{\pi}{2}\left[J_\nu^2(x) + Y_\nu^2(x)\right] \geq \frac{1}{x}$$ $\left[x \geq \nu \geq \frac{1}{2}\right]$ MO 35

2. $$|J_n(nz)| \leq 1$$ $\left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1, n \text{ a natural number}\right]$ MO 35

Relations between Bessel functions of the first, second, and third kinds

8.481 $$J_\nu(z) = \frac{Y_{-\nu}(z) - Y_\nu(z)\cos\nu\pi}{\sin\nu\pi} = H_\nu^{(1)}(z) - i\,Y_\nu(z) = H_\nu^{(2)}(z) + i\,Y_\nu(z) = \frac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right)$$

(cf. **8.403** 1, **8.405**) WA 89(1), JA

8.482 $$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} = i\,J_\nu(z) - i\,H_\nu^{(1)}(z) = i\,H_\nu^{(2)}(z) - i\,J_\nu(z) = \frac{i}{2}\left(H_\nu^{(2)}(z) - H_\nu^{(1)}(z)\right)$$

(cf. **8.403** 1, **8.405**) WA 89(3), JA

8.483

1. $$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i}\,J_\nu(z)}{i\sin\nu\pi} = \frac{Y_{-\nu}(z) - e^{-\nu\pi i}\,Y_\nu(z)}{\sin\nu\pi} = J_\nu(z) + i\,Y_\nu(z)$$ WA 89(5)

2. $$H_\nu^{(2)}(z) = \frac{e^{\nu\pi i}\,J_\nu(z) - J_{-\nu}(z)}{i\sin\nu\pi} = \frac{Y_{-\nu}(z) - e^{\nu\pi i}\,Y_\nu(z)}{\sin\nu\pi} = J_\nu(z) - i\,Y_\nu(z)$$

(cf. **8.405**) WA 89(6)

8.484

1. $$H_{-\nu}^{(1)}(z) = e^{\nu\pi i}\,H_\nu^{(1)}(z)$$ WA 89(7)

2. $$H_{-\nu}^{(2)}(z) = e^{-\nu\pi i}\,H_\nu^{(2)}(z)$$ WA 89(7)

8.485[7] $$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}$$ [ν not an integer] (see also **8.407**) WA 92(6)

8.486 Recursion formulas for the functions $I_\nu(z)$ and $K_\nu(z)$ and their consequences:

1. $$z\,I_{\nu-1}(z) - z\,I_{\nu+1}(z) = 2\nu\,I_\nu(z)$$ WA 93(1)

2. $$I_{\nu-1}(z) + I_{\nu+1}(z) = 2\frac{d}{dz}\,I_\nu(z)$$ WA 93(2)

3. $$z\frac{d}{dz}\,I_\nu(z) + \nu\,I_\nu(z) = z\,I_{\nu-1}(z)$$ WA 93(3)

4. $$z\frac{d}{dz}\,I_\nu(z) - \nu\,I_\nu(z) = z\,I_{\nu+1}(z)$$ WA 93(4)

5. $$\left(\frac{d}{z\,dz}\right)^m \{z^\nu\,I_\nu(z)\} = z^{\nu-m}\,I_{\nu-m}(z)$$ WA 93(5)

6. $$\left(\frac{d}{z\,dz}\right)^m \left\{z^{-\nu}\, I_\nu(z)\right\} = z^{-\nu-m}\, I_{\nu+m}(z)$$ WA 93(6)

7. $$I_{-n}(z) = l_n(z)$$ [n a natural number] WA 93(8)

8. $$I_2(z) = -\frac{2}{z} l_1(z) + I_0(z)$$

9. $$\frac{d}{dz}\, I_0(z) = I_1(z)$$ WA 93(7)

10. $$z\, K_{\nu-1}(z) - z\, K_{\nu+1}(z) = -2\nu\, K_\nu(z)$$ WA 93(1)

11. $$K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz}\, K_\nu(z)$$ WA 93(2)

12. $$z\frac{d}{dz}\, K_\nu(z) + \nu\, K_\nu(z) = -z\, K_{\nu-1}(z)$$ WA 93(3)

13. $$z\frac{d}{dz}\, K_\nu(z) - \nu\, K_\nu(z) = -z\, K_{\nu+1}(z)$$ WA 93(4)

14. $$\left(\frac{d}{z\,dz}\right)^m \{z^\nu\, K_\nu(z)\} = (-1)^m z^{\nu-m}\, K_{\nu-m}(z)$$ WA 93(5)

15. $$\left(\frac{d}{z\,dz}\right)^m \left\{z^{-\nu}\, K_\nu(z)\right\} = (-1)^m z^{-\nu-m}\, K_{\nu+m}(z)$$ WA 93(6)

16. $$K_{-\nu}(z) = K_\nu(z)$$ WA 93(8)

17. $$K_2(z) = \frac{2}{z}\, K_1(z) + K_0(z)$$

18. $$\frac{d}{dz}\, K_0(z) = -\, K_1(z)$$ WA 93(7)

19. $$\frac{\partial\, J_\nu(z)}{\partial \nu} = \left[\ln\frac{z}{2} - \psi(\nu+1)\right] J_\nu(z) + \frac{(z/2)^{\nu+1}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(z/2)^n\, J_{n+1}(z)}{n!(\nu+n+1)^2}$$ LUKE 360

8.486(1)[7] Differentiation with respect to order

1. $$\frac{\partial\, J_\nu(z)}{\partial \nu} = J_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}$$
[$\nu \neq n$ or $n+\frac{1}{2}$, n integer] MS 3.1.3

2. $$\frac{\partial\, J_{-\nu}(z)}{\partial \nu} = -\, J_{-\nu}(z) \ln\left(\frac{1}{2}z\right) + \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{-\nu+2k} \frac{\psi(-\nu+k+1)}{k!\,\Gamma(-\nu+k+1)}$$
[$\nu \neq n$ or $n+\frac{1}{2}$, n integer] MS 3.1.3

3. $$\frac{\partial\, Y_\nu(z)}{\partial \nu} = \cot \pi\nu \frac{\partial\, J_\nu(z)}{\partial \nu} - \operatorname{cosec} \pi\nu \frac{\partial\, J_{-\nu}(z)}{\partial \nu} - \pi \operatorname{cosec} \pi\nu\, Y_\nu(z)$$
[$\nu \neq n$ or $n+\frac{1}{2}$, n integer] MS 3.1.3

4. $$\frac{\partial\, I_\nu(z)}{\partial \nu} = I_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k!\,\Gamma(\nu+k+1)}$$ [$\nu \neq n$ or $n+\frac{1}{2}$, n integer]
MS 3.1.3

5. $$\frac{\partial K_\nu(z)}{\partial \nu} = -\pi \cot \pi\nu \, K_\nu(z) + \frac{1}{2}\pi \operatorname{cosec} \pi\nu \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]$$
$\left[\nu \neq n \text{ or } n + \tfrac{1}{2}, \quad n \text{ integer}\right]$ MS 3.1.3

6. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \frac{1}{2}\pi \, (\pm 1)^n \, Y_n(z) \pm (\pm 1)^n \, \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} J_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ MS 3.2.3

7. $$\left[\frac{\partial Y_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = -\frac{1}{2}\pi \, (\pm 1)^n \, J_n(z) \pm (\pm 1)^n \, \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} Y_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ MS 3.2.3

8. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = (-1)^{n+1} \, K_n(z) \pm (-1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ MS 3.2.3

9. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=\pm n} = \pm \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ MS 3.2.3

10. $$(-1)^n \left[\frac{\partial}{\partial \nu} \, I_\nu(z)\right]_{\nu=n} = - K_n(z) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ AS 9.6.44

11.[11] $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=n} = \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} K_k(z)}{k!(n-k)}$$
$[n = 0, 1, \ldots]$ AS 9.6.45

Special cases

12. $$\left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=0} = \tfrac{1}{2}\pi \, Y_0(z)$$ MS 3.2.3

13. $$\left[\frac{\partial Y_\nu(z)}{\partial \nu}\right]_{\nu=0} = -\tfrac{1}{2}\pi \, J_0(z)$$ MS 3.2.3

14. $$\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=0} = - K_0(z)$$ MS 3.2.3

15. $$\left[\frac{\partial K_\nu(z)}{\partial \nu}\right]_{\nu=0} = 0$$ MS 3.2.3

16. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left[\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)\right]$$ MS 3.3.3

17. $$\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left[\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)\right]$$ MS 3.3.3

18. $$\left[\frac{\partial Y_\nu(x)}{\partial \nu}\right]_{\nu=\frac{1}{2}} = \left(\tfrac{1}{2}\pi x\right)^{-1/2} \left\{\cos x \operatorname{Ci}(2x) + \sin x \left[\operatorname{Si}(2x) - \pi\right]\right\}$$ MS 3.3.3

19. $$\left[\frac{\partial Y_\nu(x)}{\partial \nu}\right]_{\nu=-\frac{1}{2}} = -\left(\tfrac{1}{2}\pi x\right)^{-1/2} \left\{\sin x \operatorname{Ci}(2x) - \cos x \left[\operatorname{Si}(2x) - \pi\right]\right\}$$ MS 3.3.3

20. $$\left[\frac{\partial I_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-1/2}\left[e^x \operatorname{Ei}(-2x) \mp e^{-x}\overline{\operatorname{Ei}}(2x)\right]$$ MS 3.3.3

21. $$\left[\frac{\partial K_\nu(x)}{\partial \nu}\right]_{\nu=\pm\frac{1}{2}} = \mp\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^x \operatorname{Ei}(-2x)$$ MS 3.3.3

8.487 Continuity with respect to the order*:

1. $\lim_{\nu\to n} Y_\nu(z) = Y_n(z)$ [n an integer] WA 76
2. $\lim_{\nu\to n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z)$ [n an integer] WA 183
3. $\lim_{\nu\to n} K_\nu(z) = K_n(z)$ [n an integer] WA 92

8.49 Differential equations leading to Bessel functions

See also **8.401**

8.491

1. $\frac{1}{z}\frac{d}{dz}(zu') + \left(\beta^2 - \frac{\nu^2}{z^2}\right)u = 0$ $\quad u = Z_\nu(\beta z)$ JA
2. $\frac{1}{z}\frac{d}{dz}(zu') + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 - \left(\frac{\nu\gamma}{z}\right)^2\right]u = 0$ $\quad u = Z_\nu\left(\beta z^\gamma\right)$ JA
3. $u'' + \frac{1-2\alpha}{z}u' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 + \frac{\alpha^2-\nu^2\gamma^2}{z^2}\right]u = 0$ $\quad u = z^\alpha Z_\nu\left(\beta z^\gamma\right)$ JA
4. $u'' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 - \frac{4\nu^2\gamma^2-1}{4z^2}\right]u = 0$ $\quad u = \sqrt{z}\, Z_\nu\left(\beta z^\gamma\right)$ JA
5. $u'' + \left(\beta^2 - \frac{4\nu^2-1}{4z^2}\right)u = 0$ $\quad u = \sqrt{z}\, Z_\nu(\beta z)$ JA
6. $u'' + \frac{1-2\alpha}{z}u' + \left(\beta^2 + \frac{\alpha^2-\nu^2}{z^2}\right)u = 0$ $\quad u = z^\alpha Z_\nu(\beta z)$ JA
7. $u'' + bz^m u = 0$ $\quad u = \sqrt{z}\, Z_{\frac{1}{m+2}}\left(\frac{2\sqrt{b}}{m+2} z^{\frac{m+2}{2}}\right)$ JA 111(5)
8. $u'' + \frac{1}{z}u' + 4\left(z^2 - \frac{\nu^2}{z^2}\right)u = 0$ $\quad u = Z_\nu\left(z^2\right)$ WA 111(6)
9. $u'' + \frac{1}{z}u' + \frac{1}{4z}\left(1 - \frac{\nu^2}{z}\right)u = 0$ $\quad u = Z_\nu\left(\sqrt{z}\right)$ WA 111(7)
10. $u'' + \frac{1-\nu}{z}u' + \frac{1}{4}\frac{u}{z} = 0$ $\quad u = z^{\frac{\nu}{2}} Z_\nu\left(\sqrt{z}\right)$ WA 111(9)a
11. $u'' + \beta^2\gamma^2 z^{2\beta-2}u = 0$ $\quad u = z^{1/2} Z_{\frac{1}{2\beta}}\left(\gamma z^\beta\right)$ WA 110(3)

*The continuity of the functions $J_\nu(z)$ and $I_\nu(z)$ follows directly from the series representations of these functions.

12. $z^2u'' + (2\alpha - 2\beta\nu + 1)zu' + \left[\beta^2\gamma^2 z^{2\beta} + \alpha(\alpha - 2\beta\nu)\right] u = 0$

$u = z^{\beta\nu-\alpha} Z_\nu \left(\gamma z^\beta\right)$ WA 112(21)

8.492

1. $u'' + \left(e^{2z} - \nu^2\right) u = 0$ $\quad u = Z_\nu\left(e^z\right)$ WA 112(22)

2. $u'' + \dfrac{e^{2/z} - \nu^2}{z^4} u = 0$ $\quad u = z\, Z_\nu\left(e^{1/z}\right)$ WA 112(22)

8.493

1. $u'' + \left(\dfrac{1}{z} - 2\tan z\right) u' - \left(\dfrac{\nu^2}{z^2} + \dfrac{\tan z}{z}\right) u = 0$ $\quad u = \sec z\, Z_\nu(z)$ JA

2. $u'' + \left(\dfrac{1}{z} + 2\cot z\right) u' - \left(\dfrac{\nu^2}{z^2} - \dfrac{\cot z}{z}\right) u = 0$ $\quad u = \operatorname{cosec} z\, Z_\nu(z)$ JA

8.494

1. $u'' + \dfrac{1}{z}u' - \left(1 + \dfrac{\nu^2}{z^2}\right) u = 0$ $\quad u = Z_\nu(iz) = C_1\, I_\nu(z) + C_2\, K_\nu(z)$ JA

2. $u'' + \dfrac{1}{z}u' - \left[\dfrac{1}{z} + \left(\dfrac{\nu}{2z}\right)^2\right] u = 0$ $\quad u = Z_\nu\left(2i\sqrt{z}\right)$ JA

3. $u'' + u' + \dfrac{1}{z^2}\left(\dfrac{1}{4} - \nu^2\right) u = 0$ $\quad u = \sqrt{z}e^{-\frac{z}{2}}\, Z_\nu\left(\dfrac{iz}{2}\right)$ JA

4.[10] $u'' + \left(\dfrac{2\nu+1}{z} - k\right) u' - \dfrac{2\nu+1}{2z} ku = 0$ $\quad u = z^{-\nu} e^{\frac{1}{2}kx}\, Z_\nu\left(\dfrac{ikz}{2}\right)$ JA

5. $u'' + \dfrac{1-\nu}{z}u' - \dfrac{1}{4}\dfrac{u}{z} = 0$ $\quad u = z^{\frac{\nu}{2}}\, Z_\nu\left(i\sqrt{z}\right)$ WA 111(8)

6. $u'' \pm \dfrac{u}{\sqrt{z}} = 0$

$u = \sqrt{z}\, Z_{\frac{2}{3}}\left(\tfrac{4}{3}z^{\frac{3}{4}}\right), \qquad u = \sqrt{z}\, Z_{\frac{2}{3}}\left(\tfrac{4}{3}iz^{\frac{3}{4}}\right)$ WA 111(10)

7. $u'' \pm zu = 0$

$u = \sqrt{z}\, Z_{\frac{1}{3}}\left(\tfrac{2}{3}z^{\frac{3}{2}}\right), \qquad u = \sqrt{z}\, Z_{\frac{1}{3}}\left(\tfrac{2}{3}iz^{\frac{3}{2}}\right)$ WA 111(10)

8. $u'' - \left(c^2 + \dfrac{\nu(\nu+1)}{z^2}\right) u = 0$ $\quad u = \sqrt{z}\, Z_{\nu+\frac{1}{2}}(icz)$ WA 108(1)

9. $u'' - \dfrac{2\nu}{z}u' - c^2 u = 0$ $\quad u = z^{\nu+\frac{1}{2}}\, Z_{\nu+\frac{1}{2}}(icz)$ WA 109(3, 4)

10. $u'' - c^2 z^{2\nu-2} u = 0$ $\quad u = \sqrt{z}\, Z_{\frac{1}{2\nu}}\left(i\dfrac{c}{\nu}z^\nu\right)$ WA 109(5, 6)

8.495

1. $u'' + \dfrac{1}{z}u' + \left(i - \dfrac{\nu^2}{z^2}\right) u = 0$ $\quad u = Z_\nu\left(z\sqrt{i}\right)$ JA

2. $u'' + \left(\frac{1}{z} \mp 2i\right) u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right) u = 0$ $\qquad u = e^{\pm iz} Z_\nu(z)$ JA

3. $u'' + \frac{1}{z} u' + se^{i\alpha} u = 0$ $\qquad u = Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right)$ JA

4. $u'' + \left(se^{i\alpha} + \frac{1}{4z^2}\right) u = 0$ $\qquad u = \sqrt{z}\, Z_0\left(\sqrt{s} z e^{\frac{i}{2}\alpha}\right)$ JA

8.496

1. $\frac{d^2}{dz^2}\left(z^4 \frac{d^2u}{dz^2}\right) - z^2 u = 0$ $\qquad u = \frac{1}{z}\left\{Z_2\left(2\sqrt{z}\right) + \overline{Z_2\left(2i\sqrt{z}\right)}\right\}$ WA 122(7)

2. $\frac{d^2}{dz^2}\left(z^{\frac{16}{5}} \frac{d^2u}{dz^2}\right) - z^{\frac{8}{5}} u = 0$ $\qquad u = z^{-7/10}\left\{Z_{\frac{5}{6}}\left(\frac{5}{3} z^{\frac{3}{5}}\right) + \overline{Z_{\frac{5}{6}}\left(\frac{5}{3} i z^{\frac{3}{5}}\right)}\right\}$ WA 122(8)

3. $\frac{d^2}{dz^2}\left(z^{12} \frac{d^2u}{dz^2}\right) - z^6 u = 0$

$$u = z^{-4}\left\{Z_{10}\left(2z^{-1/2}\right) + \overline{Z_{10}\left(2iz^{-1/2}\right)}\right\}$$ WA 122(9)

4. $\frac{d^4u}{dz^4} + \frac{2}{z}\frac{d^3u}{dz^3} - \frac{2\nu^2+1}{z^2}\frac{d^2u}{dz^2} + \frac{2\nu^2+1}{z^3}\frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1\right) u = 0,$

$u = A_1 J_\nu(z) + A_2 Y_\nu(z) + A_3 I_\nu(z) + A_4 K_\nu(z)$, where A_1, A_2, A_3, A_4 are constants MO 29

8.51–8.52 Series of Bessel functions

8.511 Generating functions for Bessel functions:

1. $$\exp \frac{1}{2}\left(t - \frac{1}{t}\right) z = J_0(z) + \sum_{k=1}^{\infty}\left[t^k + (-t)^{-k}\right] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k$$

$[|z| < |t|]$ KU 119(12)

2. $$\exp\left(t - \frac{1}{t}\right) z = \left\{\sum_{k=-\infty}^{\infty} t^k J_k(z)\right\}\left\{\sum_{m=-\infty}^{\infty} t^m J_m(z)\right\}$$ WA 40

3. $$\exp\left(\pm iz \sin\varphi\right) = J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\varphi \pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\varphi$$ KU 120(13)

4. $$\exp\left(iz \cos\varphi\right) = \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} (2k+1) i^k J_{k+\frac{1}{2}}(z) P_k\left(\cos\varphi\right)$$ WA 401(1)

$$= \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\varphi}$$ MO 27

$$= J_0(z) + 2\sum_{k=1}^{\infty} i^k J_k(z) \cos k\varphi$$ MO 27

5. $$\sqrt{\frac{i}{\pi}} e^{iz\cos 2\varphi} \int_{-\infty}^{\sqrt{2z}\cos\varphi} e^{-it^2}\,dt = \frac{1}{2} J_0(z) + \sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i} J_{\frac{k}{2}}(z) \cos k\varphi$$ MO 28

The series $\sum J_k(z)$

8.512

1. $$J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z) = 1$$ WA 44

2. $$\sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \qquad [n = 1, 2, \ldots]$$ WA 45

3. $$\sum_{k=0}^{\infty} \frac{(4k+1)(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}}$$

8.513

Notation: In formulas **8.513** $\quad Q_k^{(p)} = \sum_{m=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^m \binom{k}{m} (k-2m)^p}{2^k k!}$

1. $$\sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^{p} Q_{2k}^{(2p)} z^{2k} \qquad [p = 1, 2, 3, \ldots]$$ WA 46(1)

2. $$\sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^{p} Q_{2k+1}^{(2p+1)} z^{2k+1} \qquad [p = 0, 1, 2, 3, \ldots]$$ WA 46(2)

In particular:

3. $$\sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2}\left(z + z^3\right)$$ WA 47(4)

4. $$\sum_{k=1}^{\infty} (2k)^2 J_{2k}(z) = \frac{1}{2} z^2$$ WA 47(4)

5. $$\sum_{k=1}^{\infty} 2k(2k+1)(2k+2) J_{2k+1}(z) = \frac{1}{2} z^3$$ WA 47(4)

8.514

1. $$\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) = \frac{\sin z}{2}$$ WH

2. $$J_0(z) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(z) = \cos z$$ WH

3. $$\sum_{k=1}^{\infty} (-1)^{k+1} (2k)^2 J_{2k}(z) = \frac{z \sin z}{2}$$ WA 32(9)

4. $$\sum_{k=0}^{\infty}(-1)^k(2k+1)^2 J_{2k+1}(z) = \frac{z\cos z}{2}$$ WA 32(10)

5. $$J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos 2k\theta = \cos(z\sin\theta)$$ KU 120(14), WA 32

6. $$\sum_{k=0}^{\infty} J_{2k+1}(z)\sin(2k+1)\theta = \frac{\sin(z\sin\theta)}{2}$$ KU 120(15), WA 32

7. $$\sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2}\int_0^x J_0(t)\,dt \qquad [x \text{ is real}]$$ WA 638

8.515

1. $$\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\left(\frac{2z+t}{2z}\right)^k J_{\nu+k}(z) = \left(\frac{z}{z+t}\right)^{\nu} J_\nu(z+t)$$ AD (9140)

2. $$\sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}\left(x^2\right) = S(x)$$ MO 127a

3. $$\sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}\left(x^2\right) = C(x)$$ MO 127a

8.516 $$\sum_{k=0}^{\infty}\frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z\sin\theta) = (z\sin\theta)^{2n}$$ WA 47

The series $\sum A_k\, J_k(kx)$ and $\sum A_k\, J'_k(kx)$

8.517

1. $$\sum_{k=1}^{\infty} J_k(kz) = \frac{z}{2(1-z)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1\right]$$ WA 615(1)

2. $$\sum_{k=1}^{\infty}(-1)^k J_k(kz) = -\frac{z}{2(1+z)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1\right]$$ WA 622(1)

3. $$\sum_{k=1}^{\infty} J_{2k}(2kz) = \frac{z^2}{2\left(1-z^2\right)} \qquad \left[\left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1\right]$$ MO 58

8.518

1.[11] $$\sum_{k=1}^{\infty}\frac{J'_k(kx)}{k} = \frac{1}{2} + \frac{x}{4} \qquad [0 \le x < 1]$$ MO 58

2.[11] $$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{J'_k(kx)}{k} = \frac{1}{2} - \frac{x}{4} \qquad [0 \le x < 1]$$ MO 58

3. $$\sum_{k=1}^{\infty} k\, J'_k(kx) = \frac{1}{2(1-x)^2} \qquad [0 \le x < 1]$$ MO 58

4. $$\sum_{k=1}^{\infty}(-1)^{k-1} J_k'(kx)k = \frac{1}{2(1+x)^2} \qquad [0 \le x < 1]$$ MO 58

The series $\sum A_k J_0(kx)$

8.519 If, on the interval $[0 \le x \le \pi]$, a function $f(x)$ possesses a continuous derivative with respect to x that is of bounded variation, then

1. $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \qquad [0 < x < \pi]$$

where

2. $$a_0 = 2f(0) + \frac{2}{\pi}\int_0^{\pi} du \int_0^{\pi/2} uf'(u \sin\varphi)\, d\varphi$$

3. $$a_n = \frac{2}{\pi}\int_0^{\pi} du \int_0^{\pi/2} uf'(u \sin\varphi) \cos nu\, d\varphi$$ WH

8.521 Examples:

1. $$\sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2\sum_{m=1}^{n} \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \qquad [2n\pi < x < 2(n+1)\pi]$$ MO 59

2. $$\sum_{k=1}^{\infty}(-1)^{k+1} J_0(kx) = \frac{1}{2} \qquad [0 < x < \pi]$$ KU 124(12)

3. $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\{(2k-1)x\} \quad \frac{\pi^2}{8} - \frac{|x|}{2} \qquad [-\pi < x < \pi]$$ KU 124
$$= \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos\frac{\pi}{x} \qquad [\pi < x < 2\pi]$$ MO 59

4. $$\sum_{k=1}^{\infty} e^{-kz} J_0\left(k\sqrt{x^2+y^2}\right)$$
$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty}\left\{\frac{1}{\sqrt{(2ki\pi + z)^2 + x^2 + y^2}} - \frac{1}{\sqrt{(2ki\pi - z)^2 + x^2 + y^2}}\right\}$$
$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1}\left(\frac{z}{r}\right) \qquad [0 < r < 2\pi]$$ MO 59

where $r = \sqrt{x^2+y^2+z^2}$ and where the radical indicates the square root with a positive real part. In formula **8.521** 4, the first equation holds when x and y are real and $\operatorname{Re} z > 0$; the second equation holds when x, y, and z are all real.

The series $\sum A_k \, Z_0(kx) \sin kx$ and $\sum A_k \, Z_0(kx) \cos kx$

8.522

1. $$\sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^{m} \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^{n} \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}$$
MO 59

2. $$\sum_{k=1}^{\infty} J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^{n} \frac{1}{l} - \sum_{l=1}^{m} \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$
MO 59

3. $$\sum_{k=1}^{\infty} Y_0(kx) \cos kxt = -\frac{1}{\pi} \left(\boldsymbol{C} + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{n} \frac{1}{l} \right\} - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}$$
MO 60

In formulas **8.522**, $x > 0, 0 \le t < 1, 2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.523

1. $$\sum_{k=1}^{\infty} (-1)^k \, J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^{m} \frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} + \sum_{l=1}^{n} \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}$$
MO 60

2. $$\sum_{k=1}^{\infty} (-1)^k \, J_0(kx) \sin kxt \quad \frac{1}{2\pi} \left\{ \sum_{l=1}^{n} \frac{1}{l} - \sum_{l=1}^{m} \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$
MO 60

3. $$\sum_{k=1}^{\infty}(-1)^k\, Y_0(kx)\cos kxt \quad -\frac{1}{\pi}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)+\frac{1}{2\pi}\left\{\sum_{l=1}^{m}\frac{1}{l}+\sum_{l=1}^{n}\frac{1}{l}\right\} - \sum_{l=m+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi+tx]^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi-tx]^2-x^2}}-\frac{1}{2l\pi}\right\}$$ MO 60

In formulas **8.523**, $x>0, 0\le t<1$, $(2m-1)\pi<x(1-t)<(2m+1)\pi$, $(2n-1)\pi<x(1+t)<(2n+1)\pi$, m and n are natural numbers.

8.524

1. $$\sum_{k=1}^{\infty}J_0(kx)\cos kxt=-\frac{1}{2}+\sum_{l=m+1}^{n}\frac{1}{\sqrt{x^2-(2l\pi-tx)^2}}$$ MO 60

2. $$\sum_{k=1}^{\infty}J_0(kx)\sin kxt\sum_{l=0}^{m}\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}+\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi+tx)^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}-\frac{1}{2l\pi}\right\}+\frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l}$$ MO 60

3.[6] $$\sum_{k=1}^{\infty}Y_0(kx)\cos kxt \quad -\frac{1}{\pi}\left(\boldsymbol{C}+\ln\frac{x}{4\pi}\right)-\sum_{l=0}^{m}\frac{1}{\sqrt{(2\pi l-tx)^2-x^2}}+\frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi+tx)^2-x^2}}-\frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{(2l\pi-tx)^2-x^2}}-\frac{1}{2l\pi}\right\}$$ MO 61

In formulas **8.524**, $x>0, t>1, 2m\pi<x(t-1)<2(m+1)\pi$, $2n\pi<x(t+1)<2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.525

1. $$\sum_{k=1}^{\infty}(-1)^k\, J_0(kx)\cos kxt=-\frac{1}{2}+\sum_{l=m+1}^{n}\frac{1}{\sqrt{x^2-[(2l-1)\pi-tx]^2}}$$ MO 61

2. $$\sum_{k=1}^{\infty}(-1)^k J_0(kx)\sin kxt = \sum_{l=1}^{m} \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} + \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

3. $$\sum_{k=1}^{\infty}(-1)^k Y_0(kx)\cos kxt = -\frac{1}{\pi}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{1}{2\pi}\sum_{l=1}^{n}\frac{1}{l} - \sum_{l=1}^{m}\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi}\right\} - \sum_{l=n+1}^{\infty}\left\{\frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

In formulas **8.525**, $x > 0, t > 1$, $(2m-1)\pi < x(t-1) < (2m+1)\pi$, $(2n-1)\pi < x(t+1) < (2n+1)\pi$, m and n are natural numbers.

8.526

1. $$\sum_{k=1}^{\infty} K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi}\right\} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi}\right\}$$

MO 61

2. $$\sum_{k=1}^{\infty}(-1)^k K_0(kx)\cos kxt = \frac{1}{2}\left(\boldsymbol{C} + \ln\frac{x}{4\pi}\right) + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi}\right\} + \frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi}\right\}$$

$[x > 0, \quad t \text{ real}]$ (see also **8.66**) MO 62

8.53 Expansion in products of Bessel functions

"Summation theorems"

8.530 Suppose that $r > 0, \varrho > 0, \varphi > 0$, and $R = \sqrt{r^2 + \varrho^2 - 2r\varrho\cos\varphi}$; that is, suppose that r, ϱ, and R are the sides of a triangle such that the angle between the sides r and ϱ is equal to φ. Suppose also that ϱ <r and that ψ is the angle opposite the side ϱ, so that

1. $$0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}$$

When these conditions are satisfied, we have the "summation theorem" for Bessel functions:

1. $$e^{i\nu\psi} Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho)\, Z_{\nu+k}(mr) e^{ik\varphi}$$ [m is an arbitrary complex number] WA 394(6)

For $Z_\nu = J_\nu$ and ν an integer, the restriction ϱ <r is superfluous. MO 31

8.531 Special cases:

1. $$J_0(mR) = J_0(m\varrho)\, J_0(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)\, J_k(mr)\cos k\varphi$$ WA 391(1)

2. $$H_0^{(1,2)}(mR) = J_0(m\varrho)\, H_0^{(1,2)}(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)\, H_k^{(1,2)}(mr)\cos k\varphi$$ MO 31

3. $$J_0(z\sin\alpha) = J_0^2\left(\frac{z}{2}\right) + 2\sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right)\cos 2k\alpha$$
$$= \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z)\, P_{2k}(\cos\alpha)$$ MO 31

8.532 The term "summation theorem" is also applied to the formula

1. $$\frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu}\,\Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) \frac{J_{\nu+k}(m\varrho)}{\varrho^\nu} \frac{Z_{\nu+k}(mr)}{r^\nu}\, C_k^\nu(\cos\varphi)$$

[$\nu \neq -1, -2, -3, \ldots$; the conditions on r, ϱ, R, φ, and m are the same as in formula **8.530**; for $Z_\nu = J_\nu$ and ν an integer, formula **8.532** 1 is valid for arbitrary r, ϱ, and φ]. WA 398(4)

8.533 Special cases:

1. $$\frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, H_{k+\frac{1}{2}}^{(1)}(mr)\, P_k(\cos\varphi)$$ MO 31

2. $$\frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, H_{k+\frac{1}{2}}^{(2)}(mr)\, P_k(\cos\varphi)$$ MO 31

8.534 A degenerate addition theorem ($r \to \infty$):

$$e^{im\varrho\cos\varphi} = \sqrt{\frac{\pi}{2m\varrho}} \sum_{k=0}^{\infty} i^k (2k+1)\, J_{k+\frac{1}{2}}(m\varrho)\, P_k(\cos\varphi) \qquad \text{WA 401(1)}$$

$$= 2^\nu\, \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) i^k (m\varrho)^{-\nu}\, J_{\nu+k}(m\varrho)\, C_k^\nu(\cos\varphi) \qquad [\nu \neq 0, -1, -2, \dots] \qquad \text{WA 401(2)}$$

8.535 The term "product theorem" is also applied to the formula

$$Z_\nu(\lambda z) = \lambda^\nu \sum_{k=0}^{\infty} \frac{1}{k!}\, Z_{\nu+k}(z) \left(\frac{1-\lambda^2}{2} z\right)^k \qquad \left[|1-\lambda|^2 < 1\right]$$

For $Z_\nu = J_\nu$, it is valid for all values of λ and z. MO 32

8.536

1. $$\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0] \qquad \text{WA 47(1)}$$

2. $$2 \sum_{k=n}^{\infty} \frac{k\,\Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \qquad [n > 0] \qquad \text{WA 47(2)}$$

3. $$J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z) = 1 \qquad \text{WA 41(3)}$$

8.537

1. $$\sum_{k=-\infty}^{\infty} Z_{\nu-k}(t)\, J_k(z) = Z_\nu(z+t) \qquad [|z| < |t|] \qquad \text{WA 158(2)}$$

2. $$\sum_{k=-\infty}^{\infty} J_k(z)\, J_{n-k}(z) = J_n(2z) \qquad \text{WA 41}$$

8.538

1. $$\sum_{k=-\infty}^{\infty} (-1)^k\, J_{-\nu+k}(t)\, J_k(z) = J_{-\nu}(z+t) \qquad [|z| < |t|] \qquad \text{WA 159}$$

2. $$\sum_{k=-\infty}^{\infty} Z_{\nu+k}(t)\, J_k(z) = Z_\nu(t-z) \qquad [|z| < |t|] \qquad \text{WA 159(5)}$$

8.54 The zeros of Bessel functions

8.541 For arbitrary real ν, the function $J_\nu(z)$ has infinitely many real zeros. For $\nu > -1$, all its zeros are real. WA 526, 530

A Bessel function $Z_\nu(z)$ has no multiple zeros except possibly the coordinate origin. WA 528

8.542 All zeros of the function $Y_0(z)$ with positive real parts are real. WA 531

8.543 If $-(2s+2) < \nu < -(2s+1)$, where s is a natural number or 0, then $J_\nu(z)$ has exactly $4s+2$ complex roots, two of which are purely imaginary. If $-(2s+1) < \nu < -2s$, where s is a natural number, then the function $J_\nu(z)$ has exactly $4s$ complex zeros, none of which are purely imaginary. WA 532

8.544 If x_ν and x'_ν are, respectively, the smallest positive zeros of the functions $J_\nu(z)$ and $J'_\nu(z)$ for $\nu > 0$, then $x_\nu > \nu$ and $x'_\nu > \nu$. Suppose also that y_ν is the smallest positive zero of the function $Y_\nu(z)$. Then, $x_\nu < y_\nu < x'_\nu$. WA 534, 536

Suppose that $z_{\nu,m}$ (for $m = 1, 2, 3, \dots$) are the zeros of the function $z^{-\nu} J_\nu(z)$, numbered in order of the absolute value of their real parts. Here, we assume that $\nu \neq -1, -2, -3, \dots$. Then, for arbitrary z

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right). \qquad \text{WA 550}$$

8.545[8] The number of zeros of the function $z^{-\nu} J_\nu(z)$ that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + \tfrac{1}{2}\operatorname{Re}\nu + \tfrac{1}{4}\right)\pi, \qquad \text{WA 497}$$

is exactly m.

8.546 For $\nu \geq 0$, the number of zeros of the function $K_\nu(z)$ that occur in the region $\operatorname{Re} z < 0$, $|\arg z| < \pi$ is equal to the even number closest to $\nu - \frac{1}{2}$. WA 562

8.547 Large zeros of the functions $J_\nu(z)\cos\alpha - Y_\nu(z)\sin\alpha$, where ν and α are real numbers, are given by the asymptotic expansion

$$\begin{aligned} x_{\nu,m} \sim & \left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]} \\ & - \frac{\left(4\nu^2 - 1\right)\left(28\nu^2 - 31\right)}{384\left[\left(m + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha\right]^3} - \dots \end{aligned} \qquad \text{KU 109(24), WA 558}$$

8.548 In particular, large zeros of the function $J_0(z)$ are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m-1) + \frac{1}{2\pi(4m-1)} - \frac{31}{6\pi^3(4m-1)^3} + \frac{3779}{15\pi^5(4m-1)^5} - \dots \qquad \text{KU 109(25), WA 556}$$

This series is suitable for calculating all (except the smallest x_{01}) zeros of the function $J_0(z)$ correctly to at least five digits.

8.549 To calculate the roots $x_{\nu,m}$ of the function $J_\nu(z)$ of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}. \qquad \text{KU 112(27)a, WA 554}$$

8.55 Struve functions

8.550 Definitions:

1. $$\mathbf{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)} \qquad \text{WA 358(2)}$$

2. $$\mathbf{L}_\nu(z) = -ie^{-i\nu\frac{\pi}{2}}\,\mathbf{H}_\nu\left(ze^{i\frac{\pi}{2}}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right)\Gamma\left(\nu + m + \frac{3}{2}\right)} \qquad \text{WA 360(11)}$$

8.551 Integral representations:

1. $$\mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 \left(1 - t^2\right)^{\nu - \frac{1}{2}} \sin zt\, dt = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi/2} \sin\left(z\cos\varphi\right)\left(\sin\varphi\right)^{2\nu} d\varphi$$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 358(1)

2. $$\mathbf{L}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{\pi/2} \sinh(z\cos\varphi)(\sin\varphi)^{2\nu}\,d\varphi$$

$\left[\operatorname{Re}\nu > -\frac{1}{2}\right]$ WA 360(11)

8.552 Special cases:

1.[6] $$\mathbf{H}_n(z) = \frac{1}{\pi}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma\left(n+\frac{1}{2}-m\right)} - \mathbf{E}_n(z)$$

$[n=1,2,\ldots]$ EH II 40(66), WA 337(1)

2.[6] $$\mathbf{H}_{-n}(z) = (-1)^{n+1}\frac{1}{\pi}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{\Gamma\left(n-m-\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)} - \mathbf{E}_{-n}(z)$$

$[n=1,2,\ldots]$ EH II 40(67), WA 337(2)

3. $$\mathbf{H}_{n+\frac{1}{2}}(z) = Y_{n+\frac{1}{2}}(z) + \frac{1}{\pi}\sum_{m=0}^{n} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)}$$

$[n=0,1,\ldots]$ EH II 39(64)

4. $\mathbf{H}_{-\left(n+\frac{1}{2}\right)}(z) = (-1)^n J_{n+\frac{1}{2}}(z)$ $[n=0,1,\ldots]$ EH II 39(65)

5. $\mathbf{L}_{-\left(n+\frac{1}{2}\right)}(z) = I_{n+\frac{1}{2}}(z)$ $[n=0,1,\ldots]$ EH II 39(65)

6. $$\mathbf{H}_{\frac{1}{2}}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}}(1-\cos z)$$ EH II 39, WA 364(3)

7. $$\mathbf{H}_{\frac{3}{2}}(z) = \left(\frac{z}{2\pi}\right)^{1/2}\left(1+\frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{1/2}\left(\sin z + \frac{\cos z}{z}\right)$$ WA 364(3)

8.553 Functional relations:

1. $\mathbf{H}_\nu\left(ze^{im\pi}\right) = e^{i\pi(\nu+1)m}\,\mathbf{H}_\nu(z)$ $[m=1,2,3,\ldots]$ WA 362(5)

2. $$\frac{d}{dz}\left[z^\nu\,\mathbf{H}_\nu(z)\right] = z^\nu\,\mathbf{H}_{\nu-1}(z)$$ WA 358

3. $$\frac{d}{dz}\left[z^{-\nu}\,\mathbf{H}_\nu(z)\right] = 2^{-\nu}\pi^{-1/2}\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1} - z^{-\nu}\,\mathbf{H}_{\nu+1}(z)$$ WA 359

4. $$\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{H}_\nu(z) + \pi^{-1/2}\left(\frac{z}{2}\right)^\nu\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}$$ WA 359(5)

5. $$\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\,\mathbf{H}'_\nu(z) - \pi^{-1/2}\left(\frac{z}{2}\right)^\nu\left[\Gamma\left(\nu+\frac{3}{2}\right)\right]^{-1}$$ WA 359(6)

8.554 Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = Y_\nu(\xi) + \frac{1}{\pi}\sum_{m=0}^{p-1} \frac{\Gamma\left(m+\frac{1}{2}\right)\left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu+\frac{1}{2}-m\right)} + O\left(|\xi|^{\nu-2p-1}\right)$$

$[|\arg\xi| < \pi]$ EH II 39(63), WA 363(2)

For the asymptotic representation of $Y_\nu(\xi)$, see **8.451** 2.

8.555 The differential equation for Struve functions:

$$z^2 y'' + zy' + \left(z^2 - \nu^2\right) y = \frac{1}{\sqrt{\pi}} \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\Gamma\left(\nu + \frac{1}{2}\right)}$$ WA 359(10)

8.56 Thomson functions and their generalizations

$\operatorname{ber}_\nu(z)$, $\operatorname{bei}_\nu(z)$, $\operatorname{her}_\nu(z)$, $\operatorname{hei}_\nu(z)$, $\operatorname{ker}_\nu(z)$, $\operatorname{kei}_\nu(z)$

8.561

1. $$\operatorname{ber}_\nu(z) + i \operatorname{bei}_\nu(z) = J_\nu\left(ze^{\frac{3}{4}\pi i}\right)$$ WA 96(6)

2. $$\operatorname{ber}_\nu(z) - i \operatorname{bei}_\nu(z) = J_\nu\left(ze^{-\frac{3}{4}\pi i}\right).$$ WA 96(6)

8.562

1. $$\operatorname{her}_\nu(z) + i \operatorname{hei}_\nu(z) = H^\nu_{(1)}\left(ze^{\frac{3}{4}\pi i}\right)$$ (see also **8.567**) WA 96(7)

2. $$\operatorname{her}_\nu(z) - i \operatorname{hei}_\nu(z) = H^\nu_{(1)}\left(ze^{-\frac{3}{4}\pi i}\right)$$ (see also **8.567**) WA 96(7)

8.563

1. $$\operatorname{ber}_0(z) \equiv \operatorname{ber}(z); \quad \operatorname{bei}_0(z) \equiv \operatorname{bei}(z)$$ WA 96(8)

2. $$\operatorname{ker}(z) \equiv -\frac{\pi}{2} \operatorname{hei}_0(z); \quad \operatorname{kei}(z) \equiv \frac{\pi}{2} \operatorname{hei}_0(z)$$ WA 96(8)

For integral representations, see **6.251**, **6.536**, **6.537**, **6.772** 4, **6.777**.

Series representation

8.564

1. $$\operatorname{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k}\left[(2k)!\right]^2}$$ WA 96(3)

2. $$\operatorname{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2}\left[(2k+1)!\right]^2}$$ WA 96(4)

3. $$\operatorname{ker}(z) = \left(\ln\frac{2}{z} - \boldsymbol{C}\right) \operatorname{ber}(z) + \frac{\pi}{4} \operatorname{bei}(z) + \sum_{k=1}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k}\left[(2k)!\right]^2} \sum_{m=1}^{2k} \frac{1}{m}$$ WA 96(9)a, DW

4. $$\operatorname{kei}(z) = \left(\ln\frac{2}{z} - \boldsymbol{C}\right) \operatorname{bei}(z) - \frac{\pi}{4} \operatorname{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2}\left[(2k+1)!\right]^2} \sum_{m=1}^{2k+1} \frac{1}{m}$$ WA 96(10)a, DW

8.565 $$\operatorname{ber}^2_\nu(z) + \operatorname{bei}^2_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2\nu+4k}}{k!\,\Gamma(\nu+k+1)\,\Gamma(\nu+2k+1)}$$ WA 163(6)

Asymptotic representation

8.566

1. $$\operatorname{ber}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos\beta(z) \qquad \left[|\arg z| < \frac{\pi}{4}\right]$$ WA 227(1)

2. $$\operatorname{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin\beta(z) \qquad \left[|\arg z| < \frac{\pi}{4}\right]$$ WA 227(1)

3. $$\operatorname{ker}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos\beta(-z) \qquad \left[|\arg z| < \frac{5}{4}\pi\right]$$ WA 227(2)

4. $$\operatorname{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin\beta(-z) \qquad \left[|\arg z| < \frac{5}{4}\pi\right],$$ WA 227(2)

where

$$\alpha(z) \sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots,$$
$$\beta(z) \sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots$$

8.567 Functional relations

1. $$\operatorname{ker}(z) + i\operatorname{kei}(z) = K_0\left(z\sqrt{i}\right)$$ (see **8.562**) WA 96(5), DW

2. $$\operatorname{ker}(z) - i\operatorname{kei}(z) = K_0\left(z\sqrt{-i}\right)$$ (see **8.562**) WA 96(5), DW

For integrals of Thomson's functions, see **6.87**.

8.57 Lommel functions

8.570 Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$:

1. $$s_{\mu,\nu}(z) = \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2-\nu^2]\,[(\mu+3)^2-\nu^2]\dots\left[(\mu+2m+1)^2-\nu^2\right]}$$
$$= z^{\mu-1}\sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)}$$
[$\mu \pm \nu$ is not a negative odd integer] EH II 40(69), WA 377(2)

2.[11] $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)$$
$$\times \frac{\cos\left[\frac{1}{2}(\mu-\nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu+\nu)\pi\right] J_\nu(z)}{\sin\nu\pi}$$ EH II 40(71), WA 379(2)
$$= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)$$
$$\times \left\{\sin\left[\frac{1}{2}(\mu-\nu)\pi\right] J_\nu(z) - \cos\left[\frac{1}{2}(\mu-\nu)\pi\right] Y_\nu(z)\right\}$$ EH II 41(71), WA 379(3)

Integral representations

8.571 $s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z z^\mu J_\nu(z)\,dz - J_\nu(z)\int_0^z z^\mu Y_\nu(z)\,dz\right]$ WA 378(9)

8.572 $s_{\mu,\nu}(z)$

$$= 2^\mu\left(\frac{z}{2}\right)^{\frac{1}{2}(1+\nu+\mu)}\Gamma\left(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu\right)\int_0^{\pi/2} J_{\frac{1}{2}(1+\mu-\nu)}(z\sin\theta)(\sin\theta)^{\frac{1}{2}(1+\nu-\mu)}(\cos\theta)^{\nu+\mu}\,d\theta$$

$[\operatorname{Re}(\nu+\mu+1)>0]$ EH II 42(86)

8.573 Special cases:

1. $S_{1,2n}(z) = zO_{2n}(z)$ WA 382(1)
2. $S_{0,2n+1}(z) = \frac{z}{2n+1}\,O_{2n+1}(z)$ WA 382(1)
3. $S_{-1,2n}(z) = \frac{1}{4n}\,S_{2n}(z)$ WA 382(2)
4. $S_{0,2n+1}(z) = \frac{1}{2}\,S_{2n+1}(z)$ WA 382(2)
5. $S_{\nu,\nu}(z) = \Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}2^{\nu-1}\,\mathbf{H}_\nu(z)$ EH II 42(84)
6. $S_{\nu,\nu}(z) = [\mathbf{H}_\nu(z) - Y_\nu(z)]\,2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)$ EH II 42(84)

8.574 Connections with other special functions:

1. $\mathbf{J}_\nu(z) = \frac{1}{\pi}\sin(\nu\pi)\left[s_{0,\nu}(z) - \nu\, s_{-1,\nu}(z)\right]$ EH II 41(82)
2. $\mathbf{E}_\nu(z) = -\frac{1}{\pi}\left[(1+\cos\nu\pi)\,s_{0,\nu}(z) + \nu\,(1-\cos\nu\pi)\,s_{-1,\nu}(z)\right]$ EH II 42(83)

A connection with a hypergeometric function

3. $s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\;{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{z^2}{4}\right)$

EH II 40(69), WA 378(10)

8.575 Functional relations:

1. $s_{\mu+2,\nu}(z) = z^{\mu+1} - \left[(\mu+1)^2-\nu^2\right]s_{\mu,\nu}(z)$ EH II 41(73), WA 380(1)

2.[8] $s'_{\mu,\nu}(z) + \left(\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z)$ EH II 41(74), WA 380(2)

3. $s'_{\mu,\nu}(z) - \left(\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(75), WA 380(3)

4. $\left(2\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(76), WA 380(4)

5.[8] $2\,s'_{\mu,\nu}(z) = (\mu+\nu-1)\,s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)\,s_{\mu-1,\nu+1}(z)$ EH II 41(77), WA 380(5)

In formulas **8.575** 1–5, $s_{\mu,\nu}(z)$ can be replaced with $S_{\mu,\nu}(z)$.

8.576 Asymptotic expansion of $S_{\mu,\nu}(z)$.

In the case in which $\mu \pm \nu$ is not a positive odd integer, $S_{\mu,\nu}(z)$ has the following asymptotic expansion:

$$S_{\mu,\nu}(z) \sim z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1-\mu+\nu}{2}\right)_m \left(\frac{1-\mu-\nu}{2}\right)_m \left(\frac{z}{2}\right)^{-2m}$$

$[|z| \to \infty, \quad |\arg z| < \pi]$ WA 347, 352

The series terminates and is equal to $S_{\mu,\nu}(z)$ when $\mu \pm \nu$ is a positive odd integer.

8.577 Lommel functions satisfy the following differential equation:

$$z^2 w'' + z w' + \left(z^2 - \nu^2\right) w = z^{\mu+1}$$ WA 377(1), EH II 40(68)

8.578 Lommel functions of two variables $U_\nu(w,z)$ and $V_\nu(w,z)$:

Definition

1. $$U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z)$$ EH II 42(87), WA 591(5)

2. $$V_\nu(w,z) = \cos\left[\frac{1}{2}\left(w + \frac{z^2}{w} + \nu\pi\right)\right] + U_{-\nu+2}(w,z)$$ EH II 42(88), WA 591(6)

Particular values:

3. $U_0(z,z) = V_0(z,z) = \frac{1}{2}\left\{J_0(z) + \cos z\right\}$ WA 591(9)

4. $U_1(z,z) = -V_1(z,z) = \frac{1}{2}\sin z$ WA 591(10)

5. $$U_{2n}(z,z) = \frac{(-1)^n}{2}\left\{\cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z)\right\}$$

$$[n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$$ WA 591(11)

6. $$U_{2n+1}(z,z) = \frac{(-1)^n}{2}\left\{\sin z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m+1} J_{2m+1}(z)\right\}$$

$$[n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0 \end{cases}$$ WA 591(12)

7. $$V_n(w,z) = (-1)^n U_n\left(\frac{z^2}{w}, z\right)$$

8. $$U_\nu(w,0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right)$$ WA 593(9)

9. $$V_{-\nu+2}(w,0) = \frac{\left(\frac{w}{2}\right)^{1/2}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{w}{2}\right)$$ WA 593(10)

8.579 Functional relations:

1. $$2\frac{\partial}{\partial w} U_\nu(w,z) = U_{\nu-1}(w,z) + \left(\frac{z}{w}\right)^2 U_{\nu+1}(w,z)$$ WA 593(2)

2. $$2\frac{\partial}{\partial w} V_\nu(w,z) = V_{\nu+1}(w,z) + \left(\frac{z}{w}\right)^2 V_{\nu-1}(w,z)$$ WA 593(4)

3. The function $U_\nu(w,z)$ is a particular solution of the differential equation

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z}\frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z)$$ WA 592(2)

4. The function $V_\nu(w,z)$ is a particular solution of the differential equation

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z}\frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z)$$ WA 592(3)

8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.580 Definitions:

1. The Anger function $\mathbf{J}_\nu(z)$:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\, d\theta$$ WA 336(1), EH II 35(32)

2. The Weber function $\mathbf{E}_\nu(z)$:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)\, d\theta$$ WA 336(2), EH II 35(32)

8.581 Series representations:

1.
$$\mathbf{J}_\nu(z) = \cos\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} + \sin\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}$$

EH II 36(36), WA 337(3)

2.
$$\mathbf{E}_\nu(z) = \sin\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)} - \cos\frac{\nu\pi}{2}\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right)\Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}$$

EH II 36(37), WA 338(4)

8.582 Functional relations:

1.[6] $2\,\mathbf{J}'_\nu(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z)$ EH II 36(40), WA 340(2)

2.[6] $2\,\mathbf{E}'_\nu(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z)$ EH II 36(41), WA 340(6)

3.[6] $\mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{J}_\nu(z) - 2(\pi z)^{-1}\sin(\nu\pi)$ EH II 36(42), WA 340(1)

4.[6] $\mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{E}_\nu(z) - 2(\pi z)^{-1}(1-\cos\nu\pi)$ EH II 36(43), WA 340(5)

8.583 Asymptotic expansions:

1.[6] $$\mathbf{J}_\nu(z) = J_\nu(z) + \frac{\sin\nu\pi}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)}\frac{\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n} + O\left(|z|^{-2p}\right) - \nu\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1} + \nu\, O\left(|z|^{-2p-1}\right)\right]$$
$[|\arg z| < \pi]$ EH II 37(47), WA 344(1)

2. $$\mathbf{E}_\nu(z) = -\,Y_\nu(z) - \frac{1+\cos(\nu\pi)}{\pi z}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+\frac{1+\nu}{2}\right)\Gamma\left(n+\frac{1-\nu}{2}\right)}{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}z^{-2n} + O\left(|z|^{-2p}\right)\right] - \frac{\nu\left(1-\cos\nu\pi\right)}{z\pi}\left[\sum_{n=0}^{p-1}(-1)^n 2^{2n}\frac{\Gamma\left(n+1+\frac{1}{2}\nu\right)\Gamma\left(n+1-\frac{1}{2}\nu\right)}{\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\nu\right)}z^{-2n-1} + O\left(|z|^{-2p-1}\right)\right]$$
WA344(2), EH II 37(48)

For the asymptotic expansion of $J_\nu(z)$ and $Y_\nu(z)$, see **8.451**.

8.584 The Anger and Weber functions satisfy the differential equation

$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = f(\nu, z),$$

where $f(\nu, z) = \dfrac{z-\nu}{\pi z^2}\sin\nu\pi$ for $\mathbf{J}_\nu(z)$ WA 341(9), EH II 37(44)

and $f(\nu, z) = -\dfrac{1}{\pi z^2}\left[z + \nu + (z-\nu)\cos\nu\pi\right]$ for $\mathbf{E}_\nu(z)$ EH II 37(45), WA 341(10)

8.59 Neumann's and Schläfli's polynomials: $O_n(z)$ and $S_n(z)$

8.590 Definition of Neumann's polynomials

1. $O_n(z) = \dfrac{1}{4}\displaystyle\sum_{m=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{n(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n-1}$ $[n \geq 1]$ WA 299(2), EH II 33(6)

2. $O_{-n}(z) = (-1)^n\, O_n(z)$ $[n \geq 1]$ WA 303(8)

3. $O_0(z) = \dfrac{1}{z}$ WA 299(3), EH II 33(7)

4. $O_1(z) = \dfrac{1}{z^2}$ EH II 33(7)

5. $O_2(z) = \dfrac{1}{z} + \dfrac{4}{z^3}$ EH II 33(7)

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$.

8.591 Functional relations:

1. $O_0'(z) = -\,O_1(z)$ EH II 33(9), WA 301(3)

2. $2\,O_n'(z) = O_{n-1}(z) - O_{n+1}(z)$ $[n \geq 1]$ EH II 33(10), WA 301(2)

3. $$(n-1)\,O_{n+1}(z) + (n+1)\,O_{n-1}(z) - 2z^{-1}\left(n^2-1\right)O_n(z) = 2nz^{-1}\left(\sin n\frac{\pi}{2}\right)^2$$
$[n \geq 1]$ EH II 33(11), WA 301(1)

4. $$nz\,O_{n-2}(z) - \left(n^2-1\right)O_n(z) = (n-1)z\,O_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2$$ EH II 33(12), WA 303(4)

5. $$nz\,O_{n+1}(z) - \left(n^2-1\right)O_n(z) = -(n+1)z\,O_n'(z) + n\left(\sin n\frac{\pi}{2}\right)^2$$ EH II 33(13), WA 303(5)a

8.592 The generating function:

$$\frac{1}{z-\xi} = J_0(\xi)z^{-1} + 2\sum_{n=1}^{\infty} J_n(\xi)\,O_n(z) \qquad [|\xi| < |z|]$$ EH II 32(1), WA 298(1)

8.593 The integral representation:

$$O_n(z) = \int_0^\infty \frac{\left[u+\sqrt{u^2+z^2}\right]^n + \left[u-\sqrt{u^2+z^2}\right]^n}{2z^{n+1}} e^{-u}\,du$$

See also **3.547** 6, 8, **3.549** 1, 2. EH II 32(3), WA 305(1)

8.594 The inequality

$$|O_n(z)| \leq 2^{n-1} n! |z|^{-n-1} e^{\frac{1}{4}|z|^2} \qquad [n > 1]$$ EH II 33(8), WA 300(8)

8.595 Neumann's polynomial $O_n(z)$ satisfies the differential equation

$$z^2\frac{d^2y}{dz^2} + 3z\frac{dy}{dz} + \left(z^2+1-n^2\right)y = z\left(\cos n\frac{\pi}{2}\right)^2 + n\left(\sin n\frac{\pi}{2}\right)^2$$ EH II 33(14), WA 303(1)

8.596 Schläfli's polynomials $S_n(z)$. These are the functions that satisfy the formulas

1. $$S_0(z) = 0$$ EH II 34(18), WA 312(2)

2. $$S_n(z) = \frac{1}{n}\left[2zO_n(z) - 2\left(\cos n\frac{\pi}{2}\right)^2\right] \qquad [n \geq 1]$$ EH II 34(19), WA 312(3)

$$= \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n} \qquad [n \geq 1]$$ EH II 34(18)

3. $$S_{-n}(z) = (-1)^{n+1}\,S_n(z)$$ WA 313(6)

8.597 Functional relations:

1. $$S_{n-1}(z) + S_{n+1}(z) = 4\,O_n(z)$$ WA 313(7)

Other functional relations may be obtained from **8.591** by replacing $O_n(z)$ with the expression for $S_n(z)$ given by **8.596** 2.

8.6 Mathieu Functions

8.60 Mathieu's equation

$$\frac{d^2y}{dz^2} + \left(a - 2k^2\cos 2z\right)y = 0, \quad k^2 = q$$ MA

8.61 Periodic Mathieu functions

8.610 In general, Mathieu's equation **8.60** does not have periodic solutions. If k is a real number, there exist infinitely many *eigenvalues* a, not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z).$$

If k is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions.*

8.611 Mathieu's equation has four series of distinct periodic solutions:

1. $$\mathrm{ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz$$ MA

2. $$\mathrm{ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z$$ MA

3. $$\mathrm{se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z$$ MA

4. $$\mathrm{se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z$$ MA

5. The coefficients A and B depend on q. The eigenvalues a of the functions ce_{2n}, ce_{2n+1}, se_{2n}, se_{2n+1} are denoted by a_{2n}, a_{2n+1}, b_{2n}, b_{2n+1}.

8.612 The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2\, dx = \pi$$ MO 65

8.613

1. $$\lim_{q\to 0} \mathrm{ce}_0(x) = \frac{1}{\sqrt{2}}$$

2. $$\lim_{q\to 0} \mathrm{ce}_n(x) = \cos nx \qquad [n \neq 0]$$

3. $$\lim_{q\to 0} \mathrm{se}_n(x) = \sin nx$$ MO 65

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$

8.621

1. $$aA_0^{(2n)} - qA_2^{(2n)} = 0$$ MA

2. $$(a-4)A_2^{(2n)} - q\left(A_4^{(2n)} + 2A_0^{(2n)}\right) = 0$$ MA

3. $$\left(a-4r^2\right)A_{2r}^{(2n)} - q\left(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}\right) = 0 \qquad [r \geq 2]$$ MA

8.622

1. $(a-1-q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0$ MA

2. $\left[a-(2r+1)^2\right] A_{2r+1}^{(2n+1)} - q\left(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)}\right) = 0 \qquad [r \geq 1]$ MA

8.623

1. $(a-1+q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0$ MA

2. $\left[a-(2r+1)^2\right] B_{2r+1}^{(2n+1)} - q\left(B_{2r+3}^{(2n+1)} + B_{2r-1}^{(2n+1)}\right) = 0$

$[r \geq 1]$ MA

8.624

1. $(a-4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0$ MA

2.[11] $\left(a-4r^2\right) B_{2r}^{(2n+2)} - q\left(B_{2r+2}^{(2n+2)} + B_{2r-2}^{(2n+2)}\right) = 0 \qquad [r \geq 2]$ MA

8.625 We can determine the coefficients A and B from equations **8.612**, **8.613** and **8.621-8.624** provided a is known. Suppose, for example, that we need to determine the coefficients $A_{2r}^{(2n)}$ for the function $\mathrm{ce}_{2n}(z,q)$. From the recursion formulas, we have

1. $$\begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a-4 & -q & 0 & 0 & \dots \\ 0 & -q & a-16 & -q & 0 & \dots \\ 0 & 0 & -q & a-36 & -q & \\ 0 & 0 & 0 & -q & a-64 & \\ \vdots & \vdots & \vdots & & & \ddots \end{vmatrix} = 0$$ ST

For given q in equation **8.625** 1, we may determine the eigenvalues

2. $a = A_0, A_2, A_4, \dots \qquad [|A_0| \leq |A_2| \leq |A_4| \leq \dots]$ no absolute values

If we now set $a = A_{2n}$, we can determine the coefficients $A_{2r}^{(2n)}$ from the recursion formulas **8.621** up to a proportionality coefficient. This coefficient is determined from the formula

3. $$2\left[A_0^{(2n)}\right]^2 + \sum_{r=1}^{\infty}\left[A_{2r}^{(2n)}\right]^2 = 1,$$ MA

which follows from the conditions of normalization.

