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BONDS INTERSECTING CYCLES IN A GRAPH

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Let G be a k-connected graph G having circumference $c \geq 2k$. It is shown that for $k \geq 2$, then there is a bond B which intersects every cycle of length $c-k+2$ or greater.

1. Introduction

It was shown by [7] that for a loopless 2-connected graph G with circumference c and cocircumference c^*, it holds that $|E(G)| \leq \frac{1}{2}cc^*$. Recently, Lemos and Oxley [2] showed that this bound holds not only for graphs but for connected matroids in general. They showed that if M is a connected matroid with at least 2 elements, and M has circumference c, and cocircumference c^*, then $|e(M)| \leq \frac{1}{2}cc^*$.

Oxley [5] posed the following conjecture:

Conjecture 1.1. For any connected matroid M with at least 2 elements, one can find a collection of at most $c^*(M)$ cycles which cover each element of M at least twice.

In [4], Neumann-Lara et al showed that the above conjecture holds for cographic matroids. They used the following lemma which appears in Wu [7].

Lemma 1.1. *Let G be a 2-connected graph. Then there is a bond which intersects every cycle of length c or $c-1$.*

In [3], a corresponding result for cycles was proven, namely:

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Theorem 1.1. *Let G be a k-connected graph with cocircumference c^*. Then for $k \geq 2$, there is a cycle which intersects every bond of size $c^* - k + 2$ or greater.*

The object of this paper is to dualize this result. We first prove the following theorem which constitutes the bulk of this paper.

Theorem 1.2. *For a k-connected graph where $k \geq 2$ and $c = c(G) \geq 2k$, if C_1 and C_2 are a pair of cycles which intersect in at most one vertex, then it holds that $|V(C_1)| + |V(C_2)| \leq 2(c - k + 1)$.*

Using this result, we shall prove:

Theorem 1.3. *For any k-connected graph G where $k \geq 2$ and having circumference $c \geq 2k$, there is a bond B which intersects every cycle of length $c - k + 2$ or greater.*

2. Disjoint path lemmas

A useful tool for k-connected graphs is the so-called 'Fan Lemma' (see [1]). We shall use the following variant of this lemma:

Lemma 2.1. *Let G be a k-connected graph where $k \geq 1$, let X and Y be disjoint subsets of vertices of a graph G where $|X| \geq k$, and $|Y| \geq k$. There exist k vertex-disjoint paths $P_1, \ldots, P_k$ each originating at a vertex in X and terminating at a vertex in Y, and each path intersecting X and Y only at its terminal vertices.*

We shall need a modified version of the above lemma, namely:

Lemma 2.2. *Let G be a k-connected graph where $k \geq 1$, let X and Y be disjoint sets of vertices where $0 < |X| \leq k$ and $|Y| \geq k$. Assume $X = \{u_1, u_2, \ldots, u_s\}$ and let $w_1, \ldots, w_s$ be positive integers such that $\sum_{i=1}^{s} w_i = k$. Then there exist k internally vertex-disjoint paths from X to Y such that for each i, exactly w_i of these paths originate at u_i. Moreover, no two paths terminate at the same vertex in Y, and each path intersects X and Y only at its terminal vertices.*

3. Finding large independent sets

In this section, we shall show that if the circumference of a k-connected graph is 'small', then it must contain a 'large' independent set of vertices.

Lemma 3.1. *Let G be a k-connected graph ($k \geq 2$) with circumference $c = 2k + \beta$, where $0 < \beta < \frac{2}{3}k$. Assuming $|V(G)| \geq 2k + 2\beta + 1$, then G has an independent set with at least $|V(G)| - k - \beta$ vertices.*

