Coloring the Square of a Planar Graph

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Abstract: We prove that for any planar graph G with maximum degree Δ, it holds that the chromatic number of the square of G satisfies $\chi(G^2) \leq 2\Delta + 25$. We generalize this result to integer labelings of planar graphs involving constraints on distances one and two in the graph.

Keywords: *planar graph; chromatic number; labeling of a graph*

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1. INTRODUCTION

Throughout this article, $V = V(G)$ and $E = E(G)$ shall denote the set of vertices and the set of edges, respectively, of a graph G. For vertices u and v in G, we let $\mathrm{dist}_G(u, v)$ denote the *distance* between u and v, which is the length of the shortest path joining them. For integers $p, q \geq 0$, a labeling of a graph $\varphi : V(G) \longrightarrow \{0, 1, \ldots, n\}$, for a certain $n \geq 0$, is called an *$L(p, q)$-labeling* if it satisfies:

$$|\varphi(u) - \varphi(v)| \geq p, \quad \text{if } \mathrm{dist}_G(u, v) = 1;$$
$$|\varphi(u) - \varphi(v)| \geq q, \quad \text{if } \mathrm{dist}_G(u, v) = 2.$$

The *p, q-span* of a graph G, denoted $\lambda(G; p, q)$, is the minimum n for which an $L(p, q)$-labeling exists. The problem of determining $\lambda(G; p, q)$ for certain graphs or classes of graphs (or at least finding good lower or upper bounds) has been studied before, see for example, [6–10,16]. The main inspiration for $L(p, q)$-labelings in those articles comes from problems related to the *Frequency Assignment Problem* on large-scale telecommunication networks.

Determining $\lambda(G; 1, 0)$ amounts to finding the chromatic number $\chi(G)$ and for the planar graphs, we have the famous 4-Color Theorem.

Theorem 1.1 (Appel and Haken [2], Appel et al. [3], Robertson et al. [15]). *If G is a planar graph, then $\chi(G) \leq 4$.*

For general p, the above is easily seen to yield the following upper bound.

Corollary 1.1. *If G is a planar graph, then $\lambda(G; p, 0) \leq 3p$.*

Now we shall look at the case when $q \geq 1$. The problem of finding an $L(1, 1)$-labeling amounts to finding a proper coloring of the *square of G*. The square of a graph G (denoted G^2) is defined such that $V(G^2) = V(G)$, and two vertices u and v are adjacent in G^2 iff $\mathrm{dist}_G(u, v) \in \{1, 2\}$. It is easy to see that we have the relation $\chi(G^2) = \lambda(G; 1, 1) + 1$.

The question of finding the best possible upper bound for the chromatic number of the square of a planar graph seems to first have been put forward in Wegner [17] in 1977. Wegner conjectured the following.

Conjecture 1.1. (Wegner [17]). *Let G be a planar graph with maximum degree Δ, then*

$$\chi(G^2) \leq \begin{cases} \Delta + 5, & \text{if } 4 \leq \Delta \leq 7; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner also gave examples illustrating that these upper bounds are best possible and proved that the square of a planar graph with $\Delta = 3$ can be colored with 8 colors. He conjectured that in fact 7 colors should suffice. More information and problems relating coloring and distances in graphs can be found in Jensen and Toft [12, Section 2.18].

This may be a good point to note that one has to be careful when analyzing straightforward greedy algorithms to obtain bounds on $\chi(G^2)$ (or $\lambda(G;p,q)$ for that matter). For instance, it is well known that one can order the vertices of a planar graph G as $v_1,\ldots,v_n$ such that v_i has at most 5 neighbors in $\{v_1,\ldots,v_{i-1}\}$. In a greedy labeling of G using this order, one would give the label 0 to v_1 and then assign to each vertex the smallest available label. One would like to argue that, since the vertex v_i has at most 5 neighbors in $\{v_1,\ldots,v_{i-1}\}$, it also has at most $5\,(\Delta-1)$ vertices at distance two in $\{v_1,\ldots,v_{i-1}\}$ (which would prove a bound $\chi(G^2)\le 5\,\Delta+1$). But this is not necessarily the case, since vertices at distance two from v_i in $\{v_1,\ldots,v_{i-1}\}$ are not necessarily adjacent to a neighbor of v_i in $\{v_1,\ldots,v_{i-1}\}$. So the number of vertices in $\{v_1,\ldots,v_{i-1}\}$ at distance two from v_i in G can be much larger than $5\,(\Delta-1)$.

Nevertheless, it is possible to obtain the following upper bounds. The ideas from the proof can be found in Jonas [13].

Theorem 1.2. *If G is a planar graph with maximum degree $\Delta\ge 5$, then $\chi(G^2)\le 9\,\Delta-19$.*

Proof. Order the vertices of the planar graph G as $v_1,\ldots,v_n$, in such a way that each v_i has at most 5 neighbors in $\{v_1,\ldots,v_{i-1}\}$. We greedily assign colors to $v_1,\ldots,v_n$ in that order. So we must show that every vertex v_i has at most $9\,\Delta-20$ vertices at distance one or two in G in $\{v_1,\ldots,v_{i-1}\}$. Suppose v_i has k neighbors in $\{v_1,\ldots,v_{i-1}\}$, for some $0\le k\le 5$, hence there are k vertices at distance one from v_i in $\{v_1,\ldots,v_{i-1}\}$. Suppose w is a vertex in $\{v_1,\ldots,v_{i-1}\}$ at distance two from v_i, so there is a path v_iuw in G.