8.63 Mathieu functions with a purely imaginary argument

8.630 If, in equation **8.60**, we replace z with iz, we arrive at the differential equation

1.[11] $$\frac{d^2y}{dz^2} + (-a + 2q\cosh 2z)\,y = 0$$

We can find the solutions of this equation if we replace the argument z with iz in the functions $\mathrm{ce}_n(z,q)$ and $\mathrm{se}_n(z,q)$. The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

1. $\mathrm{Ce}_{2n}(z,q), \quad \mathrm{Ce}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+1}(z,q), \quad \mathrm{Se}_{2n+2}(z,q)$

8.631

1. $$\mathrm{Ce}_{2n}(z,q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2rz$$ MA

2. $$\mathrm{Ce}_{2n+1}(z,q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh(2r+1)z$$ MA

3. $$\mathrm{Se}_{2n+1}(z,q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh(2r+1)z$$ MA

4. $$\mathrm{Se}_{2n+2}(z,q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh(2r+2)z$$ MA

8.64 Non-periodic solutions of Mathieu's equation

Along with each periodic solution of equation **8.60**, there exists a second non-periodic solution that is linearly independent. The non-periodic solutions are denoted as follows:

$$\mathrm{fe}_{2n}(z,q), \quad \mathrm{fe}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+1}(z,q), \quad \mathrm{ge}_{2n+2}(z,q).$$

Analogously, the second solutions of equation **8.630** 1 are denoted by

$$\mathrm{Fe}_{2n}(z,q), \quad \mathrm{Fe}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+1}(z,q), \quad \mathrm{Ge}_{2n+2}(z,q).$$

8.65 Mathieu functions for negative q

8.651 If we replace the argument z in equation **8.60** with$\pm\left(\frac{\pi}{2} \pm z\right)$, we get the equation

$$\frac{d^2y}{dz^2} + (a + 2q\cos 2z)\,y = 0.$$ MA

This equation has the following solutions:

8.652

1. $\mathrm{ce}_{2n}(z,-q) = (-1)^n\,\mathrm{ce}_{2n}\left(\frac{1}{2}\pi - z, q\right)$ MA
2. $\mathrm{ce}_{2n+1}(z,-q) = (-1)^n\,\mathrm{se}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA
3. $\mathrm{se}_{2n+1}(z,-q) = (-1)^n\,\mathrm{ce}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA
4. $\mathrm{se}_{2n+2}(z,-q) = (-1)^n\,\mathrm{se}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ MA
5. $\mathrm{fe}_{2n}(z,-q) = (-1)^{n+1}\,\mathrm{fe}_{2n}\left(\frac{1}{2}\pi - z, q\right)$ MA
6. $\mathrm{fe}_{2n+1}(z,-q) = (-1)^n\,\mathrm{ge}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA
7. $\mathrm{ge}_{2n+1}(z,-q) = (-1)^n\,\mathrm{fe}_{2n+1}\left(\frac{1}{2}\pi - z, q\right)$ MA
8. $\mathrm{ge}_{2n+2}(z,-q) = (-1)^n\,\mathrm{ge}_{2n+2}\left(\frac{1}{2}\pi - z, q\right)$ MA

8.653 Analogously, if we replace z with $\frac{\pi}{2}i + z$ in equation **8.630** 1, we get the equation

$$\frac{d^2y}{dz^2} - (a + 2q\cosh z)\,y = 0.$$

It has the following solutions:

8.654

1. $\operatorname{Ce}_{2n}(z, -q) = (-1)^n \operatorname{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right)$ MA

2. $\operatorname{Ce}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA

3. $\operatorname{Se}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ce}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA

4. $\operatorname{Se}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Se}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA

5. $\operatorname{Fe}_{2n}(z, -q) = (-1)^n \operatorname{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right)$ MA

6.[11] $\operatorname{Fe}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ge}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA

7.[11] $\operatorname{Ge}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Fe}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right)$ MA

8.[11] $\operatorname{Ge}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Ge}_{2n+2}\left(\frac{1}{2}\pi i + z, q\right)$ MA

8.66 Representation of Mathieu functions as series of Bessel functions

8.661

1. $$\operatorname{ce}_{2n}(z, q) = \frac{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, J_{2r}\left(2k\cos z\right)$$ MA
$$= \frac{\operatorname{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, I_{2r}\left(2k\sin z\right)$$ MA

2. $$\operatorname{ce}_{2n+1}(z, q) = -\frac{\operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)}\, J_{2r+1}\left(2k\cos z\right)$$ MA
$$= \frac{\operatorname{ce}_{2n+1}(0, q)}{kA_1(2n+1)} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)}\, I_{2r+1}\left(2k\sin z\right)$$ MA

3. $$\operatorname{se}_{2n+1}(z, q) = \frac{\operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)}\, J_{2r+1}\left(2k\cos z\right)$$ MA
$$= \frac{\operatorname{se}'_{2n+1}(0, q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)}\, I_{2r+1}\left(2k\sin z\right)$$ MA

4. $$\operatorname{se}_{2n+2}(z, q) = \frac{-\operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)}\, J_{2r+2}(2k\ \cos_z)$$ MA
$$= \frac{\operatorname{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \cot z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)}\, I_{2r+2}(2k\sin z)$$ MA

8.662

1. $$\operatorname{fe}_{2n}(z, q) = -\frac{\pi \operatorname{fe}'_{2n}(0, q)}{2\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \operatorname{Im}\left[J_r\left(ke^{iz}\right) Y_r\left(ke^{-iz}\right)\right]$$ MA

2. $$\mathrm{fe}_{2n+1}(z,q) = \frac{\pi k\, \mathrm{fe}'_{2n+1}(0,q)}{2\,\mathrm{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)} \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \operatorname{Im}\left[J_r\left(ke^{iz}\right) Y_{r+1}\left(ke^{-iz}\right) + J_{r+1}\left(ke^{iz}\right) Y_r\left(ke^{-iz}\right)\right]$$

MA

3. $$\mathrm{ge}_{2n+1}(z,q) = -\frac{\pi k\, \mathrm{ge}_{2n+1}(0,q)}{2\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},q\right)} \times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \operatorname{Re}\left[J_r\left(ke^{iz}\right) Y_{r+1}\left(ke^{-iz}\right) - J_{r+1}\left(ke^{iz}\right) Y_r\left(ke^{-iz}\right)\right]$$

MA

4. $$\mathrm{ge}_{2n+2}(z,q) = -\frac{\pi k^2\, \mathrm{ge}_{2n+2}(0,q)}{2\,\mathrm{se}'_{2n+2}\left(\frac{1}{2}\pi,q\right)} \times \sum_{r=0}^{\infty} (-1)^r \operatorname{Re}\left[J_k\left(ke^{iz}\right) Y_{r+2}\left(ke^{-iz}\right) - J_{r+2}\left(ke^{iz}\right) Y_r\left(ke^{-iz}\right)\right]$$

MA

The expansions of the functions Fe_n and Ge_n as series of the functions Y_ν are denoted, respectively, by Fey_n and Gey_n, and the expansions of these functions as series of the functions K_ν are denoted, respectively, by Fek_n and Gek_n.

8.663

1. $$\mathrm{Fey}_{2n}(z,q) = \frac{\mathrm{ce}_{2n}(0,q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)}\, Y_{2r}(2k\sinh z)$$

$k^2 = q\, [|\sinh z| > 1, \quad \operatorname{Re} z > 0]$ MA

$$= \frac{\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, Y_{2r}(2k\cosh z)$$

$[|\cosh z| > 1]$ MA

$$= \frac{\mathrm{ce}_{2n}(0,q)\,\mathrm{ce}_{2n}\left(\frac{\pi}{2},q\right)}{\left[A_0^{(2n)}\right]^2} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, J_r\left(ke^{-z}\right) Y_r\left(ke^{z}\right)$$

MA

2. $$\operatorname{Fey}_{2n+1}(z,q) = \frac{\operatorname{ce}_{2n+1}(0,q)\coth z}{kA_1(2n+1)} \sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(2n+1)}\, Y_{2r+1}(2k\sinh z),$$

$$k^2 = q, \quad [|\sinh z| > 1, \quad \operatorname{Re} z > 0]$$ MA

$$= -\frac{\operatorname{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\, Y_{2r+1}(2k\cosh z)$$

$$[|\cosh z| > 1]$$ MA

$$= -\frac{\operatorname{ce}_{2n+1}(0,q)\operatorname{ce}'_{2n+1}\left(\frac{\pi}{2},q\right)}{k\left[A_1^{(2n+1)}\right]^2}$$

$$\times \sum_{r=0}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}\left[J_r\left(ke^{-z}\right) Y_{r+1}\left(ke^{z}\right) + J_{r+1}\left(ke^{-z}\right) Y_r\left(ke^{z}\right)\right]$$

MA

3. $$\operatorname{Gey}_{2n+1}(z,q) = \frac{\operatorname{se}'_{2n+1}(0,q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}\, Y_{2r+1}(2k\sinh z)$$

$$[|\sinh z| > 1, \quad \operatorname{Re} z > 0]$$ MA

$$= \frac{\operatorname{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{kB_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty}(-1)^r(2r+1)B_{2r+1}^{(2n+1)}\, Y_{2r+1}(2k\cosh z)$$

$$[|\cosh z| > 1]$$ MA

$$= \frac{\operatorname{se}_{2n+1}(0,q)\operatorname{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{k\left[B_1^{(2n+1)}\right]^2} \sum_{r=0}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}$$

$$\times \left[J_r\left(ke^{-z}\right) Y_{r+1}\left(ke^{z}\right)\right] J_{r+1}\left(ke^{-z}\right) Y_r\left(ke^{z}\right)$$

MA

4. $$\operatorname{Gey}_{2n+2}(z,q) = \frac{\operatorname{se}'_{2n+2}(0,q)}{k^2 B_2^{(2n+2)}} \coth z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)}\, Y_{2r+2}(2k \sinh z)$$

$[|\sinh z| > 1, \quad \operatorname{Re} z > 0]$ MA

$$= -\frac{\operatorname{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)}\, Y_{2r+2}(2k \cosh z)$$

$[|\cosh z| > 1]$ MA

$$= \frac{\operatorname{se}'_{2n+2}(0,q)\, \operatorname{se}'_{2n+2\left(\frac{\pi}{2},q\right)}}{k^2 \left[B_2^{(2n+2)}\right]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)}$$

$$\times \left[J_r\left(ke^{-z}\right) Y_{r+2}\left(ke^{z}\right)\right] - J_{r+2}\left(ke^{-z}\right) Y_r\left(ke^{z}\right)$$

MA

8.664

1. $$\operatorname{Fek}_{2n}(z,q) = \frac{\operatorname{ce}_{2n}(0,q)}{\pi A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}\, K_{2r}(-2ik \sinh z)$$

$k^2 = q, \qquad [|\sinh z| > 1, \quad \operatorname{Re} z > 0]$ MA

2. $$\operatorname{Fek}_{2n+1}(z,q) = \frac{\operatorname{ce}_{2n+1}(0,q)}{\pi k A_1^{(2n+1)}} \coth z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)}\, K_{2r+1}(-2ik \sinh z)$$

$k^2 = q \qquad [|\sinh z| > 1, \quad \operatorname{Re} z > 0]$ MA

3. $$\operatorname{Gek}_{2n+1}(z,q) = \frac{\operatorname{se}_{2n+1}\left(\frac{\pi}{2},q\right)}{\pi k B_1^{(2n+1)}} \tanh z \sum_{r=0}^{\infty} (2r+1) B_{2r+1}^{(2n+1)}\, K_{2r+1}(-2ik \cosh z)$$ MA

4. $$\operatorname{Gek}_{2n+2}(z,q) = \frac{\operatorname{se}'_{2n+2}\left(\frac{\pi}{2},q\right)}{\pi k^2 B_2^{(2n+2)}} \tanh z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)}\, K_{2r+2}(-2ik \cosh z)$$ MA

8.67 The general theory

If $i\mu$ is not an integer, the general solution of equation **8.60** can be found in the form

8.671

1. $$y = Ae^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} + Be^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rzi}$$ MA

The coefficients c_{2r} can be determined from the homogeneous system of linear algebraic equations

2.[11] $c_{2r} + \xi_{2r}\left(c_{2r+2} + c_{2r-2}\right) = 0, \qquad r = \ldots, -2, -1, 0, 1, 2, \ldots,$ MA

where

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}$$

The condition that this system be compatible yields an equation that μ must satisfy:

3.[7] $$\Delta(i\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$ MA

This equation can also be written in the form

4. $\cosh\mu\pi = 1 - 2\Delta(0)\sin^2\left(\frac{\pi\sqrt{a}}{2}\right)$, where $\Delta(0)$ is the value that is assumed by the determinant of the preceding article if we set $\mu = 0$ in the expressions for ξ_{2r}.

5. If the pair (a, q) is such that $|\cosh\mu\pi| < 1$, then $\mu = i\beta, \operatorname{Im}\beta = 0$, and the solution **8.671** 1 is bounded on the real axis.

6. If $|\cosh\mu\pi| > 1, \mu$ may be real or complex, and the solution **8.671** 1 will not be bounded on the real axis.

7. If $\cosh\mu\pi = \pm 1$, then $i\mu$ will be an integer. In this case, one of the solutions will be of period π or 2π (depending on whether n is even or odd). The second solution is non-periodic (see **8.61** and **8.64**).

8.7–8.8 Associated Legendre Functions

8.70 Introduction

8.700 An *associated Legendre function* is a solution of the differential equation

1. $$\left(1 - z^2\right)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1 - z^2}\right]u = 0,$$

in which ν and μ are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see **9.151**). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable z that lie in the interval $[-1, 1]$ and, on the other hand, in solutions corresponding to an arbitrary complex number z such that $\operatorname{Re} z > 1$. These are multiple-valued in the z-plane. To separate these functions into single-valued branches, we make a cut along the real axis from $-\infty$ to $+1$. We are also interested in those solutions of equation **8.700** 1 for which ν or μ or both are integers. Of special significance is the case in which $\mu = 0$.

8.701 In connection with this, we shall use the following notations:

The letter z will denote *an arbitrary complex variable*; the letter x will denote a *real* variable that varies over the interval $[-1, +1]$. We shall sometimes set $x = \cos\varphi$, where φ is a real number.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular for $|z| < 1$ and, in particular, uniquely determined for $z = x$.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation **8.700** 1 that are single-valued and regular forRe $z > 1$. When these functions cannot be unrestrictedly extended without violating their single-valuedness, we make a cut along the real axis to the left of the point $z = 1$. The values of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ on the upper and lower boundaries of that portion of the cuts lying between the points -1 and $+1$ are denoted, respectively, by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters n and m denote natural numbers or zero. The letters ν and μ denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z)$$

The *linearly independent* functions

8.702 $$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1-\mu; \quad \frac{1-z}{2}\right)$$

$$\left[\arg\frac{z+1}{z-1} = 0, \text{ if } z \text{ is real and greater than 1 and}\right]$$ MO 80, WH

8.703 $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)\,\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(z^2-1\right)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

[$\arg\left(z^2-1\right) = 0$ when z is real and greater than 1; $\arg z = 0$ when z is real and greater than zero] which are solutions of the differential equation **8.700** 1, are called *associated Legendre functions* (or *spherical functions*) of *the first* and *second kinds*, respectively. They are uniquely defined, respectively, in the intervals $|1-z| < 2$ and $|z| > 1$, with the portion of the real axis that lies between $-\infty$ and $+1$ excluded. They can be extended by means of hypergeometric series to the entire z-plane where the above-mentioned cut was made. These expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ lose their meaning when $1-\mu$ and $\nu+\frac{3}{2}$ are non-positive integers, respectively. MO 80

When z is a real number lying on the interval $[-1, +1]$, so that $(z = x = \cos\varphi)$, we take the following functions as linearly independent solutions of the equation:

8.704 $$P_\nu^\mu(x) = \tfrac{1}{2}\left[e^{\frac{1}{2}\mu\pi i}\,P_\nu^\mu(\cos\varphi + i0) + e^{-\frac{1}{2}\mu\pi i}\,P_\nu^\mu(\cos\varphi - i0)\right]$$ EH I 143(1)

$$= \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$ EH I 143(6)

8.705 $$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-\mu\pi i}\left[e^{-\frac{1}{2}\mu\pi i}\,Q_\nu^\mu(x+i0) + e^{\frac{1}{2}\mu\pi i}\,Q_\nu^\mu(x-i0)\right]$$ EH I 143(2)

$$= \frac{\pi}{2\sin\mu\pi}\left[P_\nu^\mu(x)\cos\mu\pi - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\,P_\nu^{-\mu}(x)\right]$$ (cf. **8.732** 5)

If $\mu = \pm m$ is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

8.706

1. $$Q_\nu^m(x) = (-1)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}\,Q_\nu(x)$$ (cf. **8.752** 1) EH I 149(7)

2.[11] $$Q_\nu^{-m}(x) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\,Q_\nu^m(x)$$ EH I 144(18)

The functions $Q_\nu^\mu(z)$ are not defined when $\nu+\mu$ is equal to a negative integer. Therefore, we must exclude the cases when $\nu+\mu = -1, -2, -3, \ldots$ for these formulas.

The functions

$$P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

are *linearly independent solutions* of the differential equation for $\nu + \mu \neq 0, \pm 1, \pm 2, \ldots$.

8.707 Nonetheless, two linearly independent solutions can always be found. Specifically, for $\nu \pm \mu$ not an integer, the differential equation **8.700** 1 has the following solutions:

1. $P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$

   respectively, for $z = x = \cos\varphi$,

2. $P_{\nu}^{\pm\mu}(\pm x), \quad Q_{\nu}^{\pm\mu}(\pm x), \quad P_{-\nu-1}^{\pm\mu}(\pm x), \quad Q_{-\nu-1}^{\pm\mu}(\pm x)$.

   If $\nu \pm \mu$ is not an integer, the solutions

3. $P_{\nu}^{\mu}(z), \quad Q_{\nu}^{\mu}(z)$, respectively, and $P_{\nu}^{\mu}(x), \quad Q_{\nu}^{\mu}(x)$

   are linearly independent. If $\nu \pm \mu$ is an integer but μ itself is not an integer, the following functions are linearly independent solutions of equation **8.700** 1:

4. $P_{\nu}^{\mu}(z), \quad P_{\nu}^{-\mu}(z)$, respectively, and $P_{\nu}^{\mu}(x), \quad P_{\nu}^{-\mu}(x)$.

   If $\mu = \pm m, \nu = n$, or $\nu = -n-1$, the following functions are linearly independent solutions of equation **8.700** 1 for $n \geq$m:

5. $P_{n}^{m}(z), \quad Q_{n}^{m}(z)$, respectively, and $P_{n}^{m}(x), \quad Q_{n}^{m}(x)$,

   and for $n <$m, the following functions will be linearly independent solutions

6. $P_{n}^{-m}(z), \quad Q_{n}^{m}(z)$, respectively, and $P_{n}^{-m}(x), \quad Q_{n}^{m}(x)$.

8.71 Integral representations

8.711

1. $$P_{\nu}^{-\mu}(z) = \frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{2^{\mu}\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)} \int_{-1}^{1} \frac{\left(1-t^2\right)^{\mu-\frac{1}{2}}}{\left(z+t\sqrt{z^2-1}\right)^{\mu-\nu}}\,dt \qquad \left[\operatorname{Re}\mu > -\tfrac{1}{2}, \quad |\arg(z\pm 1)| < \pi\right]$$

   MO 88

2. $$P_{\nu}^{m}(z) = \frac{(\nu+1)(\nu+2)\ldots(\nu+m)}{\pi} \int_0^{\pi} \left[z+\sqrt{z^2-1}\cos\varphi\right]^{\nu} \cos m\varphi\,d\varphi$$
   $$= (-1)^m \frac{\nu(\nu-1)\ldots(\nu-m+1)}{\pi} \int_0^{\pi} \frac{\cos m\varphi\,d\varphi}{\left[z+\sqrt{z^2-1}\cos\varphi\right]^{\nu+1}}$$
   $$\left[|\arg z| < \frac{\pi}{2}, \quad \arg\left(z+\sqrt{z^2-1}\cos\varphi\right) = \arg z \text{ for } \varphi = \frac{\pi}{2}\right]$$

   (cf. **8.822** 1) SM 483(15), WH

3. $$Q_{\nu}^{\mu}(z) = \sqrt{\pi} \frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)}{2^{\mu}\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu+1)} \left(z^2-1\right)^{\frac{\mu}{2}} \int_0^{\infty} \frac{\sinh^{2\mu} t\,dt}{\left(z+\sqrt{z^2-1}\cosh t\right)^{\nu+\mu+1}}$$
   $$[\operatorname{Re}(\nu\pm\mu) > -1, \quad |\arg(z\pm 1)| < \pi]$$

   (cf. **8.822** 2) MO 88

4. $$Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)} \int_0^{\infty} \frac{\cosh\mu t\,dt}{\left(z+\sqrt{z^2-1}\cosh t\right)^{\nu+1}}$$
   $$[\operatorname{Re}(\nu+\mu) > -1, \nu \neq -1, -2, -3, \ldots, \quad |\arg(z\pm 1)| < \pi]$$

   WH, MO 88

5. $$\int_{-1}^{1} P_l^2(x)\,P_l^0(x)\,dx = -\frac{l!}{(l-2)!}\frac{1}{2l+1} = -\frac{l(l-1)}{2l+1}$$

8.712 $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\,\Gamma(\nu+1)}\left(z^2-1\right)^{-\frac{\mu}{2}}\int_{-1}^{1}\left(1-t^2\right)^{\nu}(z-t)^{-\nu-\mu-1}\,dt$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}\mu > -1, \quad |\arg(z\pm 1)| < \pi]$ (cf. **8.821** 2) MO 88a, EH I 155(5)a

8.713

1. $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2\pi}}\left(z^2-1\right)^{\frac{\mu}{2}}\left\{\int_0^\pi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(z-\cos t)^{\mu+\frac{1}{2}}} - \cos\nu\pi\int_0^\infty \frac{e^{-\left(\nu+\frac{1}{2}\right)t}\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}\right\}$$
$\left[\operatorname{Re}\mu > -\frac{1}{2}, \quad \operatorname{Re}(\nu+\mu) > -1, \quad |\arg(z\pm 1)| < \pi\right]$ MO 89

2. $$P_\nu^{-\mu}(z) = \frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{2^\nu\,\Gamma(\mu-\nu)\,\Gamma(\nu+1)}\int_0^\infty \frac{\sinh^{2\nu+1} t}{(z+\cosh t)^{\nu+\mu+1}}\,dt$$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+1) > 0, \quad \operatorname{Re}(\mu-\nu) > 0]$ MO 89

3. $$P_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}\int_0^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(z+\cosh t)^{\mu+\frac{1}{2}}}$$
$[\operatorname{Re} z > -1, \quad |\arg(z\pm 1)| < \pi, \quad \operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$ MO 89

8.714

1. $$P_\nu^\mu(\cos\varphi) = \sqrt{\frac{2}{\pi}}\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\varphi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(\cos t-\cos\varphi)^{\mu+\frac{1}{2}}}$$
$\left[0 < \varphi < \pi, \quad \operatorname{Re}\mu < \frac{1}{2}\right]$; (cf. **8.823**) MO 87

2. $$P_\nu^{-\mu}(\cos\varphi) = \frac{\Gamma(2\mu+1)\sin^\mu\varphi}{2^\mu\,\Gamma(\mu+1)\,\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}\int_0^\infty \frac{t^{\nu+\mu}\,dt}{(1+2t\cos\varphi+t^2)^{\mu+\frac{1}{2}}}$$
$[\operatorname{Re}(\nu+\mu) > -1, \quad \operatorname{Re}(\mu-\nu) > 0]$ MO 89

3. $$Q_\nu^\mu(\cos\varphi) = \frac{1}{2^{\mu+1}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)} \times\int_0^\infty\left[\frac{\sinh^{2\mu} t}{(\cos\varphi+i\sin\varphi\cosh t)^{\nu+\mu+1}} + \frac{\sinh^{2\mu} t}{(\cos\varphi-i\sin\varphi\cosh t)^{\nu+\mu+1}}\right]dt$$
$\left[\operatorname{Re}(\nu+\mu+1) > 0, \quad \operatorname{Re}(\nu-\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ MO 89

4. $$P_\nu^\mu(\cos\varphi) = \frac{i}{2^\mu}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)} \times\int_0^\infty\left[\frac{\sinh^{2\mu} t}{(\cos\varphi+i\sin\varphi\cosh t)^{\nu+\mu+1}} - \frac{\sinh^{2\mu} t}{(\cos\varphi-i\sin\varphi\cosh t)^{\nu+\mu+1}}\right]dt$$
$\left[\operatorname{Re}(\nu\pm\mu+1) > 0, \quad \operatorname{Re}\mu > -\frac{1}{2}\right]$ MO 89

8.715

1. $$P_\nu^\mu(\cosh\alpha) = \frac{\sqrt{2}\sinh^\mu\alpha}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\alpha \frac{\cosh\left(\nu+\frac{1}{2}\right)t\,dt}{(\cosh\alpha-\cosh t)^{\mu+\frac{1}{2}}}$$
$$\left[\alpha>0, \quad \operatorname{Re}\mu<\tfrac{1}{2}\right]$$ MO 87

2. $$Q_\nu^\mu(\cosh\alpha) = \sqrt{\frac{\pi}{2}}\frac{e^{\mu\pi i}\sinh^\mu\alpha}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\alpha^\infty \frac{e^{-\left(\nu+\frac{1}{2}\right)t}\,dt}{(\cosh t-\cosh\alpha)^{\mu+\frac{1}{2}}}$$
$$\left[\alpha>0, \quad \operatorname{Re}\mu<\tfrac{1}{2}, \quad \operatorname{Re}(\nu+\mu)>-1\right]$$ MO 87

See also **3.277** 1, 4, 5, 7, **3.318**, **3.516** 3, **3.518** 1, 2, **3.542** 2, **3.663** 1, **3.894**, **3.988** 3, **6.622** 3, **6.628** 1, 4–7, and also **8.742**.

8.72 Asymptotic series for large values of $|\nu|$

8.721[6] For real values of $\mu, |\nu| \gg 1, |\nu| \gg |\mu|, |\arg\nu| < \pi$, we have:

1. $$P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}}\Gamma(\nu+\mu+1)\sum_{k=0}^\infty \frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)}\frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi+\frac{\pi}{4}(2k-1)+\frac{\mu\pi}{2}\right]}{k!\,\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$$
$$\Big[\nu+\mu\neq-1,-2,-3,\ldots; \quad \nu\neq-\tfrac{3}{2},-\tfrac{5}{2},\tfrac{7}{2}\ldots; \text{ for } \frac{\pi}{6}<\varphi<\frac{5\pi}{6}$$
This series also converges for complex values of ν and μ. In the remaining cases, it is an asymptotic expansion for
$$|\nu|\gg|\mu|, |\nu|\gg 1, \text{if } \nu>0, \mu>0 \text{ and } 0<\varepsilon\leq\varphi\leq\pi-\varepsilon\Big]$$
MO 92

2.[6] $$Q_\nu^\mu(\cos\varphi) = \sqrt{\pi}\,\Gamma(\nu+\mu+1)$$
$$\times\sum_{k=0}^\infty(-1)^k\frac{\Gamma\left(\mu+k+\frac{1}{2}\right)}{\Gamma\left(\mu-k+\frac{1}{2}\right)}\frac{\cos\left[\left(\nu+k+\frac{1}{2}\right)\varphi-\frac{\pi}{4}(2k-1)+\frac{\mu\pi}{2}\right]}{k!\,\Gamma\left(\nu+k+\frac{3}{2}\right)(2\sin\varphi)^{k+\frac{1}{2}}}$$
$$\Big[\nu+\mu\neq-1,-2,-3,\ldots; \quad \nu\neq-\tfrac{3}{2},-\tfrac{5}{2},-\tfrac{7}{2},\ldots; \text{ for } \frac{\pi}{6}<\varphi<\frac{5}{6}\pi$$
This series also converges for complex values of ν and μ. In the remaining cases, it is an asymptotic expansion for
$$|\nu|\gg|\mu|, \quad |\nu|\gg 1, \text{if } \nu>0, \quad \mu>0, \quad 0<\varepsilon\leq\varphi\leq\pi-\varphi\Big]$$
EH I 147(6), MO 92

3. $$P_\nu^\mu(\cos\varphi) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{\cos\left[\left(\nu+\frac{1}{2}\right)\varphi-\frac{\pi}{4}+\frac{\mu\pi}{2}\right]}{\sqrt{2\sin\varphi}}\left[1+O\left(\frac{1}{\nu}\right)\right]$$
$$\left[0<\varepsilon\leq\varphi\leq\pi-\varepsilon, \quad |\nu|\gg\frac{1}{\varepsilon}\right]$$ MO 92

For $\nu>0, \mu>0$ and $\nu>\mu$, it follows from formulas **8.721** 1 and **8.721** 2 that

4. $$\nu^{-\mu} P_\nu^\mu(\cos\varphi) = \sqrt{\frac{2}{\nu\pi\sin\varphi}}\cos\left[\left(\nu+\frac{1}{2}\right)\varphi - \frac{\pi}{4} + \frac{\mu\pi}{2}\right] + O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

5. $$\nu^{-\mu} Q_\nu^\mu(\cos\varphi) = \sqrt{\frac{\pi}{2\nu\sin\varphi}}\cos\left[\left(\nu+\frac{1}{2}\right)\varphi + \frac{\pi}{4} + \frac{\mu\pi}{2}\right] O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

$$\left[0 < \varepsilon \le \varphi \le \pi - \varepsilon; \quad \nu \gg \frac{1}{\varepsilon}\right]$$ MO 92

8.722 If φ is sufficiently close to 0 or π that $\nu\varphi$ or $\nu(\pi-\varphi)$ is small in comparison with 1, the asymptotic formulas **8.721** become unsuitable. In this case, the following asymptotic representation is applicable for $\mu \le 0, \nu \gg 1$, and *small* values of φ:

1. $$\left[\left(\nu+\frac{1}{2}\right)\cos\frac{\varphi}{2}\right]^\mu P_\nu^{-\mu}(\cos\varphi) = J_\mu(\eta) + \sin^2\frac{\varphi}{2}\left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6}J_{\mu+3}(\eta)\right] + O\left(\sin^4\frac{\varphi}{2}\right)$$

where $\eta = (2\nu+1)\sin\frac{\varphi}{2}$. In particular, it follows that

1. $$\lim_{\nu\to\infty} \nu^\mu P_\nu^{-\mu}\left(\cos\frac{x}{\nu}\right) = J_\mu(x) \qquad [x \ge 0, \mu \ge 0]$$ MO 93

8.723 We can see how the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ behave for large $|\nu|$ and real values of $z > \frac{3}{2\sqrt{2}}$:

1. $$P_\nu^\mu(\cosh\alpha) = \frac{2^\mu}{\sqrt{\pi}}\left\{\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}\frac{e^{(\mu-\nu)\alpha}\sinh^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}} F\left(\mu+\frac{1}{2}, -\mu+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{1-e^{2\alpha}}\right)\right.$$
$$\left. + \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{e^{(\nu+\mu+1)\alpha}\sinh^\mu\alpha}{(e^{2\alpha}-1)^{\mu+\frac{1}{2}}} F\left(\mu+\frac{1}{2}, -\mu+\frac{1}{2}; -\nu+\frac{1}{2}; \frac{1}{1-e^{2\alpha}}\right)\right\}$$
$$\left[\nu \ne \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \dots; \quad a > \tfrac{1}{2}\ln 2\right]$$ MO 94

2. $$Q_\nu^\mu(\cosh\alpha) = e^{\mu\pi i} 2^\mu \sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{e^{-(\nu+\mu+1)\alpha}}{(1-e^{-2\alpha})^{\mu+\frac{1}{2}}}\sinh^\mu\alpha$$
$$\times F\left(\mu+\frac{1}{2}, -\mu+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{1-e^{2\alpha}}\right)$$
$$\left[\mu+\nu+1 \ne 0, -1, -2, \dots; \quad \alpha > \tfrac{1}{2}\ln 2\right]$$ MO 94

See also **8.776**.

8.724 For the inequalities in **8.776** 1–4, ν and μ are arbitrary real numbers satisfying the inequalities $\nu \ge 1$, $\nu-\mu+1 > 0$, and $\mu \ge 0$:

1. $$\left|P_\nu^{\pm\mu}(\cos\varphi)\right| < \sqrt{\frac{8}{\nu\pi}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

2. $$\left|Q_\nu^{\pm\mu}(\cos\varphi)\right| < \sqrt{\frac{2\pi}{\nu}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

3. $$\left|P_\nu^{\pm\mu}(\cos\varphi)\right| < \frac{2}{\sqrt{\nu\pi}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

4. $$\left|Q_\nu^{\pm\mu}(\cos\varphi)\right| < \sqrt{\frac{\pi}{\nu}}\frac{\Gamma(\nu\pm\mu+1)}{\Gamma(\nu+1)}\frac{1}{\sin^{\mu+\frac{1}{2}}\varphi}$$ MO 91-92

5.[8] $$\left|\sqrt{\sin\varphi}\,P_n^m(\cos\varphi)\right| < \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n-m+1)}2^{(m+n)^2/n}\sup_{0<t<\infty}\left|\sqrt{t}\,J_m(t)\right|$$

[uniformly $0 \le m \le n$]

8.725[10] For fixed z and ν and $\operatorname{Re}\mu\to\infty$, with z not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$, the following are asymptotic expansions in which the upper and lower signs are taken according to whether $\operatorname{Im} z$ is greater than or less than 0:

1. $$P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)}{\pi\,\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\sin\mu\pi\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\right.$$
$$\left.-\frac{\sin\nu\pi}{\sin\mu\pi}e^{\mp i\mu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$
AS 8.10.1

2. $$Q_\nu^\mu(z) = \frac{1}{2}e^{i\mu\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\Gamma(\mu-\nu)\left[F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}+\frac{1}{2}z\right)\right.$$
$$\left.-e^{\mp i\nu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F\left(-\nu,\nu+1;1+\mu;\frac{1}{2}-\frac{1}{2}z\right)\right]$$
AS 8.10.2

3. $$Q_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi}\operatorname{cosec}\left[\pi(\nu-\mu)\right]}{2\pi\,\Gamma(1+\mu)}\left[e^{\mp i\nu\pi}\left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\,\nu+1;\,1+\mu;\,\frac{1}{2}-\frac{1}{2}z\right)\right.$$
$$\left.-\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}\mu}F\left(-\nu,\,\nu+1;\,1+\mu;\,\frac{1}{2}+\frac{1}{2}z\right)\right]$$
AS 8.10.3

8.73–8.74 Functional relations

8.731

1. $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = (\nu-\mu+1)\,P_{\nu+1}^\mu(z)-(\nu+1)z\,P_\nu^\mu(z)$$
(cf. **8.832** 1, **8.914** 2)
EH I 161(10), MO 81

1(1)[9] $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = \nu z\,P_\nu^\mu(z)-(\nu+\mu)\,P_{\nu-1}^\mu(z)$$ AS 8.5.4

1(2) $$\left(z^2-1\right)\frac{d\,P_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)\sqrt{z^2-1}\,P_\nu^{\mu-1}(z)-\mu z\,P_\nu^\mu(z)$$ AS 8.5.2

2. $$(2\nu+1)z\,P_\nu^\mu(z) = (\nu-\mu+1)\,P_{\nu+1}^\mu(z) + (\nu+\mu)\,P_{\nu-1}^\mu(z)$$
(cf. **8.832** 2, **8.914** 1) EH I 160(2), MO 81

3. $$P_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}\,P_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)\,P_\nu^\mu(z)$$
MO 82, EH I 160(1)

3(1)[9] $$P_\nu^{\mu+1}(z) = \left(z^2-1\right)^{-1/2}\left[(\nu-\mu)z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)\right]$$
AS 8.5.1

4. $$P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu+1)\sqrt{z^2-1}\,P_\nu^{\mu-1}(z)$$
EH I 160(3), MO 82

4(1)[9] $$(\nu-\mu+1)\,P_{\nu+1}^\mu(z) = (2\nu+1)z\,P_\nu^\mu(z) - (\nu+\mu)\,P_{\nu-1}^\mu(z)$$
AS 334(8.5.3)

4(2)[9] $$P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)\left(z^2-1\right)^{1/2}P_\nu^{\mu-1}(z)$$
AS 334(8.5.5)

5. $$P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$$
(cf. **8.820**, **8.832** 4) EH I 140(1), MO 82

8.732

1. $$\left(z^2-1\right)\frac{d\,Q_\nu^\mu(z)}{dz} = (\nu-\mu+1)\,Q_{\nu+1}^\mu(z) - (\nu+1)z\,Q_\nu^\mu(z)$$
(cf. **8.832** 3) MO 82

2.[10] $$(2\nu+1)z\,Q_\nu^\mu(z) = (\nu-\mu+1)\,Q_{\nu+1}^\mu(z) + (\nu+\mu)\,Q_{\nu-1}^\mu(z)$$
(cf. **8.832** 4) MO 82

3. $$Q_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}\,Q_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)\,Q_\nu^\mu(z)$$
MO 82

4. $$Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu+1)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$$
MO 82a

5. $$e^{-\mu\pi i}\,Q_\nu^\mu\,(x\pm i0) = e^{\pm\frac{1}{2}\mu\pi i}\left[Q_\nu^\mu(x) \mp i\frac{\pi}{2}\,P_\nu^\mu(x)\right]$$
MO 83

8.733

1. $$\begin{aligned}\left(1-x^2\right)\frac{d\,P_\nu^\mu(x)}{dx} &= P_\nu^\mu(x) - (\nu-\mu+1)\,P_{\nu+1}^\mu(x) \qquad \text{(cf. } \mathbf{8.731}\ 1)\\ &= -\nu x\,P_\nu^\mu(x) + (\nu+\mu)\,P_{\nu-1}^\mu(x)\\ &= -\sqrt{1-x^2}\,P_\nu^{\mu+1}(x) - \mu x\,P_\nu^\mu(x);\\ &= (\nu-\mu+1)(\nu+\mu)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x) + \mu x\,P_\nu^\mu(x)\end{aligned}$$
MO 82

2. $$(2\nu+1)x\,P_\nu^\mu(x) = (\nu-\mu+1)\,P_{\nu+1}^\mu(x) + (\nu+\mu)\,P_{\nu-1}^\mu(x)$$
(cf. **8.731** 2) MO 82

3.[11] $$P_\nu^{\mu+2}(x) + 2(\mu+1)\frac{x}{\sqrt{1-x^2}}\,P_\nu^{\mu+1}(x) + (\nu-\mu)(\nu+\mu+1)\,P_\nu^\mu(x) = 0$$
(cf. **8.731** 3) MO 82

4. $$P_{\nu-1}^\mu(x) - P_{\nu+1}^\mu(x) = (2\nu+1)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$$
(cf. **8.731** 4) MO 82

5. $$P_{-\nu-1}^\mu(x) = P_\nu^\mu(x)$$
(cf. **8.731** 5)

8.734

1. $(\nu+\mu+1)z\,Q_\mu^\nu(z)+\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu+1)\,Q_{\nu+1}^\mu(z)$ MO 82

2. $(\nu+\mu)\,Q_{\nu-1}^\mu(z)+\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu)z\,Q_\nu^\mu(z)$ MO 82

3. $Q_{\nu-1}^\mu(z)-z\,Q_\nu^\mu(z)=-(\nu-\mu+1)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$ MO 82

4. $z\,Q_\nu^\mu(z)-Q_{\nu+1}^\mu(z)=-(\nu+\mu)\sqrt{z^2-1}\,Q_\nu^{\mu-1}(z)$ MO 82

5. $(\nu+\mu)(\nu+\mu+1)\,Q_{\nu-1}^\mu(z)+(2\nu+1)\sqrt{z^2-1}\,Q_\nu^{\mu+1}(z)=(\nu-\mu)(\nu-\mu+1)\,Q_{\nu+1}^\mu(z)$ MO 82

8.735

1. $(\nu+\mu+1)x\,P_\nu^\mu(x)+\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)=(\nu-\mu+1)\,P_{\nu+1}^\mu(x)$ MO 83

2. $(\nu-\mu)x\,P_\nu^\mu(x)-(\nu+\mu)\,P_{\nu-1}^\mu(x)=\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)$ MO 83

3. $P_{\nu-1}^\mu(x)-x\,P_\nu^\mu(x)=(\nu-\mu+1)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$ MO 83

4. $x\,P_\nu^\mu(x)-P_{\nu+1}^\mu(x)=(\nu+\mu)\sqrt{1-x^2}\,P_\nu^{\mu-1}(x)$ MO 83

5. $(\nu-\mu)(\nu-\mu+1)\,P_{\nu+1}^\mu(x)=(\nu+\mu)(\nu+\mu+1)\,P_{\nu-1}^\mu(x)+(2\nu+1)\sqrt{1-x^2}\,P_\nu^{\mu+1}(x)$ MO 83

8.736

1. $P_\nu^{-\mu}(z)=\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z)-\frac{2}{\pi}e^{-\mu\pi i}\sin\mu\pi\,Q_\nu^\mu(z)\right]$ MO 83

2. $P_\nu^\mu(-z)=e^{\nu\pi i}\,P_\nu^\mu(z)-\frac{2}{\pi}\sin[(\nu+\mu)\pi]e^{-\mu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im} z<0]$ (cf. **8.833** 1) MO 83

3. $P_\nu^\mu(-z)=e^{-\nu\pi i}\,P_\nu^\mu(z)-\frac{2}{\pi}\sin[(\nu+\mu)\pi]e^{-\mu\pi i}\,Q_\nu^\mu(z)$
$[\operatorname{Im} z>0]$ (cf. **8.833** 2) MO 83

4. $Q_\nu^{-\mu}(z)=e^{-2\mu\pi i}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,Q_\nu^\mu(z)$ MO 82

5. $Q_\nu^\mu(-z)=-e^{-\nu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im} z<0]$ MO 82

6. $Q_\nu^\mu(-z)=-e^{\nu\pi i}\,Q_\nu^\mu(z)$ $[\operatorname{Im} z>0]$ MO 82

7.[6] $Q_\nu^\mu(z)\sin[(\nu+\mu)\pi]-Q_{-\nu-1}^\mu(z)\sin[(\nu-\mu)\pi]=\pi e^{\mu\pi i}\cos\nu\pi\,P_\nu^\mu(z)$ MO 83

8.737

1. $P_\nu^{-\mu}(x)=\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[\cos\mu\pi\,P_\nu^\mu(x)-\frac{2}{\pi}\sin(\mu\pi)\,Q_\nu^\mu(x)\right]$ MO 84

2. $P_\nu^\mu(-x)=\cos[(\nu+\mu)\pi]\,P_\nu^\mu(x)-\frac{2}{\pi}\sin[(\nu+\mu)\pi]\,Q_\nu^\mu(x)$ MO 84

3. $Q_\nu^\mu(-x)=-\cos[(\nu+\mu)\pi]\,Q_\nu^\mu(x)-\frac{\pi}{2}\sin[(\nu+\mu)\pi]\,P_\nu^\mu(x)$ MO 83, EH I 144(15)

4. $Q_{-\nu-1}^\mu(x)=\frac{\sin[(\nu+\mu)\pi]}{\sin[(\nu-\mu)\pi]}\,Q_\nu^\mu(x)-\frac{\pi\cos\nu\pi\cos\mu\pi}{\sin[(\nu-\mu)\pi]}\,P_\nu^\mu(x)$ MO 84

8.738

1.[11] $$Q_\nu^\mu(i\cot\varphi) = \exp\left[i\pi\left(\mu - \frac{\nu+1}{2}\right)\right]\sqrt{\pi}\,\Gamma(\nu+\mu+1)\sqrt{\frac{1}{2}\sin\varphi}\,P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi)$$
$$\left[0 < \varphi < \frac{\pi}{2}\right]$$ MO 83

2.[6] $$P_\nu^\mu(i\cot\varphi) = \sqrt{\frac{2}{\pi}}\exp\left[i\pi\left(\nu + \frac{1}{4}\right)\right]\frac{\sqrt{\sin\varphi}}{\Gamma(-\nu-\mu)}\,Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\cos\varphi - i0)$$
$$\left[0 < \varphi < \frac{\pi}{2}\right]$$ MO 83

8.739 $$e^{-\mu\pi i}\,Q_\nu^\mu(\cosh\alpha) = \frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{\sqrt{2\sinh\alpha}}\,P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(\coth\alpha) \qquad [\operatorname{Re}(\cosh\alpha) > 0]$$ MO 83

8.741

1. $$P_\nu^{-\mu}(x)\frac{d\,P_\nu^\mu(x)}{dx} - P_\nu^\mu(x)\frac{d\,P_\nu^{-\mu}(x)}{dx} = \frac{2\sin\mu\pi}{\pi\left(1-x^2\right)}$$ MO 83

2. $$P_\nu^\mu(x)\frac{d\,Q_\nu^\mu(x)}{dx} - Q_\nu^\mu(x)\frac{d\,P_\nu^\mu(x)}{dx} = \frac{2^{2\mu}}{1-x^2}\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)}$$ MO 83

8.742

1. $$\frac{\Gamma(\nu-\mu-1)}{\Gamma(\nu+\mu+1)}\left\{\cos\mu\pi\,P_\nu^\mu(\cos\varphi) - \frac{2}{\pi}\sin\mu\pi\,Q_\nu^\mu(\cos\varphi)\right\} = \sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\varphi \frac{\cos\left(\nu+\frac{1}{2}\right)t\,dt}{(\cos t - \cos\varphi)^{\frac{1}{2}-\mu}}$$
$$\left[\operatorname{Re}\mu > -\tfrac{1}{2}\right]$$ MO 88

2. $$\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left\{\cos\nu\pi\,P_\nu^\mu(\cos\varphi) - \frac{2}{\pi}\sin\nu\pi\,Q_\nu^\mu(\cos\varphi)\right\}$$
$$= \sqrt{\frac{2}{\pi}}\frac{\operatorname{cosec}^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_\varphi^\pi \frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]\,dt}{(\cos\varphi - \cos t)^{\frac{1}{2}-\mu}}$$
$$\left[\operatorname{Re}\mu > -\tfrac{1}{2}\right]$$ MO 88

3. $$P_\nu^\mu(\cos\varphi)\cos(\nu+\mu)\pi - \frac{2}{\pi}\,Q_\nu^\mu(\cos\varphi)\sin(\nu+\mu)\pi = \sqrt{\frac{2}{\pi}}\frac{\sin^\mu\varphi}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\varphi^\pi \frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right]\,dt}{(\cos\varphi - \cos t)^{\mu+\frac{1}{2}}}$$
$$\left[\operatorname{Re}\mu < \tfrac{1}{2}\right]$$ MO 88

4. $$\cos\mu\pi\,P_\nu^\mu(\cos\varphi) - \frac{2}{\pi}\sin\mu\pi\,Q_\nu^\mu(\cos\varphi)$$
$$= \frac{1}{2^\mu\sqrt{\pi}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\frac{\sin^\mu\varphi}{\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^\pi \frac{\sin^{2\mu}t\,dt}{(\cos\varphi \pm i\sin\varphi\cos t)^{\nu-\mu}}$$
$$\left[\operatorname{Re}\mu > -\tfrac{1}{2}, \quad 0 < \varphi < \pi\right]$$ MO 38

For integrals of Legendre functions, see **7.11–7.21**.

8.75 Special cases and particular values

8.751

1. $$P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu+m+1)\left(1-x^2\right)^{\frac{m}{2}}}{2^m\,\Gamma(\nu-m+1)m!}\,F\left(m-\nu, m+\nu+1; m+1; \frac{1-x}{2}\right)$$ MO 84

2. $$P_\nu^m(z) = \frac{\Gamma(\nu+m+1)\left(z^2-1\right)^{\frac{m}{2}}}{2^m m!\,\Gamma(\nu-m+1)}\,F\left(m-\nu, m+\nu+1; m+1; \frac{1-z}{2}\right)$$ MO 84

3.[8] $$Q_{n+\frac{1}{2}}^\mu(z) = \frac{e^{\mu\pi i}\,\Gamma\left(\mu+n+\frac{3}{2}\right)}{2^{n+\frac{3}{2}}(n+1)!}\left(z^2-1\right)^{\frac{\mu}{2}}\pi^{1/2}z^{-n-\mu-3/2}\,F\left(\frac{\mu+n+\frac{5}{2}}{2}, \frac{\mu+n+\frac{3}{2}}{2}; n+2; \frac{1}{z^2}\right)$$ MO 84

8.752

1. $$P_\nu^m(x) = (-1)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_\nu(x)$$ WH, MO 84, EH I 148(6)

2. $$P_\nu^{-m}(x) = (-1)^m\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}P_\nu^m(x) = \left(1-x^2\right)^{-\frac{m}{2}}\int_x^1\ldots\int_x^1 P_\nu(x)(dx)^m$$ $[m \geq 1]$ HO 99a, MO 85, EH I 149(10)a

3. $$P_\nu^{-m}(z) = \left(z^2-1\right)^{-\frac{m}{2}}\int_1^z\ldots\int_1^z P_\nu(z)(dz)^m$$ $[m \geq 1]$ MO 85, EH I 149(8)

4. $$Q_\nu^m(z) = \left(z^2-1\right)^{\frac{m}{2}}\frac{d^m}{dz^m}Q_\nu(z)$$ WH, MO 85, EH I 148(5)

5. $$Q_\nu^{-m}(z) = (-1)^m\left(z^2-1\right)^{-\frac{m}{2}}\int_z^\infty\ldots\int_z^\infty Q_\nu(z)(dz)^m$$ $[m \geq 1]$ MO 85, EH I 149(9)

Special values of the indices

8.753

1. $$P_0^\mu(\cos\varphi) = \frac{1}{\Gamma(1-\mu)}\cot^\mu\frac{\varphi}{2}$$ MO 84

2. $$P_\nu^{-1}(\cos\varphi) = -\frac{1}{\nu(\nu+1)}\frac{d\,P_\nu(\cos\varphi)}{d\varphi}$$ MO 84

3. $$P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0$$ for $m > n$ MO 85

8.754

1. $$P_{\nu-\frac{1}{2}}^{1/2}(\cosh\alpha) = \sqrt{\frac{2}{\pi\sinh\alpha}}\cosh\nu\alpha$$ MO 85

2. $$P_{\nu-\frac{1}{2}}^{1/2}(\cos\varphi) = \sqrt{\frac{2}{\pi\sin\varphi}}\cos\nu\varphi$$ MO 85

3. $$P_{\nu-\frac{1}{2}}^{-1/2}(\cos\varphi) = \sqrt{\frac{2}{\pi\sin\varphi}}\frac{\sin\nu\varphi}{\nu}$$ MO 85

4. $$Q^{1/2}_{\nu-\frac{1}{2}}(\cosh\alpha) = i\sqrt{\frac{\pi}{2\sinh\alpha}}e^{-\nu\alpha}$$ MO 85

8.755

1. $$P^{-\nu}_{\nu}(\cos\varphi) = \frac{1}{\Gamma(1+\nu)}\left(\frac{\sin\varphi}{2}\right)^{\nu}$$ MO 85

2. $$P^{-\nu}_{\nu}(\cosh\alpha) = \frac{1}{\Gamma(1+\nu)}\left(\frac{\sinh\alpha}{2}\right)^{\nu}$$ MO 85

Special values of Legendre functions

8.756

1. $$P^{\mu}_{\nu}(0) = \frac{2^{\mu}\sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)}$$ MO 84

2. $$\frac{d\,P^{\mu}_{\nu}(0)}{dx} = \frac{2^{\mu+1}\sin\frac{1}{2}(\nu+\mu)\pi\,\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$ MO 84

3. $$Q^{\mu}_{\nu}(0) = -2^{\mu-1}\sqrt{\pi}\sin\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}$$ MO84

4. $$\frac{d\,Q^{\mu}_{\nu}(0)}{dx} = 2^{\mu}\sqrt{\pi}\cos\frac{1}{2}(\nu+\mu)\pi\frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$ MO 84

8.76 Derivatives with respect to the order

8.761 $$\frac{\partial P^{-\mu}_{\nu}(x)}{\partial\nu} = \frac{1}{\Gamma(\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}}\sum_{n=1}^{\infty}\frac{(-\nu)(1-\nu)\dots(n-1-\nu)(\nu+1)(\nu+2)\dots(\nu+n)}{(\mu+1)(\mu+2)\dots(\mu+n)1\cdot 2\dots n}$$
$$\times\left[\psi(\nu+n+1)-\psi(\nu-n+1)\right]\left(\frac{1-x}{2}\right)^{n}$$
$[\nu \neq 0, \pm 1, \pm 2, \dots; \quad \operatorname{Re}\mu > -1]$ MO 94

8.762

1. $$\left[\frac{\partial\,P_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = 2\ln\cos\frac{\varphi}{2}$$ MO 94

2. $$\left[\frac{\partial P^{-1}_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=0} = -\tan\frac{\varphi}{2} - 2\cot\frac{\varphi}{2}\ln\cos\frac{\varphi}{2}$$ MO 94

3. $$\left[\frac{\partial P^{-1}_{\nu}(\cos\varphi)}{\partial\nu}\right]_{\nu=1} = -\frac{1}{2}\tan\frac{\varphi}{2}\sin^2\frac{\varphi}{2} + \sin\varphi\ln\cos\frac{\varphi}{2}$$ MO 94

- For a connection with the polynomials $C^{\lambda}_{n}(x)$, see **8.936**.
- For a connection with a hypergeometric function, see **8.77**.

8.77 Series representation

For a representation in the form of a series, see **8.721**. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

8.771

1. $$P_\nu^\mu(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$ MO 15

2.[8] $$Q_\nu^\mu(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{z^{\nu+\mu+1}} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$ MO 15

See also **8.702**, **8.703**, **8.704**, **8.723**, **8.751**, **8.772**.

The analytic continuation for $|z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see **9.154** and **9.155**):

8.772

1. $$P_\nu^\mu(z) = \frac{\sin(\nu+\mu)\pi\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi}\cos\nu\pi\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(z^2-1\right)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$
$$+\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)} \left(z^2-1\right)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$
$$[2\nu \neq \pm1, \pm3, \pm5, \ldots; \quad |z| > 1; \quad |\arg(z\pm1)| < \pi]$$ MO 85

2. $$P_\nu^\mu(z) = \frac{\Gamma\left(-\nu-\frac{1}{2}\right)\left(z^2-1\right)^{-\frac{\nu+1}{2}}}{2^{\nu+1}\sqrt{\pi}\,\Gamma(-\nu-\mu)} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)$$
$$+\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)} \left(z^2-1\right)^{\frac{\nu}{2}} F\left(\frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}; \frac{1}{2}-\nu; \frac{1}{1-z^2}\right)$$
$$\left[2\nu \neq \pm1, \pm3, \pm5; \ldots; \quad \left|1-z^2\right| > 1; \quad |\arg(z\pm1)| < \pi\right]$$ MO 85

3. $$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^\nu F\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right)$$
$$\left[\left|\frac{z-1}{z+1}\right| < 1\right]$$ MO 86

8.773

1. $$Q_\nu^\mu(z) = e^{\mu\pi i} \frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(z^2-1\right)^{-\frac{\nu+1}{2}} F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)$$
$$\left[\nu+\mu \neq -1, -2, -3, \ldots; \quad |\arg(z\pm1)| < \pi; \quad \left|1-z^2\right| > 1\right]$$ MO 86

2. $$Q_\nu^\mu(z) = \frac{1}{2} e^{\mu\pi i} \left\{\Gamma(\mu) \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)\right.$$
$$\left.+\frac{\Gamma(-\mu)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu+1; \quad 1+\mu; \quad \frac{1-z}{2}\right)\right\}$$
$$[|\arg(z\pm1)| < \pi, \quad |1-z| < 2]$$ MO 86

8.774 $$P_\nu^\mu(i\cot\varphi) = \sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu-\mu)}e^{-i(\nu+1)\frac{\pi}{2}}\left(\tan\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\nu+\frac{3}{2};\sin^2\frac{\varphi}{2}\right) + \sqrt{\frac{\sin\varphi}{2\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}e^{i\nu\frac{\pi}{2}}\left(\cot\frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\frac{1}{2}-\nu;\sin^2\frac{\varphi}{2}\right)$$
$$\left[2\nu \neq \pm1,\pm3,\pm5,\ldots,\quad 0<\varphi<\frac{\pi}{2}\right]$$ MO 86

8.775

1.[6] $$P_\nu^\mu(x) = \frac{2^\mu\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\left(1-x^2\right)^{\frac{\mu}{2}} F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right) + \frac{2^{\mu+1}}{\sqrt{\pi}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}x\left(1-x^2\right)^{\frac{\mu}{2}} F\left(\frac{\nu+\mu}{2}+1,\frac{-\nu+\mu+1}{2};\frac{3}{2};x^2\right)$$ MO 87

2.[6] $$Q_\nu^\mu(x) = -\frac{\sqrt{\pi}}{2^{1-\mu}}\frac{\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\left(1-x^2\right)^{\frac{\mu}{2}} F\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right) + 2^\mu\sqrt{\pi}\frac{\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}x\left(1-x^2\right)^{\frac{\mu}{2}} F\left(\frac{\nu+\mu}{2}+1,\frac{\mu-\nu+1}{2};\frac{3}{2};x^2\right)$$ MO 87

8.776 For $|z| \gg 1$

1. $$P_\nu^\mu(z) = \left\{\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu-\mu+1)}z^\nu + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\,\Gamma(-\nu-\mu)}z^{-\nu-1}\right\}\left(1+O\left(\frac{1}{z^2}\right)\right)$$
$$[2\nu\neq\pm1,\pm3,\pm5,\ldots,\quad |\arg z|<\pi]$$ MO 87

2. $$Q_\nu^\mu(z) = \sqrt{\pi}\frac{e^{\mu\pi i}}{2^{\nu+1}}\frac{\Gamma(\mu+\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}\left(1+O\left(\frac{1}{z^2}\right)\right)$$
$$[2\nu\neq-3,-5,-7,\ldots;\quad |\arg z|<\pi]$$ MO 87

8.777 Set $\zeta = z+\sqrt{z^2-1}$. The variable ζ is uniquely defined by this equation on the entire z-plane in which a cut is made from $-\infty$ to $+1$. Here, we are considering that branch of the variable ζ for which values of ζ exceeding 1 correspond to real values of z exceeding 1. In this case,

1. $$P_\nu^\mu(z) = \frac{2^\mu\,\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu-\mu)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2}+\mu,\nu+\mu+1;\nu+\frac{3}{2};\frac{1}{\zeta^2}\right) + \frac{2^\mu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu-\mu+1)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\mu-\nu}} F\left(\frac{1}{2}+\mu,\mu-\nu;\frac{1}{2}-\nu;\frac{1}{\zeta^2}\right)$$
$$[2\nu\neq\pm1,\pm3,\pm5,\ldots;\quad |\arg(z-1)|<\pi]$$ MO 86

2. $$Q_\nu^\mu(z) = 2^\mu e^{\mu\pi i}\sqrt{\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)}\frac{\left(z^2-1\right)^{\frac{\mu}{2}}}{\zeta^{\nu+\mu+1}} F\left(\frac{1}{2}+\mu,\nu+\mu+1;\nu+\frac{3}{2};\frac{1}{\zeta^2}\right)$$
$$[|\arg(z-1)|<\pi]$$ MO 86

8.78 The zeros of associated Legendre functions

8.781 The function $P_\nu^{-\mu}(\cos\varphi)$, considered as a function of ν, has infinitely many zeros for $\mu \geq 0$. These are all simple and real. If a number ν_0 is a zero of the function $P_\nu^{-\mu}(\cos\varphi)$, the number $-\nu_0 - 1$ is also a zero of this function. MO 91

8.782 If ν and μ are both real and $\mu \leq 0$, or if ν and μ are integers, the function $P_\nu^\mu(t)$ has no *real* zeros exceeding 1. If ν and μ are both real with $\nu < \mu < 0$, the function $P_\nu^\mu(t)$ has no real zeros exceeding 1 when $\sin\mu\pi\sin(\mu-\nu)\pi > 0$, but does have one such zero when $\sin\mu\pi\sin(\mu-\nu)\pi < 0$. Finally, if $\mu \leq \nu$, the function $P_\nu^\mu(t)$ has no zeros exceeding 1 for $\lfloor\mu\rfloor$ even but does have one zero for $\lfloor\mu\rfloor$ odd.

8.783 If $\nu > -\frac{3}{2}$ and $\nu + \mu + 1 > 0$, the function $Q_\nu^\mu(t)$ has no real zeros exceeding 1. MO 91

8.784 The function $P_{-\frac{1}{2}+i\lambda}(z)$ has infinitely many zeros for real λ. All these zeros are *real* and *greater than unity.*

8.785 For n a natural number, the function $P_n(x)$ has exactly n real zeros which lie in the closed interval $-1, +1$.

8.786 The function $Q_n(z)$ has no zeros for which $|\arg(z-1)| < \pi$ if n is a natural number. The function $Q_n(\cos\varphi)$ has exactly $n+1$ zeros in the interval $0 \leq \varphi \leq \pi$. MO 91

8.787 The following approximate formula can be used to calculate the values of ν for which the equation $P_\nu^{-\mu}(\cos\varphi) = 0$ holds for given small values of φ:

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2\sin\frac{\varphi}{2}}\left\{1 - \frac{\sin^2\frac{\varphi}{2}}{6}\left(1 - \frac{4\mu^2-1}{j_\mu^2}\right) + O\left(\sin^4\frac{\varphi}{2}\right)\right\}. \qquad \text{MO 93}$$

Here, j_μ denotes an arbitrary nonzero root of the equation $J_\mu(z) = 0$ (for $\mu \geq 0$). If φ is close to π then, instead of this formula, we can use the following formulas:

1. $$\nu \approx \mu + k + \frac{\Gamma(2\mu+k+1)}{\Gamma(\mu)\,\Gamma(\mu+1)\,\Gamma(k+1)}\left(\frac{\pi-\varphi}{3}\right)^{2\mu} \qquad [\mu > 0, \quad k = 0, 1, 2, \ldots] \qquad \text{MO 93}$$

2. $$\nu \approx k + \frac{1}{2\ln\left(\frac{2}{\pi-\varphi}\right)} \qquad [\mu = 0, \quad k = 0, 1, 2, \ldots] \qquad \text{MO 93}$$

8.79 Series of associated Legendre functions

8.791

1. $$\frac{1}{z-t} = \sum_{k=0}^{\infty}(2k+1)\,P_k(t)\,Q_k(z) \qquad \left[\left|t+\sqrt{t^2-1}\right| < \left|z+\sqrt{z^2-1}\right|\right]$$

Here, t must lie inside an ellipse passing through the point z with foci at the points ± 1.

2. $$\frac{1}{\sqrt{1-2tz+t^2}}\ln\frac{z-t+\sqrt{1-2tz+t^2}}{\sqrt{z^2-1}} = \sum_{k=0}^{\infty} t^k\,Q_k(z)$$

$$[\operatorname{Re} z > 1, \quad |t| < 1] \qquad \text{MO 78}$$

8.792 $$P_\nu^{-\alpha}(\cos\varphi)\,P_\nu^{-\beta}(\cos\psi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right]P_k^{-\alpha}(\cos\varphi)\,P_k^{-\beta}(\cos\psi)$$

$$[a \geq 0, \quad \beta \geq 0, \quad \nu \text{ real}, \quad -\pi < \varphi \pm \psi < \pi] \qquad \text{MO 94}$$

8.793 $$P_\nu^{-\mu}(\cos\varphi) = \frac{\sin\nu\pi}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right)P_k^{-\mu}(\cos\varphi) \qquad [\mu \geq 0, \quad 0 < \varphi < \pi]$$ MO 94

Addition theorems

8.794

1.[11] $$\begin{aligned} P_\nu(\cos\psi_1\cos\psi_2 + \sin\psi_1\sin\psi_2\cos\varphi) &= P_\nu(\cos\psi_1)\,P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)\,P_\nu^k(\cos\psi_2)\cos k\varphi \\ &= P_\nu(\cos\psi_1)\,P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)}P_\nu^k(\cos\psi_1)\,P_\nu^k(\cos\psi_2)\cos k\varphi \end{aligned}$$ $[0 \leq \psi_1 < \pi, \quad 0 \leq \psi_2 < \pi, \quad \psi_1 + \psi_2 < \pi, \quad \varphi \text{ real}]$ (cf. **8.814**, **8.844** 1) MO 90

2. $$\begin{aligned} Q_\nu(\cos\psi_1)\cos\psi_2 + \sin\psi_1\sin\psi_2\cos\varphi &= P_\nu(\cos\psi_1)\,Q_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)\,Q_\nu k(\cos\psi_2)\cos k\varphi \end{aligned}$$ $\left[0 < \psi_1 < \frac{\pi}{2}, \quad 0 < \psi_2 < \pi, \quad 0 < \psi_1 + \psi_2 < \pi; \quad \varphi \text{ real}\right]$ (cf. **8.844** 3) MO 90

8.795

1. $$P_\nu\left(z_1z_2 - \sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\varphi\right) = P_\nu(z_1)\,P_\nu(z_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^k(z_1)\,P_\nu^{-k}(z_2)\cos k\varphi$$ $[\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0, \quad |\arg(z_1-1)| < \pi, \quad |\arg(z_2-1)| < \pi]$ MO 91

2. $$Q_\nu\left(x_1x_2 - \sqrt{x_1^2-1}\sqrt{x_2^2-1}\cos\varphi\right) = P_\nu(x_1)\,Q_\nu(x_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(x_1)\,Q_\nu^k(x_2)\cos k\varphi$$ $[1 < x_1 < x_2, \quad \nu \neq -1, -2, -3, \ldots, \quad \varphi \text{ real}]$ MO 91

3. $$Q_n\left(x_1x_2 + \sqrt{x_1^2+1}\sqrt{x_2^2+1}\cosh\alpha\right) = \sum_{k=n+1}^{\infty}\frac{1}{(k-n-1)!(k+n)!}Q_n^k(ix_1)\,Q_n^k(ix_2)\,e^{-k\alpha}$$ $[x_1 > 0, \quad x_2 > 0, \quad \alpha > 0]$ MO 91

8.796 $$\begin{aligned} P_\nu(-\cos\psi_1\cos\psi_2 - \sin\psi_1\sin\psi_2\cos\varphi) = P_\nu(-\cos\psi_1)\,P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)} \\ \times P_\nu^{-k}(-\cos\psi_1)\,P_\nu^{-k}(\cos\psi_2)\cos k\varphi \end{aligned}$$ $[0 < \psi_2 < \psi_1 < \pi, \quad \varphi \text{ real}]$ (cf. **8.844** 2) MO 91

See also **8.934** 3.