Proof. Let G be as in the statement of the lemma where $|V(G)| \geq 2k + 2\beta + 1$. Let C be a cycle of length c. We shall assume that C has an circular orientation, and for any vertices $u, v \in V(C)$ we let $C[u, v]$ denote the path along C directed from u to v. For any vertex $v \in V(C)$, let v^- and v^+ denote the vertices directly preceeding and succeeding v on C, respectively. Let $v \in V(G) \backslash V(C)$. Since G is k-connected, there are k paths $P_1, \dots, P_k$ from v to C, which meet only at v. For $i = 1, \dots, k$, let v_i be the terminal vertex of P_i on C. We can assume that $v_1, \dots, v_k$ occur in order as move along C in the direction of its orientation. For $j \geq 1$, let α_j be the number of paths $C[v_{i-1}, v_i]$, $i = 1, \dots, k$ (taking $v_0 = v_k$) having length j. Clearly $\alpha_1 = 0$, for otherwise C could be augmented to a longer cycle via v. We have

$$\alpha_2 + \alpha_3 + \sum_{j \geq 4} \alpha_j = k$$

$$2\alpha_2 + 3\alpha_3 + 4\sum_{j \geq 4} \alpha_j \leq |C| = 2k + \beta$$

From the above, we obtain $2\alpha_2 + \alpha_3 \geq 2k - \beta$. If $|C[v_{i-1}, v_i]| = 2$, then $|P_{i-1}| = |P_i| = 1$, for otherwise, $C' = C[v_i, v_{i-1}] \cup P_{i-1} \cup P_i$ would be a longer cycle than C. Thus if $|C[v_{i-1}, v_i]| = 2$, then v is adjacent to v_{i-1} and v_i. We call paths $C[v_{i-1}, v_i]$ having length j, *v j-segments.*

We have $|V(G)| \geq 2k + 2\beta + 1 = |C| + \beta + 1 > |C| + 1$. Thus there is a vertex $u \in V(G) \backslash (V(C) \cup \{v\})$. There are k paths $Q_1, \dots, Q_k$ from u to C which meet only at u. For $i = 1, \dots, k$ let u_i be the terminal vertex of Q_i on C, where we can assume that $u_1, \dots, u_k$ occur in order as we move around C. For $j \geq 1$, let γ_j denote the number of u j-segments. As with v, we have $2\gamma_2 + \gamma_3 \geq 2k - \beta$.

Suppose that no v 2-segment is a u 2-segment. If any u 2-segment shares an edge with a v 2-segment, then it is seen that C could be modified into a longer cycle via u and v. Thus no u 2-segment and v 2-segment have a common edge. Similarly, no u 2-segment can have exactly one edge in common with a v 3-segment and no v 2-segment can have exactly one edge in common with a u 3-segment. Suppose a u 2-segment $C[u_i, u_{i+1}] = u_i u_i^+ u_{i+1}$ is contained in a v 3-segment $C[v_j, v_{j+1}]$. We can assume that $u_i = v_j$, and $C[v_j, v_{j+1}] = u_i u_i^+ u_{i+1} u_{i+1}^+$. Clearly there is no path in $G \backslash V(C)$ from u to v; for otherwise, C could be modified into a longer cycle. This means that the paths $P_1, \dots, P_k$ can only intersect the paths $Q_1, \dots, Q_k$ at vertices of C.

Suppose for some $s \neq i, i+1$ and $t \neq j, j+1$ we have $u_s = v_t^+$. Then

$$C' = C[u_s, u_i] \cup C[u_{i+1}, v_t] \cup \{u_s u u_{i+1}, u_i v v_t\}$$

is a cycle with length $|C|+1$. On the other hand, if $v_t = u_s^+$, then

$$C' = C[v_t, u_{i+1}] \cup C[u_{i+1}^+, u_s] \cup \{u_s u u_{i+1}, u_{i+1}^+ v v_t\}$$

is a cycle of length $|C|$+2. From this we conclude that no such vertices u_s and v_t can exist. This in turn implies that at most one u 2-segment is contained in a v 3-segment, and likewise, at most one v 2-segment is contained in a u 3-segment.

For a u 3-segment and v 3-segment which share edges, we have that either both segments are equal or they have exactly one edge in common. In consideration of this observation and the ones preceeding it, we obtain the bound

$$\begin{aligned} 2k + \beta = |C| &\geq 2\alpha_2 + 2\gamma_2 + \alpha_3 + \gamma_3 - 1 \\ 2k + \beta &\geq 2(2k - \beta) - 1 \\ \beta &\geq \frac{2k-1}{3} \end{aligned}$$

Since β is an integer, we have $\beta \geq \lceil \frac{2k-1}{3} \rceil$. Here we reach a contradiction, since $\beta < \frac{2}{3}k$.