If $u\in\{v_1,\ldots,v_{i-1}\}$, then there can be at most $\Delta-1$ neighbors w of u in $\{v_1,\ldots,v_{i-1}\}$, hence there can be at most $k\,(\Delta-1)$ paths v_iuw with $u,w\in\{v_1,\ldots,v_{i-1}\}$.

Now consider the case $u\notin\{v_1,\ldots,v_{i-1}\}$. There are at most $\Delta-k$ of such u. Also, since u has at most 5 neighbors in $\{v_1,\ldots,v_{j-1}\}$, and one of those neighbors is v_i, u can have at most 4 neighbors in $\{v_1,\ldots,v_{i-1}\}$. So the number of paths v_iuw with $u\notin\{v_1,\ldots,v_{i-1}\}$ but $w\in\{v_1,\ldots,v_{i-1}\}$ is at most $4\,(\Delta-k)$.

Combining everything, we find at most $k+k\,(\Delta-1)+4\,(\Delta-k)$ vertices at distance one or two from v_i in $\{v_1,\ldots,v_{i-1}\}$. It is easy to see that for $\Delta\ge 5$ and $0\le k\le 5$, this number is at most $9\,\Delta-20$. ∎

Using a somewhat more involved argument, it is proved in Jonas [13] that for a planar graph G with $\Delta\ge 5$, we have $\lambda(G;2,1)\le 8\,\Delta-13$. A small modification of the proof in Jonas [13] will give the upper bound $\chi(G^2)\le 8\,\Delta-22$ for a planar graph G with $\Delta\ge 7$.

As a special case of Theorem 1.4 to be formulated later, we improve this lower bound to the following:

Theorem 1.3. *If G is a planar graph with maximum degree Δ, then $\chi(G^2)\le 2\,\Delta+25$.*

A straightforward argument shows that if G is a graph with maximum degree Δ, then we must have $\lambda(G;p,q) \geq q\,\Delta + p - q$. It is not too hard to construct planar graphs G with $\lambda(G;p,q) = \frac{3}{2}\,q\,\Delta + c_1(p,q)$, where $c_1(p,q)$ is a constant depending only on p and q. For instance, start by taking a graph consisting of three vertices v_1, v_2, v_3. Between v_1 and v_2 add k paths of length two (i.e., paths with one internal vertex); between v_1 and v_3, and between v_2 and v_3, add one edge and $k-1$ paths of length two. The resulting graph G is a simple planar graph on $3\,k+1$ vertices, with $\Delta = 2\,k$, and each pair of vertices has distance at most two. So we find, for p not too large, $\lambda(G;p,q) = (|V(G)|-1)\,q = 3\,k\,q = \frac{3}{2}\,q\,\Delta$. And in particular $\chi(G^2) = |V(G)| = \frac{3}{2}\,\Delta + 1$, showing that the second bound in Conjecture 1.1 would be sharp.

As far as upper bounds for $\lambda(G;p,q)$ are concerned, in Chang and Kuo [7], it is shown that $\lambda(G;2,1) \leq \Delta^2 + \Delta$. This suggests that for graphs in general, the best possible upper bound for $\lambda(G;p,q)$ will be of the order $q\,\Delta^2$. When G is planar, we can reduce the order of the upper bound. Using a greedy algorithm for labeling, if we assign a certain label to a vertex at distance one from a certain vertex v, then this reduces the number of labels available to v with at most $2\,p-1$, whereas assigning a label to a vertex at distance two from v can "forbid" at most $2\,q-1$ labels for v. With these observations and those from the proof of Theorem 1.2, it is easy to show that for a planar graph G with maximum degree $\Delta \geq 5$, we have

$$\begin{aligned}\lambda(G;p,q) &\leq 5\,(2\,p-1) + (9\,\Delta - 25)\,(2\,q-1)\\ &= (18\,q-9)\,\Delta + 10\,p - 50\,q + 20.\end{aligned}$$

Our main result shows this upper bound can be reduced significantly.

Theorem 1.4. *If G is a planar graph with maximum degree Δ and p,q are positive integers with $p \geq q$, then*

$$\lambda(G;p,q) \leq (4\,q-2)\,\Delta + 10\,p + 38\,q - 24.$$

Theorem 1.3 follows immediately from Theorem 1.4 by setting $p = q = 1$ and using the observation $\lambda(G;1,1) = \chi(G^2) - 1$.

The remainder of this article will form the proof of Theorem 1.4. The proof can be found in Section 2. This proof relies on two lemmas, guaranteeing existence of certain "unavoidable configurations" in planar graphs. These two lemmas will be proved in the final section.

Since the completion of the first version of this article, new results concerning coloring the square of planar graphs have come to our attention. Using a somewhat different proof technique, in Agnarsson and Halldórsson [1] a better upper bound $\chi(G^2) \leq \lfloor \frac{9}{5}\,\Delta \rfloor + 2$ is proved, provided $\Delta \geq 749$. The upper bound $\chi(G^2) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$ is proved in Borodin et al. [4,5], but then for the case $\Delta \geq 47$. A further improvement was made in Molloy and Salavatipour [14] to

$\chi(G^2) \leq \frac{5}{3}\Delta + 78$, and $\chi(G^2) \leq \frac{5}{3}\Delta + 24$ for $\Delta \geq 241$. The preprints [5,14] also contain results on $L(p,q)$-labelings of planar graphs.