8.81 Associated Legendre functions with integer indices

8.810 For *integer* values of ν and μ, the differential equation **8.700** 1. (with $|\nu| > |\mu|$) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions $P_n^m(x)$ are called *associated Legendre functions* (or *spherical functions*) *of the first kind.* The number n is called the *degree*, and the number m is called the *order* of the function $P_n^m(x)$. The functions $\{\cos m\vartheta\, P_n^m(\cos\varphi), \quad \sin m\vartheta\, P_n^m(\cos\varphi)\}$, which depend on the angles φ and ϑ, are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for $m < n$ and *sectoral harmonics* for $m = n$. These last functions are periodic with respect to the angles φ and ϑ. Their periods are, respectively, π and 2π. They are single-valued and continuous everywhere on the surface of the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (where $x_1 = \sin\varphi\cos\vartheta$, $x_2 = \sin\varphi\sin\vartheta$, $x_3 = \cos\varphi$), and they are solutions of the differential equation

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\vartheta^2} + n(n+1)Y = 0.$$

8.811 The integral representation

$$P_n^m(\cos\varphi) = \frac{(-1)^m (n+m)!}{\Gamma\left(m+\frac{1}{2}\right)(n-m)!}\sqrt{\frac{2}{\pi}}\sin^{-m}\varphi\int_0^\varphi (\cos t - \cos\varphi)^{m-\frac{1}{2}}\cos\left(n+\tfrac{1}{2}\right)t\,dt \qquad \text{MO 75}$$

8.812 The series representation:

$$P_n^m(x) = \frac{(-1)^m (n+m)!}{2^m m!(n-m)!}\left(1-x^2\right)^{\frac{m}{2}}\left\{1 - \frac{(n-m)(m+n+1)}{1!(m+1)}\frac{1-x}{2}\right.$$
$$\left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)}\left(\frac{1-x}{2}\right)^2 - \dots\right\} \qquad \text{MO 73}$$
$$= \frac{(-1)^m (2n-1)!!}{(n-m)!}\left(1-x^2\right)^{\frac{m}{2}}\left\{x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}x^{n-m-2}\right.$$
$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-m-4} - \dots\right\} \qquad \text{MO 73}$$
$$= \frac{(-1)^m (2n-1)!!}{(n-m)!}\left(1-x^2\right)^{\frac{m}{2}} x^{n-m} F\left(\frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right) \qquad \text{MO 73}$$

8.813 Special cases:

1. $P_1^1(x) = -\left(1-x^2\right)^{1/2} = -\sin\varphi$ MO 73
2. $P_2^1(x) = -3\left(1-x^2\right)^{1/2} x = -\frac{3}{2}\sin 2\varphi$ MO 73
3. $P_2^2(x) = 3\left(1-x^2\right) = \frac{3}{2}(1-\cos 2\varphi)$ MO 73
4. $P_3^1(x) = -\frac{3}{2}\left(1-x^2\right)^{1/2}\left(5x^2-1\right) = -\frac{3}{8}(\sin\varphi + 5\sin 3\varphi)$ MO 73
5. $P_3^2(x) = 15\left(1-x^2\right)x = \frac{15}{4}(\cos\varphi - \cos 3\varphi)$ MO 73
6. $P_3^3(x) = -15\left(1-x^2\right)^{3/2} = -\frac{15}{4}(3\sin\varphi - \sin 3\varphi)$ MO 73

Functional relations

For recursion formulas, see **8.731**.

8.814 $P_n(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2\cos\Theta)$

$$=P_n(\cos\varphi_1)P_n(\cos\varphi_2)+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\varphi_1)P_n^m(\cos\varphi_2)\cos m\Theta$$

$[0\le\varphi_1\le\pi,\quad 0\le\varphi_2\le\pi]$ ("addition theorem") MO 74

8.815 If

$$Y_{n_1}(\varphi,\vartheta)=A_0P_{n_1}(\cos\varphi)+\sum_{m=1}^{n_1}(a_m\cos m\vartheta+b_m\sin m\vartheta)P_{n_1}^m(\cos\varphi),$$

$$Z_{n_2}(\varphi,\vartheta)=\alpha_0P_{n_2}(\cos\varphi)+\sum_{m=1}^{n_2}(\alpha_m\cos m\vartheta+\beta_m\sin m\vartheta)P_{n_2}^m(\cos\varphi),$$

then

$$\int_0^{2\pi}d\vartheta\int_0^{\pi}\sin\varphi\,d\varphi\;Y_{n_1}(\varphi,\vartheta)\;Y_{n_2}(\varphi,\vartheta)=0,$$

$$\int_0^{2\pi}d\vartheta\int_0^{\pi}\sin\varphi\,d\varphi\;Y_n(\varphi,\vartheta)\,P_n[\cos\varphi\cos\psi+\sin\varphi\sin\psi\cos(\vartheta-\theta)]=\frac{4\pi}{2n+1}\;Y_n(\psi,\theta)$$ MO 75

8.816 $(\cos\varphi+i\sin\varphi\cos\vartheta)^n=P_n(\cos\varphi)+2\sum_{m=1}^{n}(-1)^m\frac{n!}{(n+m)!}\cos m\vartheta\,P_n^m(\cos\varphi)$ MO 75

For integrals of the functions, $P_n^m(x)$, see **7.112** 1, **7.122** 1.

8.82–8.83 Legendre functions

8.820 The differential equation

$$\frac{d}{dz}\left[\left(1-z^2\right)\frac{du}{dz}\right]+\nu(\nu+1)u=0\qquad(\text{cf. }\mathbf{8.700}\ 1),$$

where the parameter ν can be an arbitrary number, has the following two linearly independent solutions:

1. $P_\nu(z)=F\left(-\nu,\nu+1;1;\frac{1-z}{2}\right)$

2. $Q_\nu(z)=\frac{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\,\Gamma\left(\nu+\frac{3}{2}\right)}z^{-\nu-1}\,F\left(\frac{\nu+2}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{z^2}\right)$ SM 518(137)

   The functions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first* and *second kind* respectively. If ν is not an integer, the function $P_\nu(z)$ has *singularities* at $z=-1$ and $z=\infty$. However, if $\nu=n=0,1,2,\dots$, the function $P_\nu(z)$ becomes the *Legendre polynomial* $P_n(z)$ (see **8.91**) For $\nu=-n=-1,-2,\dots$, we have

$$P_{-n-1}(z)=P_n(z).$$

3. If $\nu\ne0,1,2,\dots$, the function $Q_\nu(z)$ has singularities at the points $z=\pm1$ and $z=\infty$. These points are branch points of the function. On the other hand, if $\nu=n=0,1,2,\dots$, the function $Q_n(z)$ is single-valued for $|z|>1$ and regular for $z=\infty$.

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{z-1}{z+1}\right) \qquad [\operatorname{Re} z > 0]$$

5. The function $P_\nu(z)$ is uniquely determined by equations **8.820** 1 and **8.820** 4 within a circle of radius 2 with its center at the point $z = 1$ in the right half-plane.

For $z = x = \cos\varphi$, a solution of equation **8.820** is the function

6. $$P_\nu(x) = P_\nu(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\frac{\varphi}{2}\right);$$

In general,

7. $$P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x), \text{ for } z = x$$

8. The function $Q_\nu(z)$ for $|z| > 1$ is uniquely determined by equation **8.820** 2 everywhere in the z-plane in which a cut is made from the point $z = -\infty$ to the point $z = 1$. By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut $(-1 \le x \le +1)$ of the real axis, the function $Q_\nu(x)$ is determined by the equation

9. $$Q_\nu(x) = \tfrac{1}{2}\left[Q_\nu(x+i0) + Q_\nu(x-i0)\right]$$ HO 52(53), WH

Integral representations

8.821

1. $$P_\nu(z) = \frac{1}{2\pi i}\int_A^{(1+,z+)} \frac{\left(t^2-1\right)^\nu}{2^\nu(t-z)^{\nu+1}}\,dt$$

Here, A is a point on the real axis to the right of the point $t = 1$ and to the right of z if z is real. At the point A, we set

$$\arg(t-1) = \arg(t+1) = 0 \text{ and } [|\arg(t-z)| < \pi]$$ WH

2. $$Q_\nu(z) = \frac{1}{4i\sin\nu\pi}\int_A^{(1-,1+)} \frac{\left(t^2-1\right)^\nu}{2^\nu(z-t)^{\nu+1}}\,dt$$

[ν is not an integer; the point A is at the end of the major axis of an ellipse to the right of $t = 1$ drawn in the t-plane with foci at the points ± 1 and with a minor axis sufficiently small that the point z lies outside it. The contour begins at the point A, follows the path $(1-, -1+)$, and returns to A; $|\arg z| \le \pi$ and $|\arg(z-t)| \to \arg z$ as $t \to 0$ on the contour; $\arg(t+1) = \arg(t-1) = 0$ at the point A; z does not lie on the real axis between -1 and 1.]

For $\nu = n$ an integer,

3. $$Q_n(z) = \frac{1}{2^{n+1}}\int_{-1}^{1} \left(1-t^2\right)^n (z-t)^{-n-1}\,dt$$ SM 517(134), WH

8.822

1. $$P_\nu(z) = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{\left(z+\sqrt{z^2-1}\cos\varphi\right)^{\nu+1}} = \frac{1}{\pi}\int_0^\pi \left(z+\sqrt{z^2-1}\cos\varphi\right)^\nu d\varphi$$

$$\left[\operatorname{Re} z > 0 \text{ and } \arg\left\{z+\sqrt{z^2-1}\cos\varphi\right\} = \arg z \text{ for } \varphi = \frac{\pi}{2}\right]$$ WH

2. $$Q_\nu(z) = \int_0^\infty \frac{d\varphi}{\left(z+\sqrt{z^2-1}\cosh\varphi\right)^{\nu+1}},$$
$\left[\operatorname{Re}\nu > -1; \quad \text{if } \nu \text{ is not an integer, } \left\{\left(z+\sqrt{z^2-1}\right)\cosh\varphi\right\} \text{ for } \varphi = 0 \text{ has its principal value}\right]$ WH

8.823 $$P_\nu(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos\left(\nu+\frac{1}{2}\right)\varphi}{\sqrt{2(\cos\varphi-\cos\theta)}}\,d\varphi$$ WH

8.824 $$Q_n(z) = 2^n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2-1)^{n+1}} = 2^n \int_z^\infty \frac{(t-z)^n}{(t^2-1)^{n+1}}\,dt$$
$$= \frac{(-1)^n}{(2n-1)!!}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n \int_z^\infty \frac{dt}{(t^2-1)^{n+1}}\right] \qquad [\operatorname{Re} z > 1]$$
WH, MO 78

8.825 $$Q_n(z) = \frac{1}{2}\int_{-1}^1 \frac{P_n(t)}{z-t}\,dt \qquad [|\arg(z-1)| < \pi]$$ WH, MO 78

See also **6.622** 3, **8.842**.

8.826 Fourier series:

1. $$P_n(\cos\varphi) = \frac{2^{n+2}}{\pi}\frac{n!}{(2n+1)!!}\left[\sin(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\sin(n+3)\varphi + \frac{1\cdot 3(n+1)(n+2)}{1\cdot 2(2n+3)(2n+5)}\sin(n+5)\varphi + \dots\right]$$
$[0 < \varphi < \pi]$ MO 79

2. $$Q_n(\cos\varphi) = 2^{n+1}\frac{n!}{(2n+1)!!}\left[\cos(n+1)\varphi + \frac{1}{1}\frac{n+1}{2n+3}\cos(n+3)\varphi + \frac{1\cdot 3}{1\cdot 2}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\cos(n+5)\varphi + \dots\right]$$
$[0 < \varphi < \pi]$ MO 79

The expressions for Legendre functions in terms of a hypergeometric function (see **8.820**) provide other series representations of these functions.

Special cases and particular values

8.827

1. $$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x} = \operatorname{arctanh} x$$ JA

2. $$Q_1(x) = \frac{x}{2}\ln\frac{1+x}{1-x} - 1$$ JA

3. $$Q_2(x) = \frac{1}{4}\left(3x^2-1\right)\ln\frac{1+x}{1-x} - \frac{3}{2}x$$ JA

4. $$Q_3(x) = \frac{1}{4}\left(5x^3-3x\right)\ln\frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}$$ JA

5. $$Q_4(x) = \frac{1}{16}\left(35x^4 - 30x^2 + 3\right)\ln\frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x$$ JA

6. $$Q_5(x) = \frac{1}{16}\left(63x^5 - 70x^3 + 15x\right)\ln\frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}$$ JA

8.828

1. $$P_\nu(1) = 1$$ MO 79

2. $$P_\nu(0) = -\frac{1}{2}\frac{\sin\nu\pi}{\sqrt{\pi^3}}\,\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)$$ MO 79

8.829 $$Q_\nu(0) = \frac{1}{4\sqrt{\pi}}\left(1-\cos\nu\pi\right)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)$$ MO 79

Functional relationships

8.831

1. $$Q_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left[\cos\nu\pi\,P_\nu(x) - P_\nu(-x)\right] \qquad [\nu \neq 0, \quad \pm1, \pm2, \ldots]$$ MO 76

2. $$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - W_{n-1}(x) \qquad [n = 0, 1, 2, \ldots],$$

where

3. $$W_{n-1}(x) = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{2(n-2k)-1}{(2k+1)(n-k)}P_{n-2k-1}(x) = \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)\,P_{n-k}(x)$$

and

4. $$W_{-1}(x) \equiv 0$$ (see also **8.839**) SM 516(131), MO 76

5. $$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right)P_k(\cos\varphi) = \frac{\pi}{\sin\nu\pi}P_\nu(\cos\varphi)$$

$[\nu \text{ not an integer}; \quad 0 \leq \varphi < \pi]$ MO 77

6. $$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1}\right)P_k(\cos\varphi)\,P_k(\cos\psi) = \frac{\pi}{\sin\nu\pi}P_\nu(\cos\varphi)\,P_\nu(\cos\psi)$$

$[\nu \text{ not an integer}, \quad -\pi < \varphi+\psi < \pi, \quad -\pi < \varphi-\psi < \pi]$ MO 77

See also **8.521** 4.

8.832

1. $$\left(z^2-1\right)\frac{d}{dz}P_\nu(z) = (\nu+1)\left[P_{\nu+1}(z) - z\,P_\nu(z)\right]$$ WH

2. $$(2\nu+1)z\,P_\nu(z) = (\nu+1)\,P_{\nu+1}(z) + \nu\,P_{\nu-1}(z)$$ WH

3. $$\left(z^2-1\right)\frac{d}{dz}Q_\nu(z) = (\nu+1)\left[Q_{\nu+1}(z) - z\,Q_\nu(z)\right]$$ WH

4. $$(2\nu+1)z\,Q_\nu(z)\,Q_{\nu+1}(z) + \nu\,Q_{\nu-1}(z)$$ WH

8.833

1. $P_\nu(-z) = e^{\nu\pi i}\, P_\nu(z) - \frac{2}{\pi} \sin\nu\pi\, Q_\nu(z)$ [Im $z < 0$] MO 77

2. $P_\nu(-z) = e^{-\nu\pi i}\, P_\nu(z) - \frac{2}{\pi} \sin\nu\pi\, Q_\nu(z)$ [Im $z > 0$] MO 77

3. $Q_\nu(-z) = -e^{-\nu\pi i}\, Q_\nu(z)$ [Im $z < 0$] MO 77

4. $Q_\nu(-z) = -e^{\nu\pi i}\, Q_\nu(z)$ [Im $z > 0$] MO 77

8.834

1. $Q_\nu\,(x \pm i0) = Q_\nu(x) \mp \frac{\pi i}{2}\, P_\nu(x)$ MO 77

2. $Q_n(z) = \frac{1}{2}\, P_n(z) \ln\frac{z+1}{z-1} - W_{n-1}(z)$ (see **8.831** 3) MO 77

8.835

1. $Q_\nu(z) - Q_{-\nu-1}(z) = \pi\cot\nu\pi\, P_\nu(z)$ [$\sin\nu\pi \neq 0$] MO 77

2. $Q_{-\nu-1}\,(\cos\varphi) = Q_\nu\,(\cos\varphi) - \pi\cot\nu\pi\, P_\nu\,(\cos\varphi)$ [$\sin\nu\pi \neq 0$] MO 77

3. $Q_\nu\,(-\cos\varphi) = -\cos\nu\pi\, Q_\nu\,(\cos\varphi) - \frac{\pi}{2}\sin\nu\pi\, P_\nu\,(\cos\varphi)$ MO 77

8.836

1. $Q_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n \ln\frac{z+1}{z-1}\right] - \frac{1}{2}\, P_n(z)\ln\frac{z+1}{z-1}$ MO 79

2. $Q_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n \ln\frac{1+x}{1-x}\right] - \frac{1}{2}\, P_n(x)\ln\frac{1+x}{1-x}$ MO 79

8.837

1. $P_\nu(x) = P_\nu\,(\cos\varphi) = F\left(-\nu, \nu+1; 1; \sin^2\frac{\varphi}{2}\right)$ (cf. **8.820** 6) MO 76

2. $$P_\nu(z) = \frac{\tan\nu\pi}{2^{\nu+1}\sqrt{\pi}}\frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1}\, F\left(\frac{\nu}{2}+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) + \frac{2^\nu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} z^\nu\, F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2}-\nu; \frac{1}{z^2}\right)$$ MO 78

See also **8.820**.

For integrals of Legendre functions, see **7.1–7.2**.

8.838 Inequalities ($0 \leq \varphi \leq \pi$, $\nu > 1$, and C_0 is a number that does not depend on the values of ν or φ):

1. $\left|P_\nu\,(\cos\varphi) - P_{\nu+2}\,(\cos\varphi)\right| \leq 2C_0\sqrt{\frac{1}{\nu\pi}}$ MO 78

2. $\left|Q_\nu\,(\cos\varphi) - Q_{\nu+2}\,(\cos\varphi)\right| < C_0\sqrt{\frac{\pi}{\nu}}$ MO 78

With regard to the zeros of Legendre functions of the second kind, see **8.784**, **8.785**, and **8.786**. For the expansion of Legendre functions in series of associated Legendre functions, see **8.794**, **8.795**, and **8.796**.

8.839 A differential equation leading to the functions W_{n-1} (see **8.831** 3):

$$\left(1-x^2\right)\frac{d^2W_{n-1}}{dx^2}-2x\frac{dW_{n-1}}{dx}+(n+1)nW_{n-1}=2\frac{d\,P_\nu}{dx}$$ MO 76

8.84 Conical functions

8.840 Let us set

$$\nu=-\tfrac{1}{2}+i\lambda,$$

where λ is a real parameter, in the defining differential equation **8.700** 1 for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x),\quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class—the class of conical functions. The most important of these peculiarities is the following

8.841 The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos\varphi)=1+\frac{4\lambda^2+1^2}{2^2}\sin^2\frac{\varphi}{2}+\frac{\left(4\lambda^2+1^2\right)\left(4\lambda^2+3^2\right)}{2^24^2}\sin^4\frac{\varphi}{2}+\dots$$

are real for real values of φ. Also,

$$P_{-\frac{1}{2}+i\lambda}(x)\equiv P_{-\frac{1}{2}-i\lambda}(x)$$ MO 95

8.842 Integral representations:

1. $$P_{-\frac{1}{2}+i\lambda}(\cos\varphi)=\frac{2}{\pi}\int_0^\varphi\frac{\cosh\lambda u\,du}{\sqrt{2\left(\cos u-\cos\varphi\right)}}=\frac{2}{\pi}\cosh\lambda\pi\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cos\varphi+\cosh u\right)}}$$ MO 95

2.[6] $$Q_{-\frac{1}{2}\mp\lambda i}(\cos\varphi)=\pm i\sinh\lambda\pi\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cosh u+\cos\varphi\right)}}+\int_0^\infty\frac{\cos\lambda u\,du}{\sqrt{2\left(\cosh u-\cos\varphi\right)}}$$ MO 95

Functional relations

(See also **8.73**)

8.843 $$P_{-\frac{1}{2}+i\lambda}(-\cos\varphi)=\frac{\cosh\lambda\pi}{\pi}\left[Q_{-\frac{1}{2}+i\lambda}(\cos\varphi)+Q_{-\frac{1}{2}-i\lambda}(\cos\varphi)\right]$$ MO 95

8.844

1. $$P_{-\frac{1}{2}+i\lambda}(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi)$$
$$=P_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P_{-\frac{1}{2}+i\lambda}(\cos\vartheta)+2\sum_{k=1}^\infty\frac{(-1)^k2^{2k}\,P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P^k_{-\frac{1}{2}+i\lambda}(\cos\vartheta)\cos k\varphi}{\left(4\lambda^2+1^2\right)\left(4\lambda^2+3^2\right)\cdots\left[4\lambda^2+(2k-1)^2\right]}$$
$$\left[0<\vartheta<\frac{\pi}{2},\quad 0<\psi<\pi,\quad 0<\psi+\vartheta<\pi\right]$$ (cf. **8.794** 1) MO 95

2. $$P_{-\frac{1}{2}+i\lambda}(-\cos\psi\cos\vartheta-\sin\psi\sin\vartheta\cos\varphi)$$
$$=P_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P_{-\frac{1}{2}+i\lambda}(-\cos\vartheta)+2\sum_{k=1}^\infty\frac{(-1)^k2^{2k}\,P^k_{-\frac{1}{2}+i\lambda}(\cos\psi)\,P^k_{-\frac{1}{2}+i\lambda}(-\cos\vartheta)\cos k\varphi}{\left(4\lambda^2+1\right)\left(4\lambda^2+3^2\right)\cdots\left[4\lambda^2+(2k-1)^2\right]}$$
$$\left[0<\psi<\frac{\pi}{2}<\vartheta,\quad \psi+\vartheta<\pi\right]$$ (cf. **8.796**) MO 95

3. $$Q_{-\frac{1}{2}+i\lambda}\left(\cos\psi\cos\vartheta+\sin\psi\sin\vartheta\cos\varphi\right) = P_{-\frac{1}{2}+i\lambda}\left(\cos\psi\right) Q_{-\frac{1}{2}+i\lambda}\left(\cos\vartheta\right) + 2\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k} P^k_{-\frac{1}{2}+i\lambda}\left(\cos\psi\right) Q^k_{-\frac{1}{2}+i\lambda}\left(\cos\vartheta\right)\cos k\varphi}{\left(4\lambda^2+1\right)\left(4\lambda^2+3^2\right)\cdots\left[4\lambda^2+(2k-1)^2\right]}$$
$$\left[0<\psi<\frac{\pi}{2}<\vartheta, \quad \psi+\vartheta<\pi\right]$$ (cf. **8.794** 2) MO 96

Regarding the zeros of conical functions, see **8.784**.

8.85 Toroidal functions

8.850 Solutions of the differential equation

1. $$\frac{d^2u}{d\eta^2}+\frac{\cosh\eta}{\sinh\eta}\frac{du}{d\eta}-\left(n^2-\frac{1}{4}+\frac{m^2}{\sinh^2\eta}\right)u=0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions
$$P^m_{n-\frac{1}{2}}\left(\cosh\eta\right), \quad Q^m_{n-\frac{1}{2}}\left(\sinh\eta\right)$$ MO 96
are solutions of equation **8.850** 1.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

8.851 Integral representations:

1. $$P^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=\frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\frac{(\sinh\eta)^m}{2^m\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)}\int_0^\pi\frac{\sin^{2m}\varphi\,d\varphi}{\left(\cosh\eta+\sinh\eta\cos\varphi\right)^{n+m+\frac{1}{2}}} = \frac{(-1)^m}{2\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\int_0^{2\pi}\frac{\cos m\varphi\,d\varphi}{\left(\cosh\eta+\sinh\eta\cos\varphi\right)^{n+\frac{1}{2}}}$$
MO 96

2. $$Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=(-1)^m\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)}\int_0^\infty\frac{\cosh mt\,dt}{\left(\cosh\eta+\sinh\eta\cosh t\right)^{n+\frac{1}{2}}} \quad [n\ge m]$$
$$=(-1)^m\frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^{\ln\coth\frac{\eta}{2}}\left(\cosh\eta-\sinh\eta\cosh t\right)^{n-\frac{1}{2}}\cosh mt\,dt$$
MO 96

8.852 Functional relations:

1. $$Q^m_{n-\frac{1}{2}}\left(\cosh\eta\right)=(-1)^m\frac{2^m\,\Gamma\left(n+m+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(n+1)}\sinh^m\left(\eta e^{-\left(n+m+\frac{1}{2}\right)\eta}\right) \times F\left(m+\tfrac{1}{2}, n+m+\tfrac{1}{2}; n+1; e^{-2\eta}\right)$$
MO 96

*Sometimes called *torus functions*

2. $$P^{-m}_{n-\frac{1}{2}}(\cosh\eta) = \frac{2^{-2m}}{\Gamma(m+1)}\left(1-e^{-2\eta}\right)^m e^{-\left(n+\frac{1}{2}\right)\eta} F\left(m+\tfrac{1}{2}, n+m+\tfrac{1}{2}; 2m+1; 1-e^{-2\eta}\right)$$ MO 96

8.853 An asymptotic representation $P_{n-\frac{1}{2}}(\cosh\eta)$ for large values of n:

$$P_{n-\frac{1}{2}}(\cosh\eta) = \frac{\Gamma(n)e^{\left(n-\frac{1}{2}\right)\eta}}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \times \left[\frac{2\,\Gamma^2\left(n+\frac{1}{2}\right)}{\pi n!\,\Gamma(n)}\ln\left(4e^{\eta}\right)e^{-2n\eta}F\left(\frac{1}{2}, n+\frac{1}{2}; n+1; e^{-2\eta}\right) + A + B\right],$$

where

$$A = 1 + \frac{1}{2^2}\frac{1\cdot(2n-1)}{1\cdot(n-1)}e^{-2\eta} + \frac{1}{2^4}\frac{1\cdot 3\cdot(2n-1)(2n-3)}{1\cdot 2\cdot(n-1)(n-2)}e^{-4\eta} + \cdots + \frac{1}{2^{2n-2}}\left(\frac{(2n-1)!!}{(n-1)!}\right)^2 e^{-2(n-1)\eta}$$

$$B = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi^3}\,\Gamma(n)}\sum_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(n+k+\frac{1}{2}\right)}{\Gamma(n+k+1)\,\Gamma(k+1)}\left(u_{n+k}+u_k-v_{n+k-\frac{1}{2}}-v_{k-\frac{1}{2}}\right)e^{-2(n+k)\eta}$$

Here,

$$u_r = \sum_{s=1}^{r}\frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^{r}\frac{2}{2s-1} \qquad [r \text{ is a natural number}]$$ MO 97

8.9 Orthogonal Polynomials

8.90 Introduction

8.901 Suppose that $w(x)$ is a nonnegative real function of a real variable x. Let (a,b) be a fixed interval on the x-axis. Let us suppose further that, for $n = 0, 1, 2, \ldots$, the integral

$$\int_a^b x^n w(x)\,dx$$

exists and that the integral

$$\int_a^b w(x)\,dx$$

is positive. In this case, there exists a sequence of polynomials $p_0(x), p_1(x), \ldots, p_n(x), \ldots$, that is uniquely determined by the following conditions:

1. $p_n(x)$ is a polynomial of degree n and the coefficient of x^n in this polynomial is positive.
2. The polynomials $p_0(x), p_1(x), \ldots$ are orthonormal; that is,

$$\int_a^b p_n(x)p_m(x)w(x)\,dx = \begin{cases} 0 & \text{for } n \neq m, \\ 1 & \text{for } n = m. \end{cases}$$

We say that the polynomials $p_n(x)$ constitute *a system of orthogonal polynomials on the interval* (a,b) *with the weight function* $w(x)$.

8.902 If q_n is the coefficient of x^n in the polynomial $p_n(x)$, then

1. $$\sum_{k=0}^{n} p_k(x)p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}$$ (Darboux–Christoffel formula)

EH II 159(10)

2.[11] $$\sum_{k=0}^{n} [p_k(x)]^2 = \frac{q_n}{q_{n+1}} \left[p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)\right]$$ EH II 159(11)

8.903 Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n)\, p_{n-1}(x) - C_n p_{n-2}(x) \qquad [n = 2, 3, 4, \ldots]$$

In this formula, A_n, B_n, and C_n are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2}$$ MO 102

8.904 Examples of normalized systems of orthogonal polynomials:

Notation and name		Interval	Weight
$\left(n+\frac{1}{2}\right)^{1/2} P_n(x)$	see **8.91**	$(-1, +1)$	1
$2^\lambda\, \Gamma(\lambda) \left[\frac{(n+\lambda)\, n!}{2\pi\, \Gamma(2\lambda+n)}\right]^{1/2} C_n^\lambda(x)$	see **8.93**	$(-1, +1)$	$\left(1-x^2\right)^{\lambda-\frac{1}{2}}$
$\sqrt{\frac{\varepsilon_n}{\pi}}\, T_n(x), \quad \varepsilon_0 = 1, \varepsilon_n = 2$ for $n = 1, 2, 3, \ldots$	see **8.94**	$(-1, +1)$	$\left(1-x^2\right)^{-1/2}$
$2^{-\frac{n}{2}} \pi^{-1/4} (n!)^{-1/2} H_n(x)$	see **8.95**	$(-\infty, \infty)$	e^{-x^2}
$\left[\frac{\Gamma(n+1)\, \Gamma(\alpha+\beta+1+n)(\alpha+\beta+1+2n)}{\Gamma(\alpha+1+n)\, \Gamma(\beta+1+n) 2^{\alpha+\beta+1}}\right]^{1/2} P_n^{(\alpha,\beta)}(x)$	see **8.96**	$(-1, +1)$	$(1-x)^\alpha (1+x)^\beta$
$\left[\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right]^{1/2} (-1)^n L_n^\alpha(x)$	see **8.97**	$(0, \infty)$	$x^\alpha e^{-x}$

Cf. **7.221** 1, **7.313**, **7.343**, **7.374** 1, **7.391** 1, **7.414** 3.

8.91 Legendre polynomials

8.910 Definition. The Legendre polynomials $P_n(z)$ are polynomials satisfying equation **8.700** 1 with $\mu = 0$ and $\nu = n$: that is, they satisfy the differential equation

1. $$\left(1-z^2\right) \frac{d^2u}{dz^2} - 2z\frac{du}{dz} + n(n+1)u = 0$$

This equation has a polynomial solution if, and only if, n is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree n are of the form

2. $$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left(z^2-1\right)^n$$

8.911 Legendre polynomials written in expanded form:

1. $$P_n(z) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k}$$
$$= \frac{(2n)!}{2^n (n!)^2} \left(z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} z^{n-4} - \dots \right)$$
$$= \frac{(2n-1)!!}{n!} z^n F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2} \right)$$

HO 13, AD (9001), MO 69

2. $$P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n n!} \left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots \right)$$
$$= (-1)^n \frac{(2n-1)!!}{2^n n!} F\left(-n, n + \frac{1}{2}; \frac{1}{2}; z^2 \right)$$

AD (9002), MO 69

3. $$P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n n!} \left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots \right)$$
$$= (-1)^n \frac{(2n+1)!!}{2^n n!} z F\left(-n, n + \frac{3}{2}; \frac{3}{2}; z^2 \right)$$

AD (9002), MO 69

4. $$P_n(\cos\varphi) = \frac{(2n-1)!!}{2^n n!} \left(\cos n\varphi + \frac{1}{1} \frac{n}{2n-1} \cos(n-2)\varphi \right.$$
$$+ \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\varphi$$
$$\left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\varphi - \dots \right)$$

WH

5. $$P_{2n}(\cos\varphi) = (-1)^n \frac{(2n-1)!!}{2^n n!}$$
$$\times \left\{ \sin^{2n}\varphi - \frac{(2n)^2}{2!} \sin^{2n-2}\varphi \cos^2\varphi + \dots + (-1)^n \frac{2^n n!}{(2n-1)!!} \cos^{2n}\varphi \right\}$$

AD (9011)

6. $$P_{2n+1}(\cos\varphi) = (-1)^n \frac{(2n+1)!!}{2^n n!} \cos\varphi$$
$$\times \left\{ \sin^{2n}\varphi - \frac{(2n)^2}{3!} \sin^{2n-2}\varphi \cos^2\varphi + \dots + (-1)^n \frac{2^n n!}{(2n+1)!!} \cos^{2n}\varphi \right\}$$

AD (9012)

7. $$P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{(n-k)!\,(k!)^2\, 2^{k+1}} \left[(1-z)^k + (-1)^n (1+z)^k \right]$$ WH

8.912 Special cases:

1. $P_0(x) = 1$ JA

2. $P_1(x) = x = \cos\varphi$ JA

3. $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) = \frac{1}{4}(3\cos 2\varphi + 1)$ JA

4. $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) = \frac{1}{8}(5\cos 3\varphi + 3\cos\varphi)$ JA

5. $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) = \frac{1}{64}(35\cos 4\varphi + 20\cos 2\varphi + 9)$ JA

6. $P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) = \frac{1}{128}(63\cos 5\varphi + 35\cos 3\varphi + 30\cos\varphi)$ JA

7.[10] $P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right) = \frac{1}{512}(231\cos 6\varphi + 126\cos 4\varphi + 105\cos 2\varphi + 50)$

8. $$P_7(x) = \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right) = \frac{1}{1024}(429\cos 7\varphi + 231\cos 5\varphi + 189\cos 3\varphi + 175\cos\varphi)$$

9. $$P_8(x) = \frac{1}{128}\left(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35\right) = \frac{1}{16384}(6435\cos 8\varphi - 3432\cos 6\varphi + 2772\cos 4\varphi - 2520\cos 2\varphi + 1225)$$

8.913 Integral representations:

1. $$P_n(\cos\varphi) = \frac{2}{\pi}\int_\varphi^\pi \frac{\sin\left(n + \frac{1}{2}\right)t}{\sqrt{2(\cos\varphi - \cos t)}}\,dt$$ WH

See also **3.611** 3, **3.661** 3, 4.

2.[7] Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{\left(t^2 - 1\right)^n}{2^n(t - z)^{n+1}}\,dt,$$

with C a simple contour containing z. SA 175(9)

3.[10] Laplace integral formula:

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \left[x + \left(x^2 - 1\right)^{1/2}\cos\varphi\right]^n d\varphi \qquad [|x| \le 1]$$ SA 180(19)

Functional relations

8.914 Recurrence formulas:

1. $(n+1)\,P_{n+1}(z) - (2n+1)z\,P_n(z) + n\,P_{n-1}(z) = 0$ WH

2. $$\left(z^2-1\right)\frac{d\,P_n}{dz} = n\left[z\,P_n(z) - P_{n-1}(z)\right] = \frac{n(n+1)}{2n+1}\left[P_{n+1}(z) - P_{n-1}(z)\right]$$ WH

8.915

1.[10] $$\sum_{k=0}^{n}(2k+1)\,P_k(x)\,P_k(y) = (n+1)\frac{P_n(x)\,P_{n+1}(y) - P_n(y)\,P_{n+1}(x)}{y-x}$$
(Christoffel summation formula)
MO 70

1(1)[10]. $$(y-x)\sum_{k=0}^{n}(2k+1)\,P_k(x)\,Q_k(y) = 1-(n+1)\left[P_{n+1}(x)\,Q_n(y) - P_n(x)\,Q_{n+1}(y)\right]$$
AS 335(8.9.2)

2.[7] $$\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k-1)\,P_{n-2k-1}(z) = P_n'(z)$$ (summation theorem) MO 70

3.[7] $$\sum_{k=0}^{\left\lfloor\frac{n-2}{2}\right\rfloor}(2n-4k-3)\,P_{n-2k-2}(z) = z\,P_n'(z) - n\,P_n(z)$$ SM 491(42), WH

4.[10] $$\sum_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor}(2n-4k+1)[k(2n-2k+1)-2]\,P_{n-2k}(z) = z^2\,P_n''(z) - n(n-1)\,P_n(z)$$ WH

5.[11] $$\sum_{k=0}^{m}\frac{a_{m-k}a_k a_{n-k}}{a_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right)P_{n+m-2k}(z) = P_n(z)\,P_m(z)$$
$$\left[a_k = \frac{(2k-1)!!}{k!}, \quad m \le n\right]$$ AD (9036)

8.916

1. $$P_n(\cos\varphi) = \frac{(2n-1)!!}{2^n n!}e^{\mp in\varphi}\,F\left(\frac{1}{2}, -n; \frac{1}{2}-n; e^{\pm 2i\varphi}\right)$$ MO 69

2. $$P_n(\cos\varphi) = F\left(n+1, -n; 1; \sin^2\frac{\varphi}{2}\right)$$ MO 69

3. $$P_n(\cos\varphi) = (-1)^n\,F\left(n+1, -n; 1; \cos^2\frac{\varphi}{2}\right)$$ WH

4. $$P_n(\cos\varphi) = \cos^n\varphi\,F\left(-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n; 1; -\tan^2\varphi\right)$$ HO 23

5. $$P_n(\cos\varphi) = \cos^{2n}\frac{\varphi}{2}\,F\left(-n, -n; 1; -\tan^2\frac{\varphi}{2}\right)$$ HO 23, 29, WH

See also **8.911** 1, **8.911** 2, **8.911** 3. For a connection with other functions, see **8.936** 3, **8.836**, **8.962** 2.

- For integrals of Legendre polynomials, see **7.22–7.25**.
- For the zeros of Legendre polynomials, see **8.785**.

8.917 Inequalities:

1. $P_0(x) < P_1(x) < P_2(x) < \cdots < P_n(x) < \ldots \qquad [x > 1]$ MO 71

2. For $x > -1$, $P_0(x) + P_1(x) + \cdots + P_n(x) > 0$. MO 71

3. $[P_n(\cos\varphi)]^2 > \dfrac{\sin(2n+1)\varphi}{(2n+1)\sin\varphi}$ MO 71

4. $\sqrt{n\sin\varphi}|P_n(\cos\varphi)| \le 1$. MO 71

5. $|P_n(\cos\varphi)| \le 1$. WH

6.[10] Let $n \ge 2$. The successive relative maxima of $|P_n(x)|$, when x decreases from 1 to 0, form a decreasing sequence. More precisely, if $\mu_1, \mu_2, \ldots, \mu_{\lfloor n/2 \rfloor}$ denote these maxima corresponding to decreasing values of x, we have

$$1 > \mu_1 > \mu_2 > \cdots > \mu_{\lfloor n/2 \rfloor}$$ SZ 162(7.3.1)

7.[10] Let $n \ge 2$. The successive relative maxima of $(\sin\theta)^{1/2}|P_n(\cos\theta)|$ when θ increases from 0 to $\pi/2$, form an increasing sequence. SZ 163(7.3.2)

8.[10] We have

$$(\sin\theta)^{1/2}|P_n(\cos\theta)| < (2/\pi)^{1/2} n^{-1/2} \qquad [0 \le q\theta \le q\pi]$$ SZ 163(7.3.8)

Here the constant $(2/\pi)^{1/2}$ cannot be replaced by a smaller one.

9.[10] $\max\limits_{0 \le q\theta \le q\pi} (\sin\theta)^{1/2}|P_n(\cos\theta)| \cong (2/\pi)^{1/2} n^{-\frac{1}{2}} \qquad [n \to \infty]$ SZ 164(7.3.12)

10.[10] Stieltjes' first theorem:

$$|P_n(\cos\theta)| \le \left(\frac{2}{\pi}\right)^{1/2} \frac{4}{\sqrt{n\sin\theta}} \qquad [n = 1, 2, \ldots, 0 < \theta < \pi]$$ SA 197(8)

11.[10] Stieltjes' second theorem:

$$|P_n(x) - P_{n+2}(x)| < \frac{4}{\sqrt{\pi}\sqrt{n+2}} \qquad [|x| \le 1]$$ SA 199(15)

12.[10] $\left|\dfrac{d\,P_n(x)}{dx}\right| < \dfrac{2}{\sqrt{\pi}} \dfrac{\sqrt{n}}{1-x^2} \qquad [|x| < 1, \quad n = 1, 2, \ldots]$ SA 201(18)

13.[10] $|P_{n+1}(x) + P_n(x)| < 6\left(\dfrac{2}{\pi n}\right)^{\frac{1}{2}} (1-x)^{-1/2} \qquad [|x| < 1, \quad n = 0, 1, \ldots]$ SA 201(19)

8.918[10] Asymptotic approximations:

1. $$P_n(\cos\theta) = \left(\frac{2}{\pi n \sin\varphi}\right)^{1/2} \cos\left[\left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + O\left(n^{-3/2}\right)$$
$[\varepsilon \le \theta \le \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2m]$ (Laplace's formula) SA 208(1)

2. $$P_n(\cos\theta) = \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \left\{\left(1 - \frac{1}{4n}\right)\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + \frac{1}{8n}\cos\theta\sin\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right]\right\} + O\left(n^{-5/2}\right)$$

$[\varepsilon \le \theta \le \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2]$ (Bonnet–Heine formula) SA 208(2)

8.919[10] Series of products of Legendre and Chebyshev polynomials

1. $$2\int_{-1}^{1} T_n(x)\, P_n(x)\, dx = \sum_{i,j=0}^{i+j=n} \int_{-1}^{1} P_i(x)\, P_j(x)\, P_n(x)\, dx$$

8.92 Series of Legendre polynomials

8.921 The generating function:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k P_k(z) \qquad \left[|t| < \min\left|z \pm \sqrt{z^2-1}\right|\right]$$ SM 489(31), WH

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) \qquad \left[|t| > \max\left|z \pm \sqrt{z^2-1}\right|\right]$$ MO 70

8.922

1. $$z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z)$$ MO 72

2. $$z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z)$$ MO 72

3. $$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{\frac{(2k-1)!!}{2^k k!}\right\}^2 P_{2k}(x) \qquad [|x| < 1, \quad (-1)!! \equiv 1]$$

MO 72, LA 385(15)

4. $$\frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!(2k+1)!!}{2^{2k+1} k!(k+1)!} P_{2k+1}(x)$$

$[|x| < 1, \quad (-1)!! \equiv 1]$ LA 385(17)

5. $$\sqrt{1-x^2} = \frac{\pi}{2} \left\{\frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!(2k-1)!!}{2^{2k+1} k!(k+1)!} P_{2k}(x)\right\}$$

$[|x| < 1, \quad (-1)!! \equiv 1]$ LA 385(18)

6.[10] $$\sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)} P_n(x) \qquad [-1 \le x \le 1]$$

7.[10] $$\frac{1-\rho^2}{\left(1-2\rho x+\rho^2\right)^{1/2}} = 1 + \sum_{n=0}^{\infty}(2n+1)\rho^n\, P_n(x), \qquad [|\rho|<1, \quad |x|\le 1]$$ SA 170(4)

8.923 $$\arcsin x = \frac{\pi}{2}\sum_{k=1}^{\infty}\left\{\frac{(2k-1)!!}{2^k k!}\right\}^2\left[P_{2k+1}(x)-P_{2k-1}(x)\right]+\pi x/2$$
$$[|x|<1, \quad (-1)!!\equiv 1]$$ WH

8.924

1. $$-\frac{1+\cos n\pi}{2\left(n^2-1\right)}P_0\left(\cos\theta\right) - \frac{1+\cos n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2\left(n^2-2^2\right)\dots\left[n^2-(2k)^2\right]}{\left(n^2-1^2\right)\left(n^2-3^2\right)\dots\left[n^2-(2k+3)^2\right]}P_{2k+2}\left(\cos\theta\right)$$
$$-\frac{3\left(1-\cos n\pi\right)}{2\left(n^2-2^2\right)}P_1\left(\cos\theta\right)$$
$$-\frac{1-\cos n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)\left(n^2-1^2\right)\dots\left[n^2-(2k-1)^2\right]}{\left(n^2-2^2\right)\left(n^2-4^2\right)\dots\left[n^2-(2k+2)^2\right]}P_{2k+1}\left(\cos\theta\right)=\cos n\theta$$
AD (9060.1)

2. $$\frac{-\sin n\pi}{2\left(n^2-1\right)}P_0\left(\cos\theta\right) - \frac{\sin n\pi}{2}\sum_{k=0}^{\infty}\frac{(4k+5)n^2\left(n^2-2^2\right)\dots\left[n^2-(2k)^2\right]}{\left(n^2-1^2\right)\left(n^2-3^2\right)\dots\left[n^2-(2k+3)^2\right]}P_{2k+2}\left(\cos\theta\right)$$
$$+\frac{3\sin n\pi}{2\left(n^2-2^2\right)}P_1\left(\cos\theta\right)$$
$$+\frac{\sin n\pi}{2}\sum_{k=1}^{\infty}\frac{(4k+3)\left(n^2-1^2\right)\left(n^2-3^2\right)\dots\left[n^2-(2k-1)^2\right]}{\left(n^2-2^2\right)\left(n^2-4^2\right)\dots\left[n^2-(2k+2)^2\right]}P_{2k+1}\left(\cos\theta\right)=\sin n\theta$$
AD (9060.2)

3.[3] $$\frac{2^{n-1}n!}{(2n-1)!!}P_n\left(\cos\theta\right) - n\sum_{k=1}^{\lfloor n/2\rfloor}(2n-4k+1)\frac{2^{n-2k-1}(n-k-1)!(2k-3)!!}{(2n-2k+1)!!k!}P_{n-2k}\left(\cos\theta\right)$$
$$=\cos n\theta$$
AD (9061.1)

4. $$\frac{(2n-1)!!P_{n-1}\left(\cos\theta\right)}{2^{n-1}(n-1)!} - \frac{n}{2^{n+1}}\sum_{k=0}^{\infty}\frac{(2n+2k-1)!!(2k-1)!!\,(2n+4k+3)}{2^{2k}(n+k+1)!(k+1)!}P_{n+2k+1}\left(\cos\theta\right)$$
$$=\frac{4\sin n\theta}{\pi}$$
AD (9061.2)

8.925

1. $$\sum_{k=1}^{\infty}\frac{4k-1}{2^{2k}(2k-1)^2}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k-1}\left(\cos\theta\right) = 1-\frac{2\theta}{\pi}$$

2. $$\sum_{k=1}^{\infty}\frac{4k+1}{2^{2k+1}(2k-1)(k+1)}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k}\left(\cos\theta\right) = \frac{1}{2}-\frac{2\sin\theta}{\pi}$$ AD (9062.2)

3. $$\sum_{k=1}^{\infty}\frac{k(4k-1)}{2^{2k-1}(2k-1)}\left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k-1}\left(\cos\theta\right) = \frac{2\cot\theta}{\pi}$$ AD (9062.3)

4. $$\sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!}\right]^2 P_{2k}(\cos\theta) = \frac{2}{\pi \sin\theta} - 1$$ AD (9062.4)

8.926

1. $$\sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos\theta) = \ln \frac{2\tan\frac{\pi-\theta}{4}}{\sin\theta} = -\ln\sin\frac{\theta}{2} - \ln\left(1+\sin\frac{\theta}{2}\right)$$ AD (9063.2)

2. $$\sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos\theta) = \ln \frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} - 1$$ AD (9063.1)

8.927 $$\sum_{k=0}^{\infty} \cos\left(k+\tfrac{1}{2}\right)\beta\, P_k(\cos\varphi) = \frac{1}{\sqrt{2(\cos\beta - \cos\varphi)}} \qquad [0 \le \beta < \varphi < \pi]$$

$$= 0 \qquad [0 < \varphi < \beta < \pi]$$

MO 72

8.928

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n (4k+1)\left[(2n-1)!!\right]^3}{2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\boldsymbol{K}(\sin\theta)}{\pi^2} - 1$$ AD (9064.1)

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1)\left[(2n-1)!!\right]^3}{(2n-1)(2n+2)2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\boldsymbol{E}(\sin\theta)}{\pi^2} - \frac{1}{2}$$ AD (9064.2)

- For series of products of Bessel functions and Legendre polynomials, see **8.511** 4, **8.531** 3, **8.533** 1, **8.533** 2, and **8.534**.
- For series of products of Legendre and Chebyshev polynomials, see **8.919**.

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.930 Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$\left(1 - 2t\alpha + \alpha^2\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t)\alpha^n$$ WH

Thus, the polynomials $C_n^\lambda(t)$ are a *generalization of the Legendre polynomials.*

1.[10] $C_0^\lambda(t) = 1$

2.[10] $C_1^\lambda(t) = 2\lambda t$

3.[10] $C_2^\lambda(t) = 2\lambda(\lambda+1)t^2 - \lambda$

4.[10] $C_3^\lambda(t) = \frac{1}{3}\lambda\left(4\lambda^2 + 12\lambda + 8\right)t^3 - 2\lambda(\lambda+1)t$

5.[11] $C_4^\lambda(t) = \frac{2}{3}\lambda\left(\lambda^3 + 6\lambda^2 + 11\lambda + 6\right)t^4 - 2\lambda\left(\lambda^2 + 3\lambda + 2\right)t^2 + \frac{1}{2}\lambda(\lambda+1)$

6.[10] $$C_5^\lambda(t) = \frac{1}{15}\lambda\left(4\lambda^4 + 40\lambda^3 + 140\lambda^2 + 200\lambda + 96\right)t^5$$
$$-\frac{1}{3}\lambda\left(4\lambda^3 + 24\lambda^2 + 44\lambda + 24\right)t^3 + \lambda\left(\lambda^2 + 3\lambda + 2\right)t$$

7.[10] $$C_6^\lambda(t) = \tfrac{1}{45}\lambda\left(\lambda^5+60\lambda^4+340\lambda^3+900\lambda^2+1096\lambda+480\right)t^6 \\ -\tfrac{1}{3}\lambda\left(2\lambda^4+20\lambda^3+70\lambda^2+100\lambda+48\right)t^4 \\ +\lambda\left(\lambda^3+6\lambda^2+11\lambda+6\right)t^2+\tfrac{1}{6}\lambda\left(\lambda^2+3\lambda+2\right)$$

8.931 Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)}\frac{\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(\lambda)}\int_0^\pi\left(t+\sqrt{t^2-1}\cos\varphi\right)^n\sin^{2\lambda-1}\varphi\,d\varphi$$ MO 99

See also **3.252** 11, **3.663** 2, **3.664** 4.

Functional relations

8.932 Expressions in terms of hypergeometric functions:

1. $$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(2\lambda)}\,F\left(2\lambda+n,-n;\lambda+\frac{1}{2};\frac{1-t}{2}\right)^*$$ MO 97
$$= \frac{2^n\,\Gamma(\lambda+n)}{n!\,\Gamma(\lambda)}t^n\,F\left(-\frac{n}{2},\frac{1-n}{2};1-\lambda-n;\frac{1}{t^2}\right)$$ MO 99

2. $$C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda+n)\,\mathrm{B}(\lambda,n+1)}\,F\left(-n,n+\lambda;\frac{1}{2};t^2\right)$$ MO 99

3. $$C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{\mathrm{B}(\lambda,n+1)}\,F\left(-n,n+\lambda+1;\frac{3}{2};t^2\right)$$ MO 99

8.933 Recursion formulas:

1. $$(n+2)\,C_{n+2}^\lambda(t) = 2(\lambda+n+1)t\,C_{n+1}^\lambda(t)-(2\lambda+n)\,C_n^\lambda(t)$$ Mo 98

2. $$n\,C_n^\lambda(t) = 2\lambda\left[t\,C_{n-1}^{\lambda+1}(t)-C_{n-2}^{\lambda+1}(t)\right]$$ WH

3. $$(2\lambda+n)\,C_n^\lambda(t) = 2\lambda\left[C_n^{\lambda+1}(t)-t\,C_{n-1}^{\lambda+1}(t)\right]$$ WH

4. $$n\,C_n^\lambda(t) = (2\lambda+n-1)t\,C_{n-1}^\lambda(t)-2\lambda\left(1-t^2\right)C_{n-2}^{\lambda+1}(t)$$ WH

8.934

1. $$C_n^\lambda(t) = \frac{(-1)^n}{2^n}\frac{\Gamma(2\lambda+n)\,\Gamma\left(\frac{2\lambda+1}{2}\right)}{\Gamma(2\lambda)\,\Gamma\left(\frac{2\lambda+1}{2}+n\right)}\frac{\left(1-t^2\right)^{\frac{1}{2}-\lambda}}{n!}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{\lambda+n-\frac{1}{2}}\right]$$ WH

2. $$C_n^\lambda(\cos\varphi) = \sum_{\substack{k,l=0\\k+l=n}}^{n}\frac{\Gamma(\lambda+k)\,\Gamma(\lambda+l)}{k!\,l!\,[\Gamma(\lambda)]^2}\cos(k-l)\varphi$$ MO 99

*Equation 8.932.1 defines the generalized functions $C_n^\lambda(t)$, where the subscript n can be an arbitrary number.

3. $$C_n^\lambda(\cos\psi\cos\vartheta + \sin\psi\sin\vartheta\cos\varphi) = \frac{\Gamma(2\lambda-1)}{[\Gamma(\lambda)]^2}\sum_{k=0}^{n}\frac{2^{2k}(n-k)!\,[\Gamma(\lambda+k)]^2}{\Gamma(2\lambda+n+k)}(2\lambda+2k-1)\sin^k\psi\,\mathrm{si}\,n^k\vartheta \times C_{n-k}^{\lambda+k}(\cos\psi)\,C_{n-k}^{\lambda+k}(\cos\vartheta)\,C_k^{\lambda-\frac{1}{2}}(\cos\varphi)$$

$\left[\psi, \vartheta, \varphi \text{ real}; \quad \lambda \neq \tfrac{1}{2}\right]$ ["summation theorem"] (see also **8.794–8.796**) WH

4. $$\lim_{\lambda\to 0}\Gamma(\lambda)\,C_n^\lambda(\cos\varphi) = \frac{2\cos n\varphi}{n}$$ MO 98

For orthogonality, see **8.904, 7.313**.

8.935 Derivatives:

1. $$\frac{d^k}{dt^k}C_n^\lambda(t) = 2^k\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}C_{n-k}^{\lambda+k}(t)$$ MO 99

In particular,

2.[11] $$\frac{d\,C_n^\lambda(t)}{dt} = 2\lambda\,C_{n-1}^{\lambda+1}(t)$$ WH

For integrals of the polynomials $C_n^\lambda(x)$ see **7.31–7.33**.

8.936 Connections with other functions:

1. $$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\,\Gamma(n+1)}\left\{\frac{1}{4}\left(t^2-1\right)\right\}^{\frac{1}{4}-\frac{\lambda}{2}}P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t)$$ MO 98

2. $$C_{n-m}^{m+\frac{1}{2}}(t) = \frac{1}{(2m-1)!!}\frac{d^m\,P_n(t)}{dt^m} = (-1)^m\frac{\left(1-t^2\right)^{-\frac{m}{2}}m!2^m}{(2m)!}P_n^m(t)$$

$[m+1 \text{ a natural number}]$ MO 98, WH

3. $$C_n^{1/2}(t) = P_n(t)$$

4. $$J_{\lambda-\frac{1}{2}}(r\sin\vartheta\sin\alpha)\,(r\sin\vartheta\sin\alpha)^{-\lambda+\frac{1}{2}}\,e^{-ir\cos\vartheta\cos\alpha} = \sqrt{2}\frac{\Gamma(\lambda)}{\Gamma\left(\lambda+\frac{1}{2}\right)}\sum_{k=0}^{\infty}(\lambda+k)i^{-k}\frac{\mathbf{J}_{\lambda+k}(r)\,C_k^\lambda(\cos\vartheta)\,C_k^\lambda(\cos\alpha)}{r^\lambda\,C_k^\lambda(1)}$$ MO 99

5. $$\lim_{\lambda\to\infty}\lambda^{-\frac{n}{2}}\,C_n^{\frac{\lambda}{2}}\left(t\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-\frac{n}{2}}}{n!}H_n(t)$$ MO 99a

See also **8.932**.

8.937 Special cases and particular values:

1. $$C_n^1(\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}$$ MO 99

2. $$C_0^0(\cos\varphi) = 1$$ MO 98

3. $$C_0^\lambda(t) \equiv 1$$ MO 98

4. $$C_n^\lambda(1) \equiv \binom{2\lambda+n-1}{n}$$ MO 98

8.938 A differential equation leading to the polynomials $C_n^\lambda(t)$:

$$y'' + \frac{(2\lambda+1)t}{t^2-1}y' - \frac{n(2\lambda+n)}{t^2-1}y = 0 \qquad \text{(cf. } \mathbf{9.174}\text{)}$$ WH

For series of products of Bessel functions and the polynomials $C_n^\lambda(x)$, see **8.532, 8.534**.

8.939[10] Differentiation and Rodrigues' formulas and orthogonality relation

1. $$\frac{d}{dt}\, C_n^\lambda(t) = 2\lambda\, C_{n-1}^{\lambda+1}(t)$$ MS 5.3.2

2. $$\frac{d^m}{dt^m}\, C_n^\lambda(t) = 2^m\lambda(\lambda+1)(\lambda+2)\dots(\lambda+m-1)\, C_{n-m}^{\lambda+m}(t)$$ MS 5.3.2

3. $$\frac{d}{dt}\, C_{n-1}^\lambda(t) = t\frac{d}{dt}\, C_n^\lambda(t) - n\, C_n^\lambda(t)$$ MS 5.3.2

4. $$\frac{d}{dt}\, C_{n+1}^\lambda(t) = t\frac{d}{dt}\, C_n^\lambda(t) + (2\lambda+n)\, C_n^\lambda(t)$$ MS 5.3.2

5. $$\left(1-t^2\right)\frac{d}{dt}\, C_n^\lambda(t) = (n+2\lambda-1)\, C_{n-1}^\lambda(t) - nt\, C_n^\lambda(t) = (n+2\lambda)t\, C_n^\lambda(t) - (n+1)\, C_{n+1}^\lambda(t)$$
$$= 2\lambda\left(1-t^2\right) C_{n-1}^{\lambda+1}(t)$$
MS 5.3.2

6. $$\frac{d}{dt}\left[C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)\right] = 2(n+\lambda)\, C_n^\lambda(t)$$ MS 5.3.2

7. $$C_n^\lambda(t) = \frac{(-1)^n 2\lambda(2\lambda+1)(2\lambda+2)\dots(2\lambda+n-1)\left(1-t^2\right)^{\frac{1}{2}-\lambda}}{2^n n!\left(\lambda+\frac{1}{2}\right)\left(\lambda+\frac{3}{2}\right)\dots\left(\lambda+n-\frac{1}{2}\right)}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{n+\lambda-\frac{1}{2}}\right]$$
$$= \frac{(-1)^n\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(n+2\lambda)\left(1-t^2\right)^{\frac{1}{2}-\lambda}}{2^n n!\,\Gamma(2\lambda)\,\Gamma\left(n+\lambda+\frac{1}{2}\right)}\frac{d^n}{dt^n}\left[\left(1-t^2\right)^{n+\lambda-\frac{1}{2}}\right]$$
[Rodrigues' formula] MS 5.3.2

8. $$\int_{-1}^1 C_n^\lambda(t)\, C_m^\lambda(t)\left(1-t^2\right)^{\lambda-\frac{1}{2}}\, dt = 0 \qquad n \neq m$$
$$= \frac{\pi 2^{1-2\lambda}\,\Gamma(n+2\lambda)}{n!(\lambda+n)\left[\Gamma(\lambda)\right]^2} \qquad n = m$$
$[\lambda \neq 0]$ [Orthogonality relation] MS 5.3.2

8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

8.940 Definition

1. Chebyshev's polynomials of the first kind

$$T_n(x) = \cos(n\arccos x) = \frac{1}{2}\left[\left(x+i\sqrt{1-x^2}\right)^n + \left(x-i\sqrt{1-x^2}\right)^n\right]$$
$$= x^n - \binom{n}{2}x^{n-2}\left(1-x^2\right) + \binom{n}{4}x^{n-4}\left(1-x^2\right)^2 - \binom{n}{6}x^{n-6}\left(1-x^2\right)^3 + \dots$$
NA 66, 71

2. Chebyshev's polynomials of the second kind:

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]} = \frac{1}{2i\sqrt{1-x^2}}\left[\left(x+i\sqrt{1-x^2}\right)^{n+1} - \left(x-i\sqrt{1-x^2}\right)^{n+1}\right]$$
$$= \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}\left(1-x^2\right) + \binom{n+1}{5}x^{n-4}\left(1-x^2\right)^2 - \dots$$

Functional relations

8.941 Recursion formulas:

1. $T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0$ NA 358
2. $U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0$
3. $T_n(x) = U_n(x) - x\,U_{n-1}(x)$ EH II 184(3)
4. $\left(1-x^2\right)U_{n-1}(x) = x\,T_n(x) - T_{n+1}(x)$ EH II 184(4)

For the orthogonality, see **7.343** and **8.904**.

8.942 Relations with other functions:

1. $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right)$ MO 104
2. $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!}\frac{d^n}{dx^n}\left(1-x^2\right)^{n-\frac{1}{2}}$ MO 104
3. $U_n(x) = \frac{(-1)^n(n+1)}{\sqrt{1-x^2}(2n+1)!!}\frac{d^n}{dx^n}\left(1-x^2\right)^{n+\frac{1}{2}}$ EH II 185(15)

See also **8.962** 3.

8.943[10] Special cases

1. $T_0(x) = 1$
2. $T_1(x) = x$
3. $T_2(x) = 2x^2 - 1$
4. $T_3(x) = 4x^3 - 3x$
5. $T_4(x) = 8x^4 - 8x^2 + 1$
6. $T_5(x) = 16x^5 - 20x^3 + 5x$
7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
10. $U_0(x) = 1$
11. $U_1(x) = 2x$
12. $U_2(x) = 4x^2 - 1$
13. $U_3(x) = 8x^3 - 4x$
14. $U_4(x) = 16x^4 - 12x^2 + 1$
15. $U_5(x) = 32x^5 - 32x^3 + 6x$
16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$
17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$
18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$

8.944 Particular values:

1. $T_n(1) = 1$
2. $T_n(-1) = (-1)^n$
3. $T_{2n}(0) = (-1)^n$
4. $T_{2n+1}(0) = 0$
5. $U_{2n+1}(0) = 0$
6. $U_{2n}(0) = (-1)^n$

8.945 The generating function:

1.[11] $$\frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2\sum_{k=1}^{\infty} T_k(x)t^k \qquad [|t| < 1]$$ MO 104

2.[11] $$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k \qquad [|t| < 1]$$ MO 104a, EH II 186(31)

8.946 Zeros. The polynomials $T_n(x)$ and $U_n(x)$ only have real simple zeros. All these zeros lie in the interval $(-1, +1)$.