We conclude that there is at least one v 2-segment which is also a u 2-segment. Since u and v where arbitrary vertices of $V(G) \backslash V(C)$, we have that the above holds for any 2 vertices in $V(G) \backslash V(C)$. As a consequence, no two vertices of $V(G) \backslash V(C)$ can be adjacent, for if there were two such vertices, then C could be modified into a longer cycle via these two vertices. Thus we have that $V(G) \backslash V(C)$ is an independent set of vertices.

Suppose for some i and j we have that v_i^- is adjacent to v_j^-. Then

$$C' = (C \backslash \{v_i^- v_i, v_j^- v_j\}) \cup P_i \cup P_j \cup \{v_i^- v_j^-\}$$

is a cycle with $|C'| = |C|+1$. We conclude that for $i \neq j$, it holds that v_i^- and v_j^- are non-adjacent. Letting $S = \{v_i^- : \ i = 1, \dots, k\}$, we have that S is an independent set. Suppose some vertex $u \in V(G) \backslash (V(C) \cup \{v\})$ is adjacent to a vertex $v_i^- \in S$. From before we know that there is a v 2-segment $C[v_{j-1}, v_j]$ which is also a u 2-segment (obviously $i \neq j$). Let

$$C' = C \backslash (C[v_{i-1}, v_i] \cup \{v_i^- v_i\}) \cup \{v_{j-1} u v_i, v_i v v_j\}.$$

It is seen that $|C'|=|C|+1$. We conclude that no vertex of $V(G)\backslash V(C)$ is adjacent to vertices in S, and consequently $S\cup(V(G)\backslash V(C))$ is an independent set.

We have

$$\begin{aligned}|S \cup (V(G)\backslash V(C))| &= k + |V(G)| - c \\ &= |V(G)| - k - \beta.\end{aligned}$$

∎

4. Bounding the circumference

In this section, we show that the circumference of a k-connected must be 'small', if one has two vertex-disjoint cycles the sum of whose lengths is 'large'.

Lemma 4.1. *Let G be a k-connected graph where $k \geq 4$. Let C_1 and C_2 be two vertex-disjoint cycles having lengths c_1 and c_2, respectively. Given G has circumference $c \geq 2k$ and $c_1 + c_2 \geq 2(c - k + 1)$, then $c < \frac{7}{3}k + \frac{5}{2}$.*

Proof. We shall assume that $c_1 \geq c_2$ and $c = 2k + \beta$, where $\beta \geq 0$. Let C_1 and C_2 be given circular orientations. For $i = 1$, or 2, and vertices $u, v \in V(C_i)$, we let $C_i[u,v]$ denote the path along C_i directed from u to v. We first note that

$$c_1 \geq \frac{c_1 + c_2}{2} \geq c - k + 1 \geq k.$$

We shall consider two cases:

Case 1. Suppose $c_2 \geq k$.

By Lemma 2.1, there are k vertex-disjoint paths $P_1, \ldots, P_k$ between $V(C_1)$ and $V(C_2)$ where each P_i intersects C_1 and C_2 at exactly one vertex (ie. at its terminal vertices). For $i = 1, \ldots, k$, let u_i, and v_i be the terminal vertices of P_i lying on C_1 and C_2 respectively. We may assume that $u_1, \ldots, u_k$ occur in order as we travel along C_1 in the direction of its orientation.

For some $1 \leq i \leq k$, we have $|C_1[u_i, u_{i+1}]| \leq \frac{c_1}{k}$. Here we take $u_{k+1} = u_1$. For the same i, we have that either $|C_2[v_i, v_{i+1}]| \geq \frac{c_2}{2}$ or $|C_2[v_{i+1}, v_i]| \geq \frac{c_2}{2}$. Without loss of generality, we assume the former holds. Let

$$C = C_1[u_{i+1}, u_i] \cup C_2[v_i, v_{i+1}] \cup P_i \cup P_{i+1}.$$

Then C is a cycle where $|C| \geq c_1(1 - \frac{1}{k}) + \frac{c_2}{2} + 2$. Consider the LP:

$$\min \left(1 - \frac{1}{k}\right) x_1 + \frac{x_2}{2} + 2.$$

$$x_1 \geq x_2$$
$$x_2 \geq k$$
$$x_1 + x_2 \geq 2(c-k+1)$$

The minimum occurs at
1) $x_1 = x_2 = c-k+1$,
or
2) $x_1 = 2c-3k+2$, $x_2 = k$.