2. PROOF OF THEOREM 1.4

Let G be a graph. For a vertex $v \in V$, we let $N_G(v)$ denote its neighbor set, use $d_G(v) = |N_G(v)|$ for its degree, and let E_v denote the set of edges incident to v (we omit the subscript G in most cases).

Now let G be a simple planar graph with a fixed embedding in the plane. For an edge $e \in E$, let $t(e)$ denote the number of triangular faces containing e, and for a vertex $v \in V$ let $t(v)$ be the number of triangular faces containing v.

We need two structural lemmas which give specific unavoidable configurations for planar graphs. The proofs of these lemmas can be found in Section 3. The first lemma is sufficient to prove the main theorem for graphs with maximum degree $\Delta \geq 12$.

Lemma 2.1. *Let G be a simple planar graph. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $d(v_1) \leq \cdots \leq d(v_k)$ such that one of the following is true:*

(i) $k \leq 2$;
(ii) $k = 3$ *with* $d(v_1) \leq 11$;
(iii) $k = 4$ *with* $d(v_1) \leq 7$ *and* $d(v_2) \leq 11$;
(iv) $k = 5$ *with* $d(v_1) \leq 6$, $d(v_2) \leq 7$, *and* $d(v_3) \leq 11$.

To be able to prove the main result for graphs with maximum degree less than 12, we need a second lemma, giving a different collection of unavoidable configurations.

Lemma 2.2. *Let G be a simple planar graph. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $d(v_1) \leq \cdots \leq d(v_k)$ such that one of the following is true:*

(i) $k \leq 2$;
(ii) $k = 3$ *with* $d(v_1) \leq 5$;
(iii) $k = 3$ *with* $t(vv_i) \geq 1$ *for some* i;
(iv) $k = 4$ *with* $d(v_1) \leq 4$;
(v) $k = 4$ *with* $t(vv_i) = 2$ *for some* i;
(vi) $k = 5$ *with* $d(v_i) \leq 4$ *and* $t(vv_i) \geq 1$ *for some* i;
(vii) $k = 5$ *with* $d(v_i) \leq 5$ *and* $t(vv_i) = 2$ *for some* i;
(viii) $k = 5$ *with* $d(v_1) \leq 7$ *and* $t(vv_i) \geq 1$ *for all* i;
(ix) $k = 5$ *with* $d(v_1) \leq 5$, $d(v_2) \leq 7$, *and for each* i *with* $t(vv_i) = 0$ *it holds that* $d(v_i) \leq 5$.

Proof of Theorem 1.4. Let G be a planar graph and let Δ be its maximum degree. If $\Delta \leq 5$, then the theorem can be proven using a straight-forward "greedy" coloring method. In fact, in this case the theorem holds even when the planarity condition is removed. The only essential observations are that for any vertex in a graph H with maximum degree Δ, the number of vertices at distance 1 from v is at most Δ and the number of vertices at distance 2 is at most $\Delta\,(\Delta - 1)$. Moreover, if we assign a certain label to a vertex at distance 1 from v, then this reduces the number of labels available to v with at most $2p - 1$, whereas assigning a label to a vertex at distance two from v can "forbid" at most $2q - 1$ labels for v. We leave the verification of the further details to the reader.

In the remainder, we are solely interested in the case $\Delta \geq 6$. We shall prove Theorem 1.4 by induction on the number of vertices and edges. Let G be a planar graph such that for all planar graphs H with $|V(H)| + |E(H)| < |V(G)| + |E(G)|$ the theorem is true. We note first that we can assume that G is simple and $\Delta \geq 6$.

For an edge $e \in E$ let G/e denote the graph obtained from G by contracting e. For a vertex $v \in V$, let $G * v$ denote the graph obtained by deleting v and for each $u \in N(v)$ adding an edge between u and u^- and between u and u^+ if these edges do not exists in G already. We will use Lemmas 2.1 and 2.2 to show that there is a vertex $v \in V$ such that $d(v) \leq 5$, the number of vertices at distance 2 from v is at most $2\,\Delta + 19$, and at least one of the following is true:

(a) $\Delta(G/e) \leq \Delta$ for some $e \in E_v$;
(b) $\Delta(G * v) \leq \Delta$.

The following proposition formulates the essential properties of the vertex degrees and distances after the operations G/e and $G * v$ have been performed.

Proposition 2.1. *Let G be a simple graph, v a vertex and $e = vu$ an edge in G.*

(i) *Let $H = G/e$, and let v' be the vertex in H corresponding to the edge vu. Then for each $w \in V(H)\backslash\{v'\}$ we have $d_H(w) \leq d_G(w)$, and $d_H(v') = d_G(v) + d_G(u) - 2 - t_G(vu)$.*
(ii) *Let $H = G * v$. Then for each $w \in V(H)$ we have $d_H(w) = d_G(w)$ if $w \notin N_G(v)$, and $d_H(w) = d_G(w) + 1 - t_G(vw)$ if $w \in N_G(v)$.*
(iii) *Let $H = G/e$, and let v' be the vertex in H corresponding to the edge vu. Then for any two vertices $w, w' \in V(H)\backslash\{v'\}$ it holds that* $\mathrm{dist}_H(w, w') \leq \mathrm{dist}_G(w, w')$ *and* $\mathrm{dist}_H(w, v') \leq \mathrm{dist}_G(w, u)$.
(iv) *Let $H = G * v$ and suppose $d_G(v) \leq 5$. Then for any two vertices w, $w' \in V(H)$ it holds that* $\mathrm{dist}_H(w, w') \leq \mathrm{dist}_G(w, w')$.