8.947 The functions $T_n(x)$ and $\sqrt{1-x^2}\, U_{n-1}(x)$ are two linearly independent solutions of the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0.$$ NA 69(58)

8.948 Of all polynomials of degree n with leading coefficient equal to 1, the one that deviates the least from zero on the interval $[-1, +1]$ is the polynomial $2^{-n+1}\, T_n(x)$.

8.949[10] Differentiation and Rodrigues' formulas and orthogonality relations

1. $$\frac{d}{dx}\, T_n(x) = n\, U_{n-1}(x)$$ MS 5.7.2

2. $$\frac{d^m}{dx^m}\, T_n(x) = 2^{m-1}\,\Gamma(m) n\, C_{n-m}^{m}(x)$$ MS 5.7.2

3. $$\left(1-x^2\right)\frac{d}{dx}\, T_n(x) = n\left[T_{n-1}(x) - x\, T_n(x)\right] = n\left[x\, T_n(x) - T_{n+1}(x)\right]$$ MS 5.7.2

4. $$\frac{d}{dx}\, U_n(x) = 2\, C_{n-1}^{2}(x)$$ MS 5.7.2

5. $$\frac{d^m}{dx^m}\, U_n(x) = 2^m m! C_{n-m}^{m+1}(x)$$ MS 5.7.2

6. $$\left(1-x^2\right)\frac{d}{dx}\, U_n(x) = (n+1)\, U_{n-1}(x) - nx\, U_n(x) = (n+2)x\, U_n(x) - (n+1)\, U_{n+1}(x)$$ MS 5.7.2

7. $$T_n(x) = \frac{(-1)^n \pi^{1/2}\left(1-x^2\right)^{c\frac{1}{2}}}{2^{n+1}\,\Gamma\left(n+\frac{1}{2}\right)}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n-\frac{1}{2}}\right]$$ [Rodrigues' formula] MS 5.7.2

8. $$U_n(x) = \frac{(-1)^n \pi^{1/2}(n+1)\left(1-x^2\right)^{-1/2}}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+\frac{1}{2}}\right]$$ [Rodrigues' formula] MS 5.7.2

9. $$\int_{-1}^{1} T_m(x)\, T_n(x) \left(1-x^2\right)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

[Orthogonality relation] MS 5.7.2

10. $$\int_{-1}^{1} U_m(x)\, U_n(x) \left(1-x^2\right)^{-1/2} dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n \end{cases}$$

[Orthogonality relation] MS 5.7.2

8.95 The Hermite polynomials $H_n(x)$

8.950 Definition

1. $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$ SM 567(14)

or

2. $$H_n(x) = 2^n x^n - 2^{n-1}\binom{n}{2} x^{n-2} + 2^{n-2}\cdot 1\cdot 3\cdot \binom{n}{4} x^{n-4} - 2^{n-3}\cdot 1\cdot 3\cdot 5\cdot \binom{n}{6} x^{n-6} + \dots$$ MO 105a

3.[10] $H_0(x) = 1$

4.[10] $H_1(x) = 2x$

5.[10] $H_2(x) = 4x^2 - 2$

6.[10] $H_3(x) = 8x^3 - 12x$

7.[10] $H_4(x) = 16x^4 - 48x^2 + 12$

8.[10] $H_5(x) = 32x^5 - 160x^3 + 120x$

9.[10] $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$

10.[10] $H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

11.[10] $H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$

8.951 The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-t^2}\, dt$$ MO 106a

Functional relations

8.952 Recursion formulas:

1. $$\frac{d\, H_n(x)}{dx} = 2n\, H_{n-1}(x)$$ SM 569(22)

2. $$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$$ SM 570(23)

For the orthogonality, see **7.374** 1 and **8.904**.

3.[10] $$n\, H_n(x) = -n\, H_{n-1}'(x) + x\, H_n'(x)$$ MS 5.6.2

4.[10] $H_n(x) = 2x\, H_{n-1}(x) - H'_{n-1}(x)$ MS 5.6.2

8.953 The connection with other functions:

1. $H_{2n}(x) = (-1)^n \dfrac{(2n)!}{n!}\, \Phi\left(-n, \frac{1}{2}; x^2\right)$ MO 106a

2. $H_{2n+1}(x) = (-1)^n 2 \dfrac{(2n+1)!}{n!} x\, \Phi\left(-n, \frac{3}{2}; x^2\right)$ MO 106a

- For a connection with the polynomials $C_n^\lambda(x)$, see **8.936** 5.
- For a connection with the Laguerre polynomials, see **8.972** 2 and **8.972** 3.
- For a connection with functions of a parabolic cylinder, see **9.253**.

8.954 Inequalities:

1.[10] $|H_n(x)| \le 2^{\frac{n}{2} - \lfloor \frac{n}{2} \rfloor} \dfrac{n!}{\lfloor n/2 \rfloor !} e^{2x\sqrt{\lfloor n/2 \rfloor}}$ MO 106a

2.[10] $|H_n(x)| < k\sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k \approx 1.086435$ SA 324

8.955 Asymptotic representation:

1. $H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[\cos\left(\sqrt{4n+1}x\right) + O\left(\dfrac{1}{\sqrt[4]{n}}\right)\right]$ SM 579

2. $H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1} e^{x^2/2} \left[\sin\left(\sqrt{4n+3}x\right) + O\left(\dfrac{1}{\sqrt[4]{n}}\right)\right]$ SM 579

8.956 Special cases and particular values:

1. $H_0(x) = 1$
2. $H_1(x) = 2x$
3. $H_2(x) = 4x^2 - 2$
4. $H_3(x) = 8x^3 - 12x$
5. $H_4(x) = 16x^4 - 48x^2 + 12$
6. $H_{2n}(0) = (-1)^n 2^n (2n-1)!!$ SM 570(24)
7. $H_{2n+1}(0) = 0$

Series of Hermite polynomials

8.957 The generating function:

1. $\exp\left(-t^2 + 2tx\right) = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x)$ SM 569(21)

2. $\dfrac{1}{e} \sinh 2x = \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x)$ MO 106a

3. $$\frac{1}{e}\cosh 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x)$$ MO 106a

4. $$e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x)$$ MO 106a

5. $$e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x)$$ MO 106a

8.958 "The summation theorem":

1.[11] $$\frac{\left(\sum_{k=1}^{r} a_k^2\right)^{\frac{n}{2}}}{n!} H_n\left(\frac{\sum_{k=1}^{r} a_k x_k}{\sqrt{\sum a_k^2}}\right) = \sum_{m_1+m_2+\cdots+m_r=n} \prod_{k=1}^{r} \left\{\frac{a_k^{m_k}}{m_k!} H_{m_k}(x_k)\right\}$$ MO 106a

2. A special case:

$$2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\left(x\sqrt{2}\right) H_k\left(y\sqrt{2}\right)$$ MO 107a

8.959 Hermite polynomials satisfy the differential equation

1. $$\frac{d^2u_n}{dx^2} - 2x\frac{du_n}{dx} + 2nu_n = 0;$$ SM 566(9)

A second solution of this differential equation is provided by the functions (A and B are arbitrary constants):

2. $$u_{2n} = Ax\,\Phi\left(\tfrac{1}{2} - n; \tfrac{3}{2}; x^2\right),$$

3. $$u_{2n+1} = B\,\Phi\left(-\tfrac{1}{2} - n; \tfrac{1}{2}; x^2\right)$$ MO 107

8.959(1)[10] Rodrigues' formula and orthogonality relation

1. $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right]$$ [Rodrigues' formula] MS 5.6.2

2. $$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi^{1/2} 2^n n! & \text{for } m = n \end{cases}$$ MS 5.6.2

8.96 Jacobi's polynomials

8.960 Definition

1. $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$ EH II 169(10), CO

$$= \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m$$ EH II 169(2)

8.961 Functional relations:

1.[11] $P_n^{(\alpha,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\alpha)}(x)$ EH II 169(13)

2. $$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) P_{n+1}^{(\alpha,\beta)}(x) = (2n+\alpha+\beta+1)\left[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+\alpha^2-\beta^2\right] P_n^{(\alpha,\beta)}(x) - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha,\beta)}(x)$$ EH II 169(11)

3. $$(2n+\alpha+\beta)\left(1-x^2\right)\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = n[(\alpha-\beta)-(2n+\alpha+\beta)x] P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta) P_{n-1}^{(\alpha,\beta)}(x)$$ EH II 170(15)

4.[11] $$\frac{d^m}{dx^m}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{1}{2^m}\frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m,\beta+m)}(x)$$ $[m=1,2,\dots,n]$ EH II 170(17)

5. $\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1-x) P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1) P_n^{(\alpha,\beta)}(x) - (n+1) P_{n+1}^{(\alpha,\beta)}(x)$ EH II 173(32)

6. $\left(n+\frac{1}{2}\alpha+\frac{1}{2}\beta+1\right)(1+x) P_n^{(\alpha,\beta+1)}(x) = (n+\beta+1) P_n^{(\alpha,\beta)}(x) + (n+1) P_{n+1}^{(\alpha,\beta)}(x)$ EH II 173(33)

7. $(1-x) P_n^{(\alpha+1,\beta)}(x) + (1+x) P_n^{(\alpha,\beta+1)}(x) = 2 P_n^{(\alpha,\beta)}(x)$ EH II 173(34)

8. $(2n+\alpha+\beta) P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta) P_n^{(\alpha,\beta)}(x) - (n+\beta) P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(35)

9. $(2n+\alpha+\beta) P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta) P_n^{(\alpha,\beta)}(x) + (n+\alpha) P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(36)

10. $P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x)$ EH II 173(37)

8.962 Connections with other functions:

1. $$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\,\Gamma(n+1+\beta)}{n!\,\Gamma(1+\beta)} F\left(n+\alpha+\beta+1,-n;1+\beta;\frac{1+x}{2}\right)$$ CO, EH II 170(16)
$$= \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)} F\left(n+\alpha+\beta+1,-n;1+\alpha;\frac{1-x}{2}\right)$$ EH II 170(16)
$$= \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)}\left(\frac{1+x}{2}\right)^n F\left(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}\right)$$ EH II 170(16)
$$= \frac{\Gamma(n+1+\beta)}{n!\,\Gamma(1+\beta)}\left(\frac{x-1}{2}\right)^n F\left(-n,-n-\alpha;\beta+1;\frac{x+1}{x-1}\right)$$ EH II 170(16)

2. $P_n(x) = P_n^{(0,0)}(x)$ CO, EH II 179(3)

3. $$T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)$$ CO, EH II 184(5)a

4. $$C_n^\nu(x) = \frac{\Gamma(n+2\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\,\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{(\nu-1/2,\nu-1/2)}(x)$$ MO 108a, EH II 174(4)

8.963 The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

$$R = \sqrt{1-2xz+z^2} \qquad [|z|<1]$$

EH II 172(29)

8.964 The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$\left(1-x^2\right) y'' + [\beta - \alpha - (\alpha+\beta+2)x] y' + n(n+\alpha+\beta+1) y = 0. \qquad \text{EH II 169(14)}$$

8.965 Asymptotic representation

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos\left\{\left[n+\frac{1}{2}(\alpha+\beta+1)\right]\theta - \left(\frac{1}{2}\alpha+\frac{1}{4}\right)\pi\right\}}{\sqrt{\pi n}\left(\sin\frac{1}{2}\theta\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{1}{2}\theta\right)^{\beta+\frac{1}{2}}} + O\left(n^{-3/2}\right) \qquad [\operatorname{Im}\alpha = \operatorname{Im}\beta = 0, \quad 0<\theta<\pi]$$

EH II 198(10)

8.966 A limit relationship:

$$\lim_{n\to\infty}\left[n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)\right] = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) \qquad \text{EH II 173(41)}$$

8.967 If $\alpha > -1$ and $\beta > -1$, all the zeros of the polynomial $P_n^{(\alpha,\beta)}(x)$ are simple, and they lie in the interval $(-1,1)$.

8.97 The Laguerre polynomials

8.970 Definition.

1. $$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right) \qquad \text{[Rodrigues' formula]}$$ EH II 188(5), MO 108

$$= \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}$$ MO 109, EH II 188(7)

2. $L_n^0(x) = L_n(x)$ ET I 369

3.[10] $L_0^\alpha(x) = 1$

4.[10] $L_1^\alpha(x) = -x+\alpha+1$

5.[10] $$L_2^\alpha(x) = \frac{1}{2}\left[x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2)\right]$$

6.[10] $$L_3^\alpha(x) = -\frac{1}{6}\left[x^3 - 3(\alpha+3)x^2 + 3(\alpha+2)(\alpha+3)x - (\alpha+1)(\alpha+2)(\alpha+3)\right]$$

7.[10] $$L_4^\alpha(x) = \frac{1}{24}\left[x^4 - 4(\alpha+4)x^3 + 6(\alpha+3)(\alpha+4)x^2 - 4(\alpha+2)(\alpha+3)(\alpha+4)x + (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\right]$$

8.[10] $$L_5^\alpha(x) = -\frac{1}{120}\left[x^5 - 5(\alpha+5)x^4 + 10(\alpha+4)(\alpha+5)x^3 - 10(\alpha+3)(\alpha+4)(\alpha+5)x^2 + 5(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)x - (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)\right]$$

8.971 Functional relations:

1. $$\frac{d}{dx}\left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] = L_n^\alpha(x)$$ EH II 189(16)

2.[11] $$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{n L_n^\alpha(x) - (n+\alpha) L_{n-1}^\alpha(x)}{x}$$ EH II 189(15), SM 575(42)a

3. $$x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n+\alpha) L_{n-1}^\alpha(x)$$
$$= (n+1) L_{n+1}^\alpha(x) - (n+\alpha+1-x) L_n^\alpha(x)$$
EH II 189(12), MO 109

4. $$x L_n^{\alpha+1}(x) = (n+\alpha+1) \; L_n^\alpha(x) - (n+1) L_{n+1}^\alpha(x)$$
$$= (n+\alpha) L_{n-1}^\alpha(x) - (n-x) L_n^\alpha(x)$$
SM 575(43)a, EH II 190(23)

5. $$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x)$$ SM 575(44)a, EH II 190(24)

6. $$(n+1) L_{n+1}^\alpha(x) - (2n+\alpha+1-x) L_n^\alpha(x) + (n+\alpha) L_{n-1}^\alpha(x) = 0$$
$[n = 1, 2, \ldots]$ MO 109, EH II 190(25, 24)

7.[10] $$(n+\alpha) L_n^{\alpha-1}(x) = (n+1) L_{n+1}^\alpha(x) - (n+1-x) L_n^\alpha(x)$$ MS 5.5.2

8.[10] $$n L_n^\alpha(x) = (2n+\alpha-1-x) L_{n-1}^\alpha(x) - (n+\alpha-1) L_{n-2}^\alpha(x)$$
$[n = 2, 3, \ldots]$ MS 5.5.2

8.972 Connections with other functions:

1. $$L_n^\alpha(x) = \binom{n+\alpha}{n} \Phi(-n, \alpha+1; x)$$ MO 109, FI II 189(14)

2. $$H_{2n}(x) = (-1)^n 2^{2n} n! \, L_n^{-1/2}\left(x^2\right)$$ EH II 193(2), SM 576(47)

3. $$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x \, L_n^{1/2}\left(x^2\right)$$ EH II 193(3), SM 577(48)

8.973 Special cases:

1. $$L_0^\alpha(x) = 1$$ EH II 188(6)

2. $$L_1^\alpha(x) = \alpha + 1 - x$$ EH II 188(6)

3. $$L_n^\alpha(0) = \binom{n+\alpha}{n}$$ EH II 189(13)

4. $$L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}$$ MO 109

5. $$L_1(x) = 1 - x$$

6. $$L_2(x) = 1 - 2x + \frac{x^2}{2}$$ MO 109

8.974 Finite sums:

1. $$\sum_{m=0}^{n} \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x)\, L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} \left[L_n^\alpha(x)\, L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x)\, L_n^\alpha(y)\right]$$ EH II 188(9)

2.[11] $$\sum_{m=0}^{n} \frac{\Gamma(\alpha-\beta+m)}{\Gamma(\alpha-\beta)m!} L_{n-m}^\beta(x) = L_n^\beta(x)$$ MO 110, EH II 192(39)

3. $$\sum_{m=0}^{n} L_m^\alpha(x) = L_n^{\alpha+1}(x)$$ EH II 192(38)

4.[11] $$\sum_{m=0}^{n} L_m^\alpha(x)\, L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y)$$ EH II 192(41)

8.975 Arbitrary functions:

1. $$(1-z)^{-\alpha-1} \exp\frac{xz}{z-1} = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n \qquad [|z|<1]$$ EH II 189(17), MO 109

2. $$e^{-xz}(1+z)^\alpha = \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n \qquad [|z|<1]$$ MO 110, EH II 189(19)

3. $$J_\alpha\left(2\sqrt{xz}\right) e^z (xz)^{-\frac{1}{2}\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \qquad [\alpha > -1]$$ EH II 189(18), MO 109

8.976 Other series of Laguerre polynomials:

1. $$\sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x)\, L_n^\alpha(y) z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z} \exp\left(-z\frac{x+y}{1-z}\right) I_\alpha\left(2\frac{\sqrt{xyz}}{1-z}\right)$$ $[|z|<1]$ EH II 189(20)

2. $$\sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+1} = e^x x^{-\alpha}\, \Gamma(\alpha, x) \qquad [\alpha > -1, \quad x > 0]$$ EH II 215(19)

3.[6] $$L_n^\alpha(x)^2 = \frac{\Gamma(n+\alpha+1)}{2^{2n} n!} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(2k)!}{k!} \frac{1}{\Gamma(\alpha+k+1)} L_{2k}^{2\alpha}(2x)$$ MO 110

4.[6] $$L_n^\alpha(x)\, L_n^\alpha(y) = \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^{n} \frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)} \frac{(xy)^k}{k!}$$ MO 110, EH II 192(42)

8.977 Summation theorems:

1. $$L_n^{\alpha_1+\alpha_2+\cdots+\alpha_k+k-1}(x_1+x_2+\cdots+x_k) = \sum_{i_1+i_2+\cdots+i_2=n} L_{i_1}^{\alpha_1}(x_1)\, L_{i_2}^{\alpha_2}(x_2) \cdots L_{i_k}^{\alpha_k}(x_k)$$ MO 110

2. $$L_n^\alpha(x+y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k\, L_n^{\alpha+k}(x)$$ MO 110

8.978 Limit relations and asymptotic behavior:

1. $$L_n^\alpha(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)$$ EH II 191(35)

2. $$\lim_{n\to\infty}\left[n^{-\alpha} L_n^\alpha\left(\frac{x}{n}\right)\right] = x^{-\frac{1}{2}\alpha} J_\alpha\left(2\sqrt{x}\right)$$ EH II 191(36)

3. $$L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha-\frac{1}{4}} n^{\frac{1}{2}\alpha-\frac{1}{4}} \cos\left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O\left(n^{\frac{1}{2}\alpha-\frac{3}{4}}\right)$$
$[\operatorname{Im}\alpha = 0, \quad x > 0]$ EH II 199(1)

8.979 Laguerre polynomials satisfy the following differential equation:
$$x\frac{d^2u}{dx^2} + (\alpha - x + 1)\frac{du}{dx} + nu = 0$$ EH II 188(10), SM 574(34)

8.980[11] Orthogonality relation
$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x)\,dx = \begin{cases} 0, & m \neq n \\ \Gamma(1+\alpha)\binom{n+\alpha}{n}, & m = n \end{cases}$$ MS 5.5.2

8.981[10] Behavior of relative maxima of $|L_n^\alpha(x)|$

1. Let α be arbitrary and real. The sequence formed by the relative maxima of $|L_n^\alpha(x)|$ and by the value of this function at $x = 0$, is decreasing for $x < \alpha + \frac{1}{2}$, and increasing for $x > \alpha + \frac{1}{2}$. The successive relative maxima of $|L_n^\alpha(x)|$ form a decreasing sequence for $x \leq 0$, and an increasing sequence for $x \geq 0$. SZ 174(7.6.1)

2. Let α be an arbitrary real number. The successive relative maxima of
$$e^{-x/2}x^{(\alpha+1)/2}|L_n^\alpha(x)| \text{ and } e^{-x/2}x^{\alpha/2+\frac{1}{4}}|L_n^\alpha(x)|$$
form an increasing sequence, provided $x > x_0$. In the first case
$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \dfrac{\alpha^2 - 1}{2n + \alpha + 1} & \text{if } \alpha^2 > 1 \end{cases}$$
In the second case,
$$x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq q\frac{1}{4}, \\ \left(\alpha^2 - \frac{1}{4}\right)^{\frac{1}{2}} & \text{if } \alpha^2 > \frac{1}{4} \end{cases}$$ SZ 174(7.6.2)
In the first case, we take n so large that $2n + \alpha + 1 > 0$.

8.982[10] Asymptotic and limiting behavior of $L_n^\alpha(x)$

1. Let α be arbitrary and real, c and w fixed positive constants, and let $n \to \infty$. Then
$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}}\, O\left(n^{\alpha/2-\frac{1}{4}}\right) & \text{if } cn^{-1} \leq qx \leq q\omega \\ O\left(n^\alpha\right) & \text{if } 0 \leq qx \leq qcn^{-1} \end{cases}$$
These bounds are precise as regards their orders in n. For $\alpha \geq q - \frac{1}{2}$, both bounds hold in both intervals, that is,

$$L_n^\alpha(x) = \begin{cases} x^{-\alpha/2-\frac{1}{4}} O\left(n^{\alpha/2-\frac{1}{4}}\right), \\ O\left(n^\alpha\right), \end{cases} \quad 0 < x \le q\omega, \quad \alpha \ge q - \tfrac{1}{2}$$ SZ 175(7.6.4)

2. Let α be arbitrary and real. Then for an arbitrary complex z

$$\lim_{n\to\infty} n^{-\alpha}\, L_n^\alpha(x) = z^{-\alpha/2}\, J_\alpha\left(2z^{1/2}\right),$$ SZ 191(8.1.3)

uniformly if z is bounded.

9.1 Hypergeometric Functions

9.10 Definition

9.100 A *hypergeometric series* is a series of the form

$$F(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\cdot\beta}{\gamma\cdot 1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 1\cdot 2\cdot 3}z^3 + \dots$$

9.101 A hypergeometric series terminates if α or β is equal to a negative integer or to zero. For $\gamma = -n\,(n = 0, 1, 2, \ldots)$, the hypergeometric series is indeterminate if neither α nor β is equal to $-m$ (where $m < n$ and m is a natural number). However,

1. $$\lim_{\gamma\to -n} \frac{F(\alpha,\beta;\gamma;z)}{\Gamma(\gamma)} = \frac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{(n+1)!} \times z^{n+1} F(\alpha+n+1, \beta+n+1; n+2; z)$$

EH I 62(16)

9.102 If we exclude these values of the parameters α, β, γ, a hypergeometric series converges in the unit circle $|z| < 1$. F then has a branch point at $z = 1$. Then we have the following conditions for convergence on the unit circle:

1. $1 > \operatorname{Re}(\alpha+\beta-\gamma) \ge 0$. The series converges throughout the entire unit circle, except at the point $z = 1$.
2. $\operatorname{Re}(\alpha+\beta-\gamma) < 0$. The series converges (absolutely) throughout the entire unit circle.
3. $\operatorname{Re}(\alpha+\beta-\gamma) \ge 1$. The series diverges on the entire unit circle. FI II 410, WH

9.11 Integral representations

9.111 $$F(\alpha,\beta;\gamma;z) = \frac{1}{\mathrm{B}(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt \qquad [\operatorname{Re}\gamma > \operatorname{Re}\beta > 0]$$ WH

9.112[8] $$F\left(p, n+p; n+1; z^2\right) = \frac{z^{-n}}{2\pi}\frac{\Gamma(p)n!}{\Gamma(p+n)}\int_0^{2\pi}\frac{\cos nt\,dt}{(1-2z\cos t+z^2)^p}$$
$[n = 0, 1, 2, \ldots; \quad p \ne 0, -1, -2, \ldots; \quad |z| < 1]$ WH, MO 16

9.113 $$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)}(-z)^t\,dt$$

Here, $|\arg(-z)| < \pi$ and the path of integration are chosen in such a way that the poles of the functions $\Gamma(\alpha+t)$ and $\Gamma(\beta+t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of it.

9.114 $$F\left(-m, -\frac{p+m}{2}; 1-\frac{p+m}{2}; -1\right) = \frac{(-2)^m(p+m)}{\sin p\pi}\int_0^\pi \cos^m\varphi\cos p\varphi\,d\varphi$$
$[m+1$ is a natural number; $\quad p \ne 0, \quad \pm 1, \ldots\ ]$ EH I 80(8), MO 16

See also **3.194** 1, 2, 5, **3.196** 1, **3.197** 6, 9, **3.259** 3, **3.312** 3, **3.518** 4–6, **3.665** 2, **3.671** 1, 2, **3.681** 1, **3.984** 7.

9.12 Representation of elementary functions in terms of a hypergeometric functions

9.121

1.[8] $$F(-n,\beta;\beta;-z) = (1+z)^n$$ EH I 101(4), GA 127 Ia

2. $$F\left(-\frac{n}{2},-\frac{n-1}{2};\frac{1}{2};\frac{z^2}{t^2}\right) = \frac{(t+z)^n+(t-z)^n}{2t^n}$$ GA 127 II

3. $$\lim_{\omega\to\infty} F\left(-n,\omega;2\omega;-\frac{z}{t}\right) = \left(1+\frac{z}{2t}\right)^n$$ GA 127 IIIa

4. $$F\left(-\frac{n-1}{2},-\frac{n-2}{2};\frac{3}{2};\frac{z^2}{t^2}\right) = \frac{(t+z)^n-(t-z)^n}{2nzt^{n-1}}$$ GA 127 IV

5. $$F\left(1-n,1;2;-\frac{z}{t}\right) = \frac{(t+z)^n-t^n}{nzt^{n-1}}$$ GA 127 V

6. $$F(1,1;2;-z) = \frac{\ln(1+z)}{z}$$ GA 127 VI

7. $$F\left(\frac{1}{2},1;\frac{3}{2};z^2\right) = \frac{\ln\frac{1+z}{1-z}}{2z}$$ GA 127 VII

8. $$\lim_{k\to\infty} F\left(1,k;1;\frac{z}{k}\right) = 1+z\lim_{k\to\infty} F\left(1,k;2;\frac{z}{k}\right) = 1+z+\frac{z^2}{2}\lim_{k\to\infty} F\left(1,k;3;\frac{z}{k}\right) = \cdots = e^z$$ GA 127 VIII

9. $$\lim_{\substack{k\to\infty\\k'\to\infty}} F\left(k,k';\frac{1}{2};\frac{z^2}{4kk'}\right) = \frac{e^z+e^{-z}}{2} = \cosh z$$ GA 127 IX

10. $$\lim_{\substack{k\to\infty\\k'\to\infty}} F\left(k,k';\frac{3}{2};\frac{z^2}{4kk'}\right) = \frac{e^z-e^{-z}}{2z} = \frac{\sinh z}{z}$$ GA 127 X

11. $$\lim_{\substack{k\to\infty\\k'\to\infty}} F\left(k,k';\frac{3}{2};-\frac{z^2}{4kk'}\right) = \frac{\sin z}{z}$$ GA 127 XI

12. $$\lim_{\substack{k\to\infty\\k'\to\infty}} F\left(k,k';\frac{1}{2};-\frac{z^2}{4kk'}\right) = \cos z$$ GA 127 XII

13. $$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\sin^2 z\right) = \frac{z}{\sin z}$$ GA 127 XIII

14. $$F\left(1,1;\frac{3}{2};\sin^2 z\right) = \frac{z}{\sin z\cos z}$$ GA 127 XIV

15. $$F\left(\frac{1}{2},1;\frac{3}{2};-\tan^2 z\right) = \frac{z}{\tan z}$$ GA 127 XV

16. $$F\left(\frac{n+1}{2},-\frac{n-1}{2};\frac{3}{2};\sin^2 z\right) = \frac{\sin nz}{n\sin z}$$ GA 127 XVI

17. $$F\left(\frac{n+2}{2},-\frac{n-2}{2};\frac{3}{2};\sin^2 z\right) = \frac{\sin nz}{n\sin z\cos z}$$ GA 127 XVII

18. $F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}$ GA 127 XVIII

19. $F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\tan^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}$ GA 127 XIX

20. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz$ EH I 101(11), GA 127 XX

21. $F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}$ EH I 101(11), GA 127 XXI

22. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\tan^2 z\right) = \frac{\cos nz}{\cos^n z}$ EH I 101(11), GA 127 XXII

23. $F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\tan^2 z\right) = \cos nz \cos^n z$ GA 127 XXIII

24. $F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z}$ $\left[|z| \le \frac{1}{2}; \quad |z(1-z)| \le \frac{1}{4}\right]$

25. $F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z$

26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$ (cf. **9.121** 13)

27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\arctan z}{z}$ (cf. **9.121** 15)

28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arcsinh} z}{z}$ (cf. **9.121** 26)

29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$ (cf. **9.121** 16)

30. $F\left(1+\frac{n}{2}, 1-\frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$ (cf. **9.121** 17)

31. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z)$ (cf. **9.121** 20)

32. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}$ (cf. **9.121** 21)

The representation of special functions in terms of a hypergeometric function:

- for complete elliptic integrals, see **8.113** 1 and **8.114** 1;
- for integrals of Bessel functions, see **6.574** 1, 3, **6.576** 2–5, **6.621** 1–3;
- for Legendre polynomials, see **8.911** and **8.916**. (All these hypergeometric series terminate; that is, these series are finite sums);
- for Legendre functions, see **8.820** and **8.837**;
- for associated Legendre functions, see **8.702, 8.703, 8.751, 8.77, 8.852**, and **8.853**;
- for Chebyshev polynomials, see **8.942** 1;
- for Jacobi's polynomials, see **8.962**;

- for Gegenbauer polynomials, see **8.932**;
- for integrals of parabolic cylinder functions, see **7.725** 6.

9.122 Particular values:

1. $$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad [\operatorname{Re}\gamma > \operatorname{Re}(\alpha+\beta)]$$

GA 147(48), FI II 793

2. $$F(\alpha,\beta;\gamma;1) = F\left(-\alpha,-\beta;\gamma-\alpha-\beta;1\right) \qquad [\operatorname{Re}\gamma > \operatorname{Re}(\alpha+\beta)]$$ GA 148(49)

$$= \frac{1}{F(-\alpha,\beta;\gamma-\alpha;1)} \qquad [\operatorname{Re}\gamma > \operatorname{Re}(\alpha+\beta)]$$ GA 148(50)

$$= \frac{1}{F(\alpha,-\beta;\gamma-\beta;1)} \qquad [\operatorname{Re}\gamma > \operatorname{Re}(\alpha+\beta)]$$ GA 148(51)

3. $$F\left(1,1;\frac{3}{2};\frac{1}{2}\right) = \frac{\pi}{2} \qquad (\text{cf. } \mathbf{9.121}\ 14)$$

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.130 The series $F(\alpha,\beta;\gamma;z)$ defines an analytic function that, speaking generally, has singularities at the points $z=0$, 1, and ∞. (In the general case, there are branch points.) We make a cut in the z-plane along the real axis from $z=1$ to $z=\infty$; that is, we require that $|\arg(-z)| < \pi$ for $|z| \geq 1$. Then, the series $f(\alpha,\beta;\gamma;z)$ will, in the cut plane, yield a single-valued analytic continuation, which we can obtain by means of the formulas below (provided $\gamma+1$ is not a natural number and $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers). These formulas make it possible to calculate the values of F in the given region, even in the case in which $|z|>1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α,β,γ.

Transformation formulas

9.131

1.[11] $$F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha} F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right)$$ GA 218(91)

$$= (1-z)^{-\beta} F\left(\beta,\gamma-\alpha;\gamma;\frac{z}{z-1}\right)$$ GA 218(92)

$$= (1-z)^{\gamma-\alpha-\beta} F\left(\gamma-\alpha,\gamma-\beta;\gamma;z\right)$$

2. $$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} F\left(\alpha,\beta;\alpha+\beta-\gamma+1;1-z\right) + (1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} F\left(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z\right)$$

EH I 94, MO 13

9.132

1. $$F(\alpha,\beta;\gamma;z) = \frac{(1-z)^{-\alpha}\,\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\,F\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right) + (1-z)^{-\beta}\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}\,F\left(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right)$$
MO 13

2.[11] $$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\,F\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right) + \frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\alpha)\,\Gamma(\gamma-\beta)}(-z)^{-\beta}\,F\left(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}\right)$$
$[|\arg z| < \pi, \quad \alpha-\beta \neq \pm m, \quad m=0,1,2,\ldots]$ GA 220(93)

9.133 $$F\left(2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};z\right) = F\left(\alpha,\beta;\alpha+\beta+\tfrac{1}{2};4z(1-z)\right)$$
$\left[|z| \le \tfrac{1}{2}, \quad |z(1-z)| \le \tfrac{1}{4}\right]$ WH

9.134

1. $$F(\alpha,\beta;2\beta;z) = \left(1-\frac{z}{2}\right)^{-\alpha} F\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\beta+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right)$$ MO 13, EH I 111(4)

2. $$F(2\alpha,2\alpha+1-\gamma;\gamma;z) = (1+z)^{-2\alpha}\,F\left(\alpha,\alpha+\frac{1}{2};\gamma;\frac{4z}{(1+z)^2}\right)$$ GA 225(100)

3. $$F\left(\alpha,\alpha+\frac{1}{2}-\beta;\beta+\frac{1}{2};z^2\right) = (1+z)^{-2\alpha}\,F\left(\alpha,\beta;2\beta;\frac{4z}{(1+z)^2}\right)$$ GA 225(101)

9.135 $$F\left(\alpha,\beta;\alpha+\beta+\frac{1}{2};\sin^2\varphi\right) = F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\sin^2\frac{\varphi}{2}\right)$$
$\left[x = \sin^2\frac{\varphi}{2} \text{ real}; \quad \frac{1-\sqrt{2}}{2} < x < \frac{1}{2}\right]$
MO 13

9.136[8] We set
$$A = \frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}, \qquad B = \frac{-\Gamma\left(\alpha+\beta+\frac{1}{2}\right)2\sqrt{\pi}}{\Gamma(\alpha)\,\Gamma(\beta)};$$
then

1. $$F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{1-\sqrt{z}}{2}\right) = AF\left(\alpha,\beta;\frac{1}{2};z\right) + B\sqrt{z}\,F\left(\alpha+\frac{1}{2},\beta+\frac{1}{2};\frac{3}{2};z\right)$$
GA 227(106)

2. $$F\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{1+\sqrt{z}}{2}\right) = AF\left(\alpha,\beta;\frac{1}{2};z\right) - B\sqrt{z}\,F\left(\alpha+\frac{1}{2},\beta+\frac{1}{2};\frac{3}{2};z\right)$$
GA 227(107)

3. $$\frac{\left(\alpha-\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)}{\alpha+\beta-\frac{1}{2}}A\sqrt{z}\,F\left(\alpha,\beta;\frac{3}{2};z\right) = F\left(2\alpha-1,2\beta-1;\alpha+\beta-\frac{1}{2};\frac{1+\sqrt{z}}{2}\right) - F\left(2\alpha-1,2\beta-1;\alpha+\beta-\frac{1}{2};\frac{1-\sqrt{z}}{2}\right)$$
GA 229(110)

9.137[7] Gauss' recursion functions:

1. $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z]\, F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z\, F(\alpha, \beta; \gamma + 1; z) + \gamma(\gamma - 1)(z - 1)\, F(\alpha, \beta; \gamma - 1; z) = 0$
2. $(2\alpha - \gamma - \alpha z + \beta z)\, F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)\, F(\alpha - 1, \beta; \gamma; z) + \alpha(z - 1)\, F(\alpha + 1, \beta; \gamma; z) = 0$
3. $(2\beta - \gamma - \beta z + \alpha z)\, F(\alpha, \beta; \gamma; z) + (\gamma - \beta)\, F(\alpha, \beta - 1; \gamma; z) + \beta(z - 1)\, F(\alpha, \beta + 1; \gamma; z) = 0$
4. $\gamma\, F(\alpha, \beta - 1; \gamma; z) - \gamma\, F(\alpha - 1, \beta; \gamma; z) + (\alpha - \beta)z\, F(\alpha, \beta; \gamma + 1; z) = 0$
5.[8] $\gamma(\alpha - \beta)\, F(\alpha, \beta; \gamma; z) - \alpha(\gamma - \beta)\, F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)\, F(\alpha, \beta + 1; \gamma + 1; z) = 0$
6. $\gamma(\gamma + 1)\, F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)\, F(\alpha, \beta; \gamma + 1; z) - \alpha\beta z\, F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
7. $\gamma\, F(\alpha, \beta; \gamma; z) - (\gamma - \alpha)\, F(\alpha, \beta + 1; \gamma + 1; z) - \alpha(1 - z)\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
8. $\gamma\, F(\alpha, \beta; \gamma; z) + (\beta - \gamma)\, F(\alpha + 1, \beta; \gamma + 1; z) - \beta(1 - z)\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
9. $\gamma(\gamma - \beta z - \alpha)\, F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \alpha)\, F(\alpha - 1, \beta; \gamma; z) + \alpha\beta z(1 - z)\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
10. $\gamma(\gamma - \alpha z - \beta)\, F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \beta)\, F(\alpha, \beta - 1; \gamma; z) + \alpha\beta z(1 - z)\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
11. $\gamma\, F(\alpha, \beta; \gamma; z) - \gamma\, F(\alpha, \beta + 1; \gamma; z) + \alpha z\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
12.[8] $\gamma\, F(\alpha, \beta; \gamma; z) - \gamma\, F(\alpha + 1, \beta; \gamma; z) + \beta z\, F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0$
13. $\gamma[\alpha - (\gamma - \beta)z]\, F(\alpha, \beta; \gamma; z) - \alpha\gamma(1 - z)\, F(\alpha + 1, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z\, F(\alpha, \beta; \gamma + 1; z) = 0$
14. $\gamma[\beta - (\gamma - \alpha)z]\, F(\alpha, \beta; \gamma; z) - \beta\gamma(1 - z)\, F(\alpha, \beta + 1; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)z\, F(\alpha, \beta; \gamma + 1; z) = 0$
15.[8] $\gamma(\gamma + 1)\, F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)\, F(\alpha, \beta + 1; \gamma + 1; z) + \alpha(\gamma - \beta)z\, F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
16. $\gamma(\gamma + 1)\, F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)\, F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)z\, F(\alpha + 1, \beta + 1; \gamma + 2; z) = 0$
17. $\gamma\, F(\alpha, \beta; \gamma; z) - (\gamma - \beta)\, F(\alpha, \beta; \gamma + 1; z) - \beta\, F(\alpha, \beta + 1; \gamma + 1; z) = 0$
18.[8] $\gamma\, F(\alpha, \beta; \gamma; z) - (\gamma - \alpha)\, F(\alpha, \beta; \gamma + 1; z) - \alpha\, F(\alpha + 1, \beta; \gamma + 1; z) = 0$ MO 13–14

9.14 A generalized hypergeometric series

The series

1. $${}_pF_q\,(\alpha_1, \alpha_2, \ldots, \alpha_p;\; \beta_1, \beta_2, \ldots, \beta_q;\; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\,(\alpha_2)_k \ldots (\alpha_p)_k}{(\beta_1)_k\,(\beta_2)_k \ldots (\beta_q)_k}\frac{z^k}{k!}$$ MO 14

   is called a *generalized hypergeometric series* (see also 9.210).
2. ${}_2F_1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z)$ MO 15

For integral representations, see **3.254** 2, **3.259** 2, and **3.478** 3.

9.15 The hypergeometric differential equation

9.151 A hypergeometric series is one of the solutions of the differential equation

$$z(1 - z)\frac{d^2u}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{du}{dz} - \alpha\beta u = 0,$$ WH

which is called the *hypergeometric equation*.

The solution of the hypergeometric differential equation

9.152 The hypergeometric differential equation **9.151** possesses *two linearly independent solutions.* These solutions have analytic continuations to the entire z-plane, except possibly for the three points 0, 1, and ∞. Generally speaking, the points $z = 0, 1, \infty$ are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio $w(z)$ of two linearly independent solutions satisfies the differential equation

$$2\frac{w'''}{w'} - 3\left(\frac{w''}{w'}\right)^2 = \frac{1-a_1^2}{z^2} + \frac{1-a_2^2}{(z-1)^2} + \frac{a_1^2+a_2^2-a_3^2-1}{z(z-1)},$$

where

$$a_1^2 = (1-\gamma)^2, \quad a_2^2 = (\gamma-\alpha-\beta)^2, \quad a_3^2 = (\alpha-\beta)^2.$$

If α, β, γ are real, the function $w(z)$ maps the upper (Im $z > 0$) or the lower (Im $z <0$) half-plane onto a curvilinear triangle whose angles are $\pi a_1, \pi a_2, \pi a_3$. The vertices of this triangle are the images of the points $z = 0, z = 1$, and $z = \infty$.

9.153 Within the unit circle $|z| <1$, the linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeometric differential equation are given by the following formulas:

1. If γ is not an integer,

$$u_1 = F(\alpha, \beta; \gamma; z),$$
$$u_2 = z^{1-\gamma} e\, F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z)$$

2. If $\gamma = 1$, then

$$u_1 = F(\alpha, \beta; 1; z),$$
$$u_2 = F(\alpha, \beta; 1; z)\ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k(\beta)_k}{(k!)^2} \times \{\psi(\alpha+k) - \psi(\alpha) + \psi(\beta+k) - \psi(\beta) - 2\,\psi(k+1) + 2\,\psi(1)\}$$

(see **9.14** 2)

3. If $\gamma = m+1$ (where m is a natural number), and if neither α nor β is a positive number not exceeding m, then

$$u_1 = F(\alpha, \beta; m+1; z),$$
$$u_2 = F(\alpha, \beta; m+1; z)\ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k(\beta)_k}{(1+m)_k}\{h(k) - h(0)\} - \sum_{k=1}^{m} \frac{(k-1)!(-m)_k}{(1-\alpha)_k(1-\beta)_k} z^{-k}$$

(see **9.14** 2)

where

$$h(n) = \psi(\alpha+n) + \psi(\beta+n) - \psi(m+1+n) - \psi(n+1) \qquad [n+1 \text{ is a natural number}]$$

4.[11] Suppose that $\gamma = m+1$ (where m is a natural number) and that α or β is equal to $m'+1$, where $0 \le m' < m$. Then, for example, for $\alpha = m'+1$, we obtain

$$u_1 = F\left(1+m', \beta; 1+m; z\right),$$
$$u_2 = z^{-m}\, F\left(1+m'-m, \beta-m; 1-m; z\right)$$

In this case, u_2 is a polynomial in z^{-1}.

5. If $\gamma = 1 - m$ (where m is a natural number) and if α and β are both different from the numbers $0, -1, -2, \ldots, 1-m$, then

$$u_1 = z^m F(\alpha+m, \beta+m; 1+m; z),$$
$$u_2 = z^m F(\alpha+m, \beta+m; 1+m; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha+m)_k(\beta+m)_k}{(1+m)_k k!} \{h^*(k) - h^*(0)\}$$
$$- \sum_{k=1}^{\infty} \frac{(k-1)!(-m)_k}{(1-\alpha-m)_k(1-\beta-m)_k} z^{m-n}$$

(see **9.14** 2)

where

$$h^*(n) = \psi(\alpha+m+n) + \psi(\beta+m+n) - \psi(1+m+n) - \psi(1+n)$$

We note that

$$\psi(\alpha+n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha+1} + \cdots + \frac{1}{\alpha+n-1} \qquad \text{(cf. 8.365 3)}$$

and that, for $\alpha = -\lambda$, where λ is a natural number or zero and $n = \lambda+1, \lambda+2, \ldots$ the expression

$$(\alpha)_k \left[\psi(\alpha+n) - \psi(\alpha)\right]$$

in formulas **9.153** 2–5 should be replaced with the expression

$$(-1)^\lambda \lambda!(n-\lambda-1)!$$

6. Suppose that $\gamma = 1 - m$ (where m is a natural number) and that α or β is an integer $(-m')$, where m' is one of the following numbers: $0, 1, \ldots, m-1$. Suppose, for example, that $\alpha = -m'$. Then,

$$u_1 = F\left(-m', \beta; 1-m; z\right),$$
$$u_2 = F\left(-m'+m, \beta+m; 1+m; z\right)$$

MO 18

7. For $\gamma = \frac{1}{2}(\alpha+\beta+1)$

$$u_1 = F\left(\alpha, \beta; \tfrac{1}{2}(\alpha+\beta+1); z\right),$$
$$u_2 = F\left(\alpha, \beta; \tfrac{1}{2}(\alpha+\beta+1); 1-z\right)$$

are two linearly independent solutions of the hypergeometric differential equation, provided α, β, and γ are not zero or negative integers. MO 17–19

The analytic continuation of a solution that is regular at the point $z = 0$

9.154 Formulas **9.153** make possible the analytic continuation, by means of the hypergeometric series, of the function $F(\alpha, \beta; \gamma; z)$ defined inside the circle $|z| < 1$ to the region $|z| > 1$, and $|\arg(-z)| < \pi$. Here, it is assumed that $\alpha - \beta$ is not an integer. In the event that $\alpha - \beta$ is an integer (for example, if $\beta = \alpha + m$, where m is a natural number), then, for $|z| > 1$, and $|\arg(-z)| < \pi$ we have:

1. $$\frac{\Gamma(\alpha)\,\Gamma(\alpha+m)}{\Gamma(\gamma)}\,F(\alpha,\alpha+m;\gamma;z) = \frac{\sin\pi(\gamma-\alpha)}{\pi}\left\{\sum_{k=0}^{m-1}\frac{\Gamma(\alpha+k)\,\Gamma(1-\gamma+\alpha+k)\,\Gamma(m-k)}{k!}(-z)^{-\alpha-k} + (-z)^{-\alpha-m}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+m+k)\,\Gamma(1-\gamma+\alpha+m+k)}{k!(k+m)!}g(k)z^{-k}\right\}$$

where

2. $$g(n) = \ln(-z) + \pi\cot\pi(\gamma-\alpha) + \psi(n+1) + \psi(n+m+1) - \psi(\alpha+m+n) - \psi(1-\gamma+\alpha+m+n)$$

For $m=0$, we should set $\sum_{k=0}^{m-1} = 0$.

9.155 This formula loses its meaning when α, γ, or $\alpha-\gamma+1$ is equal to one of the numbers $0, -1, -2, \ldots$. In this last case, we have

1. If α is a non-positive integer and γ is not an integer, $F(\alpha,\alpha+m;\gamma;z)$ is a polynomial in z.
2. Suppose that γ is a non-positive integer and that α is not an integer. We then set $\gamma=-\lambda$, where $\lambda=0,1,2,\ldots$. Then,
$$\frac{\Gamma(\alpha+\lambda+1)\,\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)}z^{\lambda+1}\,F(\alpha+\lambda+1,\alpha+\lambda+m+1;\lambda+2;z)$$
is a solution of the hypergeometric equation that is regular at the point $z=0$. This solution is equal to the right-hand member of formula **9.154** 1 if we replace γ with λ in this equation and in formula **9.154** 2.
3. If $\alpha-\gamma+1$ is a non-positive integer and if α and γ are not themselves integers, we may use the formula
$$F(\alpha,\alpha+m;\gamma;z) = (1-z)^{\gamma-2\alpha-m}\,F(\gamma-\alpha-m,\gamma-\alpha;\gamma;z)$$
and apply formula **9.154** 1 to its right-hand member, provided $\gamma-\alpha-m>0$. However, if $\alpha-\gamma-m\leq 0$, the right member of this expression is a polynomial taken to the $(1-z)^{\text{th}}$ power.
4. If α, β, and γ are integers, the hypergeometric differential equation always has a solution that is regular for $z=0$ and that is of the form
$$R_1(z) + \ln(1-z)R_2(z),$$
where $R_1(z)$ and $R_2(z)$ are rational functions of z. To get a solution of this form, we need to apply formulas **9.137** 1–**9.137** 3 to the function $F(\alpha,\beta;\gamma;z)$. However, if $\gamma=-\lambda$, where $\lambda+1$ is a natural number, formulas **9.137** 1 and **9.137** 2 should be applied not to $F(\alpha,\beta;\gamma;z)$ but to the function $z^{\lambda+1}\,F(\alpha+\lambda+1,\beta+\lambda+1;\lambda+2,z)$.

By successive applications of these formulas, we can reduce the positive values of the parameters to the pair, unity and zero. Furthermore, we can obtain the desired form of the solution from the formulas
$$F(1,1;2;z) = -z^{-1}\ln(1-z),$$
$$F(0,\beta;\gamma;z) = F(\alpha,0;\gamma;z) = 1$$

MO 19–20

9.16 Riemann's differential equation

9.160 The hypergeometric differential equation is a particular case of Riemann's differential equation

1.[11] $$\frac{d^2u}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]\frac{du}{dz} + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b}\ \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)} = 0$$ WH

The coefficients of this equation have poles at the points a, b, and c, and the numbers α, α'; β, β'; γ, γ' are called the indices corresponding to these poles. The indices α, α'; β, β'; γ, γ' are related by the following equation:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 1 = 0$$ WH

2. The differential equations **9.160** 1 are written diagramatically as follows:

3. $$u = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column. WH

9.161 The two following transformation formulas are valid for Riemann's P-equation:

1. $$\left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} a & b & c & \\ \alpha+k & \beta-k-1 & \gamma+l & z \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{matrix}\right\}$$ WH

2. $$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = P\left\{\begin{matrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$ WH

The first of these formulas means that if

$$u = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\},$$

then the function

$$u_1 = \left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l u$$

satisfies a second-order differential equation having the same singular points as equation **9.161** 2 and indices equal to $\alpha+k, \alpha'+k; \beta-k-l, \beta'-k-l; \gamma+l, \gamma'+l$. The second transformation formula converts a differential equation with singularities at the points a,b, and c, indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$, and an independent variable z into a differential equation with the same indices, singular points a_1, b_1, and c_1, and independent variable z_1. The variable z_1 is connected with the variable z by the fractional transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \qquad [AD - BC \neq 0]$$

The same transformation connects the points a_1, b_1, and c_1 with the points a, b, and c.

WH, MO 20

9.162 By the successive application of the two transformation formulas **9.161** 1 and **9.161** 2, we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

For $k = -\alpha, l = -\gamma$, and $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)}$, we have

1. $$u = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$
$$= \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

MO 23

Thus, this solution can be expressed as a hypergeometric series as follows:

2. $$u = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)}\right)$$

If the constants a, b, c; α, α'; β, β'; γ, γ' are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ is an integer): WH, MO 23

9.163

1. $$u_1 = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

2. $$u_2 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma} F\left\{\alpha'+\beta+\gamma, \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

3. $$u_3 = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha+\beta+\gamma', \alpha+\beta'+\gamma'; 1+\alpha-\alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

4. $$u_4 = \left(\frac{z-a}{z-b}\right)^{\alpha'}\left(\frac{z-c}{z-b}\right)^{\gamma'} F\left\{\alpha'+\beta+\gamma', \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right\}$$

9.164

1.[10] $$u_5 = \left(\frac{z-b}{z-c}\right)^{\beta}\left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta+\gamma+\alpha, \beta+\gamma'+\alpha; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

2. $$u_6 = \left(\frac{z-b}{z-c}\right)^{\beta'}\left(\frac{z-a}{z-c}\right)^{\alpha} F\left\{\beta'+\gamma+\alpha, \beta'+\gamma'+\alpha; 1+\beta'-\beta; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

3. $$u_7 = \left(\frac{z-b}{z-c}\right)^{\beta}\left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta+\gamma+\alpha', \beta+\gamma'+\alpha'; 1+\beta-\beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

4. $$u_8 = \left(\frac{z-b}{z-c}\right)^{\beta'} \left(\frac{z-a}{z-c}\right)^{\alpha'} F\left\{\beta' + \gamma + \alpha', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)}\right\}$$

9.165

1. $$u_9 = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma + \alpha + \beta, \gamma + \alpha' + \beta; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

2. $$u_{10} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta} F\left\{\gamma' + \alpha + \beta, \gamma' + \alpha' + \beta; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

3. $$u_{11} = \left(\frac{z-c}{z-a}\right)^{\gamma} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma + \alpha + \beta', \gamma + \alpha' + \beta'; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

4. $$u_{12} = \left(\frac{z-c}{z-a}\right)^{\gamma'} \left(\frac{z-b}{z-a}\right)^{\beta'} F\left\{\gamma' + \alpha + \beta', \gamma' + \alpha' + \beta'; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)}\right\}$$

9.166

1. $$u_{13} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha + \gamma + \beta, \alpha + \gamma' + \beta; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

2. $$u_{14} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta} F\left\{\alpha' + \gamma + \beta, \alpha' + \gamma' + \beta; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

3. $$u_{15} = \left(\frac{z-a}{z-c}\right)^{\alpha} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha + \gamma + \beta', \alpha + \gamma' + \beta'; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

4. $$u_{16} = \left(\frac{z-a}{z-c}\right)^{\alpha'} \left(\frac{z-b}{z-c}\right)^{\beta'} F\left\{\alpha' + \gamma + \beta', \alpha' + \gamma' + \beta'; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)}\right\}$$

9.167

1. $$u_{17} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma + \beta + \alpha, \gamma + \beta' + \alpha; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

2. $$u_{18} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha} F\left\{\gamma' + \beta + \alpha, \gamma' + \beta' + \alpha; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

3. $$u_{19} = \left(\frac{z-c}{z-b}\right)^{\gamma} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma + \beta + \alpha', \gamma + \beta' + \alpha'; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

4. $$u_{20} = \left(\frac{z-c}{z-b}\right)^{\gamma'} \left(\frac{z-a}{z-b}\right)^{\alpha'} F\left\{\gamma' + \beta + \alpha', \gamma' + \beta' + \alpha'; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)}\right\}$$

9.168

1. $$u_{21} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta + \alpha + \gamma, \beta + \alpha' + \gamma; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

2. $$u_{22} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma} F\left\{\beta' + \alpha + \gamma, \beta' + \alpha' + \gamma; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

3. $$u_{23} = \left(\frac{z-b}{z-a}\right)^{\beta} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta+\alpha+\gamma', \beta+\alpha'+\gamma'; 1+\beta-\beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$

4. $$u_{24} = \left(\frac{z-b}{z-a}\right)^{\beta'} \left(\frac{z-c}{z-a}\right)^{\gamma'} F\left\{\beta'+\alpha+\gamma', \beta'+\alpha'+\gamma'; 1+\beta'-\beta; \frac{(c-a)(z-b)}{(c-b)(z-a)}\right\}$$ WH

9.17 Representing the solutions to certain second-order differential equations using a Riemann scheme

9.171 The hypergeometric equation (see **9.151**):

$$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & z \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{matrix}\right\}$$ WH

9.172 The associated Legendre's equation defining the functions $P_n^m(z)$ for n and m integers (see **8.700** 1):

1. $$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & \dfrac{1-z}{2} \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}$$ WH

2. $$u = P\left\{\begin{matrix} 0 & \infty & 1 & \\ -\frac{1}{2}n & \frac{1}{2}m & 0 & \dfrac{1}{1-z^2} \\ \dfrac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} & \end{matrix}\right\}$$ WH

9.173 The function $P_n^m\left(1-\dfrac{z^2}{2n^2}\right)$ satisfies the equation

$$u = P\left\{\begin{matrix} 4n^2 & \infty & 0 & \\ \frac{1}{2}m & n+1 & \frac{1}{2}m & z^2 \\ -\frac{1}{2}m & -n & -\frac{1}{2}m & \end{matrix}\right\}$$ WH

The function $J_m(z)$ satisfies the limiting form of this equation obtained as $n \to \infty$.

9.174 The equation defining the Gegenbauer polynomials $C_n^{\lambda}(z)$ (see **8.938**):

$$u = P\left\{\begin{matrix} -1 & \infty & 1 & \\ \frac{1}{2}-\lambda & n+2\lambda & \frac{1}{2}-\lambda & z \\ 0 & -n & 0 & \end{matrix}\right\}$$ WH

9.175 Bessel's equation (see **8.401**) is the limiting form of the equations:

1. $$u = P\left\{\begin{matrix} 0 & \infty & c & \\ n & ic & \frac{1}{2}+ic & z \\ -n & -ic & \frac{1}{2}-ic & \end{matrix}\right\}$$ WH

2. $$u = e^{iz} P\left\{\begin{matrix} 0 & \infty & c & \\ n & \frac{1}{2} & 0 & z \\ -n & \frac{3}{2}-2ic & 2ic-1 & \end{matrix}\right\}$$ WH

3. $$u = P\left\{\begin{matrix} 0 & \infty & c^2 & \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 & z^2 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 & \end{matrix}\right\}$$ WH

as $c \to \infty$.

9.18 Hypergeometric functions of two variables

9.180

1. $$F_1\left(\alpha, \beta, \beta', \gamma; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m\left(\beta'\right)_n}{(\gamma)_{m+n} m! n!} x^m y^n$$
$[|x| < 1, \quad |y| < 1]$
EH I 224(6), AK 14(11)

2. $$F_2\left(\alpha, \beta, \beta', \gamma, \gamma'; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m\left(\beta'\right)_n}{(\gamma)_m\left(\gamma'\right)_n m! n!} x^m y^n$$
$[|x| + |y| < 1]$ EH I 224(7), AK 14(12)

3. $$F_3\left(\alpha, \alpha', \beta, \beta', \gamma; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m\left(\alpha'\right)_n(\beta)_m\left(\beta'\right)_n}{(\gamma)_{m+n} m! n!} x^m y^n$$
$[|x| < 1, \quad |y| < 1]$
EH I 224(8), AK 14(13)

4. $$F_4\left(\alpha, \beta, \gamma, \gamma'; x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m\left(\gamma'\right)_n m! n!} x^m y^n \qquad \left[\left|\sqrt{x}\right| + \left|\sqrt{y}\right| < 1\right]$$
EH I 224(9), AK 14(14)

9.181 The functions F_1, F_2, F_3, and F_4 satisfy the following systems of partial differential equations for z:

1. System of equations for $z = F_1$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma - (\alpha+\beta+1)x\right]\frac{\partial z}{\partial x} - \beta y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$ EH I 233(9)
$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x(1-y)\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma - \left(\alpha+\beta'+1\right)y\right]\frac{\partial z}{\partial x} - \beta' x\frac{\partial z}{\partial x} - \alpha\beta' z = 0$$

2. System of equations for $z = F_2$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma - (\alpha+\beta+1)x\right]\frac{\partial z}{\partial x} - \beta y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$ EH I 234(10)
$$y(1-y)\frac{\partial^2 z}{\partial y^2} - xy\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma' - \left(\alpha+\beta'+1\right)y\right]\frac{\partial z}{\partial y} - \beta' x\frac{\partial z}{\partial x} - \alpha\beta' z = 0$$

3. System of equations for $z = F_3$:
$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma - (\alpha+\beta+1)x\right]\frac{\partial z}{\partial x} - \alpha\beta z = 0,$$
$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\,\partial y} + \left[\gamma - \left(\alpha'+\beta'+1\right)y\right]\frac{\partial z}{\partial y} - \alpha'\beta' z = 0$$
EH I 234(11)

4. System of equations for $z = F_4$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + [\gamma - (\alpha+\beta+1)x]\frac{\partial z}{\partial x} - (\alpha+\beta+1)y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(12)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + [\gamma' - (\alpha+\beta+1)y]\frac{\partial z}{\partial y} - (\alpha+\beta+1)x\frac{\partial z}{\partial x} - \alpha\beta z = 0$$

AK 44

9.182 For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

1. $$F_1(\alpha,\beta,\beta',\beta+\beta';x,y) = (1-y)^{-\alpha}\,F\left(\alpha,\beta;\beta+\beta';\frac{x-y}{1-y}\right)$$ EH I 238(1), AK 24(28)

2. $$F_2(\alpha,\beta,\beta',\beta,\gamma';x,y) = (1-x)^{-\alpha}\,F\left(\alpha,\beta';\gamma';\frac{y}{1-x}\right)$$ EH I 238(2), AK 23

3. $$F_2(\alpha,\beta,\beta',\alpha,\alpha;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}\,F\left(\beta,\beta';\alpha;\frac{xy}{(1-x)(1-y)}\right)$$ EH I 238(3)

4. $$F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta,\gamma;x,y) = (1-y)^{\alpha+\beta-\gamma}\,F(\alpha,\beta;\gamma;x+y-xy)$$ EH I 238(4), AK 25(35)

5. $$F_4(\alpha,\gamma+\gamma'-\alpha-1,\gamma,\gamma';x(1-y),y(1-x)) = F(\alpha,\gamma+\gamma'-\alpha-1;\gamma;x)\,F(\alpha,\gamma+\gamma'-\alpha-1;\gamma';y)$$ EH I 238(5)

6. $$F_4\left(\alpha,\beta,\alpha,\beta;-\frac{x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right) = \frac{(1-x)^\beta(1-y)^\alpha}{(1-xy)}$$ EH I 238(6)

7. $$F_4\left(\alpha,\beta,\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right) = (1-x)^\alpha(1-y)^\alpha\,F(\alpha,1+\alpha-\beta;\beta;xy)$$ EH I 238(7)

8. $$F_4\left(\alpha,\beta,1+\alpha-\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right) = (1-y)^\alpha\,F\left[\alpha,\beta;1+\alpha-\beta;-\frac{x(1-y)}{1-x}\right]$$ EH I 238(8)

9. $$F_4\left(\alpha,\alpha+\frac{1}{2},\gamma,\frac{1}{2};x,y\right) = \frac{1}{2}(1+\sqrt{y})^{-2\alpha}\,F\left(\alpha,\alpha+\frac{1}{2};\gamma;\frac{x}{(1+\sqrt{y})^2}\right) + \frac{1}{2}(1-\sqrt{y})^{-2\alpha}\,F\left(\alpha,\alpha+\frac{1}{2};\gamma;\frac{x}{(1-\sqrt{y})^2}\right)$$ AK 23

10. $$F_1(\alpha,\beta,\beta',\gamma;x,1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta')}\,F(\alpha,\beta:\gamma-\beta';x)$$ EH I 239(10), AK 22(23)

11. $F_1(\alpha, \beta, \beta', \gamma; x, x) = F(\alpha, \beta + \beta'; \gamma; x)$ EH I 239(11), AK 23(25)

9.183 Functional relations between hypergeometric functions of two variables:

1. $$F_1(\alpha, \beta, \beta', \gamma; x, y) = (1-x)^{-\beta}(1-y)^{-\beta} F_1\left(\gamma - \alpha, \beta, \beta', \gamma; \frac{x}{x-1}, \frac{y}{y-1}\right)$$
EH I 239(1)
$$= (1-x)^{-\alpha} F_1\left(\alpha, \gamma - \beta - \beta', \beta', \gamma; \frac{x}{x-1}, \frac{y-x}{1-x}\right)$$
EH I 239(2)
$$= (1-y)^{-\alpha} F_1\left(\alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1}\right)$$
EH I 239(3)
$$= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha, \gamma - \beta - \beta', \beta', \gamma; x, \frac{x-y}{1-y}\right)$$
EH I 240(4)
$$= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma - \alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{x-y}{x-1}, y\right)$$
EH I 240(5), AK 30(5)

2.[8] $$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = (1-x)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \beta', \gamma, \gamma'; \frac{x}{x-1}, \frac{y}{1-x}\right)$$
EH I 240(6)
$$= (1-y)^{-\alpha} F_2\left(\alpha, \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{1-y}, \frac{y}{y-1}\right)$$
EH I 240(7)
$$= (1-x-y)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right)$$
EH I 240(8), AK 32(6)

3.[7] $$F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma')\,\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\,\Gamma(\beta)}(-y)^{-\alpha} F_4\left(\alpha, \alpha + 1 - \gamma', \gamma, \alpha + 1 - \beta; \frac{x}{y}, \frac{1}{y}\right)$$
$$+ \frac{\Gamma(\gamma'(\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\,\Gamma(\alpha)}(-y)^{\beta} F_4\left(\beta + 1 - \gamma', \beta, \gamma, \beta + 1 - \alpha; \frac{x}{y}, \frac{1}{y}\right)$$
EH I 240(9), AK 26(37)

9.184 Integral representations: Double integrals of the Euler type

1. $$F_1(\alpha,\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')} \times \iint\limits_{\substack{u\ge 0, v\ge 0\\ u+v\le 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha}\,du\,dv$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\beta' > 0, \quad \operatorname{Re}(\gamma-\beta-\beta') > 0]$ EH I 230(1), AK 28(1)

2. $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = \frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta)\,\Gamma(\gamma'-\beta')} \times \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha}\,du\,dv$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\beta' > 0, \quad \operatorname{Re}(\gamma-\beta) > 0, \quad \operatorname{Re}(\gamma'-\beta') > 0]$ EH I 230(2), AK 28(2)

3. $$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')} \times \iint_{\substack{u\ge 0, v\ge 0\\ u+v\le 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{-\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'}\,du\,dv$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\beta' > 0, \quad \operatorname{Re}(\gamma-\beta-\beta') > 0]$ EH I 230(3), AK 28(3)

4. $$F_4(\alpha,\beta,\gamma,\gamma';x(1-y),y(1-x)) = \frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma'-\beta)} \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} \times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1}\,du\,dv$$
$[\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\gamma-\alpha) > 0, \quad \operatorname{Re}(\gamma'-\beta) > 0]$ EH I 230(4)

9.185 Integral representations: Integrals of the Mellin–Barnes type

The functions F_1, F_2, F_3, and F_4 can be represented by means of double integrals of the following form:

$$F(x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s,t)\,\Gamma(-s)\,\Gamma(-t)(-x)^s(-y)^t\,ds\,dt$$

$\Psi(s,t)$	$F(x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)}{\Gamma(\beta')\,\Gamma(\gamma+s+t)}$	$F_1(\alpha,\beta,\beta',\gamma;x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)\,\Gamma(\gamma')}{\Gamma(\beta')\,\Gamma(\gamma+s)\,\Gamma(\gamma'+t)}$	$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)$
$\dfrac{\Gamma(\alpha+s)\,\Gamma(\alpha'+t)\,\Gamma(\beta+s)\,\Gamma(\beta'+t)}{\Gamma(\alpha')\,\Gamma(\beta')\,\Gamma(\gamma+s+t)}$	$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$
$\dfrac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(\gamma')}{\Gamma(\gamma+s)\,\Gamma(\gamma'+t)}$	$F_4(\alpha,\beta,\gamma,\gamma';x,y)$
[$\alpha,\alpha',\beta,\beta'$ may not be negative integers]	EH I 232(9–13), AK 41(33)

9.19 A hypergeometric function of several variables

$$F_A(\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;z_1,\ldots,z_n) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}\,m_1!\cdots m_n!}z_1^{m_1}z_2^{m_2}\cdots z_n^{m_n}$$

ET I 385

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.201[10] A *confluent hypergeometric function* is obtained by taking the limit as $c\to\infty$ in the solution of Riemann's differential equation

$$u = P\left\{\begin{matrix} 0 & \infty & c & \\ \frac{1}{2}+\mu & -c & c-\lambda & z \\ \frac{1}{2}-\mu & 0 & \lambda & \end{matrix}\right\}$$ WH

9.202 The equation obtained by means of this limiting process is of the form

1. $$\frac{d^2u}{dz^2}+\frac{du}{dz}+\left(\frac{\lambda}{z}+\frac{\frac{1}{4}-\mu^2}{z^2}\right)u=0$$ WH

Equation **9.202** 1 has the following two linearly independent solutions:

2. $z^{\frac{1}{2}+\mu}e^{-z}\,\Phi\left(\frac{1}{2}+\mu-\lambda,2\mu+1;z\right)$

3. $z^{\frac{1}{2}-\mu}e^{-z}\,\Phi\left(\frac{1}{2}-\mu-\lambda,-2\mu+1;z\right)$

which are defined for all values of $\mu\neq\pm\frac{1}{2},\pm\frac{2}{2},\pm\frac{3}{2},\ldots$ MO 111

9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Psi(\alpha, \gamma; z)$

9.210[10] The series

1. $$\Phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \dots$$

   is also called a *confluent hypergeometric function.*

   A second notation: $\Phi(\alpha, \gamma; z) = {}_1F_1(\alpha; \gamma; z)$.