Suppose the minimum occurs at 1). Then

$$x_1\left(1-\frac{1}{k}\right)+\frac{x_2}{2}+2=(c-k+1)\left(\frac{3}{2}-\frac{1}{k}\right)+2.$$

We have

$$(c-k+1)\left(\frac{3}{2}-\frac{1}{k}\right)+2 \leq |C| \leq c.$$

Rearranging the inequality, we obtain

$$c \leq \frac{(k-1)\left(\frac{3}{2}-\frac{1}{k}\right)-2}{\frac{1}{2}-\frac{1}{k}}.$$

Using the above inequality, we deduce that $c < \frac{7}{3}k+\frac{5}{2}$ if $k \leq 9$.

Suppose the minimum occurs at 2). Then

$$x_1\left(1-\frac{1}{k}\right)+\frac{x_2}{2}+2=(2c-3k+2)\left(1-\frac{1}{k}\right)+\frac{k}{2}+2.$$

We have that

$$(2c-3k+2)\left(1-\frac{1}{k}\right)+\frac{k}{2}+2 \leq |C| \leq c.$$

Rearranging, we get

$$c \leq \frac{\frac{5}{2}k+\frac{2}{k}-7}{1-\frac{2}{k}}.$$

From the above inequality, we conclude that $c < \frac{7}{3}k+\frac{5}{2}$ if $k \leq 27$, and combining both cases, we obtain $c < \frac{7}{3}k+\frac{5}{2}$, if $k \leq 9$.

We therefore suppose that $k \geq 10$. By considering averages, one can show that for some $1 \leq i \leq k$ it holds $|C_1[u_i, u_{i+3}]| \leq \frac{3}{k}c_1$. Now for some $i \leq j \leq i+3$ and $i \leq j' \leq i+3$ where $v_{j'}$ occurs after v_j while moving along C_2 with its orientation, we have $|C_2[v_j, v_{j'}]| \leq \frac{c_2}{4}$. Assuming without loss of generality that $j < j'$, let

$$C = C_1[u_{j'}, u_j] \cup C_2[v_{j'}, v_j] \cup P_1 \cup P_2.$$

Then $|C| \geq c_1(1 - \frac{3}{k}) + \frac{3}{4}c_2 + 2$. Consider the LP:

$$\min \left(1 - \frac{3}{k}\right) x_1 + \frac{3x_2}{4} + 2.$$

$$x_1 \geq x_2$$

$$x_2 \geq k$$

$$x_1 + x_2 \geq 2(c - k + 1)$$

The minimum occurs at
1) $x_1 = x_2 = c - k + 1$,
or
2) $x_1 = 2c - 3k + 2$, $x_2 = k$.

Suppose the minimum occurs at 1). Then

$$x_1 \left(1 - \frac{3}{k}\right) + \frac{3x_2}{4} + 2 = (c - k + 1)\left(1 - \frac{3}{k} + \frac{3}{4}\right) + 2.$$

Thus

$$(c - k + 1)\left(1 - \frac{3}{k} + \frac{3}{4}\right) + 2 \leq |C| \leq c.$$

Rearranging we obtain

$$c \leq \frac{(k-1)\left(\frac{7}{4} - \frac{3}{k}\right) - 2}{\frac{3}{4} - \frac{3}{k}}.$$

From we see that $c < \frac{7}{3}k + \frac{5}{2}$ if $k \geq 10$.

