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CIRCUITS THROUGH COCIRCUITS IN A GRAPH WITH EXTENSIONS TO MATROIDS

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We show that for any k-connected graph having cocircumference c^*, there is a cycle which intersects every cocycle of size $c^* - k + 2$ or greater. We use this to show that in a 2-connected graph, there is a family of at most c^* cycles for which each edge of the graph belongs to at least two cycles in the family. This settles a question raised by Oxley.

A certain result known for cycles and cocycles in graphs is extended to matroids. It is shown that for a k-connected regular matroid having circumference $c \geq 2k$ if C_1 and C_2 are disjoint circuits satisfying $r(C_1)+r(C_2)=r(C_1 \cup C_2)$, then $|C_1|+|C_2| \leq 2(c-k+1)$.

1. Introduction

The *circumference* of a graph (resp. matroid) is defined to be the size of its largest cycle (resp. circuit) and the *cocircumference* is the size of its largest cocycle (resp. cocircuit). We denote the circumference of a graph G (resp. matroid M) by $c(G)$ (resp. $c(M)$) and we denote the cocircumference by $c^*(G)$ (resp. $c^*(M)$). A matroid is *connected* if for any two of its elements there is a circuit which contains them. In 1991, Thomas posed the problem: does every sufficiently large connected matroid have a large circuit or cocircuit? Lovász, Schrijver, and Seymour answered this question in the affirmative by showing that if M is a connected matroid with circumference c and cocircumference c^*, then $|e(M)| \leq 2^{c+c^*-1}$. The bound was subsequently greatly improved in [8] for graphs, where is was shown that if G is a loopless 2-connected graph with circumference c and cocircumference c^*, then $|E(G)| \leq \frac{1}{2}cc^*$. Recently, Lemos and Oxley [1] showed that this bound holds

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not only for graphs but for connected matroids in general. They showed that if M is a connected matroid with at least 2 elements and M has circumference c and cocircumference c^*, then $|E(M)| \leq \frac{1}{2}cc^*$.

Assuming that one could cover the elements of a matroid M with at most $c^*(M)$ circuits so that each element was covered at least twice, we would have the bound $2|e(M)| \leq cc^*$, or $|e(M)| \leq \frac{1}{2}cc^*$. In light of this, Oxley [4] posed the following conjecture:

Conjecture 1.1. For any connected matroid M with at least 2 elements, one can find a collection of at most $c^*(M)$ circuits which cover each element of M at least twice.

In [3], Neumann-Lara et al showed that the above conjecture holds for cographic matroids. They use the following lemma which appears in Wu [8]. Following standard graph theory terminology, we shall use the term *bond* here to mean a cocycle.

Lemma 1.1. *Let G be a 2-connected graph. Then there is a bond which intersects every cycle of length c or $c-1$.*

We shall prove a corresponding result for cographic matroids, namely:

Theorem 1.1. *Let G be a k-connected graph where $k \geq 2$. Then there is a cycle which intersects every bond of size $c^* - k + 2$ or greater.*

We shall make use of an old theorem of Tutte [7]:

Theorem 1.2. *Let M be a connected matroid and let $e \in e(M)$. Then either $M \backslash e$ or M/e is connected.*

With the aid of Theorems 1.1 and 1.2, we shall give an affirmative answer to Oxley's conjecture for graphic matroids. We shall prove the following:

Theorem 1.3. *For a 2-connected graph G, there is a collection of at most $c^*(G)$ cycles which cover each edge of G at least twice.*

In the last section, we focus on matroids and prove the result described in the abstract.

2. Maximum Cycles and Bonds in a Graph

In this section, we shall prove a result about disjoint bonds in a graph. The following is a known result for graphs (see Wu [8]):

Lemma 2.1. *Let G be a 2-connected graph with circumference c. Let C_1 be a cycle of length c. If C_2 is any other cycle of length at least $c-1$, then $|V(C_1)\cap V(C_2)|\geq 2$.*

The next result is an analogue of the above lemma for bonds, which applies to k-connected graphs. For subsets X and Y of vertices, we denote the set of edges with one end in X and the other end in Y by $[X,Y]$. We shall denote the complement of X by $\overline{X}$.

Lemma 2.2. *Let G be a k-connected graph and let $B_1=[X,\overline{X}]$ and $B_2=[Z,\overline{Z}]$ be two disjoint bonds of G where $G[\overline{X}\cap\overline{Z}]$ is connected and $X\cap Z=\emptyset$. Then $|B_1|+|B_2|\leq 2(c^*-k+1)$.*