2. $$\Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma}\Phi(\alpha-\gamma+1, 2-\gamma; z)$$ EH I 257(7)

3. Bateman's function $\mathrm{k}_\nu(x)$ is defined by

   $$\mathrm{k}_\nu(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\tan\theta - \nu\theta)\, d\theta \qquad [x,\ \nu \text{ real}]$$ EH I 267

9.211 Integral representation:

1. $$\Phi(\alpha, \gamma; z) = \frac{2^{1-\gamma}e^{\frac{1}{2}z}}{\mathrm{B}(\alpha, \gamma-\alpha)}\int_{-1}^{1}(1-t)^{\gamma-\alpha-1}(1+t)^{\alpha-1}e^{\frac{1}{2}zt}\, dt$$

   $[0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma]$ MO 114

2. $$\Phi(\alpha, \gamma; z) = \frac{1}{\mathrm{B}(\alpha, \gamma-\alpha)} z^{1-\gamma}\int_0^z e^t t^{\alpha-1}(z-t)^{\gamma-\alpha-1\, dt}$$

   $[0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma]$ MO 114

3. $$\Phi(-\nu, \alpha+1; z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^z z^{-\frac{\alpha}{2}}\int_0^\infty e^{-t}t^{\nu+\frac{\alpha}{2}} J_\alpha\left(2\sqrt{zt}\right)\, dt$$

   $\left[\operatorname{Re}(\alpha+\nu+1) > 0, \quad |\arg z| < \frac{\pi}{2}\right]$ MO 115

4.[8] $$\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}\, dt$$ $[\operatorname{Re}\alpha > 0, \quad \operatorname{Re} z > 0]$ EH I 255(2)

Functional relations

9.212

1. $\Phi(\alpha, \gamma; z) = e^z\,\Phi(\gamma-\alpha, \gamma; -z)$ MO 112
2. $\frac{z}{\gamma}\,\Phi(\alpha+1, \gamma+1; z) = \Phi(\alpha+1, \gamma; z) - \Phi(\alpha, \gamma; z)$ MO 112
3. $\alpha\,\Phi(\alpha+1, \gamma+1; z) = (\alpha-\gamma)\,\Phi(\alpha, \gamma+1; z) + \gamma\,\Phi(\alpha, \gamma; z)$ MO 112
4. $\alpha\,\Phi(\alpha+1, \gamma; z) = (z+2a-\gamma)\,\Phi(\alpha, \gamma; z) + (\gamma-\alpha)\,\Phi(\alpha-1, \gamma; z)$ MO 112

9.213 $\frac{d\,\Phi}{dz} = \frac{\alpha}{\gamma}\,\Phi(\alpha+1, \gamma+1; z)$ MO 112

9.214 $\lim_{\gamma\to-n}\frac{1}{\Gamma(\gamma)}\,\Phi(\alpha, \gamma; z) = z^{n+1}\binom{\alpha+n}{n+1}\Phi(\alpha+n+1, n+2; z) \qquad [n = 0, 1, 2, \dots]$ MO 112

9.215[10]

1. $\Phi(\alpha, \alpha; z) = e^z$ MO 15

2. $\Phi(\alpha, 2\alpha; 2z) = 2^{\alpha-\frac{1}{2}} \exp\left[\frac{1}{4}(1-2\alpha)\pi i\right] \Gamma\left(\alpha+\frac{1}{2}\right) e^z z^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}\left(ze^{\frac{\pi}{2}i}\right)$ MO 112

3. $\Phi\left(p+\frac{1}{2}, 2p+1; 2iz\right) = \Gamma(p+1)\left(\frac{z}{2}\right)^{-p} e^{iz} J_p(z)$ MO 15

For a representation of special functions in terms of a confluent hypergeometric function $\Phi(\alpha, \gamma; z)$, see:

- for the probability integral, **9.236**;
- for integrals of Bessel functions, **6.631** 1;
- for Hermite polynomials, **8.953** and **8.959**;
- for Laguerre polynomials, **8.972** 1;
- for parabolic cylinder functions, **9.240**;
- for the Whittaker functions $M_{\lambda,\mu}(z)$, **9.220** 2 and **9.220** 3.

9.216 The function $\Phi(\alpha, \gamma; z)$ is a solution of the differential equation

1. $z\frac{d^2F}{dz^2} + (\gamma - z)\frac{dF}{dz} - \alpha F = 0$ MO 111

This equation has two linearly independent solutions:

2. $\Phi(\alpha, \gamma; z)$

3. $z^{1-\gamma}\,\Phi(\alpha-\gamma+1, 2-\gamma; z)$ MO 112

9.22–9.23 The Whittaker functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$

9.220 If we make the change of variable $u = e^{-\frac{z}{2}}W$ in equation **9.202** 1, we obtain the equation

1. $\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right)W = 0$ MO 115

Equation **9.220** 1 has the following two linearly independent solutions:

2. $M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}\,\Phi\left(\mu-\lambda+\frac{1}{2}, 2\mu+1; z\right)$

3.[11] $M_{\lambda,-\mu}(z) = z^{-\mu+\frac{1}{2}}e^{-z/2}\,\Phi\left(-\mu-\lambda+\frac{1}{2}, -2\mu+1; z\right)$ MO 115

To obtain solutions that are also suitable for $2\mu = \pm 1, \pm 2, \ldots$, we introduce Whittaker's function

4. $W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\lambda\right)}\,M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\lambda\right)}\,M_{\lambda,-\mu}(z)$ WH

which, for 2μ approaching an integer, is also a solution of equation **9.220** 1.

For the functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$, $z = 0$ is a branch point and $z = \infty$ is an essential singular point. Therefore, we shall examine these functions only for $|\arg z| < \pi$.

These functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are linearly independent solutions of equation **9.220** 1.

Integral representations

9.221 $$M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu}\,\mathrm{B}\left(\mu+\lambda+\frac{1}{2}, \mu-\lambda+\frac{1}{2}\right)} \int_{-1}^{1} (1+t)^{\mu-\lambda-\frac{1}{2}} (1-t)^{\mu+\lambda-\frac{1}{2}} e^{\frac{1}{2}zt}\, dt,$$ WH

if the integral converges. See also **6.631** 1 and **7.623** 3.

9.222

1.[11] $$W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} \int_0^\infty e^{-zt} t^{\mu-\lambda-\frac{1}{2}} (1+t)^{\mu+\lambda-\frac{1}{2}}\, dt$$

$$\left[\mathrm{Re}(\mu-\lambda) > -\tfrac{1}{2}, \quad |\arg z| < \frac{\pi}{2}\right]$$ MO 118

2. $$W_{\lambda,\mu}(z) = \frac{z^{\lambda} e^{-z/2}}{\Gamma\left(\mu-\lambda+\frac{1}{2}\right)} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1+\frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt$$

$$\left[\mathrm{Re}(\mu-\lambda) > -\tfrac{1}{2}, \quad |\arg z| < \pi\right]$$ WH

9.223 $$W_{\lambda,\mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(u-\lambda)\,\Gamma\left(-u-\mu+\frac{1}{2}\right)\Gamma\left(-u+\mu+\frac{1}{2}\right)}{\Gamma\left(-\lambda+\mu+\frac{1}{2}\right)\Gamma\left(-\lambda-\mu+\frac{1}{2}\right)} z^u\, du$$

[the path of integration is chosen in such a way that the poles of the function $\Gamma(u-\lambda)$ are separated from the poles of the functions $\Gamma\left(-u-\mu+\frac{1}{2}\right)$ and $\Gamma\left(-u+\mu+\frac{1}{2}\right)$.] See also **7.142**. MO 118

9.224 $$W_{\mu,\frac{1}{2}+\mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^\infty (1+t)^{2\mu} e^{-zt}\, dt = z^{-\mu} e^{\frac{1}{2}z} \int_z^\infty t^{2\mu} e^{-t}\, dt \qquad [\mathrm{Re}\, z > 0]$$ WH

9.225

1. $$W_{\lambda,\mu}(x)\, W_{-\lambda,\mu}(x) = -x \int_0^\infty \tanh^{2\lambda} \frac{t}{2} \left\{ J_{2\mu}(x \sinh t) \sin(\mu-\lambda)\pi + Y_{2\mu}(x \sinh t) \cos(\mu-\lambda)\pi \right\} dt$$

$$\left[|\mathrm{Re}\,\mu| - \mathrm{Re}\,\lambda < \tfrac{1}{2}; \quad x > 0\right]$$ MO 119

2. $$W_{\kappa,\mu}(z_1)\, W_{\lambda,\mu}(z_2) = \frac{(z_1 z_2)^{\mu+\frac{1}{2}} \exp\left[-\frac{1}{2}(z_1+z_2)\right]}{\Gamma(1-\kappa-\lambda)} \times \int_0^\infty e^{-t} t^{-\kappa-\lambda} (z_1+t)^{-\frac{1}{2}+\kappa-\mu} (z_2+t)^{-\frac{1}{2}+\lambda-\mu} \times F\left(\tfrac{1}{2}-\kappa+\mu, \tfrac{1}{2}-\lambda+\mu; 1-\kappa-\lambda; \Theta\right) dt$$

$$\Theta = \frac{t(z_1+z_2+t)}{(z_1+t)(z_2+t)}, \qquad [z_1 \neq 0, \quad z_2 \neq 0, \quad |\arg z_1| < \pi, \quad |\arg z_2| < \pi, \quad \mathrm{Re}(\kappa+\lambda) < 1]$$

MO 119

See also **3.334**, **3.381** 6, **3.382** 3, **3.383** 4, 8, **3.384** 3, **3.471** 2.

9.226 Series representations

$$M_{0,\mu}(z) = z^{\frac{1}{2}+\mu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{4k} k! (\mu+1)(\mu+2)\dots(\mu+k)} \right\}$$ WH

Asymptotic representations

9.227[7] For large values of $|z|$

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^{\lambda} \left(1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right]\left[\mu^2 - \left(\lambda - \frac{3}{2}\right)^2\right] \dots \left[\mu^2 - \left(\lambda - k + \frac{1}{2}\right)^2\right]}{k! z^k} \right)$$

$[|\arg z| \le \pi - \alpha < \pi]$ WH

9.228 For large values of $|\lambda|$

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu+1) \lambda^{-\mu-\frac{1}{4}} z^{1/4} \cos\left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi\right)$$ MO 118

9.229

1. $$W_{\lambda,\mu} \sim -\left(\frac{4z}{\lambda}\right)^{\frac{1}{4}} e^{-\lambda+\lambda\ln\lambda} \sin\left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4}\right)$$ MO 118

2. $$W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda}\right)^{\frac{1}{4}} e^{\lambda - \lambda\ln\lambda - 2\sqrt{\lambda z}}$$ MO 118

Formulas **9.228** and **9.229** are applicable for
$|\lambda| \gg 1, \quad |\lambda| \gg |z|, \quad |\lambda| \gg |\mu|, \quad z \neq 0, \quad |\arg\sqrt{z}| < \frac{3\pi}{4}$ and $|\arg\lambda| < \frac{\pi}{2}$. MO 118

Functional relations

9.231

1. $$M_{n+\mu+\frac{1}{2},\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{1}{2}z}}{(2\mu+1)(2\mu+2)\dots(2\mu+n)} \frac{d^n}{dz^n}\left(z^{n+2\mu} e^{-z}\right)$$

$[n = 0, 1, 2, \dots; \quad 2\mu \neq -1, -2, -3, \dots]$ MO 117

2. $$z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z)$$ $[2\mu \neq -1, -2, -3, \dots]$ WH

9.232

1. $$W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z)$$ MO 116

2. $$W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu + \lambda\right)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu + \lambda\right)} M_{-\lambda,-\mu}(-z)$$

$\left[|\arg(-z)| < \frac{3}{2}\pi\right]$ WH

9.233

1. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} e^{i\pi\lambda} W_{-\lambda,\mu}\left(e^{i\pi} z\right) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu + \lambda + \frac{1}{2}\right)} \exp\left[i\pi\left(\lambda - \mu - \frac{1}{2}\right)\right] W_{\lambda,\mu}(z)$$

$\left[-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi; \quad 2\mu \neq -1, -2, \dots\right]$ MO 117

2. $$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} e^{-i\pi\lambda} W_{-\lambda,\mu}\left(e^{-i\pi} z\right) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu + \lambda + \frac{1}{2}\right)} \exp\left[-i\pi\left(\lambda - \mu - \frac{1}{2}\right)\right] W_{\lambda,\mu}(z)$$

$\left[-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi; \quad 2\mu \neq -1, -2, \dots\right]$ MO 117

9.234 Recursion formulas

1. $$W_{\lambda,\mu}(z) = \sqrt{z}\, W_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(z) + \left(\tfrac{1}{2} + \mu - \lambda\right) W_{\lambda-1,\mu}(z)$$ WH

2.[11] $$W_{\lambda,\mu}(z) = \sqrt{z}\, W_{\lambda-\frac{1}{2},\mu+\frac{1}{2}}(z) + \left(\tfrac{1}{2} - \mu - \lambda\right) W_{\lambda-1,\mu}(z)$$ WH

3. $$z\frac{d}{dz}\, W_{\lambda,\mu}(z) = \left(\lambda - \tfrac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda - \tfrac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z)$$ WH

4. $$\left[\left(\mu + \frac{1-z}{2}\right) W_{\lambda,\mu}(z) - z\frac{d}{dz}\, W_{\lambda,\mu}(z)\right]\left(\mu + \tfrac{1}{2} + \lambda\right)$$
$$= \left[\left(\mu + \frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z\frac{d}{dz}\, W_{\lambda,\mu+1}(z)\right]\left(\mu + \tfrac{1}{2} - \lambda\right)$$ MO 117

5. $$\left(\tfrac{3}{2} + \lambda + \mu\right)\left(\tfrac{1}{2} + \lambda + \mu\right) z\, W_{\lambda,\mu}(z) = z(z + 2\mu + 1)\frac{d}{dz}\, W_{\lambda+1,\mu+1}(z)$$
$$+ \left[\tfrac{1}{2}z^2 + \left(\mu - \lambda - \tfrac{1}{2}\right) z + 2\mu^2 + 2\mu + \tfrac{1}{2}\right] W_{\lambda+1,\mu+1}(z)$$ MO 117

Connections with other functions

9.235

1. $$M_{0,\mu}(z) = 2^{2\mu}\, \Gamma(\mu+1)\sqrt{z}\, I_\mu\left(\frac{z}{2}\right)$$ MO 125a

2. $$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}\, K_\mu\left(\frac{z}{2}\right)$$ MO 125

9.236

1. $$\Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}}\, W_{-\frac{1}{4},\frac{1}{4}}\left(x^2\right) = \frac{2x}{\sqrt{\pi}}\, \Phi\left(\tfrac{1}{2}, \tfrac{3}{2}; -x^2\right)$$ WH, MO 126

2. $$\operatorname{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln\frac{1}{2}}}\, W_{-\frac{1}{2},0}\,(-\ln z)$$ WH

3. $$\Gamma(\alpha, x) = e^{-x}\, \Psi(1-\alpha, 1-\alpha; x)$$ EH I 266(21)

4. $$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha}\, \Phi(\alpha, \alpha+1; -x)$$ EH I 266(22)

9.237

1. $$W_{\lambda,\mu}(z) = \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)\Gamma\left(\frac{1}{2} + \mu - \lambda\right)}$$
$$\times\left\{\sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - \lambda + \frac{1}{2}\right)}{k!(2\mu+k)!} z^k \left[\Psi(k+1) + \Psi(2\mu+k+1) - \Psi\left(\mu + k - \lambda + \tfrac{1}{2}\right) - \ln z\right]\right.$$
$$\left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu-k)\,\Gamma\left(k - \mu - \lambda + \frac{1}{2}\right)}{k!}(-z)^k\right\}$$
$\left[|\arg z| < \tfrac{3}{2}\pi; \quad 2\mu + 1 \text{ is a natural number}\right]$ MO 116

2. Set $\lambda - \mu - \frac{1}{2} = l$, where $l+1$ is a natural number. Then

3. $$W_{l+\mu+\frac{1}{2},\mu}(z) = (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} (2\mu+1)(2\mu+2)\cdots(2\mu+l)\,\Phi(-l, 2\mu+1; z)$$
$$= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}\, L_l^{2\mu}(z)$$

MO 116

9.238

1. $$J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2ix\right)$$ EH I 265(9)

2. $$I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(10)

3. $$K_\nu(x) = \sqrt{\pi} e^{-x} (2x)^\nu\,\Psi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(13)

9.24–9.25 Parabolic cylinder functions $D_p(z)$

9.240 $$D_p(z) = 2^{\frac{1}{4}+\frac{p}{2}}\, W_{\frac{1}{4}+\frac{p}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-1/2}$$
$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)}\,\Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(-\frac{p}{2}\right)}\,\Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\}$$

MO 120a

are called *parabolic cylinder functions*.

Integral representations

9.241

1. $$D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz}\,dx$$ [$\operatorname{Re} p > -1$; for $x < 0$, $\arg x^p = p\pi i$]

MO 122

2. $$D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^\infty e^{-xz-\frac{x^2}{2}} x^{-p-1}\,dx$$ [$\operatorname{Re} p < 0$] (cf. **3.462** 1) MO 122

9.242

1.[10] $$D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_\infty^{(0+)} e^{-zt-\frac{1}{2}t^2} (-t)^{-p-1}\,dt$$ [$|\arg(-t)| \le \pi$] WH

2. $$D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)}\,dt$$
$\left[|\arg z| < \frac{\pi}{4}; \quad |\arg(1+t)| \le \pi\right]$ WH

3. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t\,dt$$
$\left[|\arg z| < \frac{3}{4}\pi; \quad p \text{ is not a positive integer}\right]$ WH

4. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0-)} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t\, dt$$

[for all values of $\arg z$; also, the contours encircle the poles of the function $\Gamma(-t)$, but they do not encircle the poles of the function $\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)$]. WH

9.243

1. $$D_n(z) = (-1)^\mu \left(\frac{\pi}{2}\right)^{-1/2} \left(\sqrt{n}\right)^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt + \int_0^\infty \left[e^{\frac{1}{2}n\left(1-t^2\right)} t^n - e^{-n(t-1)^2}\right] \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt - \int_{-\infty}^0 e^{-n(t-1)^2} \begin{matrix}\cos\\ \sin\end{matrix} \left(zt\sqrt{n}\right) dt \right\}$$

[n is a natural number] WH

2. $$D_n(z) = (-1)^\mu 2^{n+2} (2\pi)^{-1/2} e^{\frac{1}{4}z^2} \int_0^\infty t^n e^{-2t^2} \begin{matrix}\cos\\ \sin\end{matrix} (2zt)\, dt$$

[n is a natural number, $\mu = \left\lfloor \frac{n}{2} \right\rfloor$, and the cosine or sine is chosen accordingly as n is even or odd]

WH

9.244

1. $$D_{-p-1}[(1+i)z] = \frac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}}\,\Gamma\left(\frac{p+1}{2}\right)} \int_0^\infty \frac{e^{-ix^2z^2} x^p}{\left(1+x^2\right)^{1+\frac{p}{2}}}\, dx \qquad \left[\operatorname{Re} p > -1, \quad \operatorname{Re}\left(iz^2\right) \geq 0\right]$$ MO 122

2. $$D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma\left(-\frac{p}{2}\right)} \int_1^\infty e^{-\frac{i}{2}z^2x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}}\, dx \qquad \left[\operatorname{Re} p < 0; \quad \operatorname{Re}\left(iz^2\right) \geq 0\right]$$ MO 122

See also **3.383** 6, 7, **3.384** 2, 6, **3.966** 5, 6.

9.245

1.[10] $$D_p(x)\, D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \coth^{p+\frac{1}{2}}\left(\frac{t}{2}\right) \frac{1}{\sqrt{\sinh t}} \sin\left(\frac{x^2 \sinh t + p\pi}{2}\right) dt$$

[x is real, $\operatorname{Re} p < 0$] MO 122

2. $$D_p\left(ze^{\frac{\pi}{4}i}\right) D_p\left(ze^{-\frac{\pi}{4}i}\right) = \frac{1}{\Gamma(-p)} \int_0^\infty \coth^p t \exp\left(-\frac{z^2}{2}\sinh 2t\right) \frac{dt}{\sinh t}$$

$\left[|\arg z| < \frac{\pi}{4}; \quad \operatorname{Re} p < 0\right]$ MO 122

See also **6.613**.

9.246 Asymptotic expansions. If $|z| \gg 1$ and $|z| \gg |p|$, then

1. $$D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right)$$

$\left[|\arg z| < \frac{3}{4}\pi\right]$ MO 121

2.[11] $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right)$$

$\left[\frac{1}{4}\pi < \arg z < \frac{5}{4}\pi\right]$ MO 121

3.[11] $$D_p(z) \sim e^{-z^2/4} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z^2/4} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots\right)$$

$\left[-\frac{1}{4}\pi > \arg z > -\frac{5}{4}\pi\right]$ MO 121

Functional relations

9.247 Recursion formulas:

1. $$D_{p+1}(z) - z\, D_p(z) + p\, D_{p-1}(z) = 0$$ WH

2. $$\frac{d}{dz} D_p(z) + \frac{1}{2} z\, D_p(z) - p\, D_{p-1}(z) = 0$$ WH

3. $$\frac{d}{dz} D_p(z) - \frac{1}{2} z\, D_p(z) + D_{p+1}(z) = 0$$ MO 121

9.248 Linear relations:

1. $$\begin{aligned} D_p(z) &= \frac{\Gamma(p+1)}{\sqrt{2\pi}} \left[e^{\pi/2} D_{-p-1}(iz) + e^{-\pi p i/2} D_{-p-1}(-iz)\right] \\ &= e^{-p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-\pi(p+1)i/2} D_{-p-1}(iz) \\ &= e^{p\pi i} D_p(-z) + \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{\pi(p+1)i/2} D_{-p-1}(-iz) \end{aligned}$$

MO 121

9.249[10] $$D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp\left[-\frac{i}{2}\left(x^2 + p\frac{\pi}{2}\right)\right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it^2/4}\, dt$$

$[x \text{ real}; \quad -1 < \operatorname{Re} p < 0]$ MO 122

9.251[10] $$D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n}\left(e^{-z^2/2}\right) \qquad [n = 0, 1, 2, \dots]$$ WH

9.252 $$D_p(ax+by) = \exp\frac{(bx-ay)^2}{4} \left(\frac{a}{\sqrt{a^2+b^2}}\right)^p \sum_{k=0}^{\infty} \binom{p}{k} D_{p-k}\left(\sqrt{a^2+b^2}x\right) D_k\left(\sqrt{a^2+b^2}y\right) \left(\frac{b}{a}\right)^k$$

$[a > b > 0, \quad x > 0, \quad y > 0, \quad \operatorname{Re} p \geq 0]$ "summation theorem" MO 124

Connections with other functions

9.253[11] $$D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right)$$ MO 123a

9.254

1. $$D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right]$$ MO 123

2.[11] $$D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{\sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} - z\left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right]\right\}$$ MO 123

9.255 Differential equations leading to parabolic cylinder functions:

1. $$\frac{d^2u}{dz^2} + \left(p + \frac{1}{2} - \frac{z^2}{4}\right) u = 0$$

The solutions are $u = D_p(z)$, $D_p(-z)$, $D_{-p-1}(iz)$, and $D_{-p-1}(-iz)$.

(These four solutions are linearly dependent. See **9.248**.)

2. $$\frac{d^2u}{dz^2} + \left(z^2 + \lambda\right) u = 0, \qquad u = D_{-\frac{1+i\lambda}{2}}\left[\pm(1+i)z\right]$$ EH II 118(12,13)a, MO 123

3.[7] $$\frac{d^2u}{dz^2} + z\frac{du}{dz} + (p+1)u = 0, \qquad u = e^{-\frac{z^2}{4}} D_p(z)$$ MO 123

9.26 Confluent hypergeometric series of two variables

9.261

1.[6] $$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n} m!n!} x^m y^n \qquad [|x| < 1]$$ EH I 225(20)

2. $$\Phi_2\left(\beta, \beta', \gamma, x, y\right) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m \left(\beta'\right)_m}{(\gamma)_{m+n} m!n!} x^m y^n$$ EH I 225(21)a, ET I 385

3. $$\Phi_3(\beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n} m!n!} x^m y^n$$ EH I 225(22)

The functions Φ_1, Φ_2, Φ_3 satisfy the following systems of partial differential equations:

9.262

1. $z = \Phi_1(\alpha, \beta, \gamma, x, y)$ EH I 235(23)

$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x\,\partial y} + [\gamma - (\alpha+\beta+1)x]\frac{\partial z}{\partial x} - \beta y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$
$$y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\,\partial y} + (\gamma - y)\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} - \alpha z = 0$$

2. $z = \Phi_2\left(\beta, \beta', \gamma, x, y\right)$ EH I 235(24)

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} + (\gamma - x)\frac{\partial z}{\partial x} - \beta z = 0,$$
$$y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\,\partial y} + (\gamma - y)\frac{\partial z}{\partial y} - \beta' z = 0$$

3. $z = \Phi_3(\beta, \gamma, x, y)$ EH I 235(25)

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\,\partial y} + (\gamma - x)\frac{\partial z}{\partial x} - \beta z = 0,$$
$$y\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\,\partial y} + \gamma\frac{\partial z}{\partial y} - z = 0$$

9.3 Meijer's G-Function

9.30 Definition

9.301 $$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} x^s\,ds$$

[$0 \le m \le q$, $\quad 0 \le n \le p$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_k + s)$ for any j and k (where $j = 1, \ldots, m$; $\quad k = 1, \ldots, n$]). Besides **9.301**, the following notations are also used:

$$G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right), \qquad G^{mn}_{pq}(x), \qquad G(x)$$ EH I 207(1)

9.302 Three types of integration paths L in the right member of **9.301** can be exhibited:

1. The path L runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1 - a_k + s)$ lie to the left, and the poles of the functions $\Gamma(b_j - s)$ lie to the right of L (for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$). In this case, the conditions under which the integral **9.301** converges are of the form

$$p + q < 2(m + n), \quad |\arg x| < \left(m + n - \tfrac{1}{2}p - \tfrac{1}{2}q\right)\pi.$$ EH I 207(2)

2. L is a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \ldots, m$) once in the negative direction. All the poles of the functions $\Gamma(1 - a_k + s)$ must remain outside this loop. Then, the conditions under which the integral **9.301** converges are:

$q \ge 1$ and either $p < q$ or $p = q$ and $|x| < 1$. EH I 207(3)

3. L is a loop, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1 - a_k + s)$ (for $k = 1, 2, \ldots, n$) once in the positive direction. All the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \ldots, m$) must remain outside this loop.

The conditions under which the integral in **9.301** converges are

$p \ge 1$ and either $p > q$ or $p = q$ and $|x| > 1$. EH I 207(4)

The function $G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)$ is analytic with respect to x; it is symmetric with respect to the parameters $a_1, \ldots, a_n$ and also with respect to $a_{n+1}, \ldots, a_p$; $b_1, \ldots, b_m$; $b_{m+1}, \ldots, b_q$.
EH I 208

9.303[11] If no two b_j (for $j = 1, 2, \ldots, n$) differ by an integer, then, under the conditions that either $p < q$ or $p = q$ and $|x| < 1$,

$$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{h=1}^{m} \frac{\prod_{j=1}^{m}{}' \Gamma(b_j - b_h) \prod_{j=1}^{n} \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + b_h - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_h)} x^{b_h}$$
$$\times {}_pF_{q-1}\left[1 + b_h - a_1, \ldots, 1 + b_h - a_p; \quad 1 + b_h - b_1, \ldots \right.$$
$$\left. \ldots, *, \ldots, 1 + b_h - b_q; \quad (-1)^{p-m-n} x\right]$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when $j = h$. The asterisk in the function ${}_pF_{q-1}$ denotes the omission of the h^{th} parameter.

9.304[7] If no two a_k (for $k = 1, 2, \ldots, n$) differ by an integer then, under the conditions that $q < p$ or $q = p$ and $|x| > 1$,

$$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{h=1}^{n} \frac{\prod_{j=1}^{n}{}' \Gamma(a_h - a_j) \prod_{j=1}^{m} \Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^{p} \Gamma(a_j - a_h + 1) \prod_{j=m+1}^{q} \Gamma(a_h - b_j)} x^{a_h - 1}$$
$$\times {}_qF_{p-1}\left[1 + b_1 - a_h, \ldots, 1 + b_q - a_h; \quad 1 + a_1 - a_h, \ldots \right.$$
$$\left. \ldots, *, \ldots, 1 + a_p - a_h; \quad (-1)^{q-m-n} x^{-1}\right]$$

EH I 208(6)

9.31 Functional relations

If one of the parameters a_j (for $j = 1, 2, \ldots, n$) coincides with one of the parameters b_j (for $j = m + 1, \quad m + 2, \ldots, q$), the order of the G-function decreases. For example,

1. $$G_{pq}^{mn}\left(x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, a_1 \end{matrix}\right) = G_{p-1,q-1}^{m,n-1}\left(x \middle| \begin{matrix} a_2, \ldots, a_p \\ b_1, \ldots, b_{q-1} \end{matrix}\right)$$

$[n, p, q \geq 1]$

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \ldots, m$) coincides with one of the a_j (for $j = n + 1, \ldots, p$). In this case, it is m and not n that decreases by one unit.

The G-function with $p > q$ can be transformed into the G-function with $p < q$ by means of the relationships:

2. $$G_{pq}^{mn}\left(x^{-1} \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = G_{qp}^{nm}\left(x \middle| \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix}\right)$$ EH I 209(9)

3. $$x\frac{d}{dx}G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\b_s\end{matrix}\right)=G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1-1,a_2,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right)+(a_1-1)\,G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\b_s\end{matrix}\right)$$
$[n\geq 1]$ EH I 210(13)

4. $$G^{m+1,n}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}\mathbf{a}_p,1-r\\0,\mathbf{b}_q\end{matrix}\right)=(-1)^r\,G^{m,n+1}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}1-r,\mathbf{a}_p\\\mathbf{b}_q,1\end{matrix}\right)$$
$[r=0,1,2,\ldots]$ MS2 6 (1.2.2)

5. $$z^k\,G^{mn}_{pq}\left(z\,\middle|\,\begin{matrix}\mathbf{a}_p\\\mathbf{b}_q\end{matrix}\right)=G^{mn}_{pq}\left(z\,\middle|\,\begin{matrix}\mathbf{a}_p+k\\\mathbf{b}_q+k\end{matrix}\right)$$ MS2 7 (1.2.7)

9.32 A differential equation for the G-function

$G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\b_s\end{matrix}\right)$ satisfies the following linear q^{th}-order differential equation:

$$\left[(-1)^{p-m-n}x\prod_{j=1}^{p}\left(x\frac{d}{dx}-a_j+1\right)-\prod_{j=1}^{q}\left(x\frac{d}{dx}-b_j\right)\right]y=0\qquad[p\leq q]$$ EH I 210(1)

9.33 Series of G-functions

$$G^{mn}_{pq}\left(\lambda x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right)=\lambda^{b_1}\sum_{r=0}^{\infty}\frac{1}{r!}(1-\lambda)^r\,G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1+r,b_2,\ldots,b_q\end{matrix}\right)$$
$[|\lambda-1|<1,\quad m\geq 1,\quad \text{if } m=1 \text{ and } p<q,\ \lambda \text{ may be arbitrary}]$
EH I 213(1)

$$=\lambda^{b_q}\sum_{r=0}^{\infty}\frac{1}{r!}(\lambda-1)^r\,G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_{q-1},b_q+r\end{matrix}\right)$$
$[m<q,\quad |\lambda-1|<1]$
EH I 213(2)

$$=\lambda^{a_1-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\lambda-\frac{1}{\lambda}\right)^r G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1-r,a_2,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right)$$
$\left[n\geq 1,\quad \operatorname{Re}\lambda>\tfrac{1}{2},\quad (\text{if } n=1 \text{ and } p>q, \text{ then } \lambda \text{ may be arbitrary})\right]$
EH I 213(3)

$$=\lambda^{a_p-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{1}{\lambda}-1\right)^r G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},a_p-r\\b_1,\ldots,b_q\end{matrix}\right)$$
$\left[n<p,\quad \operatorname{Re}\gamma>\tfrac{1}{2}\right]$
EH I 213(4)

For integrals of the G-function, see **7.8**.

9.34 Connections with other special functions

1. $$J_\nu(x)x^\mu=2^\mu\,G^{10}_{02}\left(\frac{1}{4}x^2\,\middle|\,\begin{matrix}\\\frac{1}{2}\nu+\frac{1}{2}\mu,\frac{1}{2}\mu-\frac{1}{2}\nu\end{matrix}\right)$$ EH I 219(44)

2. $$Y_\nu(x)x^\mu=2^\mu\,G^{20}_{13}\left(\frac{1}{4}x^2\,\middle|\,\begin{matrix}\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\\\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}\end{matrix}\right)$$ EH I 219(46)

3. $K_\nu(x)x^\mu = 2^{\mu-1}\, G^{20}_{02}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \\ \frac{1}{2}\mu+\frac{1}{2}\nu, \frac{1}{2}\mu-\frac{1}{2}\nu \end{matrix}\right)$ EH I 219(47)

4. $K_\nu(x) = e^x\sqrt{\pi}\, G^{20}_{12}\left(2x \,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu, -\nu \end{matrix}\right)$ EH I 219(49)

5. $\mathbf{H}_\nu(x)x^\mu = 2^\mu\, G^{11}_{13}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu, \frac{1}{2}\mu-\frac{1}{2}\nu, \frac{1}{2}\mu+\frac{1}{2}\nu \end{matrix}\right)$ EH I 220(51)

6. $S_{\mu,\nu}(x) = 2^{\mu-1}\frac{1}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)}\, G^{31}_{13}\left(\frac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{matrix}\right)$ EH I 220(55)

7.[7] ${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)x}{\Gamma(a)\,\Gamma(b)}\, G^{12}_{22}\left(x \,\middle|\, \begin{matrix} -a, -b \\ -1, -c \end{matrix}\right)$ EH I 222(74)a

8. $$\begin{aligned} {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right) &= \frac{\prod_{j=1}^q \Gamma\left(b_j\right)}{\prod_{j=1}^p \Gamma\left(a_j\right)}\, G^{1,p}_{p,q+1}\left(-x \,\middle|\, \begin{matrix} 1-a_1,\ldots,1-a_p \\ 0, 1-b_1,\ldots,1-b_q \end{matrix}\right) \\ &= \frac{\prod_{j=1}^q \Gamma\left(b_j\right)}{\prod_{j=1}^p \Gamma\left(a_j\right)}\, G^{p,1}_{q+1,p}\left(-\frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1,\ldots,b_q \\ a_1,\ldots,a_p \end{matrix}\right) \end{aligned}$$
EH I 215(1)

9. $W_{k,m}(x) = \frac{2^k\sqrt{x}e^{\frac{1}{2}x}}{\sqrt{2\pi}}\, G^{40}_{24}\left(\frac{x^2}{4} \,\middle|\, \begin{matrix} \frac{1}{4}-\frac{1}{2}k, \frac{3}{4}-\frac{1}{2}k \\ \frac{1}{2}+\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\right)$ EH I 221(70)

9.4 MacRobert's E-Function

9.41 Representation by means of multiple integrals

$$\begin{aligned} E\left(p;\alpha_r:q;\varrho_s:x\right) = &\frac{\Gamma\left(\alpha_{q+1}\right)}{\Gamma\left(\varrho_1-\alpha_1\right)\Gamma\left(\varrho_2-\alpha_2\right)\cdots\Gamma\left(\varrho_q-\alpha_q\right)} \\ &\times\prod_{\mu=1}^q\int_0^\infty \lambda_\mu^{\varrho_\mu-\alpha_\mu-1}\left(1-\lambda_\mu\right)^{-\varrho_\mu}\,d\lambda_\mu \prod_{\nu=2}^{p-q-1}\int_0^\infty e^{-\lambda_{q+\nu}}\lambda_{q+\nu}^{\alpha_{q+\nu}-1}\,d\lambda_{q+\nu} \\ &\times\int_0^\infty e^{-\lambda_p}\lambda_p^{\alpha_p-1}\left[1+\frac{\lambda_{q+2}\lambda_{q+3}\cdots\lambda_p}{\left(1+\lambda_1\right)\cdots\left(1+\lambda_q\right)x}\right]^{-\alpha_{q+1}}\,d\lambda_p \end{aligned}$$

[$|\arg x| < \pi$, $p \geq q+1$, α_r and ϱ_s are bounded by the condition that the integrals on the right be convergent.] EH I 204(3)

9.42 Functional relations

1. $$\begin{aligned} \alpha_1 x\, E\left(\alpha_1,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right) &= x\, E\left(\alpha_1+1,\alpha_2,\ldots,\alpha_p:\varrho_1,\ldots,\varrho_q:x\right) \\ &\quad + E\left(\alpha_1+1,\alpha_2+1,\ldots,\alpha_p+1:\varrho_1+1,\ldots,\varrho_q+1:x\right) \end{aligned}$$
EH I 205(7)

2. $$(\varrho_1 - 1)\, x\, E(\alpha_1, \ldots, \alpha_p : \varrho_1, \ldots, \varrho_q : x) = x\, E(\alpha_1, \ldots, \alpha_p : \varrho_1 - 1, \varrho_2, \ldots, \varrho_q : x) + E(\alpha_1 + 1, \ldots, \alpha_p + 1 : \varrho_1 + 1, \ldots, \varrho_q + 1 : x)$$ EH I 205(9)

3. $$\frac{d}{dx} E(\alpha_1, \ldots, \alpha_p : \varrho_1, \ldots, \varrho_q : x) = x^{-2}\, E(\alpha_1 + 1, \ldots, \alpha_p + 1 : \varrho_1 + 1, \ldots, \varrho_q + 1 : x)$$ EH I 205(8)

9.5 Riemann's Zeta Functions $\zeta(z,q)$ and $\zeta(z)$, and the Functions $\Phi(z,s,v)$ and $\xi(s)$

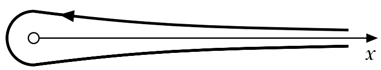
9.51 Definition and integral representations

9.511 $$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}}\, dt;$$ WH

$$= \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2\int_0^\infty \left(q^2 + t^2\right)^{-\frac{z}{2}} \left[\sin\left(z \arctan\frac{t}{q}\right)\right] \frac{dt}{e^{2\pi t} - 1}$$ $[0 < q < 1, \quad \operatorname{Re} z > 1]$ WH

9.512 $$\zeta(z,q) = -\frac{\Gamma(1-z)}{2\pi i} \int_\infty^{(0+)} \frac{(-\theta)^{z-1} e^{-q\theta}}{1 - e^{-\theta}}\, d\theta$$

This equation is valid for all values of z, except for $z = 1, 2, 3, \ldots$. It is assumed that the path of integration (see drawing below) does not pass through the points $2n\pi i$ (where n is a natural number).



See also **4.251** 4, **4.271** 1, 4, 8, **4.272** 9, 12, **4.294** 11.

9.513

1. $$\zeta(z) = \frac{1}{(1 - 2^{1-z})\,\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1}\, dt$$ $[\operatorname{Re} z > 0]$ WH

2. $$\zeta(z) = \frac{2^z}{(2^z - 1)\,\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^t}{e^{2t} - 1}\, dt$$ $[\operatorname{Re} z > 1]$ WH

3.[11] $$\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma\left(\frac{z}{2}\right)} \left[\frac{1}{z(z-1)} + \int_1^\infty \left(t^{\frac{1-z}{2}} + t^{\frac{z}{2}}\right) t^{-1} \sum_{k=1}^\infty e^{-k^2 \pi t}\, dt\right]$$ WH

4. $$\zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty \left(1 + t^2\right)^{-\frac{z}{2}} \sin(z \arctan t)\, \frac{dt}{e^{\pi t} + 1}$$ WH

5. $$\zeta(z) = \frac{2^{z-1}}{2^z - 1} \frac{z}{z-1} + \frac{2}{2^z - 1} \int_0^\infty \left(\frac{1}{4} + t^2\right)^{-z/2} \sin(z \arctan 2t)\, \frac{dt}{e^{2\pi t} - 1}$$ WH

See also **3.411** 1, **3.523** 1, **3.527** 1, 3, **4.271** 8.

9.52 Representation as a series or as an infinite product

9.521

1. $$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z} \qquad [\operatorname{Re} z > 1, \quad q \neq 0, -1, -2, \ldots]$$ WH

2. $$\zeta(z,q) = \frac{2\,\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi q n}{n^{1-z}} + \cos\frac{z\pi}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi q n}{n^{1-z}} \right]$$ $[\operatorname{Re} z < 0, \quad 0 < q \leq 1]$ WH

3.[8] $$\zeta(z,q) = \sum_{n=0}^{N} \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$$

where

$$\begin{aligned} F_n(z) &= \frac{1}{1-z} \left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}} \right) - \frac{1}{(n+1+q)^z} \\ &= z \int_n^{n+1} \frac{(t-n)\,dt}{(t+q)^{z+1}} \end{aligned}$$ WH

9.522

1. $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad [\operatorname{Re} z > 1]$$ WH

2. $$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \qquad [\operatorname{Re} z > 0]$$ WH

9.523 The following product and summation are taken over all primes p:

1.[7] $$\zeta(z) = \prod_p \frac{1}{1-p^{-z}} \qquad [\operatorname{Re} z > 1]$$ WH

2. $$\ln\zeta(z) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{kz}} \qquad [\operatorname{Re} z > 1]$$ WH

9.524[11] $$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z}, \qquad [\operatorname{Re} z > 1]$$

where $\Lambda(k) = 0$ when k is not a power of a prime and $\Lambda(k) = \ln p$ when k is a power of a prime p. WH

9.53 Functional relations

9.531 $$\zeta(-n,q) = -\frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1}$$ [n is a nonnegative integer] see EH I 27 (11) WH

9.532 $$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \, \zeta(k,q) = \ln \frac{e^{-Cz}\,\Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)} \qquad [|z| < q]$$ WH

9.533

1. $$\lim_{z\to 1}\frac{\zeta(z,q)}{\Gamma(1-z)} = -1$$ WH

2. $$\lim_{z\to 1}\left\{\zeta(z,q) - \frac{1}{z-1}\right\} = -\,\Psi(q)$$ WH

3. $$\left\{\frac{d}{dz}\,\zeta(z,q)\right\}_{z=0} = \ln\Gamma(q) - \frac{1}{2}\ln 2\pi$$ WH

9.534 $\zeta(z,1) = \zeta(z)$

9.535

1. $$\zeta(z) = \frac{1}{2^z-1}\,\zeta\left(z,\tfrac{1}{2}\right) \qquad [\operatorname{Re} z > 1]$$ WH

2.[11] $$2^z\,\Gamma(1-z)\,\zeta(1-z)\sin\left(\frac{z\pi}{2}\right) = \pi^{1-z}\,\zeta(z)$$ WH

3. $$2^{1-z}\,\Gamma(z)\,\zeta(z)\cos\frac{z\pi}{2} = \pi^z\,\zeta(1-z)$$ WH

4. $$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\,\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{\frac{z-1}{2}}\,\zeta(1-z)$$ WH

9.536 $$\lim_{z\to 1}\left\{\zeta(z) - \frac{1}{z-1}\right\} = \boldsymbol{C}$$

9.537 Set $z = \frac{1}{2} + it$. Then, $\Xi(t) = \dfrac{(z-1)\,\Gamma\left(\frac{z}{2}+1\right)}{\sqrt{\pi^z}}\,\zeta(z) = \Xi(-t)$ is an even function of t with real coefficients in its expansion in powers of t^2. JA

9.54 Singular points and zeros

9.541[7]

1. $z = 1$ is the only singular point of the function $\zeta(z)$ WH

2. The function $\zeta(z)$ has simple zeros at the points $-2n$, where n is a natural number. All other zeros of the function $\zeta(z)$ lie in the strip $0 \le \operatorname{Re} z < 1$.

3.[8] Riemann's hypothesis: All zeros of the function $\zeta(z)$ lie on the straight line $\operatorname{Re} z = \frac{1}{2}$. It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first 1,500,000,001 zeros lying in $0 < \operatorname{Im} z < 545,439,823.215$ are known to have $\operatorname{Re} z = \frac{1}{2}$. WH

9.542 Particular values:

1. $$\zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!} \qquad [m \text{ is a natural number}]$$ WH

2. $$\zeta(1-2m) = -\frac{B_{2m}}{2m} \qquad [m \text{ is a natural number}]$$ WH

3. $$\zeta(-2m) = 0 \qquad [m \text{ is a natural number}]$$ WH

4. $$\zeta'(0) = -\tfrac{1}{2}\ln 2\pi$$ WH

9.55 The Lerch function $\Phi(z, s, v)$

9.550 Definition:

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \qquad [|z| < 1, \quad v \neq 0, -1, \ldots]$$ EH I 27(1)

Functional relations

9.551 $$\Phi(z, s, v) = z^m \, \Phi(z, s, m+v) + \sum_{n=0}^{m-1} (v+n)^{-s} z^n \qquad [m = 1, 2, 3, \ldots, \quad v \neq 0, -1, -2, \ldots]$$ EH I 27(1)

9.552 $$\Phi(z, s, v) = iz^{-v} (2\pi)^{s-1} \, \Gamma(1-s) \left[e^{-i\pi \frac{s}{2}} \, \Phi\left(e^{-2\pi i v}, 1-s, \frac{\ln z}{2\pi i}\right) - e^{i\pi\left(\frac{s}{2} - 2v\right)} \, \Phi\left(e^{2\pi i v}, 1-s, 1 - \frac{\ln z}{2\pi i}\right)\right]$$ EH I 29(7)

Series representation

9.553 $$\Phi(z, s, v) = z^{-v} \, \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\ln z + 2\pi n i)^{s-1} \, e^{2\pi n v i}$$
$$[0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\ln z + 2\pi n i)| \leq \pi]$$ EH I 28(6)

9.554 $$\Phi(z, m, v) = z^{-v} \left\{ {\sum_{n=0}^{\infty}}' \zeta(m-n, v) \frac{(\ln z)^n}{n!} + \frac{(\ln z)^{m-1}}{(m-1)!} \left[\Psi(m) - \Psi(v) - \ln\left(\ln \frac{1}{z}\right)\right]\right\}^{*}$$
$$[m = 2, 3, 4, \ldots, \quad |\ln z| < 2\pi, \quad v \neq 0, -1, -2, \ldots]$$ EH I 30(9)

9.555 $$\Phi(z, -m, v) = \frac{m!}{z^v} \left(\ln \frac{1}{z}\right)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v) \, (\ln z)^r}{r!(m+r+1)} \qquad [|\ln z| < 2\pi]$$ EH I 30(11)

Integral representation

9.556 $$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} \, dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t} \, dt}{e^t - z}$$
$$[\operatorname{Re} v > 0, \text{ or } |z| \leq 1, \quad z \neq 1, \quad \operatorname{Re} s > 0, \text{ or } z = 1, \quad \operatorname{Re} s > 1]$$ EH I 27(3)

Limit relationships

9.557 $$\lim_{z \to 1} (1-z)^{1-s} \, \Phi(z, s, v) = \Gamma(1-s) \qquad [\operatorname{Re} s < 1]$$ EH I 30(12)

9.558 $$\lim_{z \to 1} \frac{\Phi(z, 1, v)}{-\ln(1-z)} = 1$$ EH I 30(13)

A connection with a hypergeometric function

9.559 $$\Phi(z, 1, v) = v^{-1} \, {}_2F_1(1, v; 1+v; z) \qquad [|z| < 1]$$ EH I 30(10)

*In 9.554 the prime on the symbol $\sum$ means that the term corresponding to $n = m - 1$ is omitted.

9.56 The function $\xi(s)$

9.561 $\xi(s) = \frac{1}{2} s(s-1) \frac{\Gamma\left(\frac{1}{2}s\right)}{\pi^{\frac{1}{2}s}} \zeta(s)$ EH III 190(10)

9.562 $\xi(1-s) = \xi(s)$ EH III 190(11)

9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions $\nu(x)$, $\nu(x,\alpha)$, $\mu(x,\beta)$, $\mu(x,\beta,\alpha)$, $\lambda(x,y)$ and Euler Polynomials

9.61 Bernoulli numbers

9.610 The numbers B_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad [0 < |t| < 2\pi],$$

are called *Bernoulli* numbers. Thus, the function $\frac{t}{e^t-1}$ is a generating function for the Bernoulli numbers. GE 48(57), FI II 520

9.611 Integral representations

1. $$B_{2n} = (-1)^{n-1} 4n \int_0^\infty \frac{x^{2n-1}}{e^{2\pi x} - 1}\, dx \qquad [n = 1, 2, \ldots] \qquad (\text{cf. } \mathbf{3.411}\ 2,\ 4)$$ FI II 721a

2. $$B_{2n} = (-1)^{n-1} \pi^{-2n} \int_0^\infty \frac{x^{2n}}{\sinh^2 x}\, dx \qquad [n = 1, 2, \ldots]$$

3. $$B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^\infty x^{2n-2} \ln\left(1 - e^{-2\pi x}\right) dx \qquad [n = 1, 2, \ldots]$$

4.* $$B_n = \lim_{x \to 0} \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1}\right)$$

See also **3.523** 2, **4.271** 3.

Properties and functional relations

9.612[8] A symbolic notation:

$$(B + \alpha)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} B_k \alpha^{n-k} \qquad [n \geq 2]$$

in particular

$$B_n = (B+1)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} B_k \qquad [n \geq 2]$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!} \qquad [n \geq 2]$$

9.613 All the Bernoulli numbers are rational numbers.

9.614 Every number B_n can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where C_n is an integer and the sum is taken over all $k > 0$ such that $k+1$ is a prime and k is a divisor of n. GE 64

9.615[11] All the Bernoulli numbers with odd index are equal to zero, except that $B_1 = -\frac{1}{2}$; that is, $B_{2n+1} = 0$ for n a natural number. GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1;\ k \text{ even}}^{n-1} \frac{2n(2n-1)\dots(2n-2k+2)}{(2k)!} B_{k/2} \qquad [n \geq 1]$$

9.616 $B_{2n} = \dfrac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \qquad [n \geq 0]$ (cf. **9.542**) GE 56(79), FI II 721a

9.617[7] $B_{2n} = (-1)^{n-1} \dfrac{2(2n)!}{(2\pi)^{2n}} \dfrac{1}{\prod_{p=2}^{\infty} \left(1 - \frac{1}{p^{2n}}\right)} \qquad [n \geq 1]$ (cf. **9.523**)

(where the product is taken over all primes p).

- For a connection with Riemann's zeta function, see **9.542**.
- For a connection with the Euler numbers, see **9.635**.
- For a table of values of the Bernoulli numbers, see **9.71**

9.619 An inequality

$$\left|(B-\theta)^{[n]}\right| \leq |B_n| \qquad [0 < \theta < 1]$$

9.62 Bernoulli polynomials

9.620 The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$ GE 51(62)

or symbolically, $B_n(x) = (B+x)^{[n]}$. GE 52(68)

9.621 The generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \qquad [0 < |t| < 2\pi] \qquad \text{(cf. 1.213)}$$ GE 65(89)a

9.622 Series representation

1.[7] $$B_n(x) = -2 \frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos\left(2\pi k x - \frac{1}{2}\pi n\right)}{k^n}$$

$[n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0]$ AS 805(23.1.16)

2.[7] $$B_{2n-1}(x) = 2\frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}}$$

$[n > 1, \quad 1 \geq x \geq 0; \quad n = 1, \quad 1 > x > 0]$ AS 805(23.1.17)

3.[10] $$B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}}$$ $[0 \leq x \leq 1, \quad n = 1, 2, \ldots]$ GE 71

9.623 Functional relations and properties:

1. $$B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m$$

[n and m are natural numbers] (see also **0.121**) GE 51(65)

2. $B_n(x+1) - B_n(x) = nx^{n-1}$ GE 65(90)

3. $B_n'(x) = n\, B_{n-1}(x)$ $[n = 1, 2, \ldots]$ GE 66

4. $B_n(1-x) = (-1)^n\, B_n(x)$ GE 66

5.[10] $(-1)^n\, B_n(-x) = B_n(x) + nx^{n-1}$ $[n = 0, 1, \ldots]$ AS 804(23.1.9)

9.624[7] $$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right)$$

$[m = 1, 2, \ldots n = 0, 1, \ldots]$; "summation theorem" GE 67

9.625 For n odd, the differences

$$B_n(x) - B_n$$

vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1. They change sign at the point $x = \frac{1}{2}$. For n even, these differences vanish at the end points of the interval $[0, 1]$. Within this interval, they do not change sign, and their greatest absolute value occurs at the point $x = \frac{1}{2}$.

9.626 The polynomials

$$B_{2n}(x) - B_{2n} \text{ and } B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval $(0, 1)$. GE 87

9.627 Special cases:

1. $B_1(x) = x - \frac{1}{2}$ GE 70

2. $B_2(x) = x^2 - x + \frac{1}{6}$ GE 70

3. $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ GE 70

4. $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ GE 70

5. $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$ GE 70

9.628 Particular values:

1. $B_n(0) = B_n$

2. $B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n$ $[n \neq 1]$ GE 76

9.63 Euler numbers

9.630 The numbers E_n, representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad \left[|t| < \frac{\pi}{2}\right],$$

are known as the *Euler numbers*. Thus, the function $\frac{1}{\cosh t}$ is a generating function for the Euler numbers. CE 330

9.631 A recursion formula

$$(E+1)^{[n]} + (E-1)^{[n]} = 0 \qquad [n \geq 1], \qquad E_0 = 1$$ CE 329

Properties of the Euler numbers

9.632 The Euler numbers are integers.

9.633 The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0.$$ CE 329

9.634 If $\alpha, \beta\gamma, \ldots$ are the divisors of the number $n-m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha+1, 2\beta+1, 2\gamma+1, \ldots$ that are primes.

9.635 A connection with the Bernoulli numbers (symbolic notation):

1.[11] $$E_{n-1} + 4(-1)^n \left(3^{n-1} - 1\right) B_1 = \frac{(4B-1)^{[n]} - (4B-3)^{[n]}}{2n} + 4(-1)^{n+1}\left(3^{n-1}-1\right) B_1$$ CE 330

2. $$B_n = \frac{n(E+1)^{[n-1]}}{2^n\left(2^n-1\right)} \qquad [n \geq 2]$$ CE 330

3.[6] $$\left(B + \tfrac{1}{4}\right)^{[2n+1]} = -4^{-2n-1}(2n+1)E_{2n} \qquad [n \geq 0]$$ CE 341

4. $$E_{n-1} = \frac{(4B+3)^{[n]} - (4B+1)^{[n]}}{2n} \qquad [n \geq 1]$$

For a table of values of the Euler numbers, see **9.72**.