Suppose the minimum occurs at 2). Then

$$x_1 \left(1 - \frac{3}{k}\right) + \frac{3x_2}{4} + 2 = (2c - 3k + 2)\left(1 - \frac{3}{k}\right) + \frac{3k}{4} + 2.$$

Thus

$$(2c - 3k + 2)\left(1 - \frac{3}{k}\right) + \frac{3k}{4} + 2 \leq |C| \leq c.$$

Rearranging we obtain

$$c \leq \frac{3\left(k - \frac{2}{3}\right)\left(1 - \frac{3}{k}\right) - \frac{3k}{4} - 2}{1 - \frac{6}{k}}.$$

From this inequality, we have that $c<\frac{7}{3}k+\frac{5}{2}$ if $k\geq 10$.

Combining both cases, we obtain that $c<\frac{7}{3}k+\frac{5}{2}$ if $k\geq 10$. This completes the proof for Case 1.

Case 2. Suppose $c_2<k$.

Let $c_2=k-\gamma$, and let $c_1+c_2=2(c-k+1)+\alpha$ where $\gamma>0$ and $\alpha\geq 0$. If $c_2\leq 4$, then $c\geq c_1\geq 2(c-k+1)-4$. From this we obtain that $c\leq 2k+2<2k+\frac{5}{2}$. Thus we may assume that $c_2\geq 5$. We pick a vertex $v\in V(C_2)$. By Lemma 2.2, we can find k internally vertex-disjoint paths $P_1,\dots,P_k$ from C_1 to C_2 where exactly $\gamma+1$ of these paths, say $P_1,\dots,P_{\gamma+1}$, terminate at v, and the remaining $k-\gamma-1$ terminate at each of the vertices of $V(C_2)\backslash\{v\}$. Let $u_1,\dots,u_{\gamma+1}$ be the respective terminal vertices on C_1 of the paths $P_1,\dots,P_{\gamma+1}$. Let v^- and v^{--} be the first and second vertices, respectively, on C_2 preceeding v, and let v^+ and v^{++} be the first and second vertices succeeding v. We let P^{--},P^-,P^+, and P^{++} denote the paths terminating at the vertices v^{--},v^-,v^+, and v^{++}, respectively, and we denote their respective terminal vertices on C_1 by u_{--},u_-,u_+, and u_{++}.

Suppose that $|C_1[u_{--},u_{++}]|\leq\beta+\alpha-1$. Let

$$C=C_1[u_{++},u_{--}]\cup C_2[v^{++},v^{--}]\cup P^{++}\cup P^{--}.$$

$$\begin{aligned}|C|&\geq c_1+c_2-(\beta+\alpha-1)-4+2\\&\geq 2(c-k+1)+\alpha-\beta-\alpha-1\\&=2k+\beta+1>c.\end{aligned}$$

This yields a contradiction. We conclude that $|C_1[u_{--},u_{++}]|\geq\beta+\alpha$. The above argument also shows that for any different vertices $u,u'\in\{u_{--},u_-,u_+,u_{++}\}$ it holds that $|C_1[u,u']|\geq\beta+\alpha$. Moreover, for any $u_i\in\{u_1,\dots,u_{\gamma+1}\}$ and $u\in\{u_{--},u_-,u_+,u_{++}\}$, it holds that $|C_1[u,u_i]|\geq\beta+\alpha$. From this, we obtain the bound $c_1\geq 5(\alpha+\beta)+\gamma$. Let $c_1=5(\alpha+\beta)+\gamma+\delta$, where $\delta\geq 0$. Then

$$\begin{aligned}c_1+c_2&=2(c-k+1)+\alpha\\5(\alpha+\beta)+\gamma+\delta+k-\gamma&=2k+2\beta+2+\alpha\\3\beta&=k+2-4\alpha-\delta\\\beta&\leq\frac{k}{3}+\frac{2}{3}\end{aligned}$$

Thus $c=2k+\beta\leq\frac{7}{3}k+\frac{2}{3}<\frac{7}{3}k+\frac{5}{2}$. This completes the proof for Case 2.