Proof. Let $Y=\overline{X}\cap\overline{Z}$. Since G is k-connected, there are k internally vertex-disjoint paths $P_1,\dots,P_k$ between X and Z (which we assume only meet X and Z at their endpoints). Because B_1 and B_2 are disjoint, G has no edges with one end in X and the other in Z. For $i=1,\dots,k$, let Q_i be the path which is the portion of P_i restricted to $G[Y]$. Since $G[Y]$ is connected, it has a spanning tree T which contains the paths Q_i, $i=1,\dots,k$. We can partition $V(T)$ into k subsets $Y_1,\dots,Y_k$ where $G[Y_i]$ is connected and $Q_i\subseteq G[Y_i]$. Since T is a tree, there are exactly $k-1$ edges of T which connect the sets $Y_1,\dots,Y_k$. Let E' be the set of these edges. Since T is bipartite, we can find a subset $S\subset\{1,\dots,k\}$ such that each edge of E' joins a vertex of Y_i to a vertex of Y_j for some i in S, and some j in $\overline{S}$. Let

$$
\begin{aligned}
B_1' &= [X\cup(\cup_{i\in S}Y_i), Z\cup(\cup_{i\in\overline{S}}Y_i)]\\
B_2' &= [X\cup(\cup_{i\in\overline{S}}Y_i), Z\cup(\cup_{i\in S}Y_i)].
\end{aligned}
$$

By the choice of $Y_1,\dots,Y_k$ and the fact that $G[X]$ and $G[Y]$ are connected, one sees that B_1' and B_2' are both bonds of G. Furthermore, $B_1'\cup B_2'=B_1\cup B_2\cup E'$.

It follows that

$$
\begin{aligned}
2c^* \geq |B_1'|+|B_2'| &= |B_1|+|B_2|+2|E'|\\
&= |B_1|+|B_2|+2(k-1).
\end{aligned}
$$

So $|B_1|+|B_2|\leq 2(c^*-k+1)$. ∎

It would seem plausible that a dual result for Lemma 2.2 holds for circuits. In [2] we prove that:

Theorem 2.1. *For a k-connected graph G ($k\geq 2$) having circumference $c\geq 2k$, for any pair of cycles C_1 and C_2 which intersect in at most one vertex, it holds that $|V(C_1)|+|V(C_2)|\leq 2(c-k+1)$.*

In the last section, we shall show that Lemma 2.2 and 2.1 can be unified into one result for regular matroids (Theorem 5.3). Theorem 2.1 implies (see [2]):

Theorem 2.2. *For any k-connected graph G $(k \geq 2)$ having circumference $c \geq 2k$, there is a bond B which intersects every cycle of length $c-k+2$ or greater.*

3. A Cycle Intersecting Bonds

In this section, we shall show that, for any k-connected graph G $(k \geq 2)$ with cocircumference c^*, there is a cycle intersecting every bond of size c^*-k+2 or greater.

Proof of Theorem 1.1. Let C_1 be a cycle which intersects the greatest number of bonds of size at least c^*-k+2. Suppose C_1 does not intersect a bond B_1 of size at least c^*-k+2. Let $B_1=[X_1,Y_1]$ where $X_1 \cup Y_1 = V(G)$. We can assume that $V(C_1) \subseteq Y_1$. Then X_1 is contained in a component K of $G \backslash V(C_1)$. Let $v_1, v_2, \ldots, v_q$ be the neighbours of K lying on C_1, enumerated in order as they occur along C_1.

Claim 3.1. *For some t, $1 \leq t \leq q$, there are vertices v_t and v_{t+1} $(v_{q+1}=v_1)$ having neighbours v'_t and v'_{t+1} in K such that there is a path P from v_t to v_{t+1} in $G[K \cup \{v_t, v_{t+1}, v_t v'_t, v_{t+1} v'_{t+1}\}]$ intersecting B_1.*

Proof of claim. If t is as in the claim, then we shall say that it is *good*. Suppose first that among the vertices $v_1, \ldots, v_q$ there are vertices which are neighbours of X_1. If $K=G[X_1]$, then the claim is easily seen to hold. So we may assume that $K \neq G[X_1]$. Since B_1 is a bond, $G[\overline{X}_1]$ is connected and consequently there is at least one vertex among $v_1, \ldots, v_q$ which is a neighbour of $K \backslash X_1$. Thus there is a t, $1 \leq t \leq q$, where either $v_t \in N_G(K \backslash X_1)$ and $v_{t+1} \in N_G(X_1)$, or $v_{t+1} \in N_G(K \backslash X_1)$ and $v_t \in N_G(X_1)$. Let v'_t and v'_{t+1} be neighbours of v_t and v_{t+1} in K, respectively, where v'_t and v'_{t+1} are separated by the edges $[X_1, V(K) \backslash X_1]$ in K. Since K is connected, there is a path P' from v'_t to v'_{t+1} in K and such a path must intersect $[X_1, V(K) \backslash X_1]$. Then $v_t v'_t P v'_{t+1} v_{t+1}$ is a path which intersects B_1. In this case, t is good.

We may assume that none of the vertices $v_1, \ldots, v_q$ are neighbours of X_1. Let $Z = V(K) \backslash X_1$. Since G is 2-connected, there are 2 internally vertex-disjoint paths P_1 and P_2 from X_1 to C_1. Let v_{t_1} and v_{t_2} be the terminal vertices of P_1 and P_2, respectively on C_1. We may assume that $t_1 < t_2$. We shall show by induction on $t_2 - t_1$ that there is a good t where $t_1 \leq t \leq t_2$.