Now define a vertex $v \in V(G)$, possibly an edge $e \in E(G)$, and a graph H as follows:

(2.1) If $\Delta \geq 12$, then let v be as described in Lemma 2.1, and set $e = vv_1$ and $H = G/e$.

(2.2) If $6 \leq \Delta \leq 11$ and one of Lemma 2.2 (i), (ii), or (iv) holds, then let v be as described, and set $e = vv_1$ and $H = G/e$.
(2.3) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (iii) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) \geq 1$, and set $H = G/e$.
(2.4) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (v) holds, then let v be as described, set $e = vv_i$ with $t(vv_i) = 2$ and set $H = G/e$.
(2.5) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (vi) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leq 4$ and $t(vv_i) \geq 1$, and set $H = G/e$.
(2.6) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (vii) holds, then let v be as described, set $e = vv_i$ with $d(v_i) \leq 5$ and $t(vv_i) = 2$, and set $H = G/e$.
(2.7) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (viii) holds, then let v be as described and set $H = G * v$.
(2.8) If $6 \leq \Delta \leq 11$ and Lemma 2.2 (ix) holds, then let v be as described and set $H = G * v$.

In the cases (2.1)–(2.6), identify the end vertex of e different from v with the vertex in H corresponding to the contracted edge e. Then using Proposition 2.1, we find that in cases (2.1)–(2.7), $d_H(w) \leq d_G(w)$ for all $w \in V(H)$, hence $\Delta(H) \leq \Delta(G) = \Delta$. In case (2.8), we can have $d_H(w) = d_G(w) + 1$ for a vertex $w \in N(v)$ with $t(vw) = 0$, but then $d_G(w) \leq 5$, and we still find $\Delta(H) \leq \Delta$. By induction, this means

$$\lambda(H; p, q) \leq (4q - 2)\Delta + 10p + 38q - 24.$$

Set $n = (4q - 2)\Delta + 10p + 38q - 24$ and let $\varphi_H : V(H) \longrightarrow \{0, 1, \ldots, n\}$ be an $L(p, q)$-labeling of H. Again using Proposition 2.1, for any two vertices $w, w' \in V(H)$ it holds that $\mathrm{dist}_H(w, w,')\leq \mathrm{dist}_G(w, w')$. Therefore, to find an $L(p, q)$-labeling for G, we need only extend φ_H to G by giving v an appropriate color. For each $w \in V(H)$ let $\varphi(w) = \varphi_H(w)$.

For any vertex $v \in V(G)$, the number of vertices at distance 2 from v is equal to

$$\sum_{u \in N(v)} d(u) - d(v) - 2\,t(v). \tag{1}$$

Since v was chosen according to (2.1)–(2.8), $d(v) \leq 5$ and Equation (1) gives that there are at most $2\Delta + 19$ vertices at distance 2 from v. So, since

$$n = (4q - 2)\Delta + 10p + 38q - 23 = 5 \cdot (2p - 1) + (2\Delta + 19) \cdot (2q - 1),$$

we can choose a color $\varphi(v) \in \{0, 1, \ldots, n\}$ such that

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\geq p, \quad \text{if } \mathrm{dist}_G(u, v) = 1; \\ |\varphi(u) - \varphi(v)| &\geq q, \quad \text{if } \mathrm{dist}_G(u, v) = 2. \end{aligned}$$

Choosing such a color for v, we see that φ is an $L(p,q)$-labeling for G. It now follows that

$$\lambda(G;p,q) \leq n = (4\,q-2)\,\Delta + 10p + 38\,q - 24,$$

which completes the induction step. ■

Concerning algorithmic aspects of our bound, the proof above implies that it is possible to label the vertices in a greedy fashion. If $\Delta \leq 5$, then the greedy algorithm described in the first paragraph of the proof will suffice. Otherwise, we can order the vertices as $v_1, v_2, \ldots, v_n$ in the following manner. Letting $G_0 = G$ and $G_i = G\backslash\{v_1, \ldots, v_i\}$ for $i \geq 1$, at each step i, we choose a vertex v_i such that $d_{G_{i-1}}(v_i) \leq 5$ and either

(a) $\Delta(G_{i-1}/e) \leq \Delta$ for some $e \in E_{v_i}$; or
(b) $\Delta(G_{i-1} * v_i) \leq \Delta$.

We label the vertices in a greedy fashion beginning with v_n and working backwards, labeling each vertex with the smallest available label. For each vertex, the number of forbidden labels will be at most

$$5 \cdot (2p-1) + (2\,\Delta + 19) \cdot (2\,q-1) = (4\,q-2)\,\Delta + 10p + 38\,q - 24.$$

3. DISCHARGING AND UNAVOIDABLE CONFIGURATIONS

In this section, we shall give proofs for Lemmas 2.1 and 2.2. We use the well-known method of "discharging," already used in Heawood [11] to prove the 5-Color Theorem, and later used extensively in Appel and Haken [2] and Appel et al. [3] in the proof of the 4-Color Theorem. Discharging is a method by which one can establish the existence of small, so-called "unavoidable configurations" in certain planar graphs. This method is tantamount to the proof of the 4-Color Theorem, and we refer the reader to a recent proof of this theorem in Robertson et al. [15] which follows the same strategy as the Appel and Haken proof, but is vastly simpler in the discharging phase. To our knowledge, the method of discharging has not been previously used in bounding $\lambda(G;p,q)$ for planar graphs G.