9.64 The functions $\nu(x)$, $\nu(x,\alpha)$, $\mu(x,\beta)$, $\mu(x,\beta,\alpha)$, and $\lambda(x,y)$

9.640

1. $$\nu(x) = \int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}$$ EH III 217(1)

2. $$\nu(x,\alpha) = \int_0^\infty \frac{x^{\alpha+t}\,dt}{\Gamma(\alpha+t+1)}$$ EH III 217(1)

3. $$\mu(x,\beta) = \int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(t+1)}$$ EH III 217(2)

4. $$\mu(x,\beta,\alpha) = \int_0^\infty \frac{x^{\alpha+t} t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(\alpha+t+1)}$$ EH III 217(2)

5. $$\lambda(x,y) = \int_0^y \frac{\Gamma(u+1)\,du}{x^u}$$ MI 9

9.65[10] Euler polynomials

9.650 The Euler polynomials are defined by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$$ AS 804 (23.1.7)

9.651 The generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$ AS 804 (23.1.1)

9.652 Series representation:

1. $$E_n(x) = 4\frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)\pi x - \frac{1}{2}\pi n\right)}{(2k+1)^{n+1}}$$
$[n > 0, \quad 1 \ge x \ge 0, \quad n = 1, \quad 1 > x > 0]$ AS 804 (23.1.16)

2.[10] $$E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}}$$ $[n = 1, 2, \ldots, \quad 1 \ge x \ge 0]$
AS 804 (23.1.17)

3. $$E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}}$$
$[n > 0, \quad 1 \ge x \ge 0, \quad n = 0, \quad 1 > x > 0]$ AS 804 (23.1.18)

9.653 Functional relations and properties:

1. $$E_m(n+1) = 2\sum_{k=1}^{n} (-1)^{n-k} k^m + (-1)^{n+1} E_m(0),$$ [m and n are natural numbers]
AS 804 (23.1.4)

2. $E'_n(x) = nE_{n-1}(x).$ $[n = 1, 2, \ldots]$ AS 804 (23.1.5)

3. $E_n(x+1) + E_n(x) = 2x^n$ $[n = 0, 1, \ldots]$ AS 804 (23.1.6)

4.[8] $$E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x - \frac{k}{m}\right)$$ $[n = 0, 1, \ldots, m = 1, 3, \ldots]$
AS 804 (23.1.10)

5. $$E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right)$$ $[n = 0, 1, \ldots, m = 2, 4, \ldots]$
AS 804 (23.1.10)

9.654 Special cases:

1. $E_1(x) = x - \frac{1}{2}$

2. $E_2(x) = x^2 - x$

3. $E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$

4. $E_4(x) = x^4 - 2x^3 + x$

5. $E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}$

9.655 Particular values:

1. $E_{2n+1} = 0.$ $[n = 0, 1, \ldots]$ AS 805 (23.1.19)

2. $E_n(0) = -E_n(1) = -2(n+1)^{-1}\left(2^{n+1} - 1\right)B_{n+1}$ $[n = 1, 2, \ldots]$ AS 805 (23.1.20)

3. $E_n\left(\frac{1}{2}\right) = 2^{-n}E_n$ $[n = 0, 1, \ldots]$ AS 805 (23.1.21)

4. $E_{2n-1}\left(\frac{1}{3}\right) = -E_{2n-1}\left(\frac{2}{3}\right) = -(2n)^{-1}\left(1 - 3^{1-2n}\right)\left(2^{2n} - 1\right)B_{2n}$ $[n = 1, 2, \ldots]$ AS 806 (23.1.22)

9.7 Constants

9.71 Bernoulli numbers

- $B_0 = 1$
- $B_1 = -1/2$
- $B_2 = 1/6$
- $B_4 = -1/30$
- $B_6 = 1/42$
- $B_8 = -1/30$
- $B_{10} = 5/66$
- $B_{12} = -691/2730$
- $B_{14} = 7/6$
- $B_{16} = -3617/510$
- $B_{18} = 43\,867/798$
- $B_{20} = -174\,611/330$
- $B_{22} = 854\,513/138$
- $B_{24} = -236\,364\,091/2730$
- $B_{26} = 8\,553\,103/6$
- $B_{28} = -23\,749\,461\,029/870$
- $B_{30} = 8\,615\,841\,276\,005/14\,322$
- $B_{32} = -7\,709\,321\,041\,217/510$
- $B_{34} = 2\,577\,687\,858\,367/6$

9.72 Euler numbers

- $E_0 = 1$
- $E_2 = -1$
- $E_4 = 5$
- $E_6 = -61$
- $E_8 = 1385$
- $E_{10} = -50\,521$
- $E_{12} = 2\,702\,765$
- $E_{14} = -199\,360\,981$
- $E_{16} = 19\,391\,512\,145$
- $E_{18} = -2\,404\,879\,675\,441$
- $E_{20} = 370\,371\,188\,237\,525$

The Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

9.73 Euler's and Catalan's constants

Euler's constant

$\boldsymbol{C} = 0.577\,215\,664\,901\,532\,860\,606\,512\ldots$ (cf. **8.367**)

Catalan's constant

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\,965\,594\ldots$$

9.74[10] Stirling numbers

9.740 The **Stirling number of the first kind** $S_n^{(m)}$ is defined by the requirement that $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles. AS 824 (23.1.3)

9.741 Generating functions:

1. $$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m$$ AS 824 (24.1.3)

2. $$\{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}$$ $[|x| < 1]$ AS 824 (24.1.3)

9.742 Recurrence relations:

1.[8] $$S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}; \quad S_n^{(0)} = \delta_{0n}; \quad S_n^{(1)} = (-1)^{n-1}(n-1)!; \quad S_n^{(n)} = 1$$ $[n \ge m \ge 1]$ AS 824 (24.1.3)

2. $$\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)}$$ $[n \ge m \ge r]$ AS 824 (24.1.3)

9.743 Functional relations and properties

1. $$x(x-h)(x-2h)\cdots(x-mh+h) = \frac{h^m\,\Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}-m+1\right)} = h^m \sum_{k=1}^{m} \left(\frac{x}{h}\right)^k S_k^{(m)}$$

2. $$[(x+1)(x+2)\cdots(x+m)]^{-1} = \left[\binom{x+m}{m} m!\right]^{-1} = \left[\sum_{k=1}^{p} (x+m)^k S_k^{(m)}\right]^{-1}$$

3. $$[(x+h)(x+2h)\cdots(x+mh)]^{-1} = \frac{\Gamma\left(\frac{x}{h}+1\right)}{h^m\,\Gamma\left(\frac{x}{h}+m+1\right)} = \left[h^m \sum_{k=1}^{m} \left(\frac{x}{h}+m\right)^k S_k^{(m)}\right]^{-1}$$

9.744 The Stirling number of the second kind $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

9.745 Generating functions:

1. $$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1)$$ AS 824 (24.1.4)

2. $$(e^x - 1)^m = m! \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}$$ AS 824 (24.1.4)

3. $$[(1-x)(1-2x)\cdots(1-mx)]^{-1} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} x^{n-m} \qquad \left[|x| < m^{-1}\right]$$ AS 824 (24.1.4)

9.746 Closed form expression:

1. $$\mathfrak{S}_n^{(m)} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n$$ AS 824 (24.1.4)

9.747 Recurrence relations:

1.[8] $$\mathfrak{S}_{n+1}^{(m)} = m\mathfrak{S}_n^{(m)} + \mathfrak{S}_n^{(m-1)}, \quad \mathfrak{S}_n^{(0)} = \delta_{0n}, \quad \mathfrak{S}_n^{(1)} = \mathfrak{S}_n^{(n)} = 1 \qquad [n \ge m \ge 1]$$ AS 825(24.1.4)

2. $$\binom{m}{r} \mathfrak{S}_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} \mathfrak{S}_{n-k}^{(r)} \mathfrak{S}_k^{(m-r)} \qquad [n \ge m \ge r]$$ AS 825 (24.1.4)

3. $$S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathfrak{S}_{n-m+k}^{(k)}$$ AS 824 (24.1.3)

9.748[7] Particular values:

Stirling numbers of the first kind $S_n^{(m)}$

m	$S_1^{(m)}$	$S_2^{(m)}$	$S_3^{(m)}$	$S_4^{(m)}$	$S_5^{(m)}$	$S_6^{(m)}$	$S_7^{(m)}$	$S_8^{(m)}$	$S_9^{(m)}$
1	1	-1	2	-6	24	-120	720	-5040	40320
2		1	-3	11	-50	274	-1764	13068	-109584
3			1	-6	35	-225	1624	-13132	118121
4				1	-10	85	-735	6769	-67284
5					1	-15	175	-1960	22449
6						1	-21	332	-4536
7							1	-28	546
8								1	-36
9									1

Stirling numbers of the second kind $\mathfrak{S}_n^{(m)}$

m	$\mathfrak{S}_1^{(m)}$	$\mathfrak{S}_2^{(m)}$	$\mathfrak{S}_3^{(m)}$	$\mathfrak{S}_4^{(m)}$	$\mathfrak{S}_5^{(m)}$	$\mathfrak{S}_6^{(m)}$	$\mathfrak{S}_7^{(m)}$	$\mathfrak{S}_8^{(m)}$	$\mathfrak{S}_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

9.749[8] Relationship between Stirling numbers of the first kind and derivatives of $(\ln x)^{-m}$:

1. $$\frac{d^n}{dx^n}\left(\frac{1}{\ln^m x}\right) = \frac{1}{\ln^m x}\sum_{k=1}^{n}\frac{(-1)^k (m)_k S_n^{(k)}}{x^n \ln^k x}$$

where $(m)_k = \Gamma(m+k)/\,\Gamma(m)$, $[m, n$ are positive integers$]$

10 Vector Field Theory

10.1–10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_2)$ be arbitrary vectors, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the set of orthogonal unit vectors in terms of which the components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written $\mathbf{a} \cdot \mathbf{b}$, and the vector product, written either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$. Their properties are as follows:

1. $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (scalar product)

2. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ (vector product)

3. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (triple scalar product)

4. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ (triple vector product)

10.12 Properties of scalar product

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutative)

2. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}$.

   Note: $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is also written $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; thus (2) may also be written

3. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

10.13 Properties of vector product

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutative)

2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$

3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

10.14 Differentiation of vectors

If $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$, $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, $\phi(t)$ is a scalar and all functions of t are differentiable, then

1. $$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$$
2. $$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$
3. $$\frac{d}{dt}(\phi\mathbf{a}) = \frac{d\phi}{dt}\mathbf{a} + \phi\frac{d\mathbf{a}}{dt}$$
4. $$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$$
5. $$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$
6. $$\frac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \frac{d\mathbf{c}}{dt}$$
7. $$\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right)$$

10.21 Operators grad, div, and curl

In cartesian coordinates $O\{x_1, x_2, x_3\}$, in which system it is convenient to denote the triad of unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the vector operator ∇, called either "del" or "nabla," has the form

1. $$\nabla \equiv \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}$$

   If $\Phi(x, y, z)$ is any differentiable scalar function, the gradient of Φ, written grad Φ, is

2. $$\operatorname{grad}\Phi \equiv \nabla\,\Phi = \frac{\partial\Phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\Phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\Phi}{\partial x_3}\mathbf{e}_3$$

   The divergence of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written $\operatorname{div}\mathbf{f}$, is

3. $$\operatorname{div} f \equiv \nabla\cdot\mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

   The curl, or rotation, of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written either curl $\mathbf{f}$ or rot $\mathbf{f}$, is

4. $$\operatorname{curl}\mathbf{f} \equiv \operatorname{rot}\mathbf{f} \equiv \nabla\times\mathbf{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{e}_3,$$

   or equivalently,

$$\operatorname{curl}\mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

10.31 Properties of the operator ∇

Let $\Phi(x_1, x_2, x_3)$, $\Psi(x_1, x_2, x_3)$ be any two differentiable scalar functions, $\mathbf{f}(x_1, x_2, x_3)$, $\mathbf{g}(x_1, x_2, x_3)$ any two differentiable vector functions, and $\mathbf{a}$ an arbitrary vector. Define the scalar operator ∇^2, called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

Then, in terms of the operator ∇, we have the following: MF I 114

1. $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$
2. $\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$
3. $\nabla(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\nabla)\,\mathbf{g} + (\mathbf{g}\cdot\nabla)\,\mathbf{f} + \mathbf{f}\times(\nabla\times\mathbf{g}) + \mathbf{g}\times(\nabla\times\mathbf{f})$
4. $\nabla\cdot(\Phi\mathbf{f}) = \Phi(\nabla\cdot\mathbf{f}) + \mathbf{f}\cdot\nabla\Phi$
5. $\nabla\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\nabla\times\mathbf{f}) - \mathbf{f}\cdot(\nabla\times\mathbf{g})$
6. $\nabla\times(\Phi\mathbf{f}) = \Phi(\nabla\times\mathbf{f}) + (\nabla\Phi)\times\mathbf{f}$
7. $\nabla\times(\mathbf{f}\times\mathbf{g}) = \mathbf{f}(\nabla\cdot\mathbf{g}) - \mathbf{g}(\nabla\cdot\mathbf{f}) + (\mathbf{g}\cdot\nabla)\,\mathbf{f} - (\mathbf{f}\cdot\nabla)\,\mathbf{g}$
8. $\nabla\times(\nabla\times\mathbf{f}) = \nabla(\nabla\cdot\mathbf{f}) - \nabla^2\mathbf{f}$
9. $\nabla\times(\nabla\Phi) \equiv \mathbf{0}$
10. $\nabla\cdot(\nabla\times\mathbf{f}) \equiv 0$
11.[10] $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2(\nabla\Phi)\cdot(\nabla\Psi) + \Psi\nabla^2\Phi$

The equivalent results in terms of grad, div, and curl are as follows:

1. $\operatorname{grad}(\Phi + \Psi) = \operatorname{grad}\Phi + \operatorname{grad}\Psi$
2. $\operatorname{grad}(\Phi\Psi) = \Phi\operatorname{grad}\Psi + \Psi\operatorname{grad}\Phi$
3. $\operatorname{grad}(\mathbf{f}\cdot\mathbf{g}) = (\mathbf{f}\cdot\operatorname{grad})\,\mathbf{g} + (\mathbf{g}\cdot\operatorname{grad})\,\mathbf{f} + \mathbf{f}\times\operatorname{curl}\mathbf{g} + \mathbf{g}\times\operatorname{curl}\mathbf{f}$
4. $\operatorname{div}(\Phi\mathbf{f}) = \Phi\operatorname{div}\mathbf{f} + \mathbf{f}\cdot\operatorname{grad}\Phi$
5. $\operatorname{div}(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot\operatorname{curl}\mathbf{f} - \mathbf{f}\cdot\operatorname{curl}\mathbf{g}$
6. $\operatorname{curl}(\Phi\mathbf{f}) = \Phi\operatorname{curl}\mathbf{f} + \operatorname{grad}\Phi\times\mathbf{f}$
7. $\operatorname{curl}(\mathbf{f}\times\mathbf{g}) = \mathbf{f}\operatorname{div}\mathbf{g} - \mathbf{g}\operatorname{div}\mathbf{f} + (\mathbf{g}\cdot\operatorname{grad})\,\mathbf{f} - (\mathbf{f}\cdot\operatorname{grad})\,\mathbf{g}$
8. $\operatorname{curl}(\operatorname{curl}\mathbf{f}) = \operatorname{grad}(\operatorname{div}\mathbf{f}) - \nabla^2\mathbf{f}$
9. $\operatorname{curl}(\operatorname{grad}\Phi) \equiv \mathbf{0}$
10. $\operatorname{div}(\operatorname{curl}\mathbf{f}) \equiv 0$
11. $\nabla^2(\Phi\Psi) = \Phi\nabla^2\Psi + 2\operatorname{grad}\Phi\cdot\operatorname{grad}\Psi + \Psi\nabla^2\Phi$

The expression $(\mathbf{a}\cdot\nabla)$ or, equivalently $(\mathbf{a}\cdot\operatorname{grad})$, defined by

$$(\mathbf{a}\cdot\nabla) \equiv a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector $\mathbf{a}$.

10.41 Solenoidal fields

A vector field $\mathbf{f}$ is said to be solenoidal if div $\mathbf{f} \equiv 0$. We have the following representation:

10.411 *Representation theorem for vector Helmholtz equation.* If u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and $\mathbf{m}$ is a constant unit vector, then the vectors

$$\mathbf{X} = \operatorname{curl}(\mathbf{m}u), \qquad \mathbf{Y} = \frac{1}{\lambda}\operatorname{curl}\mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector $\mathbf{H}$. The general solution of the equation is

$$\mathbf{H} = \operatorname{curl}(\mathbf{m}u) + \frac{1}{\lambda}\operatorname{curl}\operatorname{curl}(\mathbf{m}u).$$

10.51–10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates $O\{x_1, x_2, x_3\}$ to the general orthogonal curvilinear coordinates $O\{u_1, u_2, u_3\}$:

$$x_1 = x_1(u_1, u_2, u_3), \qquad x_2 = x_2(u_1, u_2, u_3), \qquad x_3 = x_3(u_1, u_2, u_3)$$

Then,

1. $$dx_i = \frac{\partial x_i}{\partial u_1}\,du_1 + \frac{\partial x_i}{\partial u_2}\,du_2 + \frac{\partial x_i}{\partial u_3}\,du_3 \qquad (i = 1, 2, 3),$$

and the length element dl may be determined from

2. $$dl^2 = g_{11}\,du_1^2 + g_{22}\,du_2^2 + g_{33}\,du_3^2 + 2g_{23}\,du_2\,du_3 + 2g_{31}\,du_3\,du_1 + 2g_{12}\,du_1\,du_2,$$

where

3.[3] $$g_{ij} = \frac{\partial x_1}{\partial u_i}\frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i}\frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i}\frac{\partial x_3}{\partial u_j} = g_{ji}, \qquad g_{ij} = 0, \quad i \neq j,$$

provided the Jacobian of the transformation

4. $$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

does not vanish (see **14.313**).

Define the metrical coefficients

5. $$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

then the volume element dV in orthogonal curvilinear coordinates is

6. $$dV = h_1 h_2 h_3\,du_1\,du_2\,du_3,$$

and the surface elements of area ds_i on the surfaces $u_i =$ constant, for $i = 1, 2, 3$, are

7. $$ds_1 = h_2 h_3\,du_2\,du_3, \quad ds_2 = h_1 h_3\,du_1\,du_3, \quad ds_3 = h_1 h_2\,du_1\,du_2$$

Denote by $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ the triad of orthogonal unit vectors that are tangent to the u_1, u_2, and u_3 coordinate lines through any given point P, and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components f_{u_1}, f_{u_2}, and f_{u_3} of $\mathbf{f}$ along the coordinate line,

8. $\mathbf{f} = f_{u_1}\mathbf{e}_1 + f_{u_2}\mathbf{e}_2 + f_{u_3}\mathbf{e}_3$ MF I 115

10.611 $\nabla\Phi$, $\operatorname{div}\mathbf{f}$, $\operatorname{curl}\mathbf{f}$, *and* ∇^2 *in general orthogonal curvilinear coordinates.*

1. $$\operatorname{grad}\Phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

2.[3] $$\operatorname{div}\mathbf{f} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(h_2h_3f_{u_1}\right) + \frac{\partial}{\partial u_2}\left(h_3h_1f_{u_2}\right) + \frac{\partial}{\partial u_3}\left(h_1h_2f_{u_3}\right)\right)$$

3. $$\operatorname{curl}\mathbf{f} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1f_{u_1} & h_2f_{u_2} & h_3f_{u_3} \end{vmatrix}$$

4. $$\nabla^2 \equiv \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right)$$ MF I 21-31

10.612 *Cylindrical polar coordinates.* In terms of the coordinates $O\{r,\phi,z\}$, that is, $u_1 = r$, $u_2 = \phi$, $u_3 = z$, where $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $x_3 = z$ for $-\pi < \phi \le \pi$, it follows that

1. $h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$

and

2. $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z,$$

3. $$\operatorname{div}\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}\left(rf_r\right) + \frac{1}{r}\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_z}{\partial z},$$

4. $$\operatorname{curl}\mathbf{f} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ f_r & rf_\phi & f_z \end{vmatrix},$$

5. $$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$ MF I 116

10.613 *Spherical polar coordinates.* In terms of the coordinates $O\{r,\theta,\phi\}$, that is, $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, where $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$, for $0 \le \theta \le \pi, -\pi < \phi \le \pi$, we have

1. $h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta,$

and

2.[10] $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi,$$

3. $$\operatorname{div}\mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(f_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi},$$

4. $$\operatorname{curl}\mathbf{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_r & rf_\theta & r\sin\theta f_\phi \end{vmatrix},$$

5. $$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$ MF I 116

Special Orthogonal Curvilinear Coordinates and their Metrical Coefficients h_1, h_2, h_3

10.614 *Elliptic cylinder coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_1 u_2, \qquad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \qquad x_3 = u_3$

2. $h_1 = \sqrt{\dfrac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = 1$ MF I 657

10.615 *Parabolic cylinder coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \dfrac{1}{2}\left(u_1^2 - u_2^2\right), \qquad x_2 = u_1 u_2, \qquad x_3 = u_3$

2. $h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = 1$ MF I 658

10.616 *Conical coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \dfrac{u_1}{a}\sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \qquad x_2 = \dfrac{u_1}{b}\sqrt{(b^2 + u_2^2)(b^2 - u_3^2)}, \qquad x_3 = \dfrac{u_1 u_2 u_3}{ab}$

   with $a^2 + b^2 = 1$

2. $h_1 = 1, \qquad h_2 = u_1\sqrt{\dfrac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \qquad h_3 = u_1\sqrt{\dfrac{u_2^2 + u_3^2}{(a^2 + u_3^2)(b^2 - u_3^2)}}$ MF I 659

10.617 *Rotational parabolic coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_1 u_2 u_3, \qquad x_2 = u_1 u_2 \sqrt{1 - u_3^2}, \qquad x_3 = \dfrac{1}{2}\left(u_1^2 - u_2^2\right)$

2. $h_1 = \sqrt{u_1^2 + u_2^2}, \qquad h_2 = \sqrt{u_1^2 + u_2^2}, \qquad h_3 = \dfrac{u_1 u_2}{\sqrt{1 - u_3^2}}$ MF I 660

10.618 *Rotational prolate spheroidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)(1 - u_3^2)}, \qquad x_3 = u_1 u_2$

2. $h_1 = \sqrt{\dfrac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 - a^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{\dfrac{(u_1^2 - a^2)(1 - u_2^2)}{1 - u_3^2}}$ MF I 661

10.619 *Rotational oblate spheroidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $x_1 = u_3\sqrt{(u_1^2 + a^2)(1 - u_2^2)}, \qquad x_2 = \sqrt{(u_1^2 + a^2)(1 - u_2^2)(1 - u_3^2)}, \qquad x_3 = u_1 u_2$

2. $h_1 = \sqrt{\dfrac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \qquad h_2 = \sqrt{\dfrac{u_1^2 + a^2 u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{\dfrac{(u_1^2 + a^2)(1 - u_2^2)}{1 - u_3^2}}$ MF I 662

10.620 *Ellipsoidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}}, \quad x_3 = \frac{u_1 u_2 u_3}{ab}$$

2. $$h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \qquad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 663

10.621 *Paraboloidal coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \qquad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}},$$
$$x_3 = \tfrac{1}{2}\left(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2\right)$$

2. $$h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \qquad h_2 = u_2\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = u_3\sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}$$

MF I 664

10.622 *Bispherical coordinates* $O\{u_1, u_2, u_3\}$.

1. $$x_1 = a u_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \quad x_2 = a\frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \qquad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2}$$

2. $$h_1 = \frac{a}{(u_1 - u_2)\sqrt{u_1^2 - 1}},$$
$$h_2 = \frac{a}{(u_1 - u_2)\sqrt{1 - u_2^2}}, \qquad h_3 = \left(\frac{a}{u_1 - u_2}\right)\sqrt{\frac{1 - u_2^2}{1 - u_3^2}}$$ MF I 665

10.71–10.72 Vector integral theorems

10.711 *Gauss's divergence theorem.* Let V be a volume bounded by a simple closed surface S and let $\mathbf{f}$ be a continuously differentiable vector field defined in V and on S. Then, if $d\mathbf{S}$ is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f}\, dV$$ KE 39

10.712 *Green's theorems.* Let Φ and Ψ be scalar fields which, together with $\nabla^2 \Phi$ and $\nabla^2 \Psi$, are defined both in a volume V and on its surface S, which we assume to be simple and closed. Then, if ∂/∂n denotes differentiation along the outward drawn normal to S, we have

10.713 *Green's first theorem*

$$\int_S \Phi \frac{\partial \Psi}{\partial n}\, dS = \int_V \left(\Phi\, \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi\right) dV$$ KE 212

10.714 *Green's second theorem*

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \int_V \left(\Phi \, \nabla^2 \, \Psi - \Psi \, \nabla^2 \, \Phi \right) dV$$ KE 215

10.715 *Special cases*

1. $$\int_S (\Phi \operatorname{grad} \Phi) \cdot d\mathbf{S} = \int_V \left(\Phi \, \nabla^2 \, \Phi + (\operatorname{grad} \Phi)^2 \right) dV$$

2. $$\int_S \frac{\partial \Phi}{\partial n} \, dS = \int_V \nabla^2 \, \Phi \, dV$$ MV 81

10.716 *Green's reciprocal theorem.* If Φ and Ψ are harmonic, so that $\nabla^2 \, \Phi = \nabla^2 \, \Psi = 0$, then

3. $$\int_S \Phi \frac{\partial \Psi}{\partial n} \, dS = \int_S \Psi \frac{\partial \Phi}{\partial n} \, dS$$ MM 105

10.717 *Green's representation theorem.* If Φ and $\nabla^2 \, \Phi$ are defined within a volume V bounded by a simple closed surface S, and P is an interior point of V, then in three dimensions

4. $$\Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \, \Phi \, dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} \, dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS$$ KE 219

If Φ is harmonic within V, so that $\nabla^2 \, \Phi = 0$, then the previous result becomes

5. $$\Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} \, dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS$$

In the case of two dimensions, result (4) takes the form

6. $$\Phi(p) = \frac{1}{2\pi} \int_S \nabla^2 \, \Phi(q) \ln |p - q| \, dS + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| \, dq - \frac{1}{2\pi} \int \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) \, dq$$ MM 116

where C is the boundary of the planar region S, and result (5) takes the form

7. $$\Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| \, dq - \frac{1}{2\pi} \int_C \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) \, dq$$ VL 280

10.718 *Green's representation theorem in R^n.* If Φ is twice differentiable within a region Ω in R^n bounded by the surface Σ with outward drawn unit normal $\mathbf{n}$, then for $p \notin \Sigma$ and $n > 3$

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_\Omega \frac{\nabla^2 \, \Phi(q)}{|p - q|^{n-2}} \, d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_\Sigma \left(\frac{1}{|p - q|^{n-2}} \frac{\partial \, \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p - q|^{n-2}} \right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$ VL 279

is the area of the unit sphere in R^n.

10.719 *Green's theorem of the arithmetic mean.* If Φ is harmonic in a sphere, then the value of Φ at the center of the sphere is the arithmetic mean of its value on the surface. KE 223

10.720 *Poisson's integral in three dimensions.* If Φ is harmonic in the interior of a spherical volume V of radius R and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates (r, θ, ϕ), it satisfies the boundary condition $\Phi\,(R, \theta, \phi) = f(\theta, \phi)$, then

$$\Phi(r,\theta,\phi) = \frac{R\left(R^2-r^2\right)}{4\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{f\left(\theta',\phi'\right)\sin\theta'\,d\theta'\,d\phi'}{\left(r^2+R^2-2rR\cos\gamma\right)^{3/2}},$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi-\phi'\right). \qquad \text{KE 241}$$

10.721 *Poisson's integral in two dimensions.* If Φ is harmonic in the interior of a circular disk S of radius R and is continuous on the boundary of the disk on which, in terms of the polar coordinates (r,θ), it satisfies the boundary condition $\Phi(R,\theta) = f(\theta)$, then

$$\Phi(r,\theta) = \frac{\left(R^2-r^2\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,d\phi}{r^2+R^2-2rR\cos(\theta-\phi)}.$$

10.722 *Stokes' theorem.* Let a simple closed curve C be spanned by a surface S. Define the positive normal $\mathbf{n}$ to S, and the positive sense of description of the curve C with line element $d\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $\mathbf{S} = \mathbf{n}\,dS$,

$$\oint_C \mathbf{f}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{f}\cdot d\mathbf{S}, \qquad \text{MM 143}$$

where the line integral around C is taken in the positive sense.

10.723 *Planar case of Stokes' theorem.* If a region R in the (x,y)-plane is bounded by a simple closed curve C, and $f_1(x,y), f_2(x,y)$ are any two functions having continuous first derivatives in R and on C, then

$$\oint_C \left(f_1\,dx + f_2\,dy\right) = \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy, \qquad \text{MM 143}$$

where the line integral is taken in the counterclockwise sense.

10.81 Integral rate of change theorems

10.811 *Rate of change of volume integral bounded by a moving closed surface.* Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt}\int_{V(t)} f\,dV = \int_{V(t)}\frac{\partial f}{\partial t}\,dV + \int_{S(t)} f\mathbf{v}\cdot d\mathbf{S},$$

where $d\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\,.$$

By virtue of Gauss's theorem, this also takes the form

$$\frac{D}{Dt}\int_{V(t)} f\,dV = \int_{V(t)}\left(\frac{Df}{Dt} + f\operatorname{div}\mathbf{v}\right) dV. \qquad \text{MV 88}$$

10.812 *Rate of change of flux through a surface.* Let $\mathbf{q}$ be a vector function that may also depend on the time t, and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \int_S \mathbf{q} \cdot \mathbf{n} \, dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{q} + \operatorname{curl} (\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n} \, dS. \qquad \text{MV 90}$$

10.813 *Rate of change of the circulation around a given moving curve.* Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \int_C \mathbf{q} \cdot \, d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left(\frac{\partial \mathbf{q}}{\partial t} + (\operatorname{curl} \mathbf{q}) \times \mathbf{v} \right) \cdot \, d\mathbf{r}. \qquad \text{MV 94}$$

11 Algebraic Inequalities

11.1–11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.111 *Lagrange's identity.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum (a_k b_j - a_j b_k)^2$$ BB 3

11.112 *Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of real numbers; then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional.

MT 30

11.113 *Minkowski's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers, and let $p > 1$; then

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.$$

The equality holds if, and only if, the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional.

MT 55

11.114 *Hölder's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two sets of nonnegative real numbers, and let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$; then

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \ge \sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, the sequences $a_1^p, a_2^p, \ldots, a_n^p$ and $b_1^q, b_2^q, \ldots, b_n^q$ are proportional.

MT 50

11.115 *Chebyshev's inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers such that either $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, or $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$; then

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \cdots + b_n}{n}\right) \le \frac{1}{n}\sum_{k=1}^{n} a_k b_k.$$

The equality holds if, and only if, either $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

11.116 *Arithmetic-geometric inequality.* Let $a_1, a_2, \ldots, a_n$ be any set of positive numbers, with arithmetic mean

$$A_n = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

and geometric mean

$$G_n = (a_1 a_2 \ldots a_n)^{1/n};$$

then $A_n \geq G_n$ or, equivalently,

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq (a_1 a_2 \ldots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers a_i are equal. BB 4

11.117 *Carleman's inequality.* If $a_1, a_2, \ldots, a_n$ is any finite set of non-negative numbers, then

$$\sum_{r=1}^{n} (a_1 a_2 \ldots a_r)^{1/r} \leq e\,(a_1 + a_2 + \cdots + a_n),$$

where e is the best possible constant in this inequality. The inequality is strict except for the trivial case when $a_r = 0$ for $r = 1, 2, \ldots, n$. MT 131

11.118 *An inequality involving absolute values.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^{n} \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \qquad 0 < p \leq 2.$$

11.21 Algebraic inequalities involving complex numbers

If α, β are any two real numbers, the complex number $z = \alpha + i\beta$ with real part α and imaginary part β has for its modulus $|z|$ the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude)$\arg z$ the angle $\arg z = \theta$ such that

$$\cos\theta = \frac{\alpha}{|z|} \text{ and } \sin\theta = \frac{\beta}{|z|},$$

where $-\pi < \theta \leq \pi$. The complex number $\overline{z} = \alpha - i\beta$ is said to be the **complex conjugate** of $z = \alpha + i\beta$.

$$\text{If } z = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

then

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta),$$

and, setting $r = 1$, we have **de Moivre's theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It follows directly that, if $z = e^{i\theta}$, then

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\alpha = -\frac{i}{2}\left(z - \frac{1}{z}\right),$$

and

$$\cos r\theta = \frac{1}{2}\left(z^r + \frac{1}{z^r}\right), \qquad \sin r\theta = -\frac{i}{2}\left(z^r - \frac{1}{z^r}\right).$$

If $w = z^{p/q}$ with p, q integral, and $z = re^{i\theta}$, then the q roots of $w_0, w_1, \ldots, w_{q-1}$ of z are

$$w_k = r^{p/q}\left[\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right)\right],$$

with $k = 0, 1, 2, \ldots, q-1$.

11.211[7] *Simple properties and inequalities involving the modulus and the complex conjugate.* If the real part of z is denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$, then

$$\begin{aligned}
z + \overline{z} &= 2\operatorname{Re} z = 2\alpha,\\
z - \overline{z} &= 2\operatorname{Im} z = 2i\beta,\\
z &= \overline{(\overline{z})},\\
\frac{1}{\overline{z}} &= \overline{\left(\frac{1}{z}\right)},\\
\overline{(z^n)} &= (\overline{z})^n,\\
\left|\frac{\overline{z_1}}{\overline{z_2}}\right| &= \frac{|\overline{z_1}|}{|\overline{z_2}|},\\
\overline{(z_1 + z_2 + \cdots + z_n)} &= \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n},\\
\overline{z_1 z_2 \cdots z_n} &= \overline{z_1}\,\overline{z_2}\cdots\overline{z_n}.
\end{aligned}$$

11.212 *Inequalities for pairs of complex numbers.* If a,b are any two complex numbers, then

(i) $|a+b| \le |a| + |b|$ (triangle inequality),

(ii) $|a-b| \ge ||a| - |b||$.

11.31 Inequalities for sets of complex numbers

11.311 *Complex Cauchy–Schwarz–Buniakowsky inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of complex numbers; then

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \le \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right).$$

The equality holds if, and only if, the sequences $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}$ and $b_1, b_2, \ldots, b_n$ are proportional.

MT 42

11.312 *Complex Minkowski inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of complex numbers, and let the real number p be such that $p > 1$; then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}.$$ MT 56

11.313 *Complex Hölder inequality.* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any two arbitrary sets of complex numbers, and let the real numbers p,q be such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p}\left(\sum_{k=1}^{n} |b_k|^q\right)^{1/p} \ge \left|\sum_{k=1}^{n} a_k b_k\right|.$$

The equality holds if, and only if, the sequences

$$|a_1|^p, \quad |a_2|^p, \ldots, |a_n|^p \text{ and } |b_1|^p, |b_2|^p, \ldots |b_n|^p,$$

are proportional and $\arg a_k b_k$ is independent of k for $k = 1, 2, \ldots, n$. MT 53

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12 Integral Inequalities

12.11 Mean Value Theorems

12.111 First mean value theorem

Let $f(x)$ and $g(x)$ be two bounded functions integrable in $[a, b]$, and let $g(x)$ be of one sign in this interval. Then

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx,$$ CA 105

with $a \le \xi \le b$.

12.112 Second mean value theorem

(i) Let $f(x)$ be a bounded, monotonic decreasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx,$$

with $a \le \xi \le b$.

(ii) Let $f(x)$ be a bounded, monotonic increasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function. Then,

$$\int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx,$$

with $a \le \eta \le b$.

(iii) Let $f(x)$ be bounded and monotonic in $[a, b]$, and let $g(x)$ be a bounded integrable function which experiences only a finite number of sign changes in $[a, b]$. Then,

$$\int_a^b f(x)g(x)\,dx = f(a+0)\int_a^\xi g(x)\,dx + f(b-0)\int_\xi^b g(x)\,dx,$$ CA 107

with $a \le \xi \le b$.

12.113 First mean value theorem for infinite integrals

Let $f(x)$ be bounded for $x \ge a$, and integrable in the arbitrary interval $[a, b]$, and let $g(x)$ be of one sign in $x \ge a$ and such that $\int_a^\infty g(x)\,dx$ is finite. Then,

$$\int_a^\infty f(x)g(x)\,dx = \mu\int_a^\infty g(x)\,dx, \qquad \text{CA 123}$$

where $m \le \mu \le M$ and m, M are, respectively, the lower and upper bounds of $f(x)$ for $x \ge a$.

12.114 Second mean value theorem for infinite integrals

Let $f(x)$ be bounded and monotonic when $x \ge a$, and $g(x)$ be bounded and integrable in the arbitrary interval $[a, b]$ in which it experiences only a finite number of changes of sign. Then, provided $\int_a^\infty g(x)\,dx$ is finite,

$$\int_a^\infty f(x)g(x)\,dx = f(a+0)\int_a^\xi g(x)\,dx + f(\infty)\int_\xi^\infty g(x)\,dx, \qquad \text{CA 123}$$

with $a \le \xi \le \infty$.

12.21 Differentiation of Definite Integral Containing a Parameter

12.211 Differentiation when limits are finite

Let $\phi(\alpha)$ and $\psi(\alpha)$ be twice differentiable functions in some interval $c \le \alpha \le d$, and let $f(x, \alpha)$ be both integrable with respect to x over the interval $\phi(\alpha) \le x \le \psi(\alpha)$ and differentiable with respect to α. Then,

$$\frac{d}{d\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx = \left(\frac{d\psi}{d\alpha}\right) f\left(\psi(\alpha),\alpha\right) - \left(\frac{d\phi}{d\alpha}\right) f\left(\phi(\alpha),\alpha\right) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial\alpha}\,dx. \qquad \text{FI II 680}$$

12.212 Differentiation when a limit is infinite

Let $f(x, \alpha)$ and $\partial f/\partial\alpha$ both be integrable with respect to x over the semi-infinite region $x \ge a$, $b \le \alpha < c$. Then, if the integral

$$f(\alpha) = \int_a^\infty f(x,\alpha)\,dx$$

exists for all $b \le \alpha \le c$, and if $\int_a^\infty \frac{\partial f}{\partial\alpha}\,dx$ is uniformly convergent for α in $[b, c]$, it follows that

$$\frac{d}{d\alpha}\int_a^\infty f(x,\alpha)\,dx = \int_a^\infty \frac{\partial f}{\partial\alpha}\,dx$$

12.31 Integral Inequalities

12.311 Cauchy-Schwarz-Buniakowsky inequality for integrals

Let $f(x)$ and $g(x)$ be any two real integrable functions on $[a, b]$. Then,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right),$$

and the equality will hold if, and only if, $f(x) = kg(x)$, with k real. BB 21

12.312 Hölder's inequality for integrals

Let $f(x)$ and$g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^q$ are integrable on $[a, b]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\int_a^b f(x)g(x)\,dx \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \left(\int_a^b |g(x)|^q\,dx\right)^{1/q}.$$

The equality holds if, and only if, $\alpha|f(x)|^p = \beta|g(x)|^q$, where α and β are positive constants. BB 21

12.313 Minkowski's inequality for integrals

Let $f(x)$ and $g(x)$be any two real functions for which $|f(x)|^p$ and $|g(x)|^p$ are integrable on $[a,b]$ for $p>0$; then

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

The equality holds if, and only if, $f(x) = kg(x)$ for some real $k \ge 0$. BB 21

12.314 Chebyshev's inequality for integrals

Let $f_1, f_2, \ldots, f_n$ be nonnegative integrable functions on $[a,b]$ which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x)\,dx \int_a^b f_2(x)\,dx \ldots \int_a^b f_n(x)\,dx \le (b-a)^{n-1} \int_a^b f_1(x) f_2(x) \ldots f_n(x)\,dx \qquad \text{MT 39}$$

12.315 Young's inequality for integrals

Let $f(x)$ be a real-valued continuous strictly monotonic increasing function on the interval $[0,a]$, with $f(0) = 0$ and $b \le f(a)$. Then

$$ab \le \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

where $f^{-1}(y)$ denotes the function inverse to $f(x)$. The equality holds if, and only if, $b = f(a)$. BB 15

12.316 Steffensen's inequality for integrals

Let $f(x)$ be nonnegative and monotonic decreasing in $[a,b]$, and $g(x)$ be such that $0 \le g(x) \le 1$ in $[a,b]$. Then

$$\int_{b-k}^b f(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^{a+k} f(x)\,dx,$$

where $k = \int_a^b g(x)\,dx$. MT 107

12.317 Gram's inequality for integrals

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be real square integrable functions on $[a,b]$; then

$$\begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x)f_2(x)\,dx & \cdots & \int_a^b f_1(x)f_n(x)\,dx \\ \int_a^b f_2(x)f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x)f_n(x)\,dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x)f_1(x)\,dx & \int_a^b f_n(x)f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix} \ge 0. \qquad \text{MT 47}$$

12.318 Ostrowski's inequality for integrals

Let $f(x)$ be a monotonic function integrable on $[a,b]$, and let $f(a)f(b) \geq 0, |f(a)| \geq |f(b)|$. Then, if g is a real function integrable on $[a,b]$,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq |f(a)| \max_{a \leq \xi \leq b} \left|\int_a^\xi g(x)\,dx\right|.$$

12.41 Convexity and Jensen's Inequality

A function $f(x)$ is said to be **convex** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}.$$

A function $f(x)$ is said to be **concave** on an interval $[a,b]$ if for any two points x_1, x_2 in $[a,b]$ the function $-f(x)$ is convex in that interval.

If the function $f(x)$ possesses a second derivative in the interval $[a,b]$, then a necessary and sufficient condition for it to be convex on that interval is that $f''(x) \geq 0$ for all x in $[a,b]$.

A function $f(x)$ is said to be **logarithmically convex** on the interval $[a,b]$ if $f > 0$ and $\log f(x)$ is concave on $[a,b]$.

If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a,b]$, then the functions $f(x)+g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a,b]$. MT 17

12.411 Jensen's inequality

Let $f(x), p(x)$ be two functions defined for $a \leq x \leq b$ such that $\alpha \leq f(x) \leq \beta$ and $p(x) \geq 0$, with $p(x) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then

$$\phi\left(\frac{\int_a^b f(x)p(x)\,dx}{\int_a^b p(x)\,dx}\right) \leq \frac{\int_a^b \phi(f)\,p(x)\,dx}{\int_a^b p(x)\,dx}.$$ HL 151

12.412 Carleman's inequality for integrals

If $f(x) \geq 0$ and the integrals exist, then

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\,dx \leq e\int_0^\infty f(x)\,dx.$$

12.51 Fourier Series and Related Inequalities

The trigonometric **Fourier series** representation of the function $f(x)$ integrable on $[-\pi,\pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients** a_n and b_n of $f(x)$ are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^\pi f(x)\cos nx\,dx, \qquad b_n = \frac{1}{2\pi}\int_{-\pi}^\pi f(x)\sin nx\,dx.$$

(See **0.320–0.328** for convergence of Fourier series on $(-l,l)$.) TF 1

12.511 Riemann-Lebesgue lemma

If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\sin tx\, dx \to 0$$

and

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\cos tx\, dx \to 0.$$ TF 11

12.512 Dirichlet lemma

$$\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}\, dx = \frac{\pi}{2},$$

in which $\sin\left(n+\frac{1}{2}\right)x \,/\, 2\sin\frac{1}{2}x$ is called the **Dirichlet kernel**. ZY 21

12.513 Parseval's theorem for trigonometric Fourier series

If $f(x)$ is square integrable on $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty}\left(a_r^2 + b_r^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\, dx.$$ Y 10

12.514 Integral representation of the n^{th} partial sum

If $f(x)$ is integrable on $[-\pi, \pi]$, then the n^{th} partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^{n}\left(a_r\cos rx + b_r\sin rx\right)$$

has the following integral representation in terms of the Dirichlet kernel:

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\, dt.$$ Y 20

12.515 Generalized Fourier series

Let the set of functions $\{\phi_n\}_{n=0}^{\infty}$ form an **orthonormal set** over $[a, b]$, so that

$$\int_a^b \phi_m(x)\phi_n(x)\, dx = \begin{cases} 1 & \text{for} \quad m = n, \\ 0 & \text{for} \quad m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function $f(x)$ on $[a, b]$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n\phi_n(x),$$

where the generalized Fourier coefficients of $f(x)$ are given by

$$c_n = \int_a^b f(x)\phi_n(x)\, dx.$$

12.516 Bessel's inequality for generalized Fourier series

For any square integrable function defined on $[a, b]$,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x)\,dx,$$

where the c_n are the generalized Fourier coefficients of $f(x)$.

12.517 Parseval's theorem for generalized Fourier series

If $f(x)$ is a square integrable function defined on $[a, b]$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ is a **complete orthonormal** set of continuous functions defined on $[a, b]$, then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x)\,dx,$$

where the c_n are generalized Fourier coefficients of $f(x)$.

13 Matrices and Related Results

13.11–13.12 Special Matrices

13.111 Diagonal matrix

A square matrix **A** of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

13.112 Identity matrix and null matrix

The **identity matrix** is a diagonal matrix **I** in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

13.113 Reducible and irreducible matrices

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is said to be **reducible**, if the indices $1, 2, \dots, n$ can be divided into two disjoint non-empty sets $i_1, i_2, \dots, i_\mu; j_1, j_2, \dots, j_\nu$ with $(\mu + \nu = n)$, such that

$$a_{i_\alpha j_\beta} = 0 \qquad (\alpha = 1, 2, \dots, \mu; \quad \beta = 1, 2, \dots, \nu).$$

Otherwise, **A** will be said to be irreducible. GA 61

13.114 Equivalent matrices

An $m \times n$ matrix **A** is **equivalent** to an $m \times n$ matrix **B** if, and only if, $\mathbf{B} = \mathbf{PAQ}$ for suitable non-singular $m \times m$ and $n \times n$ matrices **P** and **Q**, respectively.

13.115 Transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix with element a_{ij} in the i^{th} row and the j^{th} column, then the transpose $\mathbf{A}^{\mathrm{T}}$ of **A** is the $n \times m$ matrix

$$\mathbf{A}^{\mathrm{T}} = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from **A** by interchanging rows and columns.

13.116 Adjoint matrix

If $\mathbf{A}$ is an $n \times n$ matrix, then its **adjoint**, denoted by $\operatorname{adj}\mathbf{A}$, is the transpose of the matrix of cofactors A_{ij} of $\mathbf{A}$, so that

$$\operatorname{adj}\mathbf{A} = [A_{ij}]^{\mathrm{T}} \qquad \text{(see } \mathbf{14.13}\text{)}.$$

13.117 Inverse matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with a nonsingular determinant $|\mathbf{A}|$, then its **inverse** $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}\mathbf{A}}{|\mathbf{A}|}.$$

13.118 Trace of a matrix

The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, written $\operatorname{tr}\mathbf{A}$, is defined to be the sum of the terms on the leading diagonal, so that

$$\operatorname{tr}\mathbf{A} = a_{11} + a_{22} + \ldots + a_{nn}.$$

13.119 Symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **symmetric** if $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, n$.

13.120 Skew-symmetric matrix

The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for $i, j = 1, 2, \ldots, n$.

13.121 Triangular matrices

An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is of **upper triangular type** if $a_{ij} = 0$ for $i > j$ and of **lower triangular type** if $a_{ij} = 0$ for $j > i$.

13.122 Orthogonal matrices

A real $n \times n$ matrix $\mathbf{A}$ is **orthogonal** if, and only if, $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$.

13.123 Hermitian transpose of a matrix

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with complex elements, then its **hermitian transpose** $\mathbf{A}^{\mathrm{H}}$ is defined to be

$$\mathbf{A}^{\mathrm{H}} = [\overline{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

13.124 Hermitian matrix

An $n \times n$ matrix $\mathbf{A}$ is **hermitian** if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$, or equivalently, if $\mathbf{A} = \overline{\mathbf{A}}^{\mathrm{T}}$, with the bar denoting the complex conjugate operation.

13.125 Unitary matrix

An $n \times n$ matrix $\mathbf{A}$ is **unitary** if $\mathbf{A}\mathbf{A}^{\mathrm{H}} = \mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$.

13.126 Eigenvalues and eigenvectors

If $\mathbf{A}$ is an $n \times n$ matrix, each eigenvector $\mathbf{x}$ corresponding to λ satisfies the equation

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{x},$$

while the **eigenvalues** λ satisfy the **characteristic equation**

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \qquad \text{(see } \mathbf{15.61}\text{)}.$$

13.127 Nilpotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **nilpotent** if $\mathbf{A}^k = \mathbf{0}$ for some k.

13.128 Idempotent matrix

An $n \times n$ matrix $\mathbf{A}$ is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

13.129 Positive definite

An $n \times n$ matrix $\mathbf{A}$ is **positive definite** if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.130 Non-negative definite

An $n \times n$ matrix $\mathbf{A}$ is **non-negative definite** if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.131 Diagonally dominant

An $n \times n$ matrix $\mathbf{A}$ is **diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i.

13.21 Quadratic Forms

A **quadratic form** involving the n real variables $x_1, x_2, \ldots, x_n$ that are associated with the real $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar expression

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

In terms of matrix notation, if $\mathbf{x}$ is the $n \times 1$ column vector with real elements $x_1, x_2, \ldots, x_n$, and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, then

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x}, \mathbf{A}\mathbf{x}).$$

If the $n \times n$ matrix $\mathbf{A}$ is hermitian, so that $\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{A}$, where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix $\mathbf{A}$ and the vector $\mathbf{x}$, which may have complex elements, is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})\,.$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})\,,$$

where $\mathbf{A} = [a_{ij}]$ is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \qquad \text{for } i = 1, 2, \ldots, n$$

and

$$a_{ij} = \tfrac{1}{2}\left(\alpha_{ij} + \alpha_{ji}\right) \qquad \text{for } i, j = 1, 2, \ldots, n \quad \text{and} \quad i \neq j.$$

13.211 Sylvester's law of inertia

When a quadratic form Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \ldots - y_r^2,$$

the number p of positive squares appearing in the reduction is an invariant of the quadratic form Q, and it does not depend on the method of reduction itself. ML 377

13.212 Rank

The **rank** of the quadratic form Q in the above canonical form is the total number r of squared terms (both positive and negative) appearing in its reduced form. ML 360

13.213 Signature

The **signature** of the quadratic form Q above is the number s of positive squared terms appearing in its reduced form. It is sometimes also defined to be $2s - r$. ML 378

13.214 Positive definite and semidefinite quadratic form

The quadratic form $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is said to be **positive definite** when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. It is said to be **positive semidefinite** if $Q(x) \geq 0$ for $x \neq 0$. ML 394

13.215 Basic theorems on quadratic forms

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.
2. A real quadratic form in n variables is positive definite if, and only if, its canonical form is

$$Q = z_1^2 + z_2^2. + \ldots + z_n^2.$$

3. A real symmetric matrix $\mathbf{A}$ is positive definite if, and only if, there exists a real nonsingular matrix $\mathbf{M}$ such that $\mathbf{A} = \mathbf{M}\mathbf{M}^{\mathrm{T}}$.
4. Any real quadratic form in n variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \ldots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form $Q = (\mathbf{x}, \mathbf{Ax})$ is positive definite if, and only if, every eigenvalue of $\mathbf{A}$ is positive; it is positive semidefinite if, and only if, all the eigenvalues of $\mathbf{A}$ are nonnegative, and it is indefinite if the eigenvalues of $\mathbf{A}$ are of both signs.

6. The necessary conditions for an hermitian matrix $\mathbf{A}$ to be positive definite are

   (i) $a_{ii} > 0$ for all i,
   (ii) $a_{ii}a_{ij} > |a_{ij}|^2$ for $i \neq j$,
   (iii) the element of largest modulus must lie on the leading diagonal,
   (iv) $|\mathbf{A}| > 0$.

7. The quadratic form $Q = (\mathbf{x}, \mathbf{Ax})$ with $\mathbf{A}$ hermitian will be positive definite if all the principal minors in the top left-hand corner of $\mathbf{A}$ are positive, so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots .$$ ML 353-379

13.31 Differentiation of Matrices

If the $n \times m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have elements that are differentiable functions of t, so that

$$\mathbf{A}(t) = [a_{ij}(t)], \qquad \mathbf{B}(t) = [b_{ij}(t)]$$

then

1. $$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{d}{dt}a_{ij}(t)\right]$$

2. $$\frac{d}{dt}\left[\mathbf{A}(t) \pm \mathbf{B}(t)\right] = \left[\frac{d}{dt}a_{ij}(t) \pm \frac{d}{dt}b_{ij}(t)\right] = \frac{d}{dt}\mathbf{A}(t) \pm \frac{d}{dt}\mathbf{B}(t).$$

3. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt}\left[\mathbf{A}(t)\mathbf{B}(t)\right] = \left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{B}(t) + \mathbf{A}(t)\left(\frac{d}{dt}\mathbf{B}(t)\right).$$

4. If the matrix product $\mathbf{A}(t)\mathbf{B}(t)$ is defined, then

$$\frac{d}{dt}\left[\mathbf{A}(t)\mathbf{B}(t)\right]^{\mathrm{T}} = \left(\frac{d}{dt}\mathbf{B}(t)\right)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(t) + \mathbf{B}^{\mathrm{T}}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)^{\mathrm{T}}.$$

5. If the square matrix $\mathbf{A}$ is nonsingular, so that $|\mathbf{A}| \neq 0$, then

$$\frac{d}{dt}\left[\mathbf{A}^{-1}\right] = -\mathbf{A}^{-1}(t)\left(\frac{d}{dt}\mathbf{A}(t)\right)\mathbf{A}^{-1}(t)$$

6. $$\int_{t_0}^{T}\mathbf{A}(\tau)\,d\tau = \left[\int_{t_0}^{T}a_{ij}(\tau)\,d\tau\right]$$

13.41 The Matrix Exponential

If $\mathbf{A}$ is a square matrix, and z is any complex number, then the matrix exponential e^{Az} is defined to be

$$e^{Az} = \mathbf{I} + \mathbf{A}z + \ldots + \frac{\mathbf{A}^n z^n}{n!} + \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r z^r.$$

3.411 Basic properties

1. $e^0 = \mathbf{I}, \quad e^{Iz} = \mathbf{I}e^z, \quad e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2},$ [when $\mathbf{A}+\mathbf{B}$ is defined and $\mathbf{AB} = \mathbf{BA}$]
   $e^{-\mathbf{A}z} = \left(e^{\mathbf{A}z}\right)^{-1}, \quad e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$
2. $\dfrac{d^r}{dz^r}\left(e^{\mathbf{A}z}\right) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z}\mathbf{A}^r.$

ML 340

3. If the square matrix $\mathbf{A}$ can be expressed in the form $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$, with $\mathbf{B}$ and $\mathbf{C}$ square matrices, then

$$e^{\mathbf{A}z} = \begin{bmatrix} e^{\mathbf{B}z} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{C}z} \end{bmatrix}.$$

14 Determinants

14.11 Expansion of Second- and Third-Order Determinants

1. $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

2. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

14.12 Basic Properties

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices. Then the following results are true:

1. If any two adjacent rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.
2. If any two rows (or columns) of a determinant are identical, the determinant is zero.
3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).
4. $|k\mathbf{A}| = k^n|\mathbf{A}|$ for any scalar k.
5. $|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$ where $\mathbf{A}^{\mathrm{T}}$ is the transpose of $\mathbf{A}$.
6. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.
7. $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$ when the inverse exists.
8. If the elements a_{ij} of $\mathbf{A}$ are functions of x, then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,j=1}^{n} \frac{da_{ij}}{dx} A_{ij} \qquad \text{(see } \mathbf{14.13}\text{)}.$$

14.13 Minors and Cofactors of a Determinant

The **minor** M_{ij} of the element a_{ij} in the n^{th}-order determinant $|\mathbf{A}|$ associated with the square $n \times n$ matrix $\mathbf{A}$ is the $(n-1)^{\text{th}}$-order determinant derived from $\mathbf{A}$ by deletion of the i^{th} row and j^{th} column. The cofactor A_{ij} of the element a_{ij} is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}.$$ ML 20

14.14 Principal Minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of $\mathbf{A}$. ML 197

14.15* Laplace Expansion of a Determinant

The n^{th}-order determinant denoted by $|\mathbf{A}|$, or det $\mathbf{A}$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ may be expanded either by elements of the i^{th} row as

$$|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

or by elements of the j^{th} column as

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{ij} A_{ij},$$

where A_{ij} is the cofactor of element a_{ij}. The cofactors A_{ij} satisfy the following n linear equations:

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \qquad \sum_{i=1}^{n} a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$

ML 21

$$\text{for } i, j, k = 1, 2, \ldots, n \text{ and } \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

14.16 Jacobi's Theorem

Let M_r be an r-rowed minor of the n^{th}-order determinant $|\mathbf{A}|$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, in which the rows $i_1, i_2, \ldots, i_r$ are represented together with the columns $k_1, k_2, \ldots, k_r$.

Define the **complementary minor** to M_r to be the $(n-k)$-rowed minor obtained from $|\mathbf{A}|$ by deleting all the rows and columns associated with M_r, and the **signed complementary minor** $M^{(r)}$ to M_r to be

$$M^{(r)} = (-1)^{i_1+i_2+\cdots+i_r+k_1+k_2+\cdots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if Δ is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and M_r and M_r' are corresponding r-rowed minors of $|\mathbf{A}|$ and Δ, it follows that

$$M_r' = |\mathbf{A}|^{r-1} M^{(r)}.$$ ML 25

Corollary. If $|\mathbf{A}| = 0$, then

$$A_{pk} A_{nq} = A_{nk} A_{pq}.$$

14.17 Hadamard's Theorem

If $|\mathbf{A}|$ is an $n \times n$ determinant with elements a_{ij} that may be complex, then $|\mathbf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

14.18 Hadamard's Inequality

Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $n \times n$ nonsingular matrix with real elements and determinant $|\mathbf{A}|$. Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}^2 \right).$$

This result is also true when $\mathbf{A}$ is hermitian. ML 418

Deductions.

1. If $M = \max |a_{ij}|$, then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$ ML 419

2. If the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is positive definite, then

$$|\mathbf{A}| \leq a_{11} a_{22} \ldots a_{nn}.$$ BL 126

3. If the real $n \times n$ matrix $\mathbf{A}$ is diagonally dominant, so that $\sum_{j\neq 1}^{n} |a_{ij}| < |a_{ii}|$ for $i = 1, 2, \ldots, n$, then $|\mathbf{A}| \neq 0$.

14.21 Cramer's Rule

If the n linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n, \end{array}$$

have a nonsingular coefficient matrix $\mathbf{A} = [a_{ij}]$, so that $|\mathbf{A}| \neq 0$, then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_j}{|\mathbf{A}|}$$

for $j = 1, 2, \ldots, n$, where A_{ij} is the cofactor of element a_{ij} in the coefficient matrix $\mathbf{A}$. ML 134

14.31 Some Special Determinants

14.311 Vandermonde's determinant (alternant)

Third order.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the n^{th}-order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}n(n-1)$ pairs of numbers taken from $x_1, x_2, \ldots, x_n$, with the order of the differences taken in the reverse order of the suffixes that are involved. ML 17

14.312 Circulants

Second order.

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

Third order.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)\left(x_1 + \omega x_2 + \omega^2 x_3\right)\left(x_1 + \omega^2 x_2 + \omega x_3\right),$$

where ω and ω^2 are the complex cube roots of 1. In general, the n^{th}-order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1}^{n} \left(x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1}\right),$$

where ω_j is an n^{th} root of 1. The eigenvalues λ (see **15.61**) of an $n \times n$ circulant matrix are

$$\lambda_j = x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_n\omega_j^{n-1},$$

where ω_j is again an n^{th} root of 1. ML 36

14.313 Jacobian determinant

If $f_1, f_2, \ldots, f_n$ are n real-valued functions which are differentiable with respect to $x_1, x_2, \ldots, x_n$, then the Jacobian $J_f(x)$ of the f_i with respect to the x_j is the determinant

$$J_f(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

The notation

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

is also used to denote the Jacobian $J_f(x)$.

14.314 Hessian determinants

The Jacobian of the derivatives $\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \ldots, \frac{\partial\phi}{\partial x_n}$ of a function $\phi(x_1, x_2, \ldots, x_n)$ with respect to x_1, x_2, $\ldots$, x_n is called the Hessian H of ϕ, so that

$$H = \begin{vmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\,\partial x_2} & \frac{\partial^2\phi}{\partial x_1\,\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_1\,\partial x_n} \\ \frac{\partial^2\phi}{\partial x_2\,\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \frac{\partial^2\phi}{\partial x_2\,\partial x_3} & \cdots & \frac{\partial^2 d2\phi}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial x_n\,\partial x_1} & \frac{\partial^2\phi}{\partial x_n\,\partial x_2} & \frac{\partial^2\phi}{\partial x_n\,\partial x_3} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{vmatrix}.$$

14.315 Wronskian determinants

Let f_1, f_2, $\ldots$, f_n be n functions each n times differentiable with respect to x in some open interval (a, b). Then the Wronskian $W(x)$ of $f_1, f_2, \ldots, f_n$ is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where $f_i^{(r)} = \frac{d^r f_i}{dx^r}$.

14.316 Properties

1. $\frac{dW}{dx}$ follows from $W(x)$ by replacing the last row of the determinant defining $W(x)$ by the n^{th} derivatives $f_1^{(n)}$, $f_2^{(n)}, \ldots, f_n^{(n)}$.
2. If constants $k_1, k_2, \ldots, k_n$ exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$$

   for all x in (a, b), then $W(x) = 0$ for all x in (a, b).
3. The vanishing of the Wronskian throughout (a, b) is necessary, but not sufficient, for the linear dependence of $f_1, f_2, \ldots, f_n$.

14.317 Gram-Kowalewski theorem on linear dependence

A necessary and sufficient condition for n functions $f_1, f_2, \ldots, f_n$ square integrable over $a \le n \le b$ to be linearly dependent in this interval is the vanishing of the Gram determinant

$$G(f_1, f_2, \ldots, f_n) = \begin{vmatrix} \int_a^b f_1^2(x)\,dx & \int_a^b f_1(x) f_2(x)\,dx & \cdots & \int_a^b f_1(x) f_n(x)\,dx \\ \int_a^b f_2(x) f_1(x)\,dx & \int_a^b f_2^2(x)\,dx & \cdots & \int_a^b f_2(x) f_n(x)\,dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b f_n(x) f_1(x)\,dx & \int_a^b f_n(x) f_2(x)\,dx & \cdots & \int_a^b f_n^2(x)\,dx \end{vmatrix}.$$

SA 2 (Theorem 3)

14.318 If the n functions $f_1, f_2, \ldots, f_n$ are square integrable over $a \le n \le b$, then the Gram determinant

$$G(f_1, f_2, \ldots, f_n) \ge 0,$$

and the equality sign holds only when the functions are linearly dependent in $a \le n \le b$.