If $t_2 - t_1 = 1$, that is, $t_2 = t_1 + 1$, then t_1 is good, since we can find a path $P_3 \subseteq G[X_1]$ between terminal vertices of P_1 and P_2 in X_1, and it follows that $P_1 \cup P_2 \cup P_3$ is a path from v_{t_1} to v_{t_2} intersecting B_1. Thus the hypothesis holds for $t_2 - t_1 = 1$. Assume the hypothesis holds for $t_2 - t_1 \leq T$.

Suppose $t_2 - t_1 = T + 1$, $T \geq 1$. Consider v_{t_1+1}. Let v'_{t_1+1} be a neighbour of v_{t_1+1} in K. Suppose there is a path in $G[K \cup \{v_{t_1+1}, v_{t_1+1}v'_{t_1+1}\}]$ from v_{t_1+1} to P_2, which avoids $V(P_1) \cap Z$. In this case, it is easily seen that t_1 is good. If this is not the case, then there is a path in $G[K \cup \{v_{t_1+1}, v_{t_1+1}v'_{t_1+1}\}]$ from v_{t_1+1} to P_1 which avoids $V(P_2) \cap Z$. In this case, we can find two internally vertex-disjoint paths from X_1 to C which terminate at v_{t_1+1} and v_{t_2}. Now $t_2 - (t_1 + 1) = T$. By the inductive assumption, there is a good t where $t_1 + 1 \leq t \leq t_2$. Thus the hypothesis holds for $t_2 - t_1 = T + 1$. The claim now follows by induction. ∎

By the above claim, there are vertices v_t and v_{t+1} having neighbours v'_t, and v_{t+1}, respectively, in K, and a path P in $G[K \cup \{v_t, v_{t+1}, v_t v'_t, v_{t+1}v'_{t+1}\}]$ which intersects B_1. Let $C_2 = C_1[v_{t+1}, v_t] \cup P$, where $C_1[v_{t+1}, v_t]$ denotes the path along C_1 between v_{t+1} and v_t. The cycle C_2 intersects B_1. By the maximality of C_1, there is a bond $B_2 = [X_2, Y_2]$ of size at least $c^* - k + 2$ which C_1 intersects but C_2 does not. We can assume that $V(C_2) \subseteq Y_2$. Since C_1 intersects B_2, we have that X_2 contains some vertices of $C_1[v_t, v_{t+1}]$. Since $G[X_2]$ is connected and no vertices of $C_1[v_t, v_{t+1}]$, apart from v_t and v_{t+1}, have neighbours in K, we conclude that $X_2 \cap V(K) = \emptyset$, (hence $X_1 \cap X_2 = \emptyset$) and moreover, $B_1 \cap B_2 = \emptyset$.

We claim that $G[Y_1 \cap Y_2]$ is connected. Let P' be the path $C_1[v_{t+1}, v_t]$. We have that $P' \subseteq G[Y_1 \cap Y_2]$. Suppose $G[Y_1 \cap Y_2]$ has at least two components: let K_1 and K_2 be two of these components where $P' \subseteq K_1$. If $K \neq G[X_1]$, then each component of $G[K \backslash X_1]$ has vertices which are neighbours of vertices in P' (since $G[Y_1]$ is connected). Thus $K \backslash X_1 \subset K_1$. This means in particular that no vertex of $K \backslash X_1$ is adjacent to vertices in K_2. The vertices of $N_G(X_1)$ can only belong $K \backslash X_1$ or P'. Thus X_1 can only be adjacent to vertices of K_1. We note that since B_2 is a bond, $G[Y_2]$ is connected, and hence every component of $G[Y_1 \cap Y_2]$ must be adjacent to vertices in $Y_2 \backslash Y_1 \subseteq X_1$. However, no vertex of K_2 is adjacent to vertices of X_1. This yields a contradiction. We conclude that $G[Y_1 \cap Y_2]$ is connected.

We have that $|B_1| + |B_2| \leq 2(c^* - k + 1)$ by Lemma 2.2. However, $|B_1| + |B_2| \geq 2(c^* - k + 2)$, yielding a contradiction. We conclude that C_1 must intersect every bond of size at least $c^* - k + 2$. ∎

4. Covering Edges With At Most c^* Cycles

In this section, we shall use Theorem 1.1 to verify Conjecture 1.1 for cographic matroids (see also [4], Question 3.13).

Proof of Theorem 1.3. By induction on $|V(G)|$. If $|V(G)|=2$, then G is a 2-cycle, and the theorem clearly holds. We suppose that $|V(G)|>2$ and the theorem holds for all 2-connected graphs with fewer vertices than G. We can assume that G is not a cycle and moreover, we may assume the theorem holds for any 2-connected graph with the same number of vertices as G but fewer edges. If G has vertex of degree 2, then we can delete it, and add an edge between its neighbours. Such a graph has cocircumference c^* and has a collection of at most c^* cycles covering each of its edges at least twice. Such a collection can easily be extended to the desired collection of cycles for G. So we may assume that G has no vertices of degree 2.