Now let G be a simple connected planar graph with a fixed embedding in the plane. Let F denote the set of faces of G. For each $f \in F$ let $d(f)$ be the number of edges belonging to f, where cut-edges are counted twice. Recall the definition of the degree $d(v)$ for the degree of a vertex v, and $t(e)$ and $t(v)$ for the number of triangular faces containing an edge e and a vertex v, respectively.

Using Euler's formula, one easily obtains that (see, e.g., Jensen and Toft [12, Section 2.9])

$$\sum_{v \in V}(d(v)-4)+\sum_{f \in F}(d(f)-4)=-8. \tag{2}$$

We shall associate a *charge* $\varphi(v)$ to each vertex $v \in V$ where $\varphi(v)=d(v)-4$. Similarly, we associate a charge $\varphi(f)=d(f)-4$ to each face $f \in F$. According to Equation (2), the sum of the charges taken over all vertices and faces will be negative. We shall transfer the charge of vertices and faces to the edges of the graph, in such a way that the total charge remains constant. There are three steps to perform:

(3.1) For an edge $e=uv \in E$, we give e a *basic charge* $\varphi_b(e)$ where, given e belongs to faces $f, g \in F$, we set

$$\varphi_b(e)=\frac{\varphi(u)}{d(u)}+\frac{\varphi(v)}{d(v)}+\frac{\varphi(f)}{d(f)}+\frac{\varphi(g)}{d(g)}. \tag{3}$$

If e belongs to only one face $f \in F$, then we give e a basic charge as in the above taking $g=f$.

(3.2) For each triangular face with vertices u, v, w, where $3 \leq d(u) \leq 5$, $d(v) \geq 6$, and $d(w) \geq 6$, do the following:
Transfer a charge of $\frac{1}{2}(\frac{\varphi(v)}{d(v)}-\frac{1}{3})$ from vw to uw.
Transfer a charge of $\frac{1}{2}(\frac{\varphi(w)}{d(w)}-\frac{1}{3})$ from vw to uv.

(3.3) For each triple u, v, v' in V, with $uv, uv' \in E_u$, $d(u)=5$, $d(v) \geq 6$, $d(v') \geq 6$, $t(uv)=2$, and $t(uv')=0$ transfer a charge of $\frac{1}{6}$ from uv' to uv.

After doing all possible charge transfers once, let $\varphi(e)$ be the resulting charge on each edge $e \in E$. Since the total charge on the edges is seen to be equal to the total charge on the vertices and faces, we have from Equation (3) that

$$\sum_{v \in V} \sum_{e \in E_v} \varphi(e)=\sum_{e \in E} 2\,\varphi(e)=\sum_{e \in E} 2\,\varphi_b(e)=-16. \tag{4}$$

The following properties, whose proofs follow by following the two discharging methods given above, are used at numerous places in the sequel, although usually implicitly.

Proposition 3.1. *Let G be a simple planar graph with a fixed embedding and let $e=uv$ be an edge in G.*

(i) *If $\varphi_b(e)<0$, then $d(u) \leq 5$ or $d(v) \leq 5$, and $\varphi(e) \geq \varphi_b(e)$.*
(ii) *If $\varphi_b(e) \geq 0$, then $\varphi(e) \geq 0$.*

Let v be a vertex and vu an edge in a simple planar graph with a fixed embedding. If the edge vw is an edge which directly precedes vu, counting the edges of E_v moving clockwise around v, then we shall denote w by u^-. If vw directly succeeds vu, then we denote w by u^+.

First common steps in the proofs of Lemmas 2.1 and 2.2. Both lemmas are proved by contradiction. So let G be a simple, planar graph with a fixed embedding in the plane, and suppose that G is a counterexample to one of the lemmas. According to Equation (4) there is a vertex $v \in V$ where $\sum_{e \in E_v} \varphi(e) < 0$. Suppose w is such a vertex and suppose w has m neighbors $w_1, \ldots, w_m$ where $d(w_1) \leq \cdots \leq d(w_k)$. Since (i) does not hold, we know $m \geq 3$.

For $j = 1, 2, \ldots, m$, let T_j be the set of edges between vertices in $\{v_j, \ldots, v_m\}$ belonging to a face containing v, and set $t_j = |T_j|$.

Proof of Lemma 2.1. In this case, we may assume that G is a 2-connected triangulation, for otherwise, we could add edges to G obtaining a triangulation G'. If none of (i)–(iv) holds for G, then clearly none of (i)–(iv) holds for G'.

Claim 1. $m \neq 3$.

Proof. Suppose $m = 3$. Because (ii) does not hold, $d(w_j) \geq 12$ for all j, hence $\varphi_b(ww_j) \geq -\frac{1}{3} + \frac{2}{3} - 2 \cdot \frac{1}{3} = -\frac{1}{3}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{6}$ units will be transfered from both $w_j w_j^-$ and $w_j w_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j) + 2 \cdot \frac{1}{6} \geq 0$, and thus $\sum_{e \in E_w} \varphi(e) \geq 0$, contradicting the choice of w. ■

Claim 2. $m \neq 4$.

