Full title

1 Section title

The energy functional in general orthogonal curvilinear coordinates (ξ, η) with respective scale factors h_ξ, h_η and Jacobian $J = h_\xi h_\eta$ is

$$\Upsilon(u) = \int\limits_\Omega \left(\gamma||\mathbf{n}|| + \frac{1}{2}\Delta\rho g u^2 + \lambda u\right) \mathrm{d}\Omega, \tag{1}$$

where $u(\xi, \eta)$ is the surface, $\mathbf{n}$ is the normal to the function $F = z - u(\xi, \eta)$, the Jacobian J is contained in the differential $\mathrm{d}\Omega$, and the rest of the quantities are as in the typical surface tension problem. Introduce a variation $u_\varepsilon = u + \varepsilon v$, $|\varepsilon| \ll 1$ with v vanishing on $\partial\Omega$ with the aim to calculate $\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Upsilon(u_\varepsilon)\right|_{\varepsilon=0} = 0$. Express the normal as $\mathbf{n} = (-\nabla u \;\; 1)^{\mathrm{T}}$, where $\nabla u = \left(u_\xi/h_\xi \;\; u_\eta/h_\eta\right)$. Then

$$\left.\frac{\mathrm{d}||\mathbf{n}||}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(1 + \nabla(u+\varepsilon v)\cdot\nabla(u+\varepsilon v)\Big)^{1/2}\right|_{\varepsilon=0} = \left.\frac{1}{||\mathbf{n}||}\left\{\frac{u_\xi + \varepsilon v_\xi}{h_\xi}\,\frac{v_\xi}{h_\xi} + \frac{u_\eta + \varepsilon v_\eta}{h_\eta}\,\frac{v_\eta}{h_\eta}\right\}\right|_{\varepsilon=0} = \frac{\nabla u \cdot \nabla v}{||\mathbf{n}||}. \tag{2a}$$

Taking $\mathbf{f} = \nabla u / ||\mathbf{n}||$ in the following form of the divergence theorem for an arbitrary differentiable vector function $\mathbf{f}$ and scalar differentiable function v

$$\int\limits_\Omega \mathbf{f}\cdot\nabla v\,\mathrm{d}\Omega = \oint\limits_{\partial\Omega} v\Big(\mathbf{f}\cdot\mathrm{d}\mathbf{s}\Big) - \int\limits_\Omega v\nabla\cdot\mathbf{f}\,\mathrm{d}\Omega \tag{2b}$$

and bearing in mind that $v \equiv 0$ on $\partial\Omega$ results in

$$\int\limits_\Omega \frac{\nabla u\cdot\nabla v}{||\mathbf{n}||}\,\mathrm{d}\Omega = -\int\limits_\Omega v\nabla\cdot\left(\frac{\nabla u}{||\mathbf{n}||}\right)\mathrm{d}\Omega. \tag{2c}$$

Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Upsilon(u_\varepsilon)\right|_{\varepsilon=0} = \int\limits_\Omega v\left\{\frac{\Delta\rho g u + \lambda}{\gamma} - \nabla\cdot\left(\frac{\nabla u}{||\mathbf{n}||}\right)\right\}\mathrm{d}\Omega = 0, \tag{2d}$$

and the expression within the curly braces is the Euler-Lagrange equation.

1.1 Subsection title

The parabolic coordinates are defined by

$$x = \xi\eta, \quad y = \frac{1}{2}\Big(\eta^2 - \xi^2\Big). \tag{3}$$

The constant ξ curves are upward parabolae $y = \frac{1}{2}\Big(x^2/\xi^2 - \xi^2\Big)$ with focus at the origin and directrix $y = -\xi^2$. The constant η curves are downward parabolae $y = \frac{1}{2}\Big(\eta^2 - x^2/\eta^2\Big)$ with focus at the origin and directrix $y = \eta^2$. Figure 1 shows the mesh. The interesting feature about this system is that it allows several distinct types of domain shape. When the domain is defined by $\Big\{(\xi,\eta)\,|\,0 \leqslant \xi \leqslant \xi_{\mathrm{ex}},\; 0 \leqslant \eta \leqslant \eta_{\mathrm{ex}}\Big\}$, it is eye-shaped (strictly speaking only the right half, the left is obtained by reflexion or $-\xi_{\mathrm{ex}} \leqslant \xi \leqslant \xi_{\mathrm{ex}}$), asymmetric if $\xi_{\mathrm{ex}} \neq \eta_{\mathrm{ex}}$. When $\Big\{(\xi,\eta)\,|\,0 < \xi_{\mathrm{in}} \leqslant \xi \leqslant \xi_{\mathrm{ex}},\; 0 < \eta_{\mathrm{in}} \leqslant \eta \leqslant \eta_{\mathrm{ex}}\Big\}$, the domain is of the shape shown by the curved coloured diamond. Of course, its boundaries can be extended along any of the parabolae.

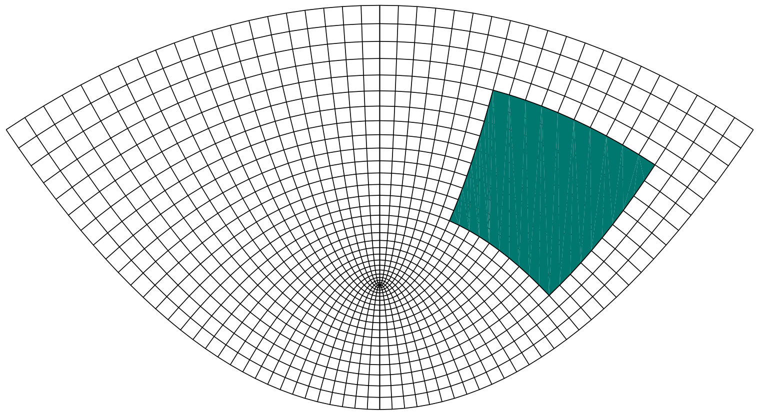

Figure 1: Parabolic coordinates: mesh and domain shape

1.1.1 Subsubsection title

The Octave code to create figure 1 is as follows.

```
clear all
figure(1); clf

xi=-2:0.1:2; lxi=length(xi);
eta=(0:0.1:3)'; leta=length(eta);
Xi=repmat(xi,leta,1);
Eta=repmat(eta,1,lxi);

X=Xi.*Eta;
Y=(Eta.^2-Xi.^2)/2;

h=plot(X,Y,X',Y');
set(h,'color',[0 0 0])
hold on

i=17:27;
j=8:18;
x=[X(i,j(1)); X(i(end),j)'; flipud(X(i,j(end))); flipud(X(i(1),j))'];
y=[Y(i,j(1)); Y(i(end),j)'; flipud(Y(i,j(end))); flipud(Y(i(1),j))'];
p1=patch(x,y,'k');
set(p1,'facecolor',[0 0.47 0.44])

axis equal
```