Suppose first that G is 3-connected. Then by Theorem 1.1, there is a cycle C for which C intersects every bond with c^* or c^*-1 edges. By Tutte's theorem (Theorem 1.2) we can partition the edges of C as $E(C)=A\cup B$, such that $G'=(G\backslash A)/B$ is 2-connected. Now it is seen that $c^*(G')\leq c^*-2$ since C intersects every bond of G with c^* or c^*-1 elements. Furthermore, $|E(G')|<|E(G)|$. Thus by the inductive assumption, there is a family $\mathcal{F}'$ of at most $c^*(G')$ cycles of G' which cover each edge of G' at least twice. For each cycle $K'\in\mathcal{F}'$ let K be a cycle of G such that $(K\backslash A)/B=K'$. Let $\mathcal{F}=\{K|K'\in\mathcal{F}'\}$. Then $\mathcal{F}\cup\{C,C\}$ is a family of cycles which covers the edges of G at least twice, and such a family has at most $c^*(G')+2\leq c^*-2+2=c^*$ cycles. This shows that the theorem holds for G.

We suppose that G is not 3-connected. Suppose G has a 2-vertex cut, say $\{x,y\}$ which separates 2 graphs G_1 and G_2 where $G_1\cup G_2=G$, $V(G_1)\cap V(G_2)=\{x,y\}$, and $|V(G_i)|\geq 4$, $i=1,2$. Let β_i, $i=1,2$ be the size of the largest bond in G_i which separates x and y. It can be shown that for any bond B of G where $B\subseteq E(G_i)$ it holds that $|B|\leq 2\beta_i$. Let $\beta=\min\{\beta_1,\beta_2\}$. We have that $\beta_1+\beta_2\leq c^*$, and consequently, $2\beta\leq c^*$. For $i=1,2$ we create a graph G_i' by adding a vertex v_i to G_i together with β edges from v_i to each of x and y. We may assume that $\beta=\beta_1$. Then $c^*(G_1')=2\beta$, and $c^*(G_2')\leq c^*$. We have that G_i' is 2-connected and $|V(G_i')|<|V(G)|$. Thus by assumption, there are $q_1\leq c^*(G_1')=2\beta$ cycles of G_1', say $C_{11},C_{12},\ldots,C_{1q_1}$ which cover each edge of G_1' at least twice. In fact, $q_1=2\beta$, since v_1 has degree 2β in G_1', and each of the cycles $C_{11},C_{12},\ldots,C_{1q_1}$ contain v_1. Furthermore, there are $q_2\leq c^*(G_2')\leq c^*$ cycles of G_2' which cover its edges at least twice. We have that $q_1=2\beta\leq q_2$. We may assume that the cycles C_{2j}, $j=1,\ldots,q_1$ contain

v_2, and for $i=1,2$ exactly p_i, $i=1,2$ of the cycles $C_{i1},\dots,C_{iq_1}$ are 2-cycles. In addition, given $p_i\geq 1$, we may assume these cycles are $C_{i(q_1-p_i+1)},\dots,C_{iq_1}$.

Suppose $p_2\leq p_1$. For $j=1,\dots,q_1-p_1$, let $C_j=(C_{1j}\backslash\{v_1\})\cup(C_{2j}\backslash\{v_2\})$. If $p_2<p_1$, then for $j=q_1-p_1+1,\dots,q_1-p_2$, let $C_j=(C_{2j}\backslash\{v_2\})\cup P$ where P is an arbitrarily chosen path from x to y in G_1. If $q_2>q_1$, then let $C_j=C_{2j}$, for $j=q_1+1,\dots,q_2$. If $q_1=q_2$, then $\{C_1,\dots,C_{q_1-p_2}\}$ is a set of at most c^*-p_2 cycles covering the edges of G at least twice. Otherwise, if $q_1<q_2$, then $\{C_1,\dots,C_{q_1-p_2},C_{q_1+1},\dots,C_{q_2}\}$ is a set of at most c^*-p_2 cycles covering the edges of G at least twice.

Suppose $p_1<p_2$. For $j=1,\dots,q_1-p_2$ let $C_j=(C_{1j}\backslash\{v_1\})\cup(C_{2j}\backslash\{v_2\})$. For $j=q_1-p_2+1,\dots,q_1-p_1$, let $C_j=(C_{1j}\backslash\{v_1\})\cup P$ where P is an arbitrarily chosen path in G_2 from x to y. If $q_2>q_1$, let $C_j=C_{2j}$ for $j=q_1+1,\dots,q_2$. Then $\{C_1,\dots,C_{q_1-p_1},C_{q_1+1},\dots,C_{q_2}\}$ is a collection of at most c^*-p_1 cycles which cover the edges of G at least twice. Otherwise, if $q_2=q_1$, then $\{C_1,\dots,C_{q_1-p_1}\}$ is a collection of at most c^*-p_1 cycles which cover the edges of G at least twice.