Proof. Suppose $m = 4$. Suppose first that $d(w_j) \geq 8$ for all j. Then $\varphi_b(ww_j) \geq 0 + \frac{1}{2} - 2 \cdot \frac{1}{3} = -\frac{1}{6}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{12}$ units will be transfered from both $w_j w_j^-$ and $w_j w_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j) + 2 \cdot \frac{1}{12} = 0$ for all j, and thus $\sum_{e \in E_w} \varphi(e) \geq 0$, contradicting the choice of w.

We conclude $d(w_1) \leq 7$. Since G does not satisfy condition (ii) in the lemma (with $v = w_1$), we know $d(w_1) \geq 4$ and hence $\varphi(ww_1) \geq \varphi_b(ww_1) \geq 0 + 0 - 2 \cdot \frac{1}{3} = -\frac{2}{3}$. It also follows that $d(w_j) \geq 12$ for all $j \geq 2$, hence $\varphi_b(ww_j) \geq 0 + \frac{2}{3} - 2 \cdot \frac{1}{3} = 0$ for all $j \geq 2$. According to the procedure for transferring charge, for each edge $w_j w_\ell \in T_2$ a charge of at least $\frac{1}{6}$ units will be transferred from $w_j w_\ell$ to both ww_j and ww_ℓ. Observing that $t_2 = 2$, we have

$$\sum_{e \in E_w} \varphi(e) \geq \varphi_b(ww_1) + \sum_{j \geq 2} \varphi_b(ww_j) + t_2 \cdot 2 \cdot \frac{1}{6} \geq 0,$$

again contradicting the choice of w. ■

Claim 3. $m \neq 5$.

Proof. Suppose $m = 5$. First suppose that $d(w_j) \geq 7$ for all j. Then $\varphi_b(ww_j) \geq \frac{1}{5} + \frac{3}{7} - 2 \cdot \frac{1}{3} = -\frac{4}{105}$. According to the procedure for transferring charge, for each j a charge of at least $\frac{1}{21}$ units is transferred from both $w_jw_j^-$ and $w_jw_j^+$ to ww_j. This gives $\varphi(ww_j) \geq \varphi_b(ww_j) + 2 \cdot \frac{1}{21} = \frac{2}{35} > 0$, and thus $\sum_{e\in E_w} \varphi(e) \geq 0$, contradicting the choice of w.

So we have that $d(w_1) \leq 6$. Again we know that $d_1(w) \geq 4$ and hence $\varphi_b(ww_1) \geq \frac{1}{5} + 0 - 2 \cdot \frac{1}{3} = -\frac{7}{15}$. If $d(w_j) \geq 8$ for all $j \geq 2$, then $\varphi_b(ww_j) \geq \frac{1}{5} + \frac{1}{2} - 2 \cdot \frac{1}{3} = \frac{1}{30}$. According to the procedure for transferring charge, for each edge $w_jw_\ell \in T_2$, we transfer a charge of at least $\frac{1}{12}$ units from w_jw_ℓ to both ww_j and ww_ℓ. Observing that $t_2 = 3$, we have

$$\sum_{e\in E_w} \varphi(e) \geq \varphi_b(ww_1) + \sum_{j\geq 2} \varphi_b(ww_j) + t_2 \cdot 2 \cdot \frac{1}{12} \geq -\frac{7}{15} + 4 \cdot \frac{1}{30} + \frac{1}{2} = \frac{1}{6} > 0,$$

contradicting the choice of w.

This means that we know $d(w_1) \leq 6$ and $d(w_2) \leq 7$, hence $d(w_j) \geq 12$ for all $j \geq 3$. Since certainly $d(w_1) \geq 4$ and $d(w_2) \geq 4$, we have $\varphi_b(ww_1) \geq -\frac{7}{15}$ and $\varphi_b(ww_2) \geq -\frac{7}{15}$. Also, $\varphi_b(ww_j) \geq \frac{1}{5} + \frac{2}{3} - 2 \cdot \frac{1}{3} = \frac{1}{5}$ for $j \geq 3$. According to the procedure for transferring basic charge, for each $w_jw_\ell \in T_3$, a charge of at least $\frac{1}{6}$ units will be transferred from w_jw_ℓ to both ww_j and ww_ℓ. Observing that $t_3 \geq 1$ we have

$$\begin{aligned}\sum_{e\in E_w} \varphi(e) &\geq \varphi_b(ww_1) + \varphi_b(ww_1) + \sum_{j\geq 3} \varphi_b(ww_j) + t_3 \cdot 2 \cdot \frac{1}{6} \\ &\geq -2 \cdot \frac{7}{15} + 3 \cdot \frac{1}{5} + \frac{1}{3} = 0,\end{aligned}$$

again contradicting the choice of w. ■

We now know that $m \geq 6$. Since the vertex w is chosen such that $\sum_{e\in E_w} \varphi(e) < 0$, there must be an edge $e \in E_w$ such that $\varphi(e) < 0$. Let $ww_a \in E_w$ be such an edge. By Lemma 3.1 (ii) this must mean that $\varphi_b(ww_a) < 0$ also. Since $d(w) = m \geq 6$, by Lemma 3.1 (i) we have that $d(w_a) \leq 5$.

Claim 4. $m \neq 6, 7$.