The proof of the lemma follows from the consideration of Cases 1 and 2 above. ∎

Lemma 4.2. *Let G be a k-connected graph where $k\geq 3$. Let C_1 and C_2 be two cycles having exactly one vertex in common, and having lengths c_1 and c_2, respectively. Given G has circumference $c\geq 2k$, if $c_1+c_2\geq 2(c-k+1)+1$, then $c<\frac{7}{3}k+\frac{5}{2}$.*

Proof. As in the previous lemma, we shall assume that $c=2k+\beta$, where $\beta\geq 0$. We shall also assume that both C_1 and C_2 have circular orientations. Let v be the vertex common to C_1 and C_2. We shall use induction on k. If $k=3$, then it is straightforward to show that $c_1+c_2\leq 2(c-2)$, and the lemma is vacuously true in this case.

We therefore suppose that the lemma is true for all graphs of connectivity k–1 or less, where $k\geq 4$. Let G be a k-connected graph. If $c_1+c_2\geq 2(c-k+1)+3$, then pick an edge $e\in E(G)\backslash E(C_1)\cup E(C_2)$ and let $G'=G\backslash\{e\}$ and $c'=c(G')$. We may choose e such that $c'=c$. Then G' is $(k-1)$-connected, and $c_1+c_2\geq 2(c-k+1)+3\geq 2(c'-(k-1)+1)+1$. By the inductive assumption, we have that $c=c'<\frac{7}{3}(k-1)+\frac{5}{2}<\frac{7}{3}k+\frac{5}{2}$.

We may therefore assume that $c_1+c_2\leq 2(c-k+1)+2$. Let v^- and v^+ be the vertices preceeding and succeeding v on C_2, respectively. Suppose there is a path P from v^- to v^+ in $G\backslash(V(C_1\cup C_2)\backslash\{v^-,v^+\})$. Then

$$C_1'=C_1,\quad C_2'=C_2[v^+,v^-]\cup P$$

would be vertex-disjoint cycles where

$$|C_1'|+|C_2'|\geq c_1+c_2-1\geq 2(c-k+1).$$

By Lemma 4.1, it would follow that $c<\frac{7}{3}k+\frac{5}{2}$. Thus we may assume that no such path P exists, and v^-, and v^+ are separated by the vertices in $S=V(C_1\cup C_2)\backslash\{v^-,v^+\}$ in G.

Since G is k-connected, there are k internally vertex-disjoint paths, say $P_1^-,\dots,P_k^-$ from v^- to S, and k internally vertex-disjoint paths, say $P_1^+,\dots,P_k^+$ from v^+ to S. We may assume that P_1^- is the edge v^-v, and P_1^+ is the edge v^+v. For all i,j the paths P_i^- and P_j^+ can only meet at their terminal vertices in S. Suppose δ_1 of the paths $P_1^-,\dots,P_k^-$ terminate at vertices in C_2, and $k-\delta_1$ paths terminate at vertices of $C_1\backslash\{v\}$. Similarly, suppose that δ_2 of the paths $P_1^+,\dots,P_k^+$ terminate at vertices in C_2, and $k-\delta_2$ paths terminate at $C_1\backslash\{v\}$. If $\delta_1+\delta_2\geq c_2$, then there is a vertex $u\in V(C_2)$ such that for some i and j, a path P_i^- terminates at u, and a path P_j^+ terminates at u^+. Let

$$C_2'=C_2[v^+,u]\cup C_2[u^+,v^-]\cup P_i^-\cup P_j^+$$
$$C_1'=C_1.$$

Then C_1' and C_2' are disjoint cycles where $|C_1'|+|C_2'|=c_1+c_2-1\geq 2(c-k+1)$. Thus by Lemma 4.1 we have $c<\frac{7}{3}k+\frac{5}{2}$. We may therefore assume that $\delta_1+\delta_2\leq c_2-1$.