We may henceforth assume that there are no such 2-cuts separating 2 vertices on each side. Create a graph G' from G in the following way: if v is a vertex in G with exactly 2 neighbours, say x and y, where there are β_1 edges between v and x and β_2 edges between v and y, then delete v and add $\beta=\max\{\beta_1,\beta_2\}$ edges between x and y. We have that $c^*(G')\leq c^*(G)$. If G' has just 2 or 3 vertices, then there is a cycle containing all vertices, and consequently this cycle would intersect all bonds of length $c^*(G')$ or $c^*(G')-1$. On the other hand, if G' has more than 3 vertices, then it is 3-connected, and hence by Theorem 1.3, there is a cycle C' of G' which intersects every bond of size $c^*(G')$ or $c^*(G')-1$. In either case, we see that there is such a cycle C'.

Let C be a cycle of G corresponding to C'. If C intersects every bond of G with c^* or c^*-1 edges, then we can proceed as in the case when G was 3-connected. Thus we can assume that there is a bond B_1 in G of size at least c^*-1 which C does not intersect. Such a bond must consist of the edges incident with a vertex, say v, where $v\in V(G)\backslash V(G')$. This means that v has exactly 2 neighbours, say v_1 and v_2. Let D be a cycle of G containing v_1, v_2, and v. We may assume that D is a largest such cycle and D does not intersect all bonds of size at least c^*-1. Thus there is a bond $B_2=[X,\overline{X}]$ where $|B_2|\geq c^*-1$, which D does not intersect. We may assume $V(D)\subseteq\overline{X}$. Now $G(\overline{X}\backslash\{v\})$ is connected since it contains the path $D\backslash\{x,y\}$ between x and y. Thus by Lemma 2.2, $|B_1|+|B_2|\leq 2c^*-2$. This means that $|B_1|=|B_2|=c^*-1$. Thus $|B_1|+|B_2|=2c^*-2$ and, upon examination of the proof of Lemma 2.2, we have that $G(\overline{X}\backslash\{v\})$ does not contain two edge-disjoint paths between v_1 and v_2. Thus there is a cut-edge $e = v_1'v_2' \in E(D)$ in $G(\overline{X}\backslash\{v\})$. The graph $G(\overline{X}\backslash\{v\})\backslash\{e\}$ contains 2 components, say K_1 and K_2 where $\{v_1, v_1'\} \subset V(K_1)$ and $\{v_2, v_2'\} \subset V(K_2)$. Suppose that any path from v to X must contain v_1' or v_2'. Then $\{v_1', v_2'\}$ is a 2-vertex cut separating v and X. In this case, we could find a cycle $D' = (D\backslash\{e\}) \cup Q$ where Q is a path between v_1' and v_2' intersecting X. Such a cycle contains v, v_1, and v_2 and is longer than D. This would contradict the choice of D. Thus $\{v_1', v_2'\}$ can not be a 2-vertex cut, and there is a path P from v to X which does not contain v_1' or v_2'. We can assume that $v_1 \in V(P)$. Let P_1 be the path representing the portion of P in K_1. We can divide the vertices of K_1 into 2 connected subgraphs K_{11} and K_{12} where $P_1 \subseteq K_{11}$ and $v_1' \in V(K_{12})$. By elementary counting arguments one deduces that $|[V(K_i), \overline{V(K_i)}]| = c^*$, for $i = 1, 2$. Now for the bond $B' = [V(K_2) \cup V(K_{12}), \overline{V(K_2) \cup V(K_{12})}]$ we have that

$$\begin{aligned} |B'| &= c^* - 1 + |[V(K_{12}), \overline{V(K_{12})}]| - 1 \\ &= c^* + |[V(K_{12}), \overline{V(K_{12})}]| - 2. \end{aligned}$$

Then $|[V(K_{12}), \overline{V(K_{12})}]| = 2$. By assumption, this means that $V(K_{12}) = \{v_1'\}$, and $d_G(v_1') = 2$. However, G is assumed to contain no vertices of degree 2. This concludes the proof of the theorem. ∎

5. Extensions To Matroids

The results given in the previous sections lead one to suspect that they have their counterparts in matroids. We present here some results representing extensions of previous results to matroids. To begin with, we briefly define some terminology to be used in this section, and refer the reader to [6] for further reference.

A matroid is *binary* if it is representable over $GF(2)$, and it is *regular* if it is representable over every field. Let M be matroid and let k be a positive integer. A partition (X, Y) of $E(M)$ is a *k-separation* if

(i) $\min\{|X|, |Y|\} \geq k$
(ii) $r(X) + r(Y) - r(M) \leq k - 1$.

The connectivity $\lambda(M)$ of M is defined to be the smallest k such that M has a k-separation, if such separations exist, and is defined to be ∞ otherwise. For $k \geq 2$, M is said to be *k-connected* if $\lambda(M) \geq k$. One consequence of this definition is that if a matroid M is k-connected and $|E(M)| \geq 2(k-1)$, then all circuits and cocircuits have size at least k. Given the conclusions of

Lemma 2.2 and Theorem 2.1, we venture the following conjecture for binary matroids:

Conjecture 5.1. Let M be a k-connected binary matroid ($k \geq 2$) having circumference $c \geq 2k$. Let C_1 and C_2 be two disjoint circuits of M where $r(C_1 \cup C_2) = r(C_1) + r(C_2)$. Then $|C_1| + |C_2| \leq 2(c - k + 1)$.