SA 4 (Corollary 1)

14.319 The rank of the matrix corresponding to the Gram determinant $G(f_1, f_2, \ldots, f_n)$ gives the maximum number of linearly independent functions $f_1,\ f_2,\ \ldots\ ,\ f_n$ in $a \le x \le b$. If the rank is r, then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

SA 3 (Theorem 4)

15 Norms

15.1–15.9 Vector Norms

15.11 General Properties

The **vector norm** $||\mathbf{x}||$ of an $n \times 1$ column vector $\mathbf{x}$ is a nonnegative number having the property that

1. $||\mathbf{x}|| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $||\mathbf{x}|| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;
2. $||k\mathbf{x}|| = |k|||\mathbf{x}||$ for any scalar k;
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

15.21 Principal Vector Norms

15.211 The norm $||\mathbf{x}||_1$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_1 = \sum_{r=1}^{n} |x_r|.$$ VA 15

15.212 The norm $||\mathbf{x}||_2$ (Euclidean or L_2 norm)

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_2 = \left(\sum_{r=1}^{n} |x_r|^2\right)^{1/2}.$$ VA 8

15.213 The norm $||\mathbf{x}||_\infty$

If $\mathbf{x}$ is a vector with complex components $x_1, x_2, \ldots, x_n$, then

$$||\mathbf{x}||_\infty = \max_i |x_i|.$$ VA 15

15.31 Matrix Norms

15.311 General properties

The **matrix norm** $||\mathbf{A}||$ of a square matrix $\mathbf{A}$ is a nonnegative number associated with $\mathbf{A}$ having the properties that

1. $||\mathbf{A}|| > 0$ when $\mathbf{A} \neq \mathbf{0}$ and $||\mathbf{A}|| = 0$ if, and only if, $\mathbf{A} = \mathbf{0}$;
2. $||k\mathbf{A}|| = |k|\,||\mathbf{A}||$ for any scalar k;
3. $||\mathbf{A}+\mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$;
4. $||\mathbf{AB}|| \leq ||\mathbf{A}||\,||\mathbf{B}||$. VA 9

The matrix norm $||\mathbf{A}||$ associated with $\mathbf{A} = [a_{ij}]$, and the vector norm $||\mathbf{x}||$ associated with the column vector $\mathbf{x}$ for which the matrix product $\mathbf{Ax}$ is defined, are said to be **compatible** if

$$||\mathbf{Ax}|| \leq ||\mathbf{A}||\,||\mathbf{x}||.$$

15.312 Induced norms

When a vector $\mathbf{z}$ with norm $||\mathbf{z}||$ exists such that the maximum is attained in the expression

$$||\mathbf{A}|| = \max_{||\mathbf{z}||=1} ||\mathbf{Az}||,$$

then $||\mathbf{A}||$ is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm $||\mathbf{z}||$. NO 428

15.313 Natural norm of unit matrix

If $\mathbf{I}$ is the unit matrix, then for any natural norm

$$||\mathbf{I}|| = 1.$$ NO 429

15.41 Principal Natural Norms

The natural matrix norms induced on matrix $\mathbf{A} = [a_{ij}]$ by the 1, 2, and ∞ vector norms are as follows:

15.411 Maximum absolute column sum norm

$$||\mathbf{A}||_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$ NO 429

15.412 Spectral norm

If $\mathbf{A}^{\mathrm{H}}$ denotes the Hermitian transpose of the square matrix $\mathbf{A} = [a_{ij}]$, so that $\mathbf{A}^{\mathrm{H}} = [\overline{a_{ji}}]$ with a bar denoting the complex conjugate operation, then

$$||\mathbf{A}||_2 = \sqrt{\text{maximum eigenvalue of } \mathbf{A}^{\mathrm{H}}\mathbf{A}},$$

or, equivalently,

$$||\mathbf{A}||_2 = \max_{||x||_2 \neq 0} \frac{||\mathbf{Ax}||_2}{||\mathbf{x}||_2}.$$ NO 429

15.413 Maximum absolute row sum norm

$$||\mathbf{A}||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$ NO 429

15.51 Spectral Radius of a Square Matrix

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with elements that may be complex, and with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the **spectral radius** $\rho(\mathbf{A})$ of $\mathbf{A}$ is the number

$$\rho(\mathbf{A}) = \max_{1 \le i \le n} |\lambda_i|.$$ VA 9

15.511 Inequalities concerning matrix norms and the spectral radius

1. $||\mathbf{A}||_2{}^2 \le ||\mathbf{A}||_1 ||\mathbf{A}||_\infty$. NO 431

2. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, and the $n \times n$ matrix $\mathbf{U}$ is unitary, so that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, with $^{\mathrm{H}}$ denoting the Hermitian transpose of $\mathbf{A}$ (see **13.123**), then

$$||\mathbf{AU}|| = ||\mathbf{UA}|| = ||\mathbf{A}||.$$ VA 15

3. If $\mathbf{A}$ is any nonsingular $n \times n$ matrix with elements that may be complex with eigenvalues λ_1, λ_2, λ_n, then

$$\frac{1}{||\mathbf{A}^{-1}||} \le |\lambda| \le ||\mathbf{A}||.$$ VA 16

4. For any square matrix $\mathbf{A}$ with spectral radius $\rho(\mathbf{A})$ and any natural norm $||\mathbf{A}||$,

$$\rho(\mathbf{A}) \le ||\mathbf{A}||.$$ NO 430

5. If the square matrix $\mathbf{A}$ is Hermitian, then

$$\rho(\mathbf{A}) = ||\mathbf{A}||.$$

6. If the square matrix $\mathbf{A}$ is Hermitian and $P_m(x)$ is any polynomial of degree m with real coefficients, then

$$||P_m(\mathbf{A})|| = \rho\left(P_m(\mathbf{A})\right).$$

7. If $\mathbf{A}$ is any arbitrary $n \times n$ matrix with elements that may be complex, then the sequence of matrices $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots$ converges to the null matrix as $n \to \infty$ if, and only if, $\rho(\mathbf{A}) < 1$.
NO 303

15.512 Deductions from Gerschgorin's theorem (see 15.814)

1. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex; then $\rho(\mathbf{A}) \le \min\left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|, \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|\right)$. VA 17

2. Let $\mathbf{A}$ be any arbitrary $n \times n$ matrix with elements that may be complex, and $x_1, x_2, \ldots, x_n$ be any set of n positive numbers; then $\rho(\mathbf{A}) \le \min\left(\max_{1\le i\le n}\left(\frac{\sum_{j=1}^{n}|a_{ij}|x_j}{x_i}\right), \max_{1\le j\le n}\left(x_j \sum_{i=1}^{n}\frac{|a_{ij}|}{x_i}\right)\right)$. VA 18

15.61 Inequalities Involving Eigenvalues of Matrices

The **eigenvalues** (**characteristic values** or **latent roots**) λ of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

When expanded, the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the **characteristic polynomial**, and it has the form

$$|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of $\mathbf{A}$. In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \qquad \text{(sum of all principal minors of } |\mathbf{A}| \text{ of order } r\text{)}.$$

It then follows that

$$b_{n-1} = (-1)^n\left(a_{11} + a_{22} + \cdots + a_{nn}\right),$$

$$b_{n-2} = (-1)^n \sum_{i<j}\left(a_{ii}a_{jj} - a_{ij}a_{ji}\right),$$

$$b_0 = |\mathbf{A}|.$$

Since the sum of the elements of the leading diagonal of $\mathbf{A}$ is called the **trace** of $\mathbf{A}$, written $\operatorname{tr}\mathbf{A}$, it follows that $b_{n-1} = (-1)^n \operatorname{tr}\mathbf{A}$. ML 198

15.611 Cayley-Hamilton theorem

Every square matrix $\mathbf{A}$ satisfies its characteristic equation, so that

$$(-1)^n\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + c_{n-2}\mathbf{A}^{n-2} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}.$$ ML 206

15.612 Corollaries

1. If $\mathbf{A}$ is nonsingular, then its adjoint, denoted by $\operatorname{adj}\mathbf{A}$, is

$$\operatorname{adj}\mathbf{A} = -\left[(-1)^n\mathbf{A}^{n-1} + c_{n-1}\mathbf{A}^{n-2} + c_{n-2}\mathbf{A}^{n-3} + \cdots + c_2\mathbf{A} + c_1\mathbf{I}\right].$$

2. If $\mathbf{A}$ is nonsingular, then the characteristic polynomial of $\mathbf{A}^{-1}$ is

$$(-1)^n\left(\lambda^n + \frac{c_1}{|\mathbf{A}|}\lambda^{n-1} + \frac{c_2}{|\mathbf{A}|}\lambda^{n-2} + \cdots + \frac{(-1)^n}{|\mathbf{A}|}\right).$$

15.71 Inequalities for the Characteristic Polynomial

The first group of inequalities that follow, which relate to the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ whose elements may be complex, refer directly to the coefficients of the polynomial when written in the form

$$P(\lambda) \equiv |\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n,$$

and only implicitly to the coefficients a_{ij} of $\mathbf{A}$ that give rise to the b_i.

15.711 Named and unnamed inequalities

The first group of inequalities relating to the eigenvalues λ satisfying $P(\lambda) = 0$ are unnamed and are as follows:

1. All the eigenvalues λ lie within or on the circle $||z|| \le r$, where r is the positive root of .

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0 \qquad \text{MG 122}$$

2. All the eigenvalues λ lie within the circle

$$|z| < 1 + \max_i |b_i|. \qquad \text{MG 123}$$

3. When $b_n \ne 0$ the eigenvalue λ of smallest modulus lies in the annulus $R \le |z| \le \dfrac{R}{2^{1/n} - 1}$, where R is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0. \qquad \text{MG 126}$$

4. All the eigenvalues λ lie on or outside the circle

$$|z| = \min_k \left[\frac{|b_n|}{(|b_n| + |b_k|)}\right]. \qquad \text{MG 126}$$

5. If the eigenvalues λ are ordered so that

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_p| > 1 \ge |\lambda_{p+1}| \ge \cdots \ge |\lambda_n|,$$

then

$$|z_1 z_2 \ldots z_p| \le N, \qquad |z_p| \le N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2. \qquad \text{MG 129}$$

6. All the eigenvalues λ lie in or on the circle

$$|z| \le \sum_{j=1}^{n} |b_j|^{1/j}. \qquad \text{MG 126}$$

7. All the eigenvalues λ lie on the disk

$$\left|z + \frac{b_1}{2}\right| \le \left|\frac{b_1}{2}\right| + |b_2|^{1/2} + |b_3|^{1/3} + \cdots + |b_n|^{1/n}. \qquad \text{MG 145}$$

8. All the eigenvalues λ lie in the annulus $m \le ||z|| \le M$, where

$$m^2 = \max\left\{0, \min_{1 \le j \le n-1}\left[1 - |b_j|, |b_n|^2\right]\right\}$$

and

$$M^2 = \max \left\{ 1 + |b_j|, |b_n|^2 + 2 \sum_{j=1}^{n-1} |b_j|^2 \right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial $P(\lambda)$. MG 145

15.712 Parodi's theorem

The eigenvalues λ satisfying $P(\lambda) = 0$ lie in the union of the disks

$$|z| \leq 1, \qquad |z + b_1| \leq \sum_{j=1}^{n} |b_j|.$$ MG 143

15.713 Corollary of Brauer's theorem

If

$$|b_1| > 1 + \sum_{j=2}^{n} |b_j|,$$

then one and only one eigenvalue satisfying $P(\lambda) = 0$ lies on the disk

$$|z + b_1| \leq \sum_{j=2}^{n} |b_j|.$$ MG 141

15.714 Ballieu's theorem

For any set $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ of positive numbers, let $\mu_0 = 0$ and

$$M_\mu = \max_{0 \leq k \leq n-1} \left[\frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}} \right].$$

Then all the eigenvalues satisfying $P(\lambda) = 0$ lie on the disk $||z|| \leq M_\mu$. MG 144

15.715 Routh-Hurwitz theorem

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0$$

determining the n eigenvalues λ of the real $n \times n$ matrix $\mathbf{A}$. Then the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \qquad \ldots, \qquad \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \ldots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \ldots & b_k \end{vmatrix}.$$ GM 230

15.81–15.82 Named Theorems on Eigenvalues

In the following theorems involving eigenvalue inequalities the elements a_{ij} of matrix $\mathbf{A}$ enter directly, and not in the form of the coefficients of the characteristic polynomial.

15.811 Schur's inequalities

If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with elements that may be complex, and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

1. $$\sum_{i=1}^{n} |\lambda_i|^2 \le \sum_{i,j=1}^{n} |a_{ij}|^2$$

2. $$\sum_{i=1}^{n} |\mathrm{Re}\,\lambda_i|^2 \le \sum_{i,j=1}^{n} \left|\frac{a_{ij} + \overline{a_{ji}}}{2}\right|^2$$

3. $$\sum_{i=1}^{n} |\mathrm{Im}\,\lambda_i|^2 \le \sum_{i,j=1}^{n} \left|\frac{a_{ij} - \overline{a_{ji}}}{2}\right|^2$$ ML 309

15.812 Sturmian separation theorem

Let $\mathbf{A}_r = [a_{ij}]$ with $i,\ j = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, N$ be a sequence of N symmetric matrices of increasing order. Then if $\lambda_k(\mathbf{A}_r)$ for $k = 1, 2, \ldots, r$ denotes the k^{th} eigenvalue of A_r, where the ordering is such that

$$\lambda_1\left(A_r\right) \ge \lambda_2\left(A_r\right) \ge \cdots \ge \lambda_r\left(A_r\right),$$

it follows that

$$\lambda_{k+1}\left(A_{i+1}\right) \le \lambda_k\left(A_i\right) \le \lambda_k\left(A_{i+1}\right).$$ BL 115

15.813 Poincare's separation theorem

Let $\left\{\mathbf{y}^k\right\}$, with $k = 1, 2, \ldots, K$, be a set of orthonormal vectors so that the inner product $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Set

$$\mathbf{x} = \sum_{k=1}^{K} u_k \mathbf{y}^k,$$

so that for any square matrix $\mathbf{A}$ for which the product $\mathbf{Ax}$ is defined, the quadratic form

$$(\mathbf{x}, \mathbf{Ax}) = \sum_{k,l=1}^{K} u_k u_l \left(\mathbf{y}^k, \mathbf{A}\mathbf{y}^l\right).$$

Then if

$$\mathbf{b}_K = \left(\mathbf{y}^k, \mathbf{A}\mathbf{y}^l\right) \text{ for } k, l = 1, 2, \ldots, K,$$

it follows that

$$\begin{aligned} \lambda_i\left(\mathbf{b}_K\right) &\le \lambda_i(\mathbf{A}) && \text{for } i = 1, 2, \ldots, K, \\ \lambda_{K-j}\left(\mathbf{b}_K\right) &\ge \lambda_{N-j}(\mathbf{A}) && \text{for } j = 0, 1, 2, \ldots, K-1. \end{aligned}$$ BL 117

15.814 Gerschgorin's theorem

Let $\mathbf{A} = [a_{ij}]$ be any arbitrary $n \times n$ matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{j=1, i \neq j}^{n} |a_{ij}| \text{ for } i = 1, 2, \ldots, n.$$

Then all of the eigenvalues λ_i of $\mathbf{A}$ lie in the union of the n disks Γ_i, where

$$\Gamma_i : \quad |z - a_{ii}| \leq \Lambda_i \text{ for } i = 1, 2, \ldots, n.$$ VA 16

15.815 Brauer's theorem

If in Gerschgorin's theorem for a given m

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all $j \neq m$, then one and only one eigenvalue of $\mathbf{A}$ lies in the disk Γ_m. MG 141

15.816 Perron's theorem

If $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ is an arbitrary set of positive numbers, then all the eigenvalues λ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ lie on the disk $|z| \leq \mathbf{M}_\mu$, where

$$\mathbf{M}_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i} |a_{ij}|.$$ MG 141

15.817 Frobenius theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients, so that $a_{ij} > 0$ for all $i, j = 1, 2, \ldots, n$, then $\mathbf{A}$ has a positive eigenvalue λ_0, and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0.$$ MG 142

15.818 Perron–Frobenius theorem

If all elements a_{ij} of an irreducible matrix $\mathbf{A}$ are nonnegative, then $R = \min M_\lambda$ is a simple eigenvalue of $\mathbf{A}$, and all the eigenvalues of $\mathbf{A}$ lie on the disk $|z| \leq R$, where, if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a set of nonnegative numbers, not all zero,

$$M_\lambda = \inf \left\{ \mu : \mu\lambda_i > \sum_{j=1}^{n} |a_{ij}| \lambda_j, 1 \leq i \leq n \right\}$$

and $R = \min M_\lambda$.

Furthermore, if $\mathbf{A}$ has exactly p eigenvalues ($p \leq n$) on the circle $|z| = R$, then the set of all its eigenvalues is invariant under rotations $2\pi/p$ about the origin. GM 69

15.819 Wielandt's theorem

If the $n \times n$ matrix $\mathbf{A}$ satisfies the conditions of the Perron–Frobenius theorem and if in the $n \times n$ matrix $\mathbf{C} = [c_{ij}]$

$$|c_{ij}| \leq a_{ij}, \qquad i, j = 1, 2, \ldots, n,$$

then any eigenvalue λ_0 of $\mathbf{C}$ satisfies the inequality $|\lambda_0| \leq R$. The equality sign holds only when there exists an $n \times n$ matrix $\mathbf{D} = [\pm\delta_{ij}]$ such that $\delta_{ii} = 1$ for all $i, \delta_{ij} = 0$ for all $i \neq j$, and

$$\mathbf{C} = (\lambda_0/R)\,\mathbf{DAD}^{-1}.$$ GM 69

15.820 Ostrowski's theorem

If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients and λ_0 is the positive eigenvalue in Frobenius' theorem, then the $n-1$ eigenvalues $\lambda_j \neq \lambda_0$ satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \qquad m = \min a_{ij} \qquad \text{for} \quad i, j = 1, 2, \ldots, n.$$ MG 145

15.821 First theorem due to Lyapunov

In order that all the eigenvalues of the real $n \times n$ matrix $\mathbf{A}$ have negative real parts, it is necessary and sufficient that if V is an $n \times n$ matrix, the equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{V} + \mathbf{VA} = -\mathbf{I}$$

has as a solution the matrix of coefficients $\mathbf{V}$ of some positive-definite quadratic form $(\mathbf{x}, \mathbf{Vx})$ (see **13.21**). GM 224

15.822 Second theorem due to Lyapunov

If all the eigenvalues of the real matrix $\mathbf{A}$ have negative real parts, then to an arbitrary negative-definite quadratic form $(\mathbf{x}, \mathbf{Wx})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive-definite quadratic form $(\mathbf{x}, \mathbf{Vx})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$$

then $(\mathbf{x}, \mathbf{Vx})$ and $(\mathbf{x}, \mathbf{Wx})$ satisfy

$$\frac{d}{dt}(\mathbf{x}, \mathbf{Vx}) = (\mathbf{x}, \mathbf{Wx}).$$

Conversely, if for some negative-definite form $(\mathbf{x}, \mathbf{Wx})$ there exists a positive-definite form $(\mathbf{x}, \mathbf{Vx})$ connected to $(\mathbf{x}, \mathbf{Wx})$ by the preceding two equations, then all the eigenvalues of $\mathbf{A}$ have negative real parts (see **13.21, 13.31**). GM 222

15.823 Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov

1. The off-diagonal Hermitian matrix $\mathbf{A}$ of rank n whose elements are given by

$$a_{jk} = (1 - \delta_{jk})\left\{1 + i\cot\left[\frac{(j-k)\,\pi}{n}\right]\right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \text{ for } s = 1, 2, \ldots, n,$$

and the corresponding eigenvectors $v^{(s)}$ have the components

$$v_j^{(s)} = \exp\left(-\frac{2\pi i s j}{n}\right) \qquad \text{for} \quad j = 1, 2, \ldots, n.$$

2. The two off-diagonal Hermitian matrices **B** and **C** whose elements are defined by the formulas

$$b_{jk} = (1 - \delta_{jk}) \sin^{-2}\left[\frac{(j-k)\,\pi}{n}\right],$$
$$c_{jk} = (1 - \delta_{jk}) \sin^{-4}\left[\frac{(j-k)\,\pi}{n}\right],$$

are related to the matrix **A** in (1) by the equations

$$\mathbf{B} = \frac{1}{2}\left(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}\right),$$
$$\mathbf{C} = -\frac{1}{6}\left(\mathbf{B}^2 - 2\left(2 + \sigma_n^{(1)}\right)\mathbf{B} - \sigma_n^{(2)}\mathbf{I}\right),$$

where **I** is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3}\left(n^2 - 1\right), \qquad \sigma_n^{(2)} = \frac{1}{45}\left(n^2 - 1\right)\left(n^2 + 11\right).$$

The eigenvalues of **B** and **C** corresponding to the eigenvector $v_j^{(s)}$ in (1) have the form

$$\lambda_s^{(b)} = \sigma_n^{(1)} - 2s(n-s) \qquad \text{for } s = 1, 2, \ldots, n,$$
$$\lambda_s^{(c)} = \sigma_n^{(2)} - 2s(n-s)\frac{s(n-s)+2}{3} \qquad \text{for } s = 1, 2, \ldots, n.$$

3. Together, the above two results imply the following diophantine summation rules:

(a) $$\sum_{k=1}^{n-1} \cot\left(\frac{k\pi}{n}\right) \sin\left(\frac{2sk\pi}{n}\right) = n - 2s \qquad \text{for } s = 1, 2, \ldots, n-1$$

(b) $$\sum_{k=1}^{n-1} \sin^{-2}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = b_s \qquad \text{for } s = 1, 2, \ldots, n-1,$$

(c) $$\sum_{k=1}^{n-1} \sin^{-4}\left(\frac{k\pi}{n}\right) \cos\left(\frac{2sk\pi}{n}\right) = c_s \qquad \text{for } s = 1, 2, \ldots, n-1,$$

(d) $$\sum_{k=1}^{n-1} \sin^{-2p}\left(\frac{k\pi}{n}\right) = \sigma_n^{(p)},$$

with $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)}\frac{2n^4 + 23n^2 + 191}{315}, \qquad \sigma_n^{(4)} = \sigma_n^{(2)}\frac{3n^4 + 10n^2 + 227}{315}$$
$$b_s = \sigma_n^{(1)} - 2s(n-s), \qquad c_s = \sigma_n^{(2)} - \frac{2}{3}s(n-s)[s(n-s)+2].$$

15.91 Variational Principles

15.911 Rayleigh quotient

If $\mathbf{A}$ is an Hermitian matrix, the Rayleigh quotient $\rho(\mathbf{x})$ is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}.$$ NO 407

15.912 Basic theorems

1. If the $n \times n$ matrix A is Hermitian and has eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, then

$$\lambda_1 \le \rho \le \lambda_n,$$

where ρ is the Rayleigh quotient for any $\mathbf{x} \ne \mathbf{0}$, and

$$\lambda_1 = \min_{x \ne 0} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \qquad \text{and} \qquad \lambda_n = \max_{x \ne 0} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})}.$$ NO 407

2. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian and has eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ corresponding to the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively, and $\mathbf{x} \ne \mathbf{0}$ is such that

$$(\mathbf{x}, \mathbf{x}_1) = (\mathbf{x}, \mathbf{x}_2) = \cdots = (\mathbf{x}, \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_{x} \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})},$$

and

$$\lambda_j \le \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \le \lambda_n.$$ NO 410

3. If the $n \times n$ matrix $\mathbf{A}$ is Hermitian, then the eigenvalue

$$\lambda_r = \max \left(\min \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \right),$$

where first the minimum over $\mathbf{x}$ is taken subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = 1, 2, \ldots, r-1$, with the $\mathbf{b}_i$ regarded as fixed vectors, and then the maximum over all possible $\mathbf{b}_i$. Also, the eigenvalue

$$\lambda_r = \min \left(\max \frac{(\mathbf{x}, \mathbf{A}\mathbf{x})}{(\mathbf{x}, \mathbf{x})} \right),$$

where now the maximum over $\mathbf{x}$ is taken first subject to $(\mathbf{b}_i, \mathbf{x}) = 0, i = r+1, r+2, \ldots, n$ for fixed $\mathbf{b}_i$, and then the minimum over all possible $\mathbf{b}_i$. NO 414

4. The $(n-1)$ eigenvalues $\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}$ obtained from the $(n-1) \times (n-1)$ matrix derived from an Hermitian matrix $\mathbf{A}$ from which the last row and column have been omitted separate the n eigenvalues of $\mathbf{A}$, so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \cdots < \lambda'_{n-1} < \lambda_n \qquad \text{(see } \mathbf{15.812}\text{)}.$$

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16 Ordinary Differential Equations

16.1–16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-Order Equations

16.111 Solution of a first-order equation

Consider the real function $f(t,x)$ that is defined and continuous in an open set $D \subset R^2$. Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t,x)$$

in the open interval $I \subset R$ is a real function $u(t)$ that is defined and is both continuous and differentiable in I, with the property that

(i) $(t, u(t)) \in D$ for $t \in I$,

(ii) $\dfrac{du}{dt} = f(t, u(t))$ for $t \in I$.

16.112 Cauchy problem

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t,x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where $(t_0, u(t_0)) \in D$, the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^{t} f(\tau, u(\tau))\, d\tau \qquad \text{(see } \mathbf{16.316}\text{)}.$$

16.113 Approximate solution to an equation

The real function $\phi(t)$ is said to be an **approximate solution**, to within the error ϵ, of the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if ϕ' is piecewise continuous, and for a given $\epsilon > 0$ and an open interval $I \subset R$,

$$|\phi'(t) - f(t, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative. HU 3

16.114 Lipschitz continuity of a function

The real function $f(t, x)$ defined and continuous in some open set $D \subset R^2$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points (t, x_1) and (t, x_2) belonging to D

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|.$$ HU 5

16.21 Fundamental Inequalities and Related Results

16.211 Gronwall's lemma

Let the three piecewise continuous, non-negative functions u, v, and w be defined in the interval $[0, a]$ and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau)\, d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau)v(\tau) \exp\left(\int_\tau^t v(\sigma)\, d\sigma\right) d\tau.$$ BB 135

16.212 Comparison of approximate solutions of a differential equation

Let f be a real function that is defined in an open set $D \subset R^2$, in which it is both continuous and Lipschitz continuous. In addition, let u_1 and u_2 be two approximate solutions of

$$\frac{dx}{dt} = f(t, x)$$

in an open set $I \subset R$ in the sense already defined, with

$$|u_1'(t) - f(t, u_1(t))| \leq \epsilon_1, \quad |u_2'(t) - f(t, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \exp\{|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)[\exp\{k|t - t_0|\} - 1].$$ HU 6

16.31 First-Order Systems

16.311 Solution of a system of equations

The **system** of n first-order differential equations

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n),$$
$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n),$$

in which the functions $f_1, f_2, \ldots, f_n$ are real and continuous in an open set $D \subset R^{n+1}$, may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where $\mathbf{x}$ and $\mathbf{f}$ are $n \times 1$ column vectors. Its solution in the open interval $I \subset R$ is the vector $\mathbf{u}(t)$ with elements $u_1(t), u_2(t), \ldots, u_n(t)$ with the property that

(i) $(t, \mathbf{u}(t)) \in D$ for $t \in I$,

(ii) $\dfrac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t))$ for $t \in I$. HU 24

16.312 Cauchy problem for a system

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector** condition

$$\mathbf{u}(t_0) = \mathbf{x}_0,$$

where $(t_0, \mathbf{u}(t_0) \in D$, the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{u}(\tau))\, d\tau.$$

16.313 Approximate solution to a system

The real vector $\boldsymbol{\phi}(t)$ is said to be an **approximate vector solution**, to within the order ϵ, of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

if the elements of $\boldsymbol{\phi}'$ are piecewise continuous, and for a given $\epsilon > 0$ and open interval $I \subset R$,

$$||\phi'(t) - \mathbf{f}(t, \phi(t))|| \leq \epsilon,$$

except at points of discontinuity of the derivative, where $||\mathbf{w}||$ denotes the supremum norm

$$||\mathbf{w}|| = \sup(|w_1|, |w_2|, \ldots, |w_n|).$$ HU 25

16.314 Lipschitz continuity of a vector

The real vector $\mathbf{f}(t, x)$ defined and continuous in some open set $D \subset R^n$ is said to be **Lipschitz continuous** with respect to x for some constant $k > 0$ if, for all points $(t, \mathbf{x}_1)$, $(t, \mathbf{x}_2)$ belonging to D,

$$||\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)|| \leq k||\mathbf{x}_1 - \mathbf{x}_2||.$$ HU 26

16.315 Comparison of approximate solutions of a system

Let $\mathbf{f}$ be a real vector defined in an open set $D \subset R \times R^n$ in which it is both continuous and Lipschitz continuous. In addition, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}\,(t, \mathbf{x})$$

in an open set $I \subset R$ in the sense already defined, with

$$|\mathbf{u}_1'(t) - \mathbf{f}\,(t, \mathbf{u}_1(t))| \le \epsilon_1, \qquad |\mathbf{u}_2'(t) - \mathbf{f}\,(t, \mathbf{u}_2(t))| \le \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$||\mathbf{u}_1\,(t_0) - \mathbf{u}_2\,(t_0)|| \le \delta,$$

it follows that

$$||\mathbf{u}_1(t) - \mathbf{u}_2(t)|| \le \delta \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)[\exp\{k|t - t_0|\} - 1]\,. \qquad \text{HU 27}$$

16.316 First-order linear differential equation

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t)\,dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)}\left(\mu\,(t_0)\,y_0 + \int_{t_0}^{t} \mu(\xi)Q(\xi)\,d\xi\right),$$

where $y_0 = y(t_0)$.

16.317 Linear systems of differential equations

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where $\mathbf{x}$ is an $n \times 1$ column vector and $\mathbf{A}(t)$ an $n \times n$ matrix. Then a **fundamental system** of solutions of this system is a set of n linearly independent solution vectors $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$, The square matrix $\mathbf{K}(t)$ whose columns comprise the vectors $\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)$ is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}\,(t_0)| \exp\left(\int_{t_0}^{t} \operatorname{tr}\mathbf{A}(\tau)\,d\tau\right).$$

Using the fundamental matrix $\mathbf{K}(t)$ defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, is

$$\phi(t) = \mathbf{K}(t)[\mathbf{K}\,(t_0)]^{-1}\mathbf{x}_0 + \mathbf{K}(t)\int_{t_0}^{t} [\mathbf{K}(\tau)]^{-1}\mathbf{b}(\tau)\,d\tau, \qquad \text{HU 43}$$

where $\mathbf{b}(t)$ is an $n \times 1$ column vector. CL 69

16.41 Some Special Types of Elementary Differential Equations

16.411 Variables separable

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y)\,dx + R(x)S(y)\,dy = 0.$$

It may then be written in the form

$$M(x)\,dx - \frac{1}{N(y)}\,dy = 0,$$

or

$$\frac{P(x)}{R(x)}\,dx + \frac{S(y)}{Q(y)}\,dy = 0,$$

provided $R(x)Q(y) \neq 0$.

16.412 Exact differential equations

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **exact** if there exists a function $h(x,y)$ such that

$$d\,[h(x,y)] = M(x,y)\,dx + N(x,y)\,dy.$$ IN 16

16.413 Conditions for an exact equation

A necessary and sufficient condition that an equation of this form is exact is that the functions $M(x,y)$ and $N(x,y)$ together with their partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$ IN 16

16.414 Homogeneous differential equations

A differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary k,

$$\frac{M(kx,ky)}{N(kx,ky)} = \frac{M(x,y)}{N(x,y)}.$$

Setting $y = sx$, it may then be expressed in the form

$$[M(1,s) + sN(1,s)]\,dx + xN(1)\,dx = 0,$$

in which the variables s and x are separable. IN 18

16.51 Second-Order Equations

16.511 Adjoint and self-adjoint equations

The linear second-order differential equation

$$L(u) \equiv a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2}\left[a(x)v\right] - \frac{d}{dx}\left[b(x)v\right] + c(x)v = 0.$$

The equation $L(u) = 0$ is said to be **self-adjoint** if $L(u) \equiv M(u)$.

A linear self-adjoint second-order differential equation defined on $[\alpha, \beta]$ can always be expressed in the form

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

where $p(x)$ and $q(x)$ are continuous on $[\alpha, \beta]$ and $p(x) > 0$. The general equation $L(u) = 0$ can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right],$$

when

$$p(x) = \exp\int\frac{b(x)}{a(x)}\,dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)}\left[\exp\int\frac{b(x)}{a(x)}\,dx\right].$$

In general, if

$$L(u) = p_0\frac{d^nu}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}}\ldots + p_{n-1}\frac{du}{dx} + p_nu,$$

then its adjoint is

$$M(v) = (-1)^n\frac{d^n}{dx^n}\left[p_0v\right] + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left[p_1v\right] + \ldots - \frac{d}{dx}\left[p_{n-1}v\right] + p_nv.$$ HI 391

16.512 Abel's identity

If $p(x)$ and $q(x)$ are continuous in $[\alpha, \beta]$ in which $p(x) > 0$, and $u(x)$ and $v(x)$ are suitably differentiable with

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0,$$

then the result

$$p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) \equiv \text{ const.}$$

is known as **Abel's identity**.

More generally, if we consider the linear n^{th}-order equation

$$p_0\frac{d^nu}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}} + \ldots + p_{n-1}\frac{du}{dx} + p_n = 0,$$

and Δ is the Wronskian of a (fundamental) set of linearly independent solutions $u_1, u_2, \ldots, u_n$, the Abel identity takes the form

$$\Delta = \Delta_0\exp\left(-\int_{x_0}^{x}\frac{p_1(x)}{p_0(x)}\,dx\right),$$

where Δ_0 is the value of Δ at $x = x_0$. IN 119

16.513 Lagrange identity

If the linear n^{th}-order equation $L(u) = 0$ is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx} \left\{ P(u, v) \right\},$$

where $M(v)$ is the adjoint of $L(u)$, is called the **Lagrange identity**. The expression $P(u, v)$, which is linear and homogeneous in

$$u, \frac{du}{dx}, \ldots, \frac{d^{n-1}u}{dx^{n-1}} \quad \text{and} \quad v, \frac{dv}{dx}, \ldots, \frac{d^{n-1}v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**. In the case of the second-order equation

$$L(u) = a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x) u = 0,$$

with adjoint $M(v)$, the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx} \left(a(x) v \frac{du}{dx} - \frac{d}{dx} (a(x) v) u + b(x) uv \right). \qquad \text{IN 124}$$

16.514 The Riccati equation

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x) z + b(x) z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{\left(p(x) \frac{du}{dx} \right)}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x) u = 0.$$

The further substitution $v = u \left(\exp \int_\alpha^x a(x)\, dx \right)$ in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x) v^2 + s(x) = 0,$$

with

$$r(x) = b(x) \exp \left(- \int_\alpha^x a(x)\, dx \right) \quad \text{and} \quad s(x) = c(x) \exp \left(\int_\alpha^x a(x)\, dx \right). \qquad \text{HI 273}$$

16.515 Solutions of the Riccati equation

If in the Riccati equation

$$\frac{dv}{dx} + r(x) v^2 + s(x) = 0,$$

$r(x) \neq 0$, while $r(x)$ and $s(x)$ are continuous on the interval $[\alpha, \beta]$, then every solution $v(x)$ may be expressed in the form

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with A, B arbitrary constants, not both zero, and the prime denoting differentiation, while u and v are linearly independent solutions of

$$\frac{d}{dx}\left(\frac{1}{r(x)}\frac{dz}{dx}\right) + s(x)z = 0.$$

Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions of this last equation and A and B are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)}\frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever $Au(x) = Bv(x) \neq 0$. IN 24

16.516 Solution of a second-order linear differential equation

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} = \phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t),$$

and the solution to the inhomogeneous equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t),$$

subject to the initial conditions $x(t_0) = x_0$ and $x'(t_0) = x_1$ may be written

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^{t} \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi)\, d\xi,$$

where the constants c_1 and c_2 are chosen such that $x(t)$ satisfies the initial conditions.

The linear combination $c_1\phi_1(t) + c_2\phi_2(t)$ is known as the **complementary function** where c_1 and c_2 are arbitrary constants.

16.61–16.62 Oscillation and Non-Oscillation Theorems for Second-Order Equations

Equations whose solutions possess an infinite number of zeros in the interval $(0, \infty)$ are said to have **oscillatory** solutions. The following theorems relate to such properties:

16.611 First basic comparison theorem

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if $\psi(x) \geq \phi(x)$ and some solutions v are non-oscillatory, then so also must some solutions u be non-oscillatory. BS 119

16.622 Second basic comparison theorem

If all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0$$

are oscillatory as $x \to \infty$, and if

$$q_2(x) \geq q_1(x),$$
$$p_2(x) \geq p_1(x) > 0,$$

then all the solutions of the self-adjoint equation

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0$$

are oscillatory. BS 120

16.623 Interlacing of zeros

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and suppose that $y_1(x)$ has at least two zeros in the interval (a, b). Then if x_1 and x_2 are two consecutive zeros of $y_1(x)$, the function $y_2(x)$ has one, and only one, zero in the interval (x_1, x_2). HI 374

16.624 Sturm separation theorem

Let $u(x)$ and $v(x)$ be two linearly independent solutions of the self-adjoint equation

q(x)y

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) = 0,$$

in which $p(x) > 0$ and $p(x), q(x)$ are continuous on $[a, b]$. Then, between any two consecutive zeros of $u(x)$ there will be one, and only one, zero of $v(x)$. IN 224

16.625 Sturm comparison theorem

Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions in the differential equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0,$$

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0.$$

Then between any two zeros of a non-trivial solution $u(x)$ of the first equation there will be at least one zero of every non-trivial solution $v(x)$ of the second equation. IN 228

16.626 Szegö's comparison theorem

Suppose, under the conditions of the Sturm comparison theorem, that $p_1(x) \equiv p_2(x), q_1(x) \not\equiv q_2(x)$, and $u(x) > 0, v(x) > 0$ for $a < x < b$, together with

$$\lim_{x\to a} p_1(x)\left(\frac{du}{dx}v - \frac{dv}{dx}u\right) = 0.$$

Then, if $u(b) = 0$, there is a point ξ in (a, b) such that $v(\xi) = 0$. HI 379

16.627 Picone's identity

Consider the equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0,$$
$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0,$$

with p_1, p_2, q_1, and q_2 positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then with $a < \alpha < \beta < b$, Picone's identity is

$$\left(\frac{u}{v}\left(p_1\frac{du}{dx}v - p_2\frac{dv}{dx}u\right)\right)_\alpha^\beta = \int_\alpha^\beta (q_2 - q_1)\, u^2\, ds + \int_\alpha^\beta (p_1 - p_2)\left(\frac{du}{ds}\right)^2 ds + \int_\alpha^\beta \frac{p_2}{v^2}\left(v\frac{du}{ds} - u\frac{dv}{ds}\right)^2 ds.$$

IN 226

16.628 Sturm-Picone theorem

Consider the self-adjoint equations

$$\frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u = 0$$

and

$$\frac{d}{dx}\left(p_2(x)\frac{dv}{dx}\right) + q_2(x)v = 0.$$

Let p_1, p_2, q_1, and q_2 be positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then, if x_1 and x_2 is a pair of consecutive zeros of $u(x)$ in $(a, b), v(x)$ has at least one zero in the open interval (a, b). (x_1,x_2). Or v(x)=\Im u(x), \Im\in{\mathbb R} (additional differences in wiki statement)) IN 225

16.629 Oscillation on the half line

Consider the self-adjoint equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = 0.$$

We then have the following results:

(i) Let $p(x) > 0$ and p, q be continuous on $[0, \infty)$. If the two improper integrals

$$\int_1^\infty \frac{dx}{p(x)} \quad \text{and} \quad \int_1^\infty q(x)\, dx$$

diverge, then every solution $u(x)$ has infinitely many zeros on the interval $[1, \infty)$. Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)} = +\infty \quad \text{and} \quad \int_0^1 q(x)\, dx = +\infty,$$

then every solution $u(x)$ has infinitely many zeros on the interval $(0, 1)$.

(ii) (Moore's theorem). Every non-trivial solution $u(x)$ has at most a finite number of zeros on the interval $[a, \infty)$ if the improper integral

$$\int_a^\infty \frac{dx}{p(x)}$$

converges, and if

$$\left| \int_a^x q(s)\,ds \right| < M \qquad \text{for} \qquad a \le x < \infty$$

with $M > 0$ a finite constant.

16.71 Two Related Comparison Theorems

16.711 Theorem 1

Consider the equations in the Sturm comparison theorem with the same assumptions on $p(x)$ and $q(x)$, and let $u(x), v(x)$ be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if $u(x)$ is increasing in $[x_1, x_2]$ and reaches a maximum at x_2, the function $v(x)$ reaches a maximum at some point x_3 such that $x_1 < x_3 < x_2$. HI 376

16.712 Theorem 2

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

in which $F(x)$ is continuous in (a, b) and such that

$$0 < m \le F(x) \le M.$$

Then, if the solution $y(x)$ has two successive zeros x_1, x_2, it follows that

$$\pi M^{-1/2} \le x_2 - x_1 \le \pi m^{-1/2}.$$

16.81–16.82 Non-Oscillatory Solutions

The real solution $y(x)$ of

$$\frac{d^2y}{dx^2} + F(x)y = 0$$

is said to be **non-oscillatory** in the wide sense in $(0, \infty)$ if there exists a finite number c such that the solution has no zeros in $[c, \infty)$. HI 376

16.811 Kneser's non-oscillation theorem

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup \left[x^2 F(x)\right] = \gamma^*,$$

$$\liminf \left[x^2 F(x)\right] = \gamma_*.$$

Then the solution $y(x)$ is non-oscillatory if $\gamma^* < \frac{1}{4}$, oscillatory if $\frac{1}{4} < \gamma_*$ and no conclusion can be drawn if either γ^* or γ_* equals $\frac{1}{4}$. HI 461

16.822 Comparison theorem for non-oscillation

Consider the differential equations

$$\frac{d^2y}{dx^2} + F(x)y = 0, \quad f(x) = x\int_x^\infty F(s)\,ds,$$

$$\frac{d^2y}{dx^2} + G(x)y = 0, \quad g(x) = x\int_x^\infty G(s)\,ds,$$

where $0 < g(x) < f(x)$. Then if the first equation is non-oscillatory in the wide sense, so also is the second. HI 460

16.823 Necessary and sufficient conditions for non-oscillation

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x\to\infty} \sup \left(x\int_x^\infty F(s)\,ds\right) = F^*,$$

$$\lim_{x\to\infty} \inf \left(x\int_x^\infty F(s)\,ds\right) = F_*,$$

it follows that:

(i) a necessary condition that the solution $y(x)$ be non-oscillatory is that $F_* \le \frac{1}{4}$ and $F^* \le 1$;

(ii) a sufficient condition that the solution $y(x)$ be non-oscillatory is that $F^* < \frac{1}{4}$.

16.91 Some Growth Estimates for Solutions of Second-Order Equations

16.911 Strictly increasing and decreasing solutions

Suppose that $G(x) > 0$ be continuous in $(-\infty, \infty)$ and such that $xG(x) \notin L(0, \infty)$. Then the equation $\frac{d^2y}{dx^2} - G(x)y = 0$ has one, and only one, solution $y_+(x)$ passing through the point $(0, 1)$, which is positive and strictly monotonic decreasing for all x, and one and only one solution $y_-(x)$ through the point $(0, 1)$, which is positive and strictly increasing for all x. The solution $y_+(x)$ has the property that

$$[G(x)]^{1/2}\, y_+(x) \in L_2(0, \infty) \text{ and } \frac{dy_+(x)}{dx} \in L_2(0, \infty).$$

If, in addition, $0 < \alpha^2 \le G(x) \le \beta^2 < \infty$, then

$$e^{-\beta x} \le y_+(x) \le e^{-\alpha x} \quad \text{for} \quad x > 0.$$

HI 359

16.912 General result on dominant and subdominant solutions

Consider the equations

$$\frac{d^2y}{dx^2} - g(x)y = 0, \quad \frac{d^2Y}{dx^2} - G(x)Y = 0,$$

where g and G are continuous on $(0, \infty)$ with $0 < g(x) < G(x)$, and $xg(x) \notin L(0, \infty)$. In addition, let y_α and Y_α be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \qquad y'_\alpha(0) = Y'_\alpha(0) = \alpha \text{ for } -\infty < \alpha < \infty.$$

Let y_ω and Y_ω be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \qquad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let y_+ and Y_+ be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while $\left[y'_+(x)\right]^2, g(x)\left[y_+(x)\right]^2, \left[Y'_+(x)\right]^2$, and $G(x)\left[Y'_+(x)\right]^2$ belong to $L(0,\infty)$. Then, if β and γ are such that $y_{-\beta} = y_+$ and $Y_{-\gamma} = Y_+$, it follows that $\beta < \gamma$ and

$$\begin{aligned} &y_\alpha(x) < Y_\alpha(x), \qquad 0 < x < \infty, \quad -\gamma \le \alpha, \\ &y_\omega(x) < Y_\omega(x), \\ &y_+(x) > Y_+(x). \end{aligned}$$ HI 440

16.913 Estimate of dominant solution

Let $G(x)$ be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G'(x) < \tfrac{5}{4}\left[G'(x)\right]^2.$$

Then there exists a **dominant solution** $y(x)$ of the fundamental solutions $Y_0(x)$ and $Y_1(x)$ of

$$\frac{d^2y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$\begin{aligned} 2Y_0(0) &= 0, \qquad Y_1(0) = 1, \\ Y'_0(0) &= 1, \qquad Y'_1(0) = 0, \end{aligned}$$

such that

$$y(x) < [G(x)]^{-1/4} \exp\left(\int_0^x [G(\xi)]^{1/2}\, d\xi\right),$$

and a positive constant C such that the normalized subdominant solution $y_+(x)$, for which $y_+(0) = 1$ and $\left[y'_+(x)\right]^2 \in L(0,\infty)$, $G(x)\left[y_+(x)\right]^2 \in L(0,\infty)$, satisfies

$$y_+(x) > CG(x)^{-1/4} \exp\left(-\int_0^x [G(\xi)]^{1/2}\, d\xi\right).$$ HI 443

16.914 A theorem due to Lyapunov

Let $y(x)$ be any solution of

$$\frac{d^2y}{dx^2} - G(x)y = 0$$

with $G(x)$ positive and continuous in $(0,\infty)$ with $xG(x) \in L(0,\infty)$. Then

$$\begin{aligned} \exp\left(-\int_0^x [G(\xi)+1]\, d\xi\right) &< [y(x)]^2 + [y'(x)]^2 \\ &< C\exp\left(\int_0^x [G(\xi)+1]\, d\xi\right), \end{aligned}$$ HI 446

where $C = [y(0)]^2 + [y'(0)]^2$.

16.92 Boundedness Theorems

16.921[6] All solutions of the equation

$$\frac{d^2u}{dx^2} + (1 + \phi(x) + \psi(x))\,u = 0$$

are bounded, provided that

(i) $\int^\infty |\phi(x)|\,dx < \infty$,

(ii) $\displaystyle\int^\infty |\psi(x)|\,dx < \infty$ and $\psi(x) \to 0$ as $x \to \infty$. BS 112

16.922 If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x) + b(x))u = 0$$

are also bounded if

$$\int^\infty |b(x)|\,dx < \infty.$$

BS 112

16.923 If $a(x) \to \infty$ monotonically as $x \to \infty$, then all solutions of

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded as $x \to \infty$. BS 113

16.924 Consider the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

in which

$$\int^\infty x|a(x)|\,dx < \infty.$$

Then $\displaystyle\lim_{x\to\infty}\left(\frac{du}{dx}\right)$ exists, and the general solution is asymptotic to $d_0 + d_1 x$ as $x \to \infty$, where d_0 and d_1 may be zero, but not simultaneously. BS 114

16.93[10] Growth of maxima of $|y|$

Sonin's theorem generalized by Pólya may be stated as follows: *Let y(x) satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

where $k(x) > 0, \phi(x) > 0$, and both functions $k(x), \phi(x)$ have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\phi(x)$ is decreasing or increasing.

SZ 164

17 Fourier, Laplace, and Mellin Transforms

17.1–17.4 Integral Transforms

17.11 Laplace transform

The **Laplace transform** of the function $f(x)$, denoted by $F(s)$, is defined by the integral

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx, \qquad \operatorname{Re} s > 0.$$

The functions $f(x)$ and $F(s)$ are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^\infty |f(x)|e^{-c|x|}\,dx$$

is finite, then the Laplace transform exists when $s = \sigma + i\tau$ is such that $\sigma \geq c$.

Setting

$$F(s) = \mathcal{L}\left[f(x); s\right]$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}\left[F(s); x\right].$$

The inversion of the Laplace transform is accomplished for analytic functions $F(s)$ of order $O\left(s^{-k}\right)$ with $k > 1$ by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{sx}\,ds,$$

where γ is a real constant that exceeds the real part of all the singularities of $F(s)$. SN 30

17.12 Basic properties of the Laplace transform

1.[8] For a and b arbitrary constants,

$$\mathcal{L}\left[af(x) + bg(x)\right] = aF(s) + bG(s) \qquad \text{(linearity)}$$

2. If $n > 0$ is an integer and $\lim_{x\to\infty} f(x)e^{-sx} = 0$, then for $x > 0$,

$$\mathcal{L}\left[f^{(n)}(x); s\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0) \quad \text{(transform of a derivative)}$$

SN 32

3.[11] If $\lim\limits_{x\to\infty}\left(e^{-sx}\int_0^x f(\zeta)\,d\zeta\right)=0$, then

$$\mathcal{L}\left[\int_0^x f(\xi)\,d\xi; s\right]=\frac{1}{s}F(s) \qquad \text{(transform of an integral)}$$ SN 37

4. $\mathcal{L}\left[e^{-ax}f(x);s\right]=F(s+a)$ (shift theorem) SU 143

5. The **Laplace convolution** $f*g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f*g(x)=\int_0^x f(x-\xi)g(\xi)\,d\xi,$$

and it has the property that $f*g=g*f$ and $f*(g*h)=(f*g)*h$. In terms of the convolution operation

$$\mathcal{L}\left[f*g(x);s\right]=F(s)G(s) \qquad \text{(convolution (Faltung) theorem).}$$ SN 30

17.13 Table of Laplace transform pairs

	$f(x)$		$F(s)$		
1	1		$1/s$		
2	$x^n,$	$n=0,1,2,\dots$	$\dfrac{n!}{s^{n+1}},$	$\operatorname{Re}s>0$	ET I 133(3)
3	$x^\nu,$	$\nu>-1$	$\dfrac{\Gamma(\nu+1)}{s^{\nu+1}},$	$\operatorname{Re}s>0$	ET I 137(1)
4	$x^{n-\frac{1}{2}}$		$\dfrac{\Gamma\left(n+\frac{1}{2}\right)}{s^{n+\frac{1}{2}}},$	$\operatorname{Re}s>0$	ET I 135(17)
5	$x^{-1/2}(x+a)^{-1},$	$\lvert\arg a\rvert<\pi$	$\pi a^{-1/2}e^{as}\operatorname{erfc}\left(a^{1/2}s^{1/2}\right),$	$\operatorname{Re}s\geq 0$	ET I 136(25)
6	$\begin{cases} x & \text{for } 0<x<1 \\ 1 & \text{for } x>1\end{cases}$		$\dfrac{1-e^{-s}}{s^2},$	$\operatorname{Re}s>0$	ET I 142(14)
7	e^{-ax}		$\dfrac{1}{s+a},$	$\operatorname{Re}s>-\operatorname{Re}a$	ET I 143(1)
8	xe^{-ax}		$\dfrac{1}{(s+a)^2},$	$\operatorname{Re}s>-\operatorname{Re}a$	ET I 144(2)
9a	$\dfrac{e^{-ax}-e^{-bx}}{b-a}$		$(s+a)^{-1}(s+b)^{-1},$	$\operatorname{Re}s>\{-\operatorname{Re}a,-\operatorname{Re}b\}$	AS 1022(29.3.12)

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	$f(x)$	$F(s)$
9b[11]	$\dfrac{\alpha e^{-ax}+\beta e^{-bx}+\gamma e^{-cx}}{(a-b)(b-c)(c-a)}$ a,b,c distinct, $\alpha=c-b$, $\beta=a-c$, $\gamma=b-a$	$(s+a)^{-1}(s+b)^{-1}(s+c)^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b, -\operatorname{Re} c\}$
10[11]	$\dfrac{ae^{-ax}-be^{-bx}}{b-a}$	$s(s+a)^{-1}(s+b)^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, -\operatorname{Re} b\}$ AS 1022(29.3.13)
11	$\dfrac{e^{ax}-1}{a}$	$s^{-1}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
12	$\dfrac{e^{ax}-ax-1}{a^2}$	$s^{-2}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
13	$\dfrac{\left(e^{ax}-\frac{1}{2}a^2x^2-ax-1\right)}{a^3}$	$s^{-3}(s-a)^{-1}$, $\operatorname{Re} s > \operatorname{Re} a$
14	$(1+ax)e^{ax}$	$\dfrac{s}{(s-a)^2}$, $\operatorname{Re} s > \operatorname{Re} a$
15	$\dfrac{1+(ax-1)e^{ax}}{a^2}$	$s^{-1}(s-a)^{-2}$, $\operatorname{Re} s > \operatorname{Re} a$
16	$\dfrac{2+ax+(ax-2)e^{ax}}{a^3}$	$s^{-2}(s-a)^{-2}$, $\operatorname{Re} s > \operatorname{Re} a$
17	$x^n e^{ax}$, $n=0,1,2,\ldots$	$n!(s-a)^{-(n+1)}$, $\operatorname{Re} s > \operatorname{Re} a$
18	$\left(x+\frac{1}{2}ax^2\right)e^{ax}$	$\dfrac{s}{(s-a)^3}$, $\operatorname{Re} s > \operatorname{Re} a$
19	$\left(1+2ax+\frac{1}{2}a^2x^2\right)e^{ax}$	$\dfrac{s^2}{(s-a)^3}$, $\operatorname{Re} s > \operatorname{Re} a$
20	$\frac{1}{6}x^3e^{ax}$	$(s-a)^{-4}$, $\operatorname{Re} s > \operatorname{Re} a$
21	$\left(\frac{1}{2}x^2+\frac{1}{6}ax^3\right)e^{ax}$	$\dfrac{s}{(s-a)^4}$, $\operatorname{Re} s > \operatorname{Re} a$
22	$\left(x+ax^2+\frac{1}{6}a^2x^3\right)e^{ax}$	$s^2(s-a)^{-4}$, $\operatorname{Re} s > \operatorname{Re} a$

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	$f(x)$	$F(s)$
23	$\left(1+3ax+\frac{3}{2}a^2x^2+\frac{1}{6}a^3x^3\right)e^{ax}$	$s^3(s-a)^{-4}, \quad \operatorname{Re} s > \operatorname{Re} a$
24	$\dfrac{ae^{ax}-be^{bx}}{a-b}$	$s(s-a)^{-1}(s-b)^{-1}, \quad \operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$
25	$\dfrac{\left(\frac{1}{a}e^{ax}-\frac{1}{b}e^{bx}+\frac{1}{b}-\frac{1}{a}\right)}{a-b}$	$s^{-1}(s-a)^{-1}(s-b)^{-1}, \quad \operatorname{Re} s > \{\operatorname{Re} a, \operatorname{Re} b\}$
26	$x^{\nu-1}e^{-ax}, \quad \operatorname{Re}\nu > 0$	$\Gamma(\nu)(s+a)^{-\nu}, \quad \operatorname{Re} s > -\operatorname{Re} a$ ET I 144(3)
27	$xe^{-x^2/(4a)}, \quad \operatorname{Re} a > 0$	$2a-2\pi^{1/2}a^{3/2}se^{as^2}\operatorname{erfc}\left(sa^{1/2}\right)$ ET I 146(22)
28	$\exp\left(-ae^x\right), \quad \operatorname{Re} a > 0$	$a^s\,\Gamma\left(-s,a\right)$ ET I 147(37)
29[8]	$x^{1/2}e^{-a/(4x)}, \quad \operatorname{Re} a \geq 0$	$\frac{1}{2}\pi^{1/2}s^{-3/2}\left(1+a^{1/2}s^{1/2}\right)\exp\left[(-as)^{1/2}\right], \quad \operatorname{Re} s > 0$ ET I 146(26)
30[8]	$x^{-1/2}e^{-a/(4x)}, \quad \operatorname{Re} a \geq 0$	$\pi^{1/2}s^{-1/2}\exp\left[(-as)^{1/2}\right], \quad \operatorname{Re} s > 0$ ET I 146(27)
31[8]	$x^{-3/2}e^{-a/(4x)}, \quad \operatorname{Re} a > 0$	$2\pi^{1/2}a^{-1/2}\exp\left[(-as)^{1/2}\right], \quad \operatorname{Re} s \geq 0$ ET I 146(28)
32	$\sin(ax)$	$a\left(s^2+a^2\right)^{-1}, \quad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 150(1)
33	$\cos(ax)$	$s\left(s^2+a^2\right)^{-1}, \quad \operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 154(3)
34	$\lvert\sin(ax)\rvert, \quad a > 0$	$a\left(s^2+a^2\right)^{-1}\coth\left(\dfrac{\pi s}{2a}\right), \quad \operatorname{Re} s > 0$ ET I 150(2)

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	$f(x)$	$F(s)$
35[11]	$\lvert\cos(ax)\rvert, \quad a > 0$	$\left(s^2+a^2\right)^{-1}\left[s + a \operatorname{cosech}\left(\frac{\pi s}{2a}\right)\right]$, $\operatorname{Re} s > 0$ ET I 155(44)
36	$\frac{1-\cos(ax)}{a^2}$	$s^{-1}\left(s^2+a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ AS 1022(29.3.19)
37	$\frac{ax-\sin(ax)}{a^3}$	$s^{-2}\left(s^2+a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ AS 1022(29.3.20)
38	$\frac{\sin(ax)-ax\cos(ax)}{2a^3}$	$\left(s^2+a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ AS 1022(29.3.21)
39	$\frac{x\sin(ax)}{2a}$	$s\left(s^2+a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 152(14)
40	$\frac{\sin(ax)+ax\cos(ax)}{2a}$	$s^2\left(s^2+a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ AS 1023(29.3.23)
41	$x\cos(ax)$	$\left(s^2-a^2\right)\left(s^2+a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 157(57)
42	$\frac{\cos(ax)-\cos(bx)}{b^2-a^2}$	$s\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Im} a\rvert, \lvert\operatorname{Im} b\rvert\}$ AS 1023(29.3.25)
43	$\frac{\left[\frac{1}{2}a^2x^2-1+\cos(ax)\right]}{a^4}$	$s^{-3}\left(s^2+a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$
44	$\frac{\left[1-\cos(ax)-\frac{1}{2}ax\sin(ax)\right]}{a^4}$	$s^{-1}\left(s^2+a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$
45	$\frac{\left[\frac{1}{b}\sin(bx)-\frac{1}{a}\sin(ax)\right]}{a^2-b^2}$	$\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Im} a\rvert, \lvert\operatorname{Im} b\rvert\}$
46[11]	$\frac{\left[1-\cos(ax)+\frac{1}{2}ax\sin(ax)\right]}{a^2}$	$s^{-1}\left(s^2+a^2\right)^{-2}\left(2s^2+a^2\right)$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$

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	$f(x)$	$F(s)$
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47	$\dfrac{a\sin(ax)-b\sin(bx)}{a^2-b^2}$	$s^2\left(s^2+a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Im}a\rvert,\lvert\operatorname{Im}b\rvert\}$
48	$\sin(a+bx)$	$(s\sin a+b\cos a)\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\lvert\operatorname{Im}b\rvert$
49	$\cos(a+bx)$	$(s\cos a-b\sin a)\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\lvert\operatorname{Im}b\rvert$
50	$\dfrac{\left[\frac{1}{a}\sinh(ax)-\frac{1}{b}\sin(bx)\right]}{a^2+b^2}$	$\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Im}b\rvert\}$
51	$\dfrac{\cosh(ax)-\cos(bx)}{a^2+b^2}$	$s\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Im}b\rvert\}$
52	$\dfrac{a\sinh(ax)+b\sin(bx)}{a^2+b^2}$	$s^2\left(s^2-a^2\right)^{-1}\left(s^2+b^2\right)^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Im}b\rvert\}$
53	$\sin(ax)\sin(bx)$	$2abs\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Im}a\rvert,\lvert\operatorname{Im}b\rvert\}$
54	$\cos(ax)\cos(bx)$	$s\left(s^2+a^2+b^2\right)\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Im}a\rvert,\lvert\operatorname{Im}b\rvert\}$
55	$\sin(ax)\cos(bx)$	$a\left(s^2+a^2-b^2\right)\left[s^2+(a-b)^2\right]^{-1}\left[s^2+(a+b)^2\right]^{-1}$, $\operatorname{Re}s>\{\lvert\operatorname{Im}a\rvert,\lvert\operatorname{Im}b\rvert\}$
56	$\sin^2(ax)$	$2a^2s^{-1}\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re}s>\lvert\operatorname{Im}a\rvert$
57	$\cos^2(ax)$	$\left(s^2+2a^2\right)s^{-1}\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re}s>\lvert\operatorname{Im}a\rvert$
58	$\sin(ax)\cos(ax)$	$a\left(s^2+4a^2\right)^{-1}$, $\operatorname{Re}s>\lvert\operatorname{Im}a\rvert$
59	$e^{-ax}\sin(bx)$	$b\left[(s+a)^2+b^2\right]^{-1}$, $\operatorname{Re}s>\{-\operatorname{Re}a,\quad\lvert\operatorname{Im}b\rvert\}$
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	$f(x)$	$F(s)$
60	$e^{-ax}\cos(bx)$	$(s+a)\left[(s+a)^2+b^2\right]^{-1}$, $\operatorname{Re} s > \{-\operatorname{Re} a, \quad \lvert\operatorname{Im} b\rvert\}$
61	$x^{-1}\sin(ax)$	$\arctan(a/s)$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 152(16)
62	$x^{-1}\left[1-\cos(ax)\right]$	$\frac{1}{2}\ln\left(1+a^2/s^2\right)$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$ ET I 157(59)
63	$\sinh(ax)$	$a\left(s^2-a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 162(1)
64	$\cosh(ax)$	$s\left(s^2-a^2\right)^{-1}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 162(2)
65	$x^{\nu-1}\sinh(ax)$, $\operatorname{Re}\nu > -1$	$\frac{1}{2}\Gamma(\nu)\left[(s-a)^{-\nu}-(s+a)^{-\nu}\right]$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 164(18)
66	$x^{\nu-1}\cosh(ax)$, $\operatorname{Re}\nu > 0$	$\frac{1}{2}\Gamma(\nu)\left[(s-a)^{-\nu}+(s+a)^{-\nu}\right]$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$ ET I 164(19)
67	$x\sinh(ax)$	$2as\left(s^2-a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
68	$x\cosh(ax)$	$\left(s^2+a^2\right)\left(s^2-a^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
69	$\sinh(ax)-\sin(ax)$	$2a^3\left(s^4-a^4\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Re} a\rvert, \lvert\operatorname{Im} a\rvert\}$ AS 1023(29.3.31)
70	$\cosh(ax)-\cos(ax)$	$2a^2s\left(s^4-a^4\right)^{-1}$, $\operatorname{Re} s > \{\lvert\operatorname{Re} a\rvert, \lvert\operatorname{Im} a\rvert\}$ AS 1023(29.3.32)
71	$\sinh(ax)+ax\cosh(ax)$	$2as^2\left(a^2-s^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$
72	$ax\cosh(ax)-\sinh(ax)$	$2a^3\left(a^2-s^2\right)^{-2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$

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	$f(x)$	$F(s)$
73	$x\sinh(ax)-\cosh(ax)$	$s\left(a^2+2a-s^2\right)\left(a^2-s^2\right)^{-2}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}a\rvert$
74	$\dfrac{\left[\frac{1}{a}\sinh(ax)-\frac{1}{b}\sinh(bx)\right]}{a^2-b^2}$	$\left(a^2-s^2\right)^{-1}\left(b^2-s^2\right)^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
75	$\dfrac{\cosh(ax)-\cosh(bx)}{a^2-b^2}$	$s\left(s^2-a^2\right)^{-1}\left(s^2-b^2\right)^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
76	$\dfrac{a\sinh(ax)-b\sinh(bx)}{a^2-b^2}$	$s^2\left(s^2-a^2\right)^{-1}\left(s^2-b^2\right)^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
77	$\sinh(a+bx)$	$(b\cosh a+s\sinh a)\left(s^2-b^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}b\rvert$
78	$\cosh(a+bx)$	$(s\cosh a+b\sinh a)\left(s^2-b^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}b\rvert$
79	$\sinh(ax)\sinh(bx)$	$2abs\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
80[8]	$\cosh(ax)\cosh(bx)$	$s\left(s^2-a^2-b^2\right)\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
81	$\sinh(ax)\cosh(bx)$	$a\left(s^2-a^2+b^2\right)\left[s^2-(a+b)^2\right]^{-1}\left[s^2-(a-b)^2\right]^{-1}$, $\quad \operatorname{Re}s>\{\lvert\operatorname{Re}a\rvert,\lvert\operatorname{Re}b\rvert\}$
82	$\sinh^2(ax)$	$2a^2s^{-1}\left(s^2-4a^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}a\rvert$
83	$\cosh^2(ax)$	$\left(s^2-2a^2\right)s^{-1}\left(s^2-4a^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}a\rvert$
84	$\sinh(ax)\cosh(ax)$	$a\left(s^2-4a^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}a\rvert$
85	$\dfrac{\cosh(ax)-1}{a^2}$	$s^{-1}\left(s^2-a^2\right)^{-1}$, $\quad \operatorname{Re}s>\lvert\operatorname{Re}a\rvert$

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	$f(x)$	$F(s)$		
86	$\dfrac{\sinh(ax) - ax}{a^3}$	$s^{-2}\left(s^2 - a^2\right)^{-1}$,	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
87	$\dfrac{\left[\cosh(ax) - \frac{1}{2}a^2x^2 - 1\right]}{a^4}$	$s^{-3}\left(s^2 - a^2\right)^{-1}$,	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
88	$\dfrac{\left[1 - \cosh(ax) + \frac{1}{2}ax\sinh(ax)\right]}{a^4}$	$s^{-1}\left(s^2 - a^2\right)^{-2}$,	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
89	$x^{1/2}\sinh(ax)$	$\left(\pi^{1/2}/4\right)\left[(s-a)^{3/2} - (s+a)^{3/2}\right]$,	$\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	
90	$\ln x$	$-s^{-1}\ln(\mathbf{C}s)$,	$\operatorname{Re} s > 0$	ET I 148(1)
91	$\ln(1 + ax)$, $\lvert\arg a\rvert < \pi$	$s^{-1}e^{s/a}\operatorname{Ei}(-s/a)$,	$\operatorname{Re} s > 0$	ET I 148(4)
92	$x^{-1/2}\ln x$	$-(\pi/s)^{1/2}\ln(4\mathbf{C}s)$,	$\operatorname{Re} s > 0$	ET I 148(9)
93	$\operatorname{H}(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases}$ (Heaviside step function)	$s^{-1}e^{-as}$,	$a \geq 0$	
94	$\delta(x)$ (Dirac delta function)	1		
95	$\delta(x-a)$	e^{-as},	$a \geq 0$	
96	$\delta'(x-a)$	se^{-as},	$a \geq 0$	
97	$\operatorname{Si}(x) \equiv \displaystyle\int_0^x \frac{\sin\xi}{\xi}\,d\xi \equiv \frac{1}{2}\pi + \operatorname{si}(x)$	$s^{-1}\operatorname{arccot} s$,	$\operatorname{Re} s > 0$	ET I 177(17)
98	$\operatorname{Ci}(x) \equiv \operatorname{ci}(x) \equiv -\displaystyle\int_x^\infty \frac{\cos\xi}{\xi}\,d\xi$	$-\dfrac{1}{2}s^{-1}\ln\left(1 + s^2\right)$,	$\operatorname{Re} s > 0$	ET I 178(19)
99[8]	$\operatorname{erf}\left(\dfrac{x}{2a}\right)$	$s^{-1}e^{a^2s^2}\operatorname{erfc}(as)$,	$\operatorname{Re} s > 0, \lvert\arg a\rvert < \pi/4$	ET I 176(2)