Let P_i^- and P_j^+ be paths which terminate at different vertices u_1 and u_2, respectively, on C_1. If $|C_1[u_1,u_2]|\leq\beta+2$, then

$$C=C_2[v^+,v^-]\cup C_1[u_2,u_1]\cup P_i^-\cup P_j^+$$

is a cycle where

$$\begin{aligned}|C|&\geq c_1+c_2-(\beta+2)\\&\geq 2(c-k+1)+1-(\beta+2)\\&=2k+\beta+1>c.\end{aligned}$$

This yields a contradiction. Thus we have $|C_1[u_1,u_2]|\geq\beta+3$, and similarly $|C_1[u_2,u_1]|\geq\beta+3$. Consequently,

$$\begin{aligned}(k-\delta_1-1)+(k-\delta_2-1)+2(\beta+3)&\leq c_1\\2k-\delta_1-\delta_2&\leq c_1-2\beta-4\end{aligned}$$

In addition, we have $\delta_1+\delta_2\leq c_2-1$. Using this, the above inequality becomes

$$\begin{aligned}2k&\leq c_1+c_2-2\beta-5\\2k+2\beta+5&\leq c_1+c_2.\end{aligned}$$

However, we are assuming that $c_1+c_2\leq 2(c-k+1)+2=2k+2\beta+4$. With this, we reach a contradiction. We conclude that no two such cycles C_1 and C_2 can exist, and thus the lemma holds for k-connected graphs as well. The proof now follows by induction. ∎

5. Proofs of the main theorems

Proof of Theorem 1.2. Let G be a k-connected graph where $c\geq 2k$. For $k=2,3,4$, the proof of the theorem is straightforward, and is left to the reader. We shall henceforth assume that $k\geq 5$, G has circumference c, and C_1 and C_2 are two cycles intersecting in at most one vertex, where $c_1=|C_1|$, and $c_2=|C_2|$. Let $c=2k+\beta$ where $\beta\geq 0$.

Suppose C_1 and C_2 are disjoint. Assuming $c_1+c_2\geq 2(c-k+1)+1$, Lemma 4.1 implies that $c<\frac{7}{3}k+\frac{5}{2}$. Since $k\geq 5$, it holds that $\frac{7}{3}k+\frac{5}{2}\leq 2k+\frac{2}{3}k$.

Now Lemma 3.1 implies that G has an independent set S with at least $|V(G)|-k-\beta$ vertices. Thus

$$|S\cap(V(C_1)\cup V(C_2))| \geq c_1+c_2-k-\beta > \frac{c_1+c_2}{2}.$$

The second inequality comes from the fact that $c_1+c_2\geq 2(c-k+1)+1=2k+2\beta+3$. On the other hand, for $i=1,2$, C_i has no independent set of size greater than $\frac{c_i}{2}$. This means that the subgraph induced by $V(C_1\cup C_2)$ has no independent set of size greater than $\frac{c_1+c_2}{2}$. Thus we reach a contradiction.

Suppose on the other hand that C_1 and C_2 intersect in exactly one vertex. Assuming again that $c_1+c_2\geq 2(c-k+1)+1$, one can show, as was done in the proof of Lemma 4.2, that C_1 and C_2 can be modified to produce two vertex-disjoint cycles C_1', C_2', where $|C_1'|+|C_2'|\geq 2(c-k+1)$. As before, we have $c<2k+\frac{2}{3}k$, and we arrive at a contradiction in the same way as before.

From consideration of the above, we conclude that $c_1+c_2\leq 2(c-k+1)$. ∎

Proof of Theorem 1.3. Let G be a graph with $k\geq 2$. Among all bonds of G, choose a bond B_1 which intersects the maximum number of cycles of length $c-k+2$ or greater. Let $B_1=[X_1,Y_1]$. Suppose there is a cycle C_1 of length at least c–k+2 which B_1 fails to intersect. We can assume $V(C_1)\subseteq Y_1$. One can choose a subset $X'\subset V(G)$ containing exactly one vertex of C_1 such that $B_2=[X_1\cup X', Y_1\backslash X']$ is a bond. Now B_2 intersects C_1. By the maximality of B_1 there must be a cycle C_2 having at least $c-k+2$ vertices which B_1 intersects, but which B_2 does not. We therefore have that $V(C_2)\subseteq X_1\cup X'$. By our choice of X', it holds that $|V(C_1)\cap V(C_2)|\leq 1$. However, Theorem 1.2 implies that $|V(C_1)|+|V(C_2)|\leq 2(c-k+1)$, yielding a contradiction. We conclude that B_1 intersects all cycles of length at least $c-k+2$. ∎

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