By Lemma 2.2 and Theorem 2.1, the above conjecture is seen to hold for graphic and cographic matroids. Our aim is to show that the conjecture holds for regular matroids, and to this end we exploit a well-known theorem of Seymour pertaining to the decomposition of regular matroids (see Oxley [6]):

Theorem 5.1. *Every regular matroid M can be constructed by means of direct sums, 2-sums, and 3-sums starting with matroids each of which is isomorphic to a minor of M and each of which is either graphic, cographic, or isomorphic to R_{10}.*

The above theorem implies that any 4-connected regular matroid is either graphic, cographic, or isomorphic to R_{10}. For R_{10}, each circuit has length at least 4. So if C_1 and C_2 are disjoint circuits satisfying $r(C_1) + r(C_2) = r(C_1 \cup C_2)$, then $r(C_1 \cup C_2) \geq 3 + 3 = 6$. However, $r(R_{10}) = 5$. This means that the above conjecture is vacuously true for R_{10}. Thus if we can show that the conjecture holds for 2- and 3-connected binary matroids, then it is true for regular matroids.

Lemma 5.1. *Let X_1 and X_2 be disjoint subsets of a matroid M where $r(X_1) + r(X_2) = r(X_1 \cup X_2)$. Then for any circuit C where $C \subset X_1 \cup X_2$, it holds that $C \subseteq X_1$, or $C \subseteq X_2$.*

Proof. Since $r(X_1) + r(X_2) = r(X_1 \cup X_2)$, the restriction $M|(X_1 \cup X_2)$ is the direct sum of $M|X_1$ and $M|X_2$. As such, no circuit of M contained in $X_1 \cup X_2$ meets both M_1 and M_2. ∎

The following corollary is an immediate consequence of the above lemma:

Corollary 5.1. *Let C_1 and C_2 be disjoint circuits in a matroid M satisfying $r(C_1) + r(C_2) = r(C_1 \cup C_2)$. If C is a circuit where $C \subset C_1 \cup C_2$, then $C = C_1$ or $C = C_2$.*

We shall first establish the truth of Conjecture 5.1 for 2-connected binary matroids. For sets X and Y, we let $X \triangle Y$ denote the symmetric difference of these sets.

Lemma 5.2. *Let M be a connected binary matroid and let C_1 and C_2 be two disjoint circuits where $r(C_1)+r(C_2)=r(C_1\cup C_2)$. Let C be a circuit intersecting both C_1 and C_2 for which $|C\backslash(C_1\cup C_2)|$ is minimum. Then $C\triangle C_1$, $C\triangle C_2$, and $C\triangle C_1\triangle C_2$ are circuits.*

Proof. Suppose that $C\triangle C_1$ is not a circuit. Let $e\in C\cap C_1$ and $f\in C\cap C_2$. There is a circuit $C_3\subseteq C\cup C_1-e$ containing f. It is seen that $C\cap C_1\neq\emptyset$, and as such C_3 intersects both C_1 and C_2. By choice of C, we have $|C\backslash(C_1\cup C_2)|=|C_3\backslash(C_1\cup C_2)|$ and $C\backslash(C_1\cup C_2)=C_3\backslash(C_1\cup C_2)$. Since $C\triangle C_3\neq\emptyset$, there is a circuit $C_4\subseteq C\triangle C_3\subseteq C_1\cup C_2$. By Corollary 5.1, we have that either $C_4=C_1$, or $C_4=C_2$. Since clearly neither of these two options is possible, we conclude that $C\triangle C_1$ must be a circuit. Using similar arguments, one can show that $C\triangle C_2$ and $C\triangle C_1\triangle C_2$ are also circuits. ∎

For subsets X_1 and X_2 of a matroid M we call a circuit C which intersects both X_1 and X_2 an (X_1,X_2)-*circuit*. We say that C is *minimum* if $|C\backslash(X_1\cup X_2)|$ is minimum amongst all (X_1,X_2)-circuits. The next lemma demonstrates that Conjecture 5.1 is true for 2-connected binary matroids.

Lemma 5.3. *Let M be a connected binary matroid with circumference c, and let C_1 and C_2 be disjoint circuits where $r(C_1)+r(C_2)=r(C_1\cup C_2)$. Then $|C_1|+|C_2|\leq 2(c-1)$.*

Proof. Let C be a minimum (C_1,C_2)-circuit. Then by Corollary 5.1, $|C\backslash C_1\cup C_2|\geq 1$. By Lemma 5.2 both $C\triangle C_1$ and $C\triangle C_2$ are circuits. Thus we obtain

$$2c\geq|C\triangle C_1|+|C\triangle C_2|=|C_1|+|C_2|+2|C\backslash(C_1\cup C_2)|.$$

Hence

$$|C_1|+|C_2|\leq 2c-2|C\backslash(C_1\cup C_2)|\leq 2c-2=2(c-1).$$

This proves the lemma. ∎

We shall now prove Conjecture 5.1 for 3-connected binary matroids.