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	$f(x)$		$F(s)$	
100	$\operatorname{erf}\left(a\sqrt{x}\right)$		$as^{-1}\left(s+a^2\right)^{-1/2}$, $\operatorname{Re} s > \left\{0, -\operatorname{Re} a^2\right\}$	ET I 176(4)
101	$\operatorname{erfc}\left(a\sqrt{x}\right)$		$s^{-1}\left(s+a^2\right)^{-\frac{1}{2}}\left[\left(s+a^2\right)^{1/2}-a\right]$, $\operatorname{Re} s > 0$	ET I 177(9)
102[8]	$\operatorname{erfc}\left(\frac{a}{\sqrt{x}}\right)$		$s^{-1}e^{-2a\sqrt{s}}$, $\operatorname{Re} s > 0, \quad \operatorname{Re} a > 0$	ET I 177(11)
103[8]	$J_\nu(ax)$,	$\operatorname{Re}\nu > -1$	$a^{-\nu}\left(\sqrt{s^2+a^2}-s\right)^\nu\left(s^2+a^2\right)^{-1/2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$,	ET I 182(1)
104	$x\,J_\nu(ax)$,	$\operatorname{Re}\nu > -2$	$a^\nu\left[s+\nu\left(s^2+a^2\right)^{1/2}\right]\left[s+\left(s^2+a^2\right)^{1/2}\right]^{-\nu} \times\left(s^2+a^2\right)^{-3/2}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$,	ET I 182(2)
105	$\frac{J_\nu(ax)}{x}$		$a^\nu\nu^{-1}\left[s+\left(s^2+a^2\right)^{1/2}\right]^{-\nu}$, $\operatorname{Re} s \geq \lvert\operatorname{Im} a\rvert$	ET I 182(5)
106	$x^n\,J_n(ax)$		$1\cdot3\cdot5\cdots(2n-1)a^n\left(s^2+a^2\right)^{-\left(n+\frac{1}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$	ET I 182(4)
107	$x^\nu\,J_\nu(ax)$,	$\operatorname{Re}\nu > -\frac{1}{2}$	$2^\nu\pi^{-1/2}\,\Gamma\left(\nu+\frac{1}{2}\right)a^\nu\left(s^2+a^2\right)^{-\left(\nu+\frac{1}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$,	ET I 182(7)
108	$x^{\nu+1}\,J_\nu(ax)$,	$\operatorname{Re}\nu > -1$	$2^{\nu+1}\pi^{-1/2}\,\Gamma\left(\nu+\frac{3}{2}\right)a^\nu s\left(s^2+a^2\right)^{-\left(\nu+\frac{3}{2}\right)}$, $\operatorname{Re} s > \lvert\operatorname{Im} a\rvert$	ET I 182(8)
109[8]	$I_\nu(ax)$,	$\operatorname{Re}\nu > -1$	$a^{-\nu}\left[s-\sqrt{s^2-a^2}\right]^\nu\left(s^2-a^2\right)^{-1/2}$, $\operatorname{Re} s > \lvert\operatorname{Re} a\rvert$	ET I 195(1)

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	$f(x)$		$F(s)$
110	$x^{\nu}\, I_{\nu}(ax)$,	$\operatorname{Re}\nu > -\frac{1}{2}$	$2^{\nu}\pi^{-1/2}\,\Gamma\left(\nu+\frac{1}{2}\right) a^{\nu}\left(s^2-a^2\right)^{-\left(\nu+\frac{1}{2}\right)}$, $\operatorname{Re}s > \lvert\operatorname{Re}a\rvert$ ET I 195(6)
111	$x^{\nu+1}\, I_{\nu}(ax)$,	$\operatorname{Re}\nu > -1$	$2^{\nu+1}\pi^{-1/2}\,\Gamma\left(\nu+\frac{3}{2}\right) a^{\nu} s\left(s^2-a^2\right)^{-\left(\nu+\frac{3}{2}\right)}$, $\operatorname{Re}s > \lvert\operatorname{Re}a\rvert$ ET I 196(7)
112	$x^{-1}\, I_{\nu}(ax)$,	$\operatorname{Re}\nu > 0$	$\nu^{-1}a^{\nu}\left[s+\left(s^2-a^2\right)^{1/2}\right]^{-\nu}$, $\operatorname{Re}s > \lvert\operatorname{Re}a\rvert$ ET I 195(4)
113	$\sin\left(2a^{1/2}x^{1/2}\right)$		$(\pi a)^{1/2}s^{-3/2}e^{-a/s}$, $\operatorname{Re}s > 0$ ET I 153(32)
114	$x^{-1/2}\cos\left(2a^{1/2}x^{1/2}\right)$		$\pi^{1/2}s^{-1/2}e^{-a/s}$, $\operatorname{Re}s > 0$ ET I 158(67)
115	$x^{-1}e^{-ax}\, I_1(ax)$		$\left[(s+2a)^{1/2}-s^{1/2}\right]\left[(s+2a)^{1/2}+s^{1/2}\right]^{-1}$, $\operatorname{Re}s > \lvert\operatorname{Re}a\rvert$ AS 1024(29.3.52)
116	$\dfrac{J_k(ax)}{x}$		$k^{-1}a^{-k}\left[\left(s^2+a^2\right)^{1/2}-s\right]^{k}$, $\operatorname{Re}s > \lvert\operatorname{Im}a\rvert, k > -1$ AS 1025(29.3.58)
117	$\left(\dfrac{x}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(ax)$		$\Gamma(k)\pi^{-1/2}\left(s^2+a^2\right)^{k}$, $\operatorname{Re}s > \lvert\operatorname{Im}a\rvert$, $k > 0$ AS 1024(29.3.57)
118	$J_0(ax) - ax\, J_1(ax)$		$s^2\left(s^2+a^2\right)^{-3/2}$, $\operatorname{Re}s > \lvert\operatorname{Im}a\rvert$
119	$I_0(ax) + ax\, I_1(ax)$		$s^2\left(s^2-a^2\right)^{-3/2}$, $\operatorname{Re}s > \lvert\operatorname{Im}a\rvert$

17.21 Fourier transform

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function $f(x)$, denoted by $F(\xi)$, is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,dx.$$

The functions $f(x)$ and $F(\xi)$ are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered. Setting $F(\xi) = \mathcal{F}\left[f(x);\xi\right]$, to emphasize the nature of the transform, we have the symbolic inverse result $f(x) = \mathcal{F}^{-1}[F(\xi); x]$. The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x}\, d\xi.$$

17.22 Basic properties of the Fourier transform

1. For a and b arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \qquad \text{(linearity)}$$

2. If $n > 0$ is an integer, and $\lim_{|x|\to\infty} f^{(r)}(x) = 0$ for $r = 0, 1, \ldots, n-1$ with $f^{(0)}(x) \equiv f(x)$, then

$$\mathcal{F}\left[f^{(n)}(x); \xi\right] = (-i\xi)^n F(\xi) \qquad \text{(transform of a derivative)}$$ SN 27

3. The **Fourier convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi,$$

and it has the property $f * g = g * f$, and $f * (g * h) = (f * g) * h$. In terms of the convolution operation,

$$\mathcal{F}[f * g(x); \xi] = F(\xi)G(\xi) \qquad \text{(convolution [Faltung] theorem).}$$ SN 24

17.23 Table of Fourier transform pairs

	$f(x)$		$F(\xi)$	
1	1		$(2\pi)^{1/2}\,\delta(\xi)$	SU 496
2[7]	$\frac{1}{x}$		$(\pi/2)^{1/2}\, i \operatorname{sign} \xi$	SU 50
3	$\delta(x)$		$(2\pi)^{-1/2}$	SU 496
4[8]	$\delta(ax + b)$,	$a, b \in \mathbb{R}, \quad a \neq 0$	$(2\pi)^{-1/2} e^{ib\xi/a}$	SU 517
5	$\begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}$,	$a > 0$	$(2/\pi)^{1/2} \xi^{-1} \sin(a\xi)$	
6[8]	$\mathrm{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$		$-\frac{1}{i\xi\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}}\,\delta(\xi)$	SN 523

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	$f(x)$		$F(\xi)$		
7	$\dfrac{1}{\lvert x\rvert^a}$,	$0<\operatorname{Re} a<1$	$\dfrac{(2/\pi)^{1/2}\,\Gamma(1-a)\sin\left(\frac{1}{2}a\pi\right)}{\lvert\xi\rvert^{1-a}}$		SN 523
8	e^{iax},	$a\in\mathbb{R}$	$(2\pi)^{1/2}\,\delta(\xi+a)$		SU 50
9	$e^{-a\lvert x\rvert}$,	$a>0$	$\dfrac{a(2/\pi)^{1/2}}{a^2+\xi^2}$		SU 50
10[7]	$xe^{-a\lvert x\rvert}$,	$a>0$	$\dfrac{2ai\xi(2/\pi)^{1/2}}{(a^2+\xi^2)^2}$,	$\xi>0$	SU 50
11	$\lvert x\rvert e^{-a\lvert x\rvert}$,	$a>0$	$\dfrac{(2/\pi)^{1/2}\left(a^2-\xi^2\right)}{(a^2+\xi^2)^2}$		SU 50
12	$\dfrac{e^{-a\lvert x\rvert}}{\lvert x\rvert^{1/2}}$,	$a>0$	$\dfrac{\left[a+\left(a^2+\xi^2\right)^{1/2}\right]^{1/2}}{x\left(a^2+\xi^2\right)^{1/2}}$		SN 523
13	$e^{-a^2x^2}$,	$a>0$	$\left(a\sqrt{2}\right)^{-1} e^{-\xi^2/4a^2}$		SU 51
14	$\dfrac{1}{a^2+x^2}$,	$\operatorname{Re} a>0$	$\dfrac{(\pi/2)^{1/2}\,e^{-a\lvert\xi\rvert}}{a}$		SU 51
15[7]	$\dfrac{x}{a^2+x^2}$,	$\operatorname{Re} a>0$	$i\operatorname{sign}\xi\,(\pi/2)^{1/2}\,e^{-a\lvert\xi\rvert}$		
16[9]	$\sin\left(ax^2\right)$		$\dfrac{1}{(2a)^{1/2}}\cos\left(\dfrac{\xi^2}{4a}+\dfrac{\pi}{4}\right)$		SN 523
17	$\cos\left(ax^2\right)$		$\dfrac{1}{(2a)^{1/2}}\cos\left(\dfrac{\xi^2}{4a}-\dfrac{\pi}{4}\right)$		SN 523
18	$e^{-a\lvert x\rvert}\cos(bx)$,	$a>0,\quad b>0$	$a(2\pi)^{-1/2}\left[\dfrac{1}{a^2+(b+\xi)^2}+\dfrac{1}{a^2+(b-\xi)^2}\right]$		
19	$e^{-\frac{1}{2}ax^2}\sin(bx)$,	$a>0,\quad b>0$	$\dfrac{1}{2}ia^{-1/2}\left\{\exp\left[-\dfrac{1}{2}\dfrac{(\xi-b)^2}{a}\right] - \exp\left[-\dfrac{1}{2}\dfrac{(\xi+b)^2}{a}\right]\right\}$		
20[9]	$\dfrac{\sinh(ax)}{\sinh(bx)}$,	$\lvert a\rvert<\lvert b\rvert$	$\dfrac{(\pi/2)^{1/2}\sin(\pi a/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$		SU 123
21[9]	$\dfrac{\cosh(ax)}{\sinh(bx)}$,	$\lvert a\rvert<\lvert b\rvert$	$\dfrac{i\,(\pi/2)^{1/2}\sinh(\pi\xi/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$		SU 123

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	$f(x)$	$F(\xi)$
22	$\dfrac{\sin(ax)}{x}$	$\begin{cases}(\pi/2)^{1/2} & \lvert\xi\rvert < a,\\ 0 & \lvert\xi\rvert > a\end{cases}$ SN 523
23[11]	$\dfrac{x}{\sinh x}$	$\dfrac{\left(2\pi^3\right)^{1/2} e^{\pi\xi}}{\left(1+e^{\pi\xi}\right)^2}$ SU 123
24[7]	$x^n \operatorname{sign} x, \quad n = 1, 2, \ldots$	$(2/\pi)^{1/2}(-i\xi)^{-(1+n)} n!$ SU 506
25[7]	$\lvert x\rvert^{\nu}$, $-1 < \nu < 0$, but not integral	$(2/\pi)^{1/2}\,\Gamma(\nu+1)\lvert\xi\rvert^{-\nu-1}\cos\left[\pi(\nu+1)/2\right]$ SU506
26[7]	$\lvert x\rvert^{\nu} \operatorname{sign} x$, $-1 < \nu < 0$, but not integral	$\dfrac{i \operatorname{sign}\xi (2/\pi)^{1/2}\sin\left[(\pi/2)(\nu+1)\right]\Gamma(\nu+1)}{\lvert\xi\rvert^{\nu+1}}$ SU 506
27	$e^{-ax}\ln\left\lvert 1-e^{-x}\right\rvert$, $-1 < \operatorname{Re} a < 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\cot(\pi a - i\xi\pi)}{a - i\xi}$ ET I 121(26)
28	$e^{-ax}\ln\left(1+e^{-x}\right)$, $-1 < \operatorname{Re} a < 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\csc(\pi a - i\xi\pi)}{a - i\xi}$ ET I 121 (27)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $1/(2\pi)^{1/2}$ employed in our definition of F has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

17.24 Table of Fourier transform pairs for spherically symmetric functions

	$f(\lvert\lvert\mathbf{r}\rvert\rvert) = \dfrac{1}{(2\pi)^{3/2}}\displaystyle\iiint E(\lvert\lvert\mathbf{k}\rvert\rvert)e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{k}$	$E(\lvert\lvert\mathbf{k}\rvert\rvert) = \dfrac{1}{(2\pi)^{3/2}}\displaystyle\iiint f(\lvert\lvert\mathbf{r}\rvert\rvert)e^{-i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}$
1	$f(r) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{r}\displaystyle\int_0^\infty E(k)\sin(kr)k\,dk$	$E(k) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{k}\displaystyle\int_0^\infty f(r)\sin(kr)r\,dr$
2	e^{-ar}	$\sqrt{\dfrac{2}{\pi}}\dfrac{2a}{\left(a^2+k^2\right)^2}$
3[11]	$\dfrac{e^{-ar}}{r}$	$\sqrt{\dfrac{2}{\pi}}\dfrac{1}{\left(a^2+k^2\right)^2}$
4[11]	1	$(2\pi)^{3/2}\,\delta(\mathbf{k})$

17.31 Fourier sine and cosine transforms

The **Fourier sine** and **cosine transforms** of the function $f(x)$, denoted by $F_s(\xi)$ and $F_c(\xi)$, respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\xi x)\, dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\xi x)\, dx.$$

The functions $f(x)$ and $F_s(\xi)$ are called a **Fourier sine transform pair**, and the functions $f(x)$ and $F_c(\xi)$ a **Fourier cosine transform pair**, and knowledge of either $F_s(\xi)$ or $F_c(\xi)$ enables $f(x)$ to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s\left[f(x); \xi\right] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c\left[f(x); \xi\right],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s{}^{-1}\left[F_s(\xi); x\right] \quad \text{and} \quad f(x) = \mathcal{F}_c{}^{-1}\left[F_c(\xi); x\right].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\xi) \sin(\xi x)\, d\xi \qquad [x \ge 0]$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\xi) \cos(\xi x)\, d\xi \qquad [x \ge 0].$$ SN 17

17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s\left[af(x) + bg(x)\right] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c\left[af(x) + bg(x)\right] = aF_c(\xi) + bG_c(\xi) \qquad \text{(linearity)}$$

2. If $\lim_{x\to\infty} f^{(r-1)}(x) = 0$ and $\lim_{x\to\infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by $F_s{}^{(r)}$ and $F_c{}^{(r)}$, respectively,

(i) $F_c{}^{(r)}(\xi) = -a_{r-1} + \xi F_s{}^{(r-1)}$.

(ii) $F_s{}^{(r)}(\xi) = -\xi F_c{}^{(r-1)}(\xi)$,

(iii) $F_c{}^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2n} F_c(\xi)$,

(iv) $F_c{}^{(2r+1)}(\xi) = -\sum_{n=0}^{r-1}{}' (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi)$,

(v) $F_s{}^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s{}^{(r-2)}(\xi)$,

(vi)[6] $F_s{}^{(2r)}(\xi) = -\sum_{n=1}^{r} (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi)$,

(vii) $F_s{}^{(2r+1)}(\xi) = -\sum_{n=1}^{r}{}' (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1}\, \xi^{2r+1} F_c(\xi)$. SN 28

3. (i) $\int_0^\infty F_s(\xi) G_s(\xi) \cos(\xi x)\, d\xi = \frac{1}{2} \int_0^\infty g(s)\, [f(s+x) + f(s-x)]\, ds,$

(ii) $\int_0^\infty F_c(\xi) G_c(\xi) \cos(\xi x)\, d\xi = \frac{1}{2} \int_0^\infty g(s)\, [f(s+x) + f(|x-s|)]\, ds$

(convolution (Faltung) theorem) SN 24

4. (i) If $F_s(\xi)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $F_s(x)$ is $f(\xi)$.

(ii) If $F_c(\xi)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $F_c(x)$ is $f(\xi)$.

(iii) If $f(x)$ is an odd function in $(-\infty, \infty)$, then the Fourier sine transform of $f(x)$ in $(0, \infty)$ is $-iF(\xi)$.

(iv) If $f(x)$ is an even function in $(-\infty, \infty)$, then the Fourier cosine transform of $f(x)$ in $(0, \infty)$ is $F(\xi)$.

(v) The Fourier sine transform of $f(x/a)$ is $aF_s(a\xi)$.

(vi) The Fourier cosine transform of $f(x/a)$ is $aF_c(a\xi)$.

(vii) $\mathcal{F}_s\,[f(x);\xi] = F_s\,(|\xi|)\,\operatorname{sign}\xi$ SU 45

17.33 Table of Fourier sine transforms

	$f(x)$		$F_s(\xi)\qquad(\xi > 0)$		
1	x^{-1}		$(\pi/2)^{1/2}$,	$\xi > 0$	ET I 64(3)
2	$x^{-\nu}$,	$0 < \operatorname{Re}\nu < 2$	$(2/\pi)^{1/2}\xi^{\nu-1}\,\Gamma(1-\nu)\cos(\nu\pi/2)$,	$\xi > 0$	ET I 68(1)
3	$x^{-1/2}$		$\xi^{-1/2}$,	$\xi > 0$	ET I 64(6)
4	$x^{-3/2}$		$2\xi^{1/2}$,	$\xi > 0$	ET I 64(9)
5	$\begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$		$(2/\pi)^{1/2}\xi^{-1}\,[1-\cos(a\xi)]$,	$\xi > 0$	ET I 63(1)
6	$\begin{cases} x^{-1} & 0 < x < a \\ 0 & x > a \end{cases}$		$(2/\pi)^{1/2}\,\operatorname{Si}(a\xi)$,	$\xi > 0$	ET I 64(4)
7	$\frac{1}{a-x}$,	$a > 0$	$(2/\pi)^{1/2}\left\{\sin(a\xi)\operatorname{Ci}(a\xi) - \cos(a\xi)\left[\frac{1}{2}\pi + \operatorname{Si}(a\xi)\right]\right\}$,	$\xi > 0$	ET I 64(11)

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	$f(x)$		$F_s(\xi) \qquad (\xi > 0)$		
8[7]	$\dfrac{1}{x^2+a^2}$,	$a > 0$	$(2\pi)^{-1/2}a^{-1}\left[e^{-a\xi}\operatorname{Ei}(a\xi) - e^{a\xi}\operatorname{Ei}(-a\xi)\right]$,	$\xi > 0$	ET I 65(14)
9	$x\left(x^2+a^2\right)^{-3/2}$,	$\operatorname{Re} a > 0$	$(2/\pi)^{1/2}\xi\, K_0(a\xi)$,	$\xi > 0$	ET I 66(27)
10	$x^{-1/2}\left(x^2+a^2\right)^{-1/2}$,	$\operatorname{Re} a > 0$	$\xi^{1/2} I_{\frac{1}{4}}\left(\frac{1}{2}a\xi\right) K_{\frac{1}{4}}\left(\frac{1}{2}a\xi\right)$,	$\xi > 0$	ET I 66(28)
11[7]	$x\left(x^2+a^2\right)^{-\nu-\frac{3}{2}}$,	$\operatorname{Re}\nu > -1,\quad \operatorname{Re} a > 0$	$\dfrac{\xi^{\nu+1}}{\sqrt{2}(2a)^\nu\,\Gamma\left(\nu+\frac{3}{2}\right)} K_\nu(a\xi)$,		
12	$\dfrac{x}{a^2+x^2}$,	$\operatorname{Re} a > 0$	$\left(\dfrac{\pi}{2}\right)^{1/2} e^{-a\xi}$,	$\xi > 0$	ET I 65(15)
13	$\dfrac{x}{\left(a^2+x^2\right)^2}$		$\sqrt{\pi/8}a^{-1}\xi e^{-a\xi}$,	$\xi > 0$	ET I 67(35)
14	$x^{-1}\left(x^2+a^2\right)^{-1}$,	$\operatorname{Re} a > 0$	$\dfrac{\sqrt{\pi/2}}{a^2}\left(1-e^{-a\xi}\right)$,	$\xi > 0$	ET I 65(20)
15	$x^{-1}e^{-ax}$,	$\operatorname{Re} a > 0$	$(2/\pi)^{1/2}\tan^{-1}\left(\dfrac{\xi}{a}\right)$,	$\xi > 0$	ET I 72(2)
16	$x^{\nu-1}e^{-ax}$,	$\operatorname{Re}\nu > -1,\quad \operatorname{Re} a > 0$	$(2/\pi)^{1/2}\,\Gamma(\nu)\left(a^2+\xi^2\right)^{-\nu/2}\sin\left[\nu\tan^{-1}\left(\dfrac{\xi}{a}\right)\right]$,	$\xi > 0$	ET I 72(7)
17	e^{-ax},	$\operatorname{Re} a > 0$	$\dfrac{\sqrt{2/\pi}\xi}{a^2+\xi^2}$,	$\xi > 0$	ET I 72(1)
18	xe^{-ax},	$\operatorname{Re} a > 0$	$\dfrac{(2/\pi)^{1/2}2a\xi}{\left(a^2+\xi^2\right)^2}$,	$\xi > 0$	ET I 72(3)
19	xe^{-ax^2},	$\lvert\arg a\rvert < \pi/2$	$(2a)^{-3/2}\xi\exp\left(\dfrac{-\xi^2}{4a}\right)$,	$\xi > 0$	ET I 73(19)
20	$\dfrac{\sin ax}{x}$,	$a > 0$	$\dfrac{1}{(2\pi)^{1/2}}\ln\left\lvert\dfrac{\xi+a}{\xi-a}\right\rvert$,	$\xi > 0$	ET I 78(1)
21	$\dfrac{\sin ax}{x^2}$,	$a > 0$	$\begin{cases}\xi\left(\frac{\pi}{2}\right)^{1/2} & 0<\xi<a\\ a\left(\frac{\pi}{2}\right)^{1/2} & a<\xi<\infty\end{cases}$,	$\xi > 0$	ET I 78(2)
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	$f(x)$		$F_s(\xi) \qquad (\xi > 0)$		
22	$\sin\left(\dfrac{a^2}{x}\right),$	$a > 0$	$a\left(\dfrac{\pi}{2}\right)^{1/2}\xi^{-1/2}J_1\left(2a\xi^{\frac{1}{2}}\right),$	$\xi > 0$	ET I 83(6)
23	$x^{-1}\sin\left(\dfrac{a^2}{x}\right),$	$a > 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}Y_0\left(2a\xi^{1/2}\right)+\left(\dfrac{2}{\pi}\right)^{1/2}K_0\left(2a\xi^{1/2}\right)$		ET I 83(7)
24	$x^{-2}\sin\left(\dfrac{a^2}{x}\right),$	$a > 0$	$\left(\dfrac{\pi}{2}\right)^{1/2}a^{-1}\xi^{1/2}J_1\left(2a\xi^{1/2}\right),$	$\xi > 0$	ET I 83(8)
25[10]	$\operatorname{cosech}(ax),$	$\operatorname{Re} a > 0$	$(\pi/2)^{1/2}a^{-1}\tanh\left(\frac{1}{2}\pi a^{-1}\xi\right),$	$\xi > 0$	ET I 88(2)
26	$\coth\left(\dfrac{1}{2}ax\right)-1,$	$\operatorname{Re} a > 0$	$(2\pi)^{1/2}a^{-1}\coth\left(\pi a^{-1}\xi\right)-\xi,$	$\xi > 0$	ET I 88(3)
27	$\left(1-x^2\right)^{-1}\sin(\pi x)$		$\begin{cases}(2/\pi)^{1/2}\sin\xi & 0 \le \xi \le \pi \\ 0 & \pi < \xi\end{cases}$		ET I 78(4)
28	$e^{-ax^2}\sin(bx),$	$\operatorname{Re} a > 0$	$(2a)^{-1/2}\exp\left[-\left(\xi^2+b^2\right)/(4a)\right]\sinh\left(b\xi/2a\right),$	$\xi > 0$	ET I 78(7)
29	$\dfrac{\sin^2(ax)}{x},$	$a > 0$	$\begin{cases}\pi^{1/2}2^{-3/2} & 0 < \xi < 2a \\ \pi^{1/2}2^{-5/2} & \xi = 2a \\ 0 & 2a < \xi\end{cases}$		ET I 78(8)
30	$\sin\left(ax^2\right),$	$a > 0$	$a^{-1/2}\left\{\cos\left(\xi^2/4a\right)C\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $+\sin\left(\xi^2/4a\right)S\left[(2\pi a)^{-1/2}\xi\right],$	$\xi > 0$	ET I 82(1)
31	$\cos\left(ax^2\right),$	$a > 0$	$a^{-1/2}\left\{\sin\left(\xi^2/4a\right)C\left[(2\pi a)^{-1/2}\xi\right]\right\}$ $-\cos\left(\xi^2/4a\right)S\left[(2\pi a)^{-1/2}\xi\right],$	$\xi > 0$	

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	$f(x)$		$F_s(\xi) \qquad (\xi > 0)$		
32	$\arctan\left(\frac{x}{a}\right)$,	$a > 0$	$(\pi/2)^{1/2}\,\xi^{-1}e^{-a\xi}$,	$\xi > 0$	ET I 87(3)
33[7]	$\arctan\left(\frac{2a}{x}\right)$,	$\operatorname{Re} a > 0$	$(2\pi)^{-1/2}e^{-a\xi}\sinh(a\xi)$,	$\xi > 0$	ET I 87(8)
34	$\frac{\ln x}{x}$		$-(\pi/2)^{1/2}\left(\boldsymbol{C} + \ln\xi\right)$,	$\xi > 0$	ET I 76(2)
35	$\ln\left\lvert\frac{x+a}{x-a}\right\rvert$,	$a > 0$	$(2\pi)^{1/2}\xi^{-1}\sin(a\xi)$,	$\xi > 0$	ET I 77(11)
36[7]	$\frac{\ln\left(1+a^2x^2\right)}{x}$,	$a > 0$	$-(2\pi)^{1/2}\operatorname{Ei}\left(-\xi/a\right)$,	$\xi > 0$	ET I 77(14)
37	$J_0(ax)$,	$a > 0$	$\begin{cases} 0 & 0 < \xi < a \\ (2/\pi)^{1/2}\left(\xi^2 - a^2\right)^{-1/2} & a < \xi < \infty \end{cases}$		ET I 99(1)
38	$J_\nu(ax)$,	$\operatorname{Re}\nu > -2, \quad a > 0$	$(2/\pi)^{1/2}\left(a^2-\xi^2\right)^{-1/2}\sin\left[\nu\sin^{-1}\left(\frac{\xi}{a}\right)\right]$ for $0 < \xi < a$; $\frac{a^\nu\cos\left(\frac{1}{2}\nu\pi\right)}{\left(\xi^2-a^2\right)^{1/2}\left[\xi+\left(\xi^2-a^2\right)^{1/2}\right]^\nu}$ for $a < \xi < \infty$		ET I 99(3)
39	$\frac{J_0(ax)}{x}$,	$a > 0$	$\begin{cases} (2/\pi)^{1/2}\sin^{-1}\left(\frac{\xi}{a}\right) & 0 < \xi < a \\ (\pi/2)^{1/2} & a < \xi < \infty \end{cases}$		ET I 99(4)
40[7]	$\left(x^2+b^2\right)^{-1}J_0(ax)$,	$a > 0, \quad \operatorname{Re} b > 0$	$(2/\pi)^{1/2}\sinh(b\xi)\,K_0(ab)/b$,	$0 < \xi < a$	ET I 100(12)
41	$x\left(x^2+b^2\right)^{-1}J_0(ax)$,	$a > 0, \quad \operatorname{Re} b > 0$	$(\pi/2)^{1/2}\,e^{-b\xi}\,I_0(ab)$,	$a < \xi < \infty$	ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_s has not been used in those tables.

17.34 Table of Fourier cosine transforms

	$f(x)$		$F_c(\xi)$		
1	$x^{-\nu},$	$0 < \operatorname{Re}\nu < 1$	$(\pi/2)^{1/2}\left[\Gamma(\nu)\right]^{-1}\sec\left(\frac{1}{2}\nu\pi\right)\xi^{\nu-1},$	$\xi > 0$	ET I 10(1)
2	$\begin{cases}1 & 0<x<a\\ 0 & x>a\end{cases}$		$(2/\pi)^{1/2}\dfrac{\sin(a\xi)}{\xi},$	$\xi > 0$	ET I 7(1)
3	$\begin{cases}0 & 0<x<a\\ 1/x & x>a\end{cases}$		$-(2/\pi)^{1/2}\operatorname{Ci}(a\xi),$	$\xi > 0$	ET I 8(3)
4	$\begin{cases}x^{-1/2} & 0<x<a\\ 0 & x>a\end{cases}$		$2\xi^{-1/2}\,C(a\xi),$	$\xi > 0$	ET I 8(5)
5	$\begin{cases}0 & 0<x<a\\ x^{-1/2} & x>a\end{cases}$		$2\xi^{-1/2}\left[\frac{1}{2} - C(a\xi)\right],$	$\xi > 0$	ET I 8(6)
6[9]	$x^{\nu-1},$	$0 < \nu < 1$	$(2/\pi)^{1/2}\,\Gamma(\nu)\xi^{-\nu}\cos\left(\frac{1}{2}\nu\pi\right),$	$0 < \nu < 1$	ET I 10(1)
7	$\dfrac{1}{x^2+a^2},$	$\operatorname{Re}a > 0$	$\dfrac{(\pi/2)^{1/2}\,e^{-a\xi}}{a},$	$\xi > 0$	ET I 11(7)
8[11]	$\dfrac{1}{\left(x^2+a^2\right)^2},$	$\operatorname{Re}a > 0$	$\dfrac{(\pi/2)^{\frac{1}{2}}\,(1+a\xi)e^{-a\xi}}{2a^3},$	$\xi > 0$	ET I 11(7)
9	$\left(x^2+a^2\right)^{-\nu-\frac{1}{2}},$	$\operatorname{Re}a > 0, \operatorname{Re}\nu > -\dfrac{1}{2}$	$\sqrt{2}\left(\dfrac{\xi}{2a}\right)^{\nu}\dfrac{K_\nu(a\xi)}{\Gamma\left(\nu+\frac{1}{2}\right)},$	$\xi > 0$	ET I 11(7)
10	$\begin{cases}\left(a^2-x^2\right)^{\nu} & 0<x<a\\ 0 & x>a\end{cases},$	$\operatorname{Re}\nu > -1$	$2^{\nu}\,\Gamma(\nu+1)(a/\xi)^{\nu+\frac{1}{2}}\,J_{\nu+\frac{1}{2}}(a\xi),$	$\xi > 0$	ET I 11(8)
11	$\begin{cases}0 & 0<x<a\\ \left(x^2-a^2\right)^{-\nu-\frac{1}{2}} & x>a\end{cases},$	$-\dfrac{1}{2} < \operatorname{Re}\nu < \dfrac{1}{2}$	$-2^{-\left(\nu+\frac{1}{2}\right)}\,\Gamma\left(\frac{1}{2}-\nu\right)(\xi/a)^{\nu}\,Y_\nu(a\xi),$	$\xi > 0$	ET I 11(9)
12	$e^{-ax},$	$\operatorname{Re}a > 0$	$(2/\pi)^{1/2}a\left(a^2+\xi^2\right)^{-1},$	$\xi > 0$	ET I 14(1)

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	$f(x)$		$F_c(\xi)$		
13	$xe^{-ax},$	$\operatorname{Re} a > 0$	$(2/\pi)^{1/2}\left(a^2-\xi^2\right)\left(a^2+\xi^2\right)^{-2},$	$\xi > 0$	ET I 15(7)
14[7]	$x^{\nu-1}e^{-ax},$	$\operatorname{Re} a > 0, \quad \operatorname{Re}\nu > a$	$(2/\pi)^{1/2}\,\Gamma(\nu)\left(a^2+\xi^2\right)^{-\nu/2}\cos\left[\nu\tan^{-1}\left(\frac{\xi}{a}\right)\right],$	$\xi > 0$	ET I 15(7)
15	$x^{-1/2}e^{-ax},$	$\operatorname{Re} a > 0$	$\left(a^2+\xi^2\right)^{-1/2}\left[\left(a^2+\xi^2\right)^{1/2}+a\right]^{1/2},$	$\xi > 0$	ET I 14(4)
16[7]	$e^{-a^2x^2},$	$\operatorname{Re} a > 0$	$2^{-1/2}\lvert a\rvert^{-1}e^{-\xi^2/4a^2},$	$\xi > 0$	ET I 15(11)
17	$x^{-1}e^{-x}\sin x$		$(2\pi)^{-1/2}\tan^{-1}\left(\frac{2}{\xi^2}\right),$	$\xi > 0$	ET I 19(7)
18	$\sin\left(ax^2\right),$	$a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right)-\sin\left(\frac{\xi^2}{4a}\right)\right],$	$\xi > 0$	ET I 23(1)
19	$\cos\left(ax^2\right),$	$a > 0$	$\frac{1}{2\sqrt{a}}\left[\cos\left(\frac{\xi^2}{4a}\right)+\sin\left(\frac{\xi^2}{4a}\right)\right],$	$\xi > 0$	ET I 24(7)
20	$\frac{\sin(ax)}{x},$	$a > 0$	$\begin{cases}(\pi/2)^{1/2} & \xi < a\\ \frac{1}{2}(\pi/2)^{1/2} & \xi = a\\ 0 & \xi > a\end{cases}$		ET I 18(1)
21[7]	$\frac{\sin^2(ax)}{x^2},$	$a > 0$	$\begin{cases}(\pi/2)^{1/2}\left(a-\frac{1}{2}\xi\right) & \xi < 2a\\ 0 & 2a < \xi\end{cases}$		ET I 19(8)
22[7]	$e^{-bx}\sin(ax), \quad a > 0,$	$\operatorname{Re} b > 0$	$(2\pi)^{-1/2}\left[\frac{a+\xi}{b^2+(a+\xi)^2}+\frac{a-\xi}{b^2+(a-\xi)^2}\right],$	$\xi > 0$	ET I 19(6)
23	$\frac{\sin\left[b\left(x^2+a^2\right)^{1/2}\right]}{\left(x^2+a^2\right)^2}$	$a > 0$	$(b/a)(\pi/2)^{1/2}e^{-a\xi},$	$\xi > 0$	ET I 26(29)
24	$\left(x^2+a^2\right)^{-1/2}\sin\left[b\left(x^2+a^2\right)^{1/2}\right],$	$a > 0$	$\begin{cases}(\pi/2)^{1/2}J_0\left[a\left(b^2-\xi^2\right)^{1/2}\right] & 0 < \xi < b\\ 0 & b < \xi\end{cases}$		ET I 26(30)

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	$f(x)$		$F_c(\xi)$	
25	$\dfrac{1-\cos(ax)}{x^2}$,	$a>0$	$\begin{cases}(\pi/2)^{1/2}(a-\xi) & \xi<a\\ 0 & a<\xi\end{cases}$	ET I 20(16)
26	$e^{-ax^2}\sin\left(bx^2\right)$,	$\operatorname{Re} a>\lvert\operatorname{Im} b\rvert$	$2^{-1/2}\left(a^2+b^2\right)^{-1/4}\exp\left\{-a\xi^2/\left[4\left(a^2+b^2\right)\right]\right\}$ $\times\sin\left[\frac{1}{2}\arctan(b/a)-\frac{1}{4}b\xi^2\left(a^2+b^2\right)^{-1}\right]$, $\xi>0$	ET I 23(5)
27	$e^{-ax^2}\cos\left(bx^2\right)$,	$\operatorname{Re} a>\lvert\operatorname{Im} b\rvert$	$2^{-1/2}\left(a^2+b^2\right)^{-1/4}\exp\left\{-a\xi^2/\left[4\left(a^2+b^2\right)\right]\right\}$ $\times\cos\left[\frac{1}{4}b\xi^2\left(a^2+b^2\right)^{-1}-\frac{1}{2}\arctan(b/a)\right]$, $\xi>0$	ET I 24(6)
28	$\dfrac{\sinh(ax)}{\sinh(bx)}$	$\lvert\operatorname{Re} a\rvert<\operatorname{Re} b$	$\left(\dfrac{\pi}{2}\right)^{1/2}\dfrac{\sin(\pi a/b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$, $\xi>0$	ET I 31(14)
29	$\dfrac{\cosh(ax)}{\cosh(bx)}$,	$\lvert\operatorname{Re} a\rvert<\operatorname{Re} b$	$\dfrac{(2\pi)^{1/2}\cos(\pi a/2b)\cosh(\pi\xi/2b)}{b\left[\cosh(\pi\xi/b)+\cos(\pi a/b)\right]}$, $\xi>0$	ET I 31(12)
30	$\operatorname{sech}(ax)$,	$\operatorname{Re} a>0$	$a^{-1}(\pi/2)^{1/2}\operatorname{sech}(\pi\xi/2a)$, $\xi>0$	ET I 30(1)
31	$\left(x^2+a^2\right)\operatorname{sech}\left(\dfrac{\pi x}{2a}\right)$,	$\operatorname{Re} a>0$	$2(2/\pi)^{1/2}a^3\operatorname{sech}^3(a\xi)$, $\xi>0$	ET I 32(19)
32	$\ln\left(1+\dfrac{a^2}{x^2}\right)$,	$\operatorname{Re} a>0$	$(2\pi)^{1/2}\xi^{-1}\left(1-e^{-a\xi}\right)$, $\xi>0$	ET I 18(10)
33[7]	$\ln\left(\dfrac{a^2+x^2}{b^2+x^2}\right)$,	$\operatorname{Re} a>0, \operatorname{Re} b>0$	$(2\pi)^{1/2}\left(e^{-b\xi}-e^{-a\xi}\right)$, $\xi>0$	ET I 18(12)
34	$\left(x^2+b^2\right)^{-1}J_0(ax)$,	$a>0,\quad \operatorname{Re} b>0$	$(\pi/2)^{1/2}b^{-1}e^{-b\xi}I_0(ab)$, $a<\xi<\infty$	ET I 45(14)

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	$f(x)$	$F_c(\xi)$
35	$x\left(x^2+b^2\right)^{-1} J_0(ax)$, $\quad a>0, \quad \operatorname{Re} b>0$	$(2/\pi)^{1/2}\cosh(b\xi)\, K_0(ab)$, $\quad 0<\xi<a$ ET I 45(15)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_c has not been used in those tables.

17.35 Relationships between transforms

The following relationships exist between transforms, and they may be used to derive further transform pairs from among the results given in Sections 17.13–17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

17.351

Fourier cosine transform and Laplace transform relationship

$$\mathcal{F}_c\left[f(x);\xi\right] = \frac{1}{\sqrt{2\pi}}\mathcal{L}\left[f(x);i\xi\right] + \frac{1}{\sqrt{2\pi}}\mathcal{L}\left[f(x);-i\xi\right].$$

17.352

Fourier sine transform and Laplace transform relationship

$$\mathcal{F}_s\left[f(x);\xi\right] = \frac{i}{\sqrt{2\pi}}\mathcal{L}\left[f(x);i\xi\right] - \frac{i}{\sqrt{2\pi}}\mathcal{L}\left[f(x);-i\xi\right].$$

17.353

Exponential Fourier transform and Laplace transform relationship

$$\mathcal{F}\left[f(x);\xi\right] = \sqrt{2\pi}\mathcal{L}\left[f(x);-i\xi\right] + \sqrt{2\pi}\mathcal{L}\left[f(-x);i\xi\right].$$

17.41[10] Mellin transform

The **Mellin transform** of the function $f(x)$, denoted by $f^*(s)$, is defined by the integral

$$f^*(s) = \int_0^\infty f(x)x^{s-1}\,dx.$$

The functions $f(x)$ and $f^*(s)$ are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

The transform exists, provided the integral

$$\int_0^\infty |f(x)|x^{k-1}\,dx$$

is bounded for some $k>0$, and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s}\, ds,$$

where $c > k$.

Setting

$$f^*(s) = \mathcal{M}\left[f(x); s\right]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = \mathcal{M}^{-1}\left[f^*(s); x\right].$$ MS 397(6)

17.42 Basic properties of the Mellin transform

1. For a and b arbitrary constants,

$$\mathcal{M}\left[af(x) + bg(x)\right] = af^*(s) + bg^*(s) \qquad \text{(linearity)}$$

2. If $\lim\limits_{x\to 0} x^{s-r-1} f^{(r)}(x) = 0, \quad r = 0, 1, \ldots, n-1,$

(i) $$\mathcal{M}\left[f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n)$$

(transform of a derivative) SU 267 (4.2.3)

(ii) $$\mathcal{M}\left[x^n f^{(n)}(x); s\right] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s)$$

(transform of a derivative) SU 267 (4.2.5)

3. Denoting the n^{th} repeated integral of $f(x)$ by $I_n\left[f(x)\right]$, where

$$I_n\left[f(x)\right] = \int_0^x I_{n-1}\left[f(u)\right] du,$$

(i) $$\mathcal{M}\left[I_n\left[f(x)\right]; s\right] = (-1)^n \frac{\Gamma(s)}{\Gamma(n+s)} f^*(s+n)$$

(transform of an integral) SU 269 (4.2.15)

(ii) $$\mathcal{M}\left[I_n^\infty\left[f(x)\right]; s\right] = \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n),$$

where

$$I_n^\infty\left[f(x)\right] = \int_x^\infty I_{n-1}^\infty\left[f(u)\right] du \qquad \text{(transform of an integral)}$$ SU 269 (4.2.18)

4. $$\mathcal{M}\left[f(x)g(x); s\right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(u) g^*(s-u)\, du$$

(Mellin convolution theorem) SU 275(4.4.1)

17.43 Table of Mellin transforms

	$f(x)$	$f^*(s)$		
1	e^{-x}	$\Gamma(s)$,	$\operatorname{Re} s > 0$	SU 521(M13)
2	e^{-x^2}	$\frac{1}{2}\Gamma\left(\frac{1}{2}s\right)$,	$\operatorname{Re} s > 0$	SU 521(M14)
3	$\cos x$	$\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)$,	$0 < \operatorname{Re} s < 1$	SU 521(M15)
4	$\sin x$	$\Gamma(s)\sin\left(\frac{1}{2}\pi s\right)$,	$0 < \operatorname{Re} s < 1$	SU 521(M16)
5	$\dfrac{1}{1-x}$	$\pi\cot(\pi s)$,	$0 < \operatorname{Re} s < 1$	SU 521(M1)
6	$\dfrac{1}{1+x}$	$\pi\operatorname{cosec}(\pi s)$,	$0 < \operatorname{Re} s < 1$	SU 521(M2)
7	$(1+x^a)^{-b}$	$\dfrac{\Gamma(s/a)\,\Gamma(b-s/a)}{a\,\Gamma(b)}$,	$0 < \operatorname{Re} s < ab$	SU 521(M3)
8	$\dfrac{T_n(x)\,\mathrm{H}(1-x)}{\sqrt{(1-x^2)}}$	$\dfrac{2^{-s}\pi\,\Gamma(s)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)}$,	$\operatorname{Re} s > 0$	SU 521(M4)
9	$\dfrac{T_n\left(x^{-1}\right)\mathrm{H}(1-x)}{\sqrt{(1-x^2)}}$	$\dfrac{2^{s-2}\Gamma\left(\frac{1}{2}n+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}n\right)}{\Gamma(s)}$,	$\operatorname{Re} s > n$	SU 521(M5)
10	$P_n(x)\,\mathrm{H}(1-x)$	$\dfrac{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\right)}{2\,\Gamma\left(\frac{1}{2}s-\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}n+1\right)}$,	$\operatorname{Re} s > 0$	SU 521(M6)
11	$P_n\left(x^{-1}\right)\mathrm{H}(1-x)$	$\dfrac{2^{s-1}\Gamma\left(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}n\right)}{\sqrt{\pi}\,\Gamma(s+1)}$,	$\operatorname{Re} s > n$	SU 521(M7)
12	$\dfrac{1+x\cos\phi}{1-2x\cos\phi+x^2}$	$\dfrac{\pi\cos(s\phi)}{\sin(s\pi)}$,	$0 < \operatorname{Re} s < 1$	SU 521(M11)
13	$\dfrac{x\sin\phi}{1-2x\cos\phi+x^2}$, $\quad -\pi < \phi < \pi$	$\dfrac{\pi\sin(s\phi)}{\sin(s\pi)}$,	$0 < \operatorname{Re} s < 1$	SU 521(M12)

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	$f(x)$	$f^*(s)$
14	$e^{-x\cos\phi}\cos(x\sin\phi)$, $\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s)\cos(s\phi)$, $\operatorname{Re} s > 0$ SU 522(M17)
15	$e^{-x\sin\phi}\sin(x\sin p\phi)$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$	$\Gamma(s)\sin(s\phi)$, $\operatorname{Re} s > -1$ SU 522(M18)
16	$x^{-\nu} J_\nu(x)$, $\nu > -\frac{1}{2}$	$\dfrac{2^{s-\nu-1}\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\nu-\frac{1}{2}s+1\right)}$, $0 < \operatorname{Re} s < 1$ SU 522(M19)
17	$Y_\nu(x)$, $\nu \in \mathbb{R}$	$-2^{s-1}\pi^{-1}\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)$ $\times\cos\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\pi$, $\lvert\nu\rvert < \operatorname{Re} s < \frac{3}{2}$ SU 522(M20)
18	$K_\nu(x)$, $\nu \in \mathbb{R}$	$2^{s-2}\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)$, $\operatorname{Re} s > \nu > 0$ SU 522(M21)
19	$\mathbf{H}_\nu(x)$, $\nu \in \mathbb{R}$	$\dfrac{2^{s-1}\tan\left(\frac{1}{2}\pi s+\frac{1}{2}\pi\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}s+1\right)}$, $-1-\nu < \operatorname{Re} s < \min\left(\frac{3}{2}, 1-\nu\right)$ SU 522(M22)
20	$\dfrac{1}{a+x^n}$, $\lvert\arg a\rvert < \pi$, $n = 1, 2, 3, \dots$,	$\pi n^{-1}\operatorname{cosec}\left(\dfrac{\pi s}{n}\right)a^{(s/n)-1}$, $0 < \operatorname{Re} s < n$ MS 453
21	$\left(1+ax^h\right)^{-\nu}$, $h > 0$, $\lvert\arg a\rvert < \pi$	$h^{-1}a^{-s/h}\,\mathrm{B}\left(s/h, \nu-(s/h)\right)$ $0 < \operatorname{Re} s < h\operatorname{Re}\nu$ MS 454
22	$\begin{cases}\left(1-x^h\right)^{\nu-1} & \text{for } 0 < x < 1\\ 0 & \text{for } x > 1\end{cases}$, $h > 0$, $\operatorname{Re}\nu > 0$	$h^{-1}\,\mathrm{B}\left(\nu, s/h\right)$ MS 454
23	$\ln(1+ax)$, $\lvert\arg a\rvert < \pi$	$\pi s^{-1}a^{-s}\operatorname{cosec}(\pi s)$, $-1 < \operatorname{Re} s < 0$ MS 454
24	$\arctan x$	$-\frac{1}{2}\pi s^{-1}\sec(\pi s/2)$, $-1 < \operatorname{Re} s < 0$ MS 454

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	$f(x)$		$f^*(s)$		
25	$\operatorname{arccot} x$		$\frac{1}{2}\pi s^{-1}\sec(\pi s/2),$	$0 < \operatorname{Re} s < 1$	MS 454
26	$\operatorname{cosech}(ax)$	$\operatorname{Re} a > 0$	$a^{-s} 2\left(1-2^{-s}\right)\Gamma(s)\,\zeta(s),$	$\operatorname{Re} s > 1$	MS 454
27	$\operatorname{sech}^2(ax),$	$\operatorname{Re} a > 0$	$4a^{-s}(1-2^{2-s})\,\Gamma(s)2^{-s}\,\zeta(s-1),$	$\operatorname{Re} s > 2$	MS 454
28	$\operatorname{cosech}^2(ax),$	$\operatorname{Re} a > 0$	$4a^{-s}\,\Gamma(s)2^{-s}\,\zeta(s-1),$	$\operatorname{Re} s > 2$	MS 454
29[11]	$\left(x^2+b^2\right)^{-\frac{1}{2}\nu} J_\nu\left[a\left(x^2+b^2\right)^{1/2}\right]$		$2^{\frac{1}{2}s-1}a^{-\frac{1}{2}s}b^{\frac{1}{2}s-\nu}\,\Gamma\left(\frac{1}{2}s\right) J_{\nu-s/2}(ab),$	$0 < \operatorname{Re} s < \frac{3}{2}+\operatorname{Re}\nu$	ET I 328
30	$\begin{cases}\left(a^2-x^2\right)^{\frac{1}{2}\nu} J_\nu\left[a\left(b^2-x^2\right)^{1/2}\right] & \text{for } 0<x<a\\ 0 & \text{for } x>a\end{cases}$	$\operatorname{Re}\nu > -1$	$2^{\frac{1}{2}s-1}\,\Gamma\left(\frac{1}{2}s\right) b^{-\frac{1}{2}s}a^{\nu+\frac{1}{2}s}\,J_{\nu+\frac{1}{2}s}(ab),$	$\operatorname{Re} s > 0$	MS 455
31	$\begin{cases}\left(a^2-x^2\right)^{-\frac{1}{2}\nu} J_\nu\left[b\left(a^2-x^2\right)^{1/2}\right] & \text{for } 0<x<a\\ 0 & \text{for } x>a\end{cases}$		$2^{1-\nu}\left[\Gamma(\nu)\right]^{-1} a^{\frac{1}{2}s-\nu}b^{-,\frac{1}{2}\nu}\; s_{\nu-1+\frac{1}{2}s,\frac{1}{2}s-\nu}(ab),$	$\operatorname{Re} s > 0$	MS 455
32	$K_\nu(\alpha x)$		$\alpha^{-s}2^{s-2}\,\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right),$	$\operatorname{Re} s > \lvert\operatorname{Re}\nu\rvert$	MS 455
33	$\left(\beta a^2+x^2\right)^{-\frac{1}{2}\nu} \times K_\nu\left[\alpha\left(\beta a^2+x^2\right)^{1/2}\right]$	$\operatorname{Re}(\alpha,\beta) > 0$	$\alpha^{-\frac{1}{2}s}2^{\frac{1}{2}s-1}\beta^{\frac{1}{2}s-\nu}\,\Gamma,\left(\frac{1}{2}s\right) K_{\nu-\frac{1}{2}s}(\alpha\beta),$	$\operatorname{Re} s > 0$	MS 455

18 The z-Transform

18.1–18.3 Definition, Bilateral, and Unilateral z-Transforms

18.1 Definitions

The ***z*-transform** converts a numerical sequence $x[n]$ into a function of the complex variable z, and it takes two different forms. The **bilateral** or **two-sided *z*-transform**, denoted here by $Z_b\{x[n]\}$, is used mainly in signal and image processing, while the **unilateral** or **one-sided *z*-transform**, denoted here by $Z_u\{x[n]\}$, is used mainly in the analysis of discrete time systems and the solution of linear difference equations.

The **bilateral *z*-transform**, $X_b(z)$ of the sequence $x[n] = \{x_n\}_{n=-\infty}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=-\infty}^{\infty} x_n z^{-n} = X_b(z),$$

and the **unilateral *z*-transform** $X_u(z)$ of the sequence $x[n] = \{x_n\}_{n=0}^{\infty}$ is defined as

$$Z_b\{x[n]\} = \sum_{n=0}^{\infty} x_n z^{-n} = X_u(z),$$

where each has its own domain of convergence (DOC). The series $X_b(z)$ is a Laurent series, and $X_u(z)$ is the principal part of the Laurent series for $X_b(z)$. When $x_n = 0$ for $n < 0$, the two z-transforms $X_b(z)$ and $X_u(z)$ are identical. In each case the sequence $x[n]$ and its associated z-transform is a called a ***z*-transform pair**.

The inverse z-transformation $x[n] = Z^{-1}\{X(z)\}$ is given by

$$x[n] = \frac{1}{2\pi i}\int_{\Gamma} X(z) z^{n-1}\,dz,$$

where $X(z)$ is either $X_b(z)$ or $X_u(z)$, and Γ is a simple closed contour containing the origin and lying entirely within the domain of convergence of $X(z)$. In many practical situations, the z-transform is either found by using a series expansion of $X(z)$ in the inversion integral or, if $X(z) = N(z)/D(z)$ where $N(z)$ and $D(z)$ are polynomials in z, by means of partial fractions and the use of an appropriate table of z-transform pairs. In order for the inverse z-transform to be unique, it is necessary to specify the domain of convergence, as can be seen by comparison of entries 3 and 4 of Table 18.2. Table 18.1 lists general properties of the bilateral z-transform, and Table 18.2 lists some bilateral z-transform pairs. In what follows, use is made of the **unit integer function** $h(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$, that is, a generalization of the Heaviside step function, and the **unit integer pulse function** $\Delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$, that is, a generalization of the delta function.

18.2 Bilateral z-transform

Table 18.1 General properties of the bilateral z-transform $X_b(n) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$.

	Term in sequence	z-Transform $X_b(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_b(z) + \beta Y_b(z)$	Intersection of DOC's of $X_b(z)$ and $Y_b(z)$ with α, β constants
2	x_{n-N}	$z^{-n} X_b(z)$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin or the point at infinity
3	nx_n	$-z\dfrac{dX_b(z)}{dz}$	DOC of $X_b(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_b\left(\dfrac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $\lvert z_0 \rvert$
5	$nz_0^n x_n$	$-z\dfrac{dX_b(z/z_0)}{dz}$	DOC of $X_b(z)$ scaled by $\lvert z_0 \rvert$ to which it may be necessary to add or delete the origin and the point at infinity
6	x_{-n}	$X_b(1/z)$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
7	nx_{-n}	$-z\dfrac{dX_b(1/z)}{dz}$	DOC of radius $1/R$, where R is the radius of convergence of DOC of $X_b(z)$
8	$\bar{x}_n$	$\overline{X_b}(\overline{z})$	The same DOC as x_n
9	$\operatorname{Re} x_n$	$\frac{1}{2}\left[X_b(z) + \overline{X_b}(\overline{z})\right]$	DOC contains the DOC of x_n
10	$\operatorname{Im} x_n$	$\frac{1}{2i}\left[X_b(z) - \overline{X_b}(\overline{z})\right]$	DOC contains the DOC of x_n
11	$\sum_{k=-\infty}^{\infty} x_k y_{n-k}$	$X_b(z)Y_b(z)$	DOC contains the intersection of the DOCs of $X_b(z)$ and $Y_b(z)$ (convolution theorem)
12	$x_n y_n$	$\dfrac{1}{2\pi i}\int_\Gamma X_b(\xi) Y_b\left(\dfrac{z}{\xi}\right) \xi^{-1}\, d\xi$	DOC contains the DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin (convolution theorem)
13	Parseval formula	$\sum_{n=-\infty}^{\infty} x_n \bar{y}_n = \dfrac{1}{2\pi i}\int_\Gamma X_b(\xi)\overline{Y_b}\left(\dfrac{z}{\overline{\xi}}\right) \xi^{-1}\, d\xi$	DOC contains the intersection of DOCs of $X_b(z)$ and $Y_b(z)$, with Γ inside the DOC and containing the origin
14	Initial value theorem for $x_n h(n)$	$x_0 = \lim_{z\to\infty} X_b(z)$	

Table 18.2 Basic bilateral z-transforms

	Term in sequence	z-Transform $X_b(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n-N)$	z^{-n}	When $N>0$ convergence is for all z except at the origin. When $N<0$ convergence is for all z except at ∞
3	$a^n h(n)$	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
4	$a^n h(-n-1)$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
5	$na^n h(n)$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > a > 0$
6	$na^n h(-n-1)$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert < a, \quad a>0$
7	$n^2 a^n h(n)$	$\dfrac{az(z+a)}{(z-a)^3}$	$\lvert z\rvert > a > 0$
8	$\left(\dfrac{1}{a^n}+\dfrac{1}{b^n}\right)h(n)$	$\dfrac{az}{az-1}+\dfrac{bz}{bz-1}$	$\lvert z\rvert > \max\left(\frac{1}{\lvert a\rvert}, \frac{1}{\lvert b\rvert}\right)$
9	$a^n h(n-N)$	$\dfrac{z\left(1-(a/z)^N\right)}{z-a}$	$\lvert z\rvert > 0$
10	$a^n h(n)\sin\Omega n$	$\dfrac{az\sin\Omega}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a > 0$
11	$a^n h(n)\cos\Omega n$	$\dfrac{z(z-a\cos\Omega)}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a > 0$
12	$e^{an}h(n)$	$\dfrac{z}{z-e^a}$	$\lvert z\rvert > e^{-a}$
13	$e^{-an}h(n)\sin\Omega n$	$\dfrac{ze^a\sin\Omega}{z^2e^{2a}-2ze^a\cos\Omega+1}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}h(n)\cos\Omega n$	$\dfrac{ze^a(ze^a-\cos\Omega)}{z^2e^{2a}-2ze^a\cos\Omega+1}$	$\lvert z\rvert > e^{-a}$

18.3 Unilateral z-transform

The relationship between the Laplace transform of a continuous function $x(t)$ sampled at $t = 0,\ T,\ 2T,$ $\ldots$ and the unilateral z-transform of the function $\hat{x}(t) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$ follows from the result

$$\begin{aligned}\mathcal{L}\{\hat{x}(t)\} &= \int_0^{\infty} \left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] e^{-st}\,dt \\ &= \sum_{k=0}^{\infty} x(kT)e^{-ksT}.\end{aligned}$$

Setting $z = e^{sT}$, this becomes:

$$\mathcal{L}\{\hat{x}(t)\} = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z),$$

showing that the unilateral z-transform $X_u(z)$ can be considered to be the Laplace transform of a continuous function $x(t)$ for $t \geq 0$ sampled at $t = 0,\ T,\ 2T,\ \ldots$.

Table 18.3 lists some general properties of the unilateral z-transform, and Table 18.4 lists some unilateral z-transform pairs.

Table 18.3 General properties of the unilateral z-transform

	Term in sequence	z-Transform $X_u(z)$	Domain of Convergence
1	$\alpha x_n + \beta y_n$	$\alpha X_u(z) + \beta Y_u(z)$	Intersection of DOC's of $X_u(z)$ and $Y_u(z)$ with α, β constants
2	x_{n+k}	$z^k X_u(z) - z^k x_0 - z^{k-1} x_1 - z^{k-2} x_2 - \cdots - z x_{k-1}$	
3	$n x_n$	$-z \dfrac{dX_u(z)}{dz}$	DOC of $X_u(z)$, to which it may be necessary to add or delete the origin and the point at infinity
4	$z_0^n x_n$	$X_u\left(\dfrac{z}{z_0}\right)$	DOC of $X_b(z)$ scaled by $\lvert z_0 \rvert$, to which it may be necessary to add or delete the origin and the point at infinity
5	$n z_0^n x_n$	$-z \dfrac{dX_u(z/z_0)}{dz}$	DOC of $X_u(z)$ scaled by $\lvert z_0 \rvert$, to which it may be necessary to add or delete the origin and the point at infinity
6	$\bar{x}_n$	$\overline{X_u}(\overline{z})$	The same DOC as x_n
7	$\operatorname{Re} x_n$	$\frac{1}{2}\left[X_u(z) + \overline{X_u}(\overline{z})\right]$	DOC contains the DOC of x_n
8	$\dfrac{\partial}{\partial \alpha} x_n(\alpha)$	$\dfrac{\partial}{\partial \alpha} X_u(z, \alpha)$	Same DOC as $x_n(\alpha)$
9	Initial value theorem	$x_0 = \lim_{z \to \infty} X_u(z)$	
10	Final value theorem	$\lim_{n \to \infty} x_n = \lim_{z \to 1} \left[\left(\dfrac{z-1}{z}\right) X_u(z)\right]$	When $X_u(z) = N(z)/D(z)$ with $N(z)$, $D(z)$ polynomials in z and the zeros of $D(z)$ inside the unit circle $\lvert z \rvert = 1$ or at $z = 1$

Table 18.4 Basic unilateral z-transforms

	Term in sequence	z-Transform $X_u(z)$	Domain of Convergence
1	$\Delta(n)$	1	Converges for all z
2	$\Delta(n-k)$	z^{-k}	Convergence for all $z \neq 0$
3	$a^n h(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
4	$na^n h(n)$	$\frac{az}{(z-az)^2}$	$\|z\| > a > 0$
5	$n^2 a^n h(n)$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > a > 0$
6	$na^{n-1} h(n)$	$\frac{z}{(z-a)^2}$	$\|z\| > a > 0$
7	$(n-1)a^n h(n)$	$\frac{z(2a-z)}{(z-a)^2}$	$\|z\| > a > 0$
8	$e^{-an} h(n)$	$\frac{ze^a}{ze^a - 1}$	$\|z\| > e^{-a}$
9	$ne^{-an} h(n)$	$\frac{ze^a}{(ze^a-1)^2}$	$\|z\| > e^{-a}$
10	$n^2 e^{-an} h(n)$	$\frac{ze^a(1+ze^a)}{(ze^a-1)^3}$	$\|z\| > e^{-a}$
11	$e^{-an} h(n) \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$\|z\| > e^{-a}$
12	$e^{-an} h(n) \cos \Omega n$	$\frac{ze^a(ze^a - \cos \Omega)}{z^2 e^{2a} - 2ze^a \cos \Omega + 1}$	$\|z\| > e^{-a}$
13	$h(n) \sinh an$	$\frac{z \sinh a}{z^2 - 2z \cosh a + 1}$	$\|z\| > e^{-a}$
14	$h(n) \cosh an$	$\frac{z(z - \cosh a)}{z^2 - 2z \cosh a + 1}$	$\|z\| > e^{-a}$
15	$h(n) a^{n-1} e^{-an} \sin \Omega n$	$\frac{ze^a \sin \Omega}{z^2 e^{2a} - 2zae^a \cos \Omega + a^2}$	$\|z\| > e^{-a}$
16	$h(n) a^n e^{-an} \cos \Omega n$	$\frac{ze^a(ze^a - a \cos \Omega)}{z^2 - 2zae^a \cos \Omega + a^2}$	$\|z\| > e^{-a}$

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