Proof. Suppose $m = 6$ or $m = 7$. We certainly can assume $d(w_a) \geq 4$, otherwise (i) or (ii) would hold. If $d(w_a) = 4$, then $d(w_a^-) \geq 12$ and $d(w_a^+) \geq 12$, otherwise (iii) holds with $v = w_a$. Then we have $\varphi_b(ww_a) \geq \frac{1}{3} + 0 - 2\frac{1}{3} = -\frac{1}{3}$. Also, according to the procedure for transferring charge, at least $\frac{1}{6}$ units are transferred from both $w_aw_a^-$ and $w_aw_a^+$ to ww_a. This means $\varphi(ww_a) \geq \varphi_b(ww_a) + 2 \cdot \frac{1}{6} \geq 0$, contradicting the choice of ww_a.

Now suppose $d(w_a) = 5$, and thus $\varphi_b(ww_a) \geq \frac{m-4}{m} + \frac{1}{5} - \frac{2}{3} = \frac{m-4}{m} - \frac{7}{15}$. Since G does not satisfy (iv) with $v = w_a$, we have that either $d(w_a^-) \geq 14 - m$ and $d(w_a^+) \geq 14 - m$, or $\max\{d(w_a^-), d(w_a^+)\} \geq 12$. In the former case we have that a charge of at least $\frac{1}{2}\left(\frac{10-m}{14-m} - \frac{1}{3}\right)$ is transfered from both $w_aw_a^-$ and $w_aw_a^+$ to ww_a.

In the latter case, a charge of at least $\frac{1}{6}$ is transfered from $w_a w_a^-$ or $w_a w_a^+$ to ww_a. So we obtain

$$\begin{aligned}\varphi(ww_a) &\geq \varphi_b(ww_a) + \min\left\{2\cdot\frac{1}{2}\left(\frac{10-m}{14-m}-\frac{1}{3}\right),\frac{1}{6}\right\}\\ &\geq \frac{m-4}{m}-\frac{7}{15}\left\{\left(\frac{10-m}{14-m}-\frac{1}{3}\right),\frac{1}{6}\right\}\geq 0,\end{aligned}$$

again contradicting the choice of ww_a. ■

Claim 5. $m \neq 8, 9, 10, 11$.

Proof. Suppose $8 \leq m \leq 11$. We can assume $d(w_a) \geq 4$, since otherwise (ii) would hold with $v = w_a$. It suffices to show $\varphi(ww_a) \geq 0$ when $d(w_a) = 4$, as $\varphi_b(ww_a) \geq \frac{1}{2}+\frac{1}{5}-2\cdot\frac{1}{3} > 0$ if $d(w_a) \geq 5$. Suppose $d(w_a) = 4$. Then $d(w_a^-) \geq 8$ and $d(w_a^+) \geq 8$, and a charge of at least $\frac{1}{12}$ is transfered from both $w_a w_a^-$ and $w_a w_a^-$ to ww_a. Hence

$$\varphi(ww_a) \geq \varphi_b(ww_a) + 2\cdot\frac{1}{12} \geq \frac{1}{2}+0-2\cdot\frac{1}{3}+\frac{1}{6} = 0,$$

contradicting the choice of ww_a. ■

To complete the proof of Lemma 2.1, we need to show that $m \geq 12$ also leads to a contradiction. Suppose $m \geq 12$. Then $d(w_a) \geq 3$, otherwise (i) would hold with $v = w_a$. It suffices to show $\varphi(ww_a) \geq 0$ when $d(w_a) = 3$, for otherwise $\varphi(ww_a) = \varphi_b(ww_a) \geq \frac{2}{3}+0-2\cdot\frac{1}{3} = 0$. If $d(w_a) = 3$, then $d(w_a^-) \geq 12$ and $d(w_a^+) \geq 12$, and a charge of at least $\frac{1}{6}$ is transfered from both $w_a w_a^-$ and $w_a w_a^-$ to ww_a. Thus we find

$$\varphi(ww_a) \geq \varphi_b(ww_a) + 2\cdot\frac{1}{6} \geq \frac{2}{3}-\frac{1}{3}-2\cdot\frac{1}{3}+\frac{1}{3} = 0,$$

the final contradiction in this proof. ■

Proof of Lemma 2.2. We use the notation and definitions from the part common with the proof of Lemma 2.1. In fact, the proof follows a line similar to the proof of the previous lemma, although the arguments are different.

Claim 1. $m \neq 3$.

Proof. Suppose $m = 3$. Since (ii) and (iii) do not hold for G, we have $d(w_j) \geq 6$ for all j, and $t(w) = 0$. Thus $\varphi(ww_j) \geq \varphi_b(ww_j) \geq -\frac{1}{3}+\frac{1}{3} = 0$ for all $ww_j \in E_w$. It follows that $\sum_{e\in E_w}\varphi(e) \geq 0$, contradicting the choice of w. ■

Claim 2. $m \neq 4$.

Proof. Suppose $m = 4$. Since (iv) does not hold for G, we have $d(w_j) \geq 5$ for all j. If $t(w) \leq 1$, then we find $\sum_{e \in E_w} \varphi(e) \geq 4 \cdot 0 + 4 \cdot \frac{1}{5} - 2 \cdot \frac{1}{3} > 0$. Thus $t(w) \geq 2$. If $t(w) \geq 3$, then $t(ww_i) = 2$ for some i, in which case (v) holds. Consequently, $t(w) = 2$ and in fact $t(ww_j) = 1$ for all j.

If $d(w_j) = 5$ for some j, then setting $v = w_j$ and $v_i = w$ we find that (vi) holds, contradicting the choice of G. Thus $d(w_j) \geq 6$ for all j, and hence $\sum_{e \in E_w} \varphi(e) \geq 4 \cdot 0 + 4 \cdot \frac{1}{3} - 4 \cdot \frac{1}{3} = 0$, contradicting the choice of w. ■

Claim 3. $m \neq 5$.