Theorem 5.2. *Let M be a 3-connected binary matroid with circumference c. Let C_1 and C_2 be disjoint circuits where $r(C_1)+r(C_2)=r(C_1\cup C_2)$. Then $|C_1|+|C_2|\leq 2(c-2)$.*

Proof. Let C be a minimum (C_1,C_2)-circuit. As in the proof of Lemma 5.3, we obtain that $|C_1|+|C_2|\leq 2c-2|C\backslash(C_1\cup C_2)|$. If $|C\backslash(C_1\cup C_2)|\geq 2$, then $|C_1|+|C_2|\leq 2c-4$, and we are done. Thus for any minimum (C_1,C_2)-circuit C we may assume that $|C\backslash(C_1\cup C_2)|=1$.

Suppose for $x \in E(M)\backslash C_1 \cup C_2$ there is a minimum (C_1, C_2)-circuit, say C, containing it. We have that $C\backslash C_1 \cup C_2 = \{x\}$, and furthermore it is not too difficult to show that any minimum (C_1, C_2)-circuit containing x must be one of the circuits $C, C\triangle C_1, C\triangle C_2$, or $C\triangle C_1\triangle C_2$. We shall refer to these circuits as *x-circuits*.

Suppose for $e, f \in E(M)\backslash(C_1 \cup C_2)$ there are e- and f-circuits, C_e and C_f respectively. Suppose $C_1 \cap (C_e\triangle C_f) \neq \emptyset, C_1$ and $C_2 \cap (C_e\triangle C_f) \neq \emptyset, C_2$. Then $C_1, C_2 \not\subseteq C_1\triangle C_e\triangle C_f$ and $C_1\triangle C_e\triangle C_f$ contains no e- or f-circuits. Now $C_1\triangle C_e\triangle C_f$ contains a (C_1, C_2)-circuit, say C. Since C is not an e- or f-circuit, it follows that $\{e, f\} \subset C$. Thus $C\triangle(C_1\triangle C_e\triangle C_f) \subseteq C_1 \cup C_2$. Now if $C\triangle(C_1\triangle C_e\triangle C_f) \neq \emptyset$, then it contains a (C_1, C_2)-circuit C', thus by Corollary 5.1 it holds that either $C' = C_1$, or $C' = C_2$. This is impossible since $C_1, C_2 \not\subseteq C_1\triangle C_e\triangle C_f$. We conclude that $C = C_1\triangle C_e\triangle C_f$ is a circuit. Similarly, it can also be shown that $C_2\triangle C_e\triangle C_f$ is a circuit. In this case we obtain

$$|C_1| + |C_2| + 4 = |C_1\triangle C_e\triangle C_f| + |C_2\triangle C_e\triangle C_f| \leq 2c.$$

Hence $|C_1| + |C_2| \leq 2(c-2)$.

Suppose $C_1 \cap (C_e\triangle C_f)$ is either empty or equal to C_1, and $C_2 \cap (C_e\triangle C_f)$ is either empty or equal to C_2. Then $C_1\triangle C_2\triangle C_e\triangle C_f$ equals $\{e, f\}, C_1 \cup \{e, f\}, C_2 \cup \{e, f\}$, or $C_1 \cup C_2 \cup \{e, f\}$. In either of the cases, one can show that $\{e, f\}$ is a circuit, contradicting the 3-connectedness of M.

In light of the above, we may assume that $C_1 \cap (C_e\triangle C_f) \neq \emptyset, C_1$ and $C_2 \cap (C_e\triangle C_f)$ is empty or equals C_2. If $C_2 \cap (C_e\triangle C_f) = \emptyset$, let $C_{ef} = C_e\triangle C_f$. Otherwise, if $C_2 \cap (C_e\triangle C_f) = C_2$, then let $C_{ef} = C_e\triangle C_f\triangle C_2$. In either case, $C_{ef} \subset C_1 \cup \{e, f\}$ and C_{ef} is a circuit containing e and f. Moreover, $C_1\triangle C_{ef}$ is also a circuit (containing e and f). Thus for any $x \in C_1$, $(C - x) \cup \{e, f\}$ contains a circuit containing e and f.

Suppose for some $g \in E(M)\backslash(C_1 \cup C_2 \cup \{e, f\})$ there is a g-circuit, say C_g. Suppose that $C_1 \cap (C_e\triangle C_g)$ is either empty or equals C_1. Then, as with C_e and C_f, we may assume that $C_2 \cap (C_e\triangle C_g) \neq \emptyset, C_2$. In this case, we deduce that $C_1 \cap (C_f\triangle C_g) \neq \emptyset, C_1$ and $C_2 \cap (C_f\triangle C_g) \neq \emptyset, C_2$. As before, one can show that $C_1\triangle C_f\triangle C_g$ and $C_2\triangle C_f\triangle C_g$ are circuits, and consequently $|C_1| + |C_2| \leq 2c - 4$. Thus we may assume that $C_1 \cap (C_e\triangle C_g) \neq \emptyset, C_1$ and similarly, $C_1 \cap (C_f\triangle C_g) \neq \emptyset, C_1$.

In general, suppose $e_1, \ldots, e_k \subseteq E(M)\backslash(C_1 \cup C_2)$ are the elements of $E(M)\backslash C_1 \cup C_2$ belonging to minimum (C_1, C_2)-circuits. For $i = 1, \ldots, k$ let C_{e_i} be an e_i-circuit. Following the discussion above, we may assume that for all $i \neq j$ it holds that $C_1 \cap (C_{e_i}\triangle C_{e_j}) \neq \emptyset, C_1$ and $C_2 \cap (C_{e_i}\triangle C_{e_j})$ is either empty or equals C_2.

Let $M' = M\backslash\{e_1,\dots,e_k\}$. If there is a (C_1,C_2)-circuit in M', then M' contains a (C_1,C_2)-circuit C' for which $|C'\backslash(C_1\cup C_2)|$ is minimum among all (C_1,C_2)-circuits of M'. As before, it holds that $C'\triangle C_1$ and $C'\triangle C_2$ are circuits and using this we obtain that

$$|C_1| + |C_2| + 2|C'\backslash(C_1\cup C_2)| = |C'\triangle C_1| + |C'\triangle C_2| \leq 2c.$$

Thus if $|C'\backslash(C_1\cup C_2)|\geq 2$, then $|C_1|+|C_2|\leq 2(c-2)$. On the other hand, if $|C'\backslash(C_1\cup C_2)|=1$, then C' is a minimum (C_1,C_2)-circuit. However, M' has no such circuit, and consequently it has no (C_1,C_2)-circuit and is therefore disconnected. For $i=1,2$, let K_i be the component of M' containing C_i. We claim that $E(M')=K_1\cup K_2$. Let $e\in M'\backslash K_1$. Since M is connected, there is a circuit C containing e and intersecting C_2. If $C\cap\{e_1,\dots,e_k\}=\emptyset$, then $e\in K_2$, a contradiction. So C intersects $\{e_1,\dots,e_k\}$ in, say, $e_1,\dots,e_l$. Now

$$C\triangle C_{e_1}\triangle\cdots\triangle C_{e_l}\subseteq C_1\cup C_2\cup(C\backslash\{e_1,\dots,e_k\}).$$

There is a circuit $C'\subseteq C\triangle C_{e_1}\triangle\cdots\triangle C_{e_k}$ containing e. One sees that C' cannot be a (C_1,C_2)-circuit since M' contains no such circuits. Thus C' intersects exactly one of C_1 or C_2 but not both. If C' intersects C_1, then $e\in K_1$, contradicting our assumption. Thus C' intersects C_2 and $e\in K_2$. Since e was arbitrarily chosen, we have that $E(M)=K_1\cup K_2$.

Let $S=K_1\cup\{e_1,\dots,e_k\}$ and $T=K_2$. We shall show that (S,T) is a 2-separation. Let $x\in C_1$. Choose a base B of M' containing C_1-x. Let $B_1=B\cap K_1$ and $B_2=B\cap K_2$. Similar to the previous discussion with e and f, we can deduce that for all $i\neq j$ there is a circuit in $(C_1-x)\cup\{e_i,e_j\}$ containing e_i and e_j. Thus $r(B_1\cup\{e_1,\dots,e_k\})\leq|B_1|+1$, and consequently, $r(S)=r(K_1\cup\{e_1,\dots,e_k\})\leq|B_1|+1$. We also have that B_2 is a maximal independent set in K_2 since $r(M')=r(K_1)+r(K_2)$. Thus $r(T)=r(K_1)=|B_2|$. We have that

$$\begin{aligned} r(S)+r(T) &\leq |B_1|+|B_2|+1\\ &= r(M')+1\\ &\leq r(M)+1\,. \end{aligned}$$

Thus (S,T) is a 2-separation, contradicting the 3-connectedness of M. This concludes the proof. ∎

To summarize the above, using Seymour's theorem, Theorems 2.1, 5.2, and Lemmas 2.2 and 5.3, we have proven Conjecture 5.1 for regular matroids.

Theorem 5.3. *Let M be a k-connected regular matroid having circumference c where $c\geq 2k$. If C_1 and C_2 are disjoint circuits where $r(C_1)+r(C_2)=r(C_1\cup C_2)$, then $|C_1|+|C_2|\leq 2(c-k+1)$.*

In light of the previous results, one might be tempted to conjecture that for any k-connected binary matroid ($k \geq 2$) having cocircumference $c^* \geq 2k$ there is a circuit C which intersects every cocircuit of size $c^* - k + 2$ or greater. It was pointed out to me by J. Oxley that this assertion is false for $AG(3,2)$.

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