Proof. Suppose $m = 5$. We first note that $d(w_j) \geq 4$ for all j, otherwise (i) or (ii) would hold. Also, $d(w_1) \leq 7$, for otherwise $\varphi(ww_j) \geq \varphi_b(ww_j) \geq \frac{1}{5} + \frac{1}{2} - 2 \cdot \frac{1}{3} > 0$ for all j. If $t(w) \leq 1$, then we find $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 5 \cdot 0 - 2 \cdot \frac{1}{3} > 0$. Thus $t(w) \geq 2$. Furthermore, since (vi) and (vii) do not hold, if $t(ww_j) = 1$ for some j, then $d(w_j) \geq 5$; and if $t(ww_\ell) = 2$ for some ℓ, then $d(w_\ell) \geq 6$.

If $t(w) = 2$, then there are at least three neighbors w_j of w with $t(ww_j) \geq 1$, and hence $d(w_j) \geq 5$. This means $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 3 \cdot \frac{1}{5} + 2 \cdot 0 - 4 \cdot \frac{1}{3} > 0$.

If $t(w) = 3$, then, since (viii) does not hold, there must be at least one neighbor w_j with $t(ww_j) = 0$. This means that in fact there are two neighbors w_j with $t(ww_j) = 1$, and hence $d(w_j) \geq 5$; and two neighbors w_ℓ with $t(ww_\ell) = 2$, and hence $d(w_\ell) \geq 6$. This gives $\sum_{e \in E_w} \varphi(e) \geq 5 \cdot \frac{1}{5} + 0 + 2 \cdot \frac{1}{5} + 2 \cdot \frac{1}{3} - 6 \cdot \frac{1}{3} > 0$.

If $t(w) \geq 4$, then for all j we find $t(ww_j) \geq 1$, which means that (viii) holds. So in all cases we contradict the choice of G or the choice of w. ■

We now know that $m \geq 6$. Since the vertex w is chosen such that $\sum_{e \in E_w} \varphi(e) < 0$, there must be an edge $e \in E_w$ such that $\varphi(e) < 0$. Let $ww_a \in E_w$ be such an edge. By Lemma 3.1 (i) this must mean that $\varphi_b(ww_a) < 0$ also, and hence

$$0 > \varphi_b(ww_a) \geq \frac{m-4}{m} + \frac{d(w_a) - 4}{d(w_a)} - t(ww_a) \cdot \frac{1}{3} \tag{5}$$

Claim 4. $m \neq 6, 7$.

Proof. Suppose $m = 6$ or $m = 7$. From Equation (5), it follows that the only possibilities for $d(w_a)$ and $t(ww_a)$ are:

$d(w_a) \leq 2$;
$d(w_a) = 3$ and $t(ww_a) \geq 1$;
$d(w_a) = 4$ and $t(ww_a) = 2$;
$d(w_a) = 5$ and $t(ww_a) = 2$.

In the first three options, we see that (i), (iii), and (v), respectively, hold, where we take $v = w_a$.

So the only possibility left is $d(w_a) = 5$ and $t(ww_a) = 2$. Let the neighbors of w_a be $\{w_a^-, w, w_a^+, u_1, u_2\}$. Then certainly $t(w_a w) \geq 1$, $t(w_a w_a^-) \geq 1$ and $t(w_a w_a^+) \geq 1$. Hence if $t(w_a u_1) \geq 1$ and $t(w_a u_2) \geq 1$, then (viii) holds with $v = w_a$.

So for at least one $p \in \{1, 2\}$, $t(w_a u_p) = 0$. Moreover, since (ix) does not hold, for at least one $p \in \{1, 2\}$ we have that $t(w_a u_p) = 0$ and $d(u_p) \geq 6$. Without loss of generality, we can assume that u_1 has these properties. Combining everything, we find that $d(w_a) = 5$, $d(w) \geq 6$, $d(u_1) \geq 6$, $t(w_a w) = 2$, and $t(w_a u_1) = 0$. This means that in the final step of the discharging process a charge of $\frac{1}{6}$ is transferred from $w_a u_1$ to $w_a w$. We find that the final charge for the edge ww_a satisfies

$$\varphi(ww_a) \geq \frac{1}{3} + \frac{1}{5} - 2 \cdot \frac{1}{3} + \frac{1}{6} > 0,$$

contradicting the choice of ww_a. ■

To complete the proof of Lemma 2.2, we only need to show that $m \geq 8$ also leads to a contradiction. Suppose $m \geq 8$. From Equation (5), it follows that the only possibilities for $d(w_a)$ and $t(ww_a)$ are:

$d(w_a) \leq 2$;
$d(w_a) = 3$ and $t(ww_a) \geq 1$;
$d(w_a) = 4$ and $t(ww_a) = 2$.

If the first possibility holds, then (i) follows; if the second holds, then (iii) holds; and the third possibility gives that (v) holds, every time taking $v = w_a$. This gives the final contradiction against the existence of a counterexample G. ■

Remark. By a more elaborate case analysis, it is possible to slightly improve Lemmas 2.1 and 2.2 in such a way that we get a slightly better bound in Theorem 1.4. But this would only improve the additive term, and not the factor $4q - 2$ in front of Δ. For this reason, we haven't tried to push our method to the limit.

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