Instructions:

1. Justify your answers and clearly show your work; organization and neatness count.
2. Record your solutions in the Examination Booklet provided.

/9

Problem 1: A point particle of mass m moves in one dimension in a potential $V(x) = \dfrac{32}{x^4} + x^2$.

(a) Write the Lagrangian for this system, in terms of the coordinate x.

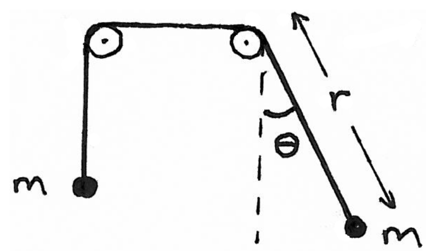
(b) Find (but do not solve) the equation of motion for the particle.

(c) Determine the position $x_0 > 0$ at which the particle is at equilibrium.

(d) The particle exhibits small oscillations about the equilibrium x_0. By linearizing the equation of motion about x_0 and changing variables to $u = x - x_0$, determine the frequency of these oscillations.

/11

Problem 2: Two point masses, each of mass m, are attached to opposite ends of a massless rope. The rope is suspended over two massless, frictionless pulleys as shown. The mass on the left moves only vertically, while the mass on the right swings as a pendulum.



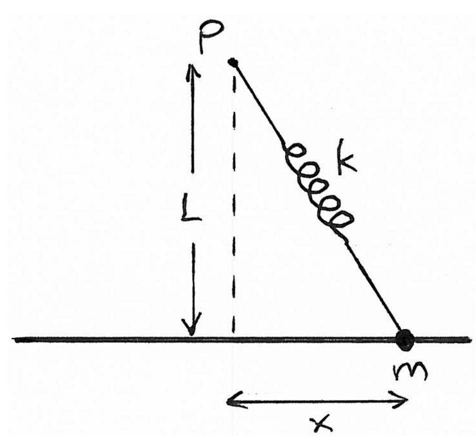
(a) Write the Lagrangian for this system, in terms of the coordinates r and θ.

(b) Write the Lagrangian equations of motion. Give an interpretation (e.g. in terms of Newton's 2nd Law) for every term in each equation.

(c) The mass on the right is released from rest with initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$ where θ_0 is small. The system exhibits small oscillations. Write linearized equations of motion for this case and solve them for the functions $r(t)$ and $\theta(t)$. Describe the motion.

/8

Problem 3: A spring (stiffness k, rest length L) pivots without friction about one end fixed at point P. The other end of the spring is attached to a bead of mass m, which slides without friction along a wire, as shown.

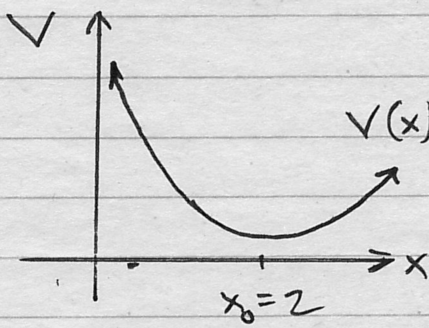


(a) Use Lagrangian constraints to determine the force exerted by the wire on the bead.

(b) Verify that your answer to (a) is exactly what you should have expected from Newtonian mechanics.

PHYS 3200 Midterm Exam 29 Feb. 2016

SOLUTIONS



1a) $T = \frac{1}{2} m \dot{x}^2$

$V = 32/x^4 + x^2$

$$\rightarrow L = T + V$$
$$= \boxed{\frac{1}{2} m \dot{x}^2 - \frac{32}{x^4} - x^2}$$

b) $\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \rightarrow \boxed{\underbrace{\frac{128}{x^5} - 2x}_{\text{force } F(x) \text{ due to potential}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}}$

c) at equilibrium position: $\ddot{x} = 0 \Rightarrow F(x_0) = 0$

$$\Rightarrow 0 = \frac{128}{x_0^5} - 2x_0 = 2x_0\left(\frac{64}{x_0^6} - 1\right)$$

$$\Rightarrow \boxed{x_0 = 64^{1/6} = 2}$$

d) to linearize eq. of motion, expand $F(x)$ in a series about $x_0 = 2$:

$$F'(x) = -\frac{640}{x^6} - 2 \Rightarrow F'(2) = -12$$

$$\therefore F(x) = \underbrace{F(2)}_{0} + \underbrace{F'(2)}_{-12} \cdot (x-2) + \text{negligible terms}$$

$$\Rightarrow F(x) \approx -12(x-2)$$

The linearized eq. of motion is:

$$m\ddot{x} = F(x) \approx -12(x-2).$$

Now let $u = x-2$:

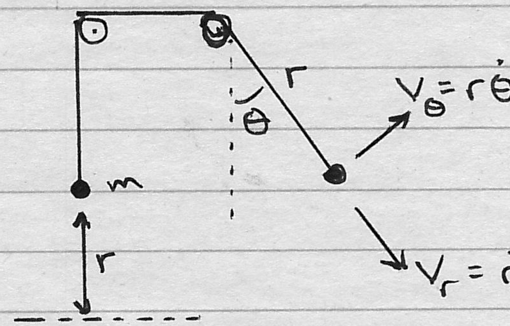
$$m\ddot{u} = -12u \rightarrow m\ddot{u} + 12u = 0$$

$$\rightarrow \ddot{u} + \underbrace{\frac{12}{m}}_{\omega^2} u = 0$$

This has oscillatory solutions $u = e^{i\omega t}$ where $\omega^2 = 12/m$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{12}{m}}}$$

2a) $T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$



$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 \right)$$

$$V = mgr - mgr\cos\theta$$

$$\Rightarrow L = T - V$$

$$= \boxed{m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr + mgr\cos\theta}$$

b) $\frac{\partial L}{\partial r} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}}$

$$\Rightarrow \underbrace{mr\dot{\theta}^2}_{\text{centrifugal force}} \underbrace{- mg + mg\cos\theta}_{\text{net gravitational force}} = \frac{d}{dt} \Big(\underbrace{2m\dot{r}}_{\text{total linear momentum}} \Big) \qquad (1)$$

$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}$

$$\Rightarrow \underbrace{-mgr\sin\theta}_{\text{torque due to gravity}} = \frac{d}{dt} \Big(\underbrace{mr^2\dot{\theta}}_{\text{angular momentum of pendulum}} \Big) \qquad (2)$$

c) employing small angle approximations:

$$\sin\theta \approx \theta \quad \text{(to 1}^{st}\text{ order)}$$
$$\cos\theta \approx 1$$

the equations become:

$$\underbrace{mr\dot{\theta}^2}_{\text{negligible (quadratic)}} = 2m\ddot{r} \Rightarrow \boxed{\ddot{r} = 0} \quad (1)$$

$$-mgr\theta = \underbrace{2mr\dot{r}\dot{\theta}}_{\text{negligible}} + mr^2\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{r}\theta = 0} \quad (2)$$

with initial conditions $r(0) = r_0$, $\dot{r}(0) = 0$, (1) gives:

$$\boxed{r(t) = r_0}$$

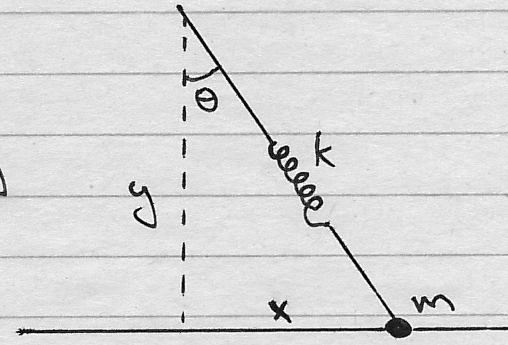
and with $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, (2) gives:

$$\boxed{\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{r_0}}\, t\right)}.$$

i.e. the mass on the left is stationary; the one on the right oscillates as a simple pendulum.

3a) $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$V = \frac{1}{2}k\left(\sqrt{x^2+y^2} - L\right)^2$ [yuck!]



$\rightarrow L = T - V$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k\left(\sqrt{x^2+y^2} - L\right)^2$

The Lagrangian constraint is $L = y \equiv f(x,y)$ so the equations of motion are:

$$\frac{\partial}{\partial x}(L + \lambda f) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$\Rightarrow -k\left(\sqrt{x^2+y^2} - L\right)\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = m\ddot{x} \qquad (1)$$

$$\frac{\partial}{\partial y}(L + \lambda f) = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

$$\Rightarrow \underbrace{-k\left(\sqrt{x^2+y^2} - L\right)\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y}_{\text{y-component of spring force}} + \underbrace{\lambda}_{\substack{\text{force due to wire} \\ (\text{downward if } \lambda > 0)}} = m\ddot{y} \qquad (2)$$

The constraint gives:

$\ddot{y} = 0 \qquad (3).$

Subbing into (2) gives: $\lambda = k(R - L)\cdot\frac{L}{R}$

where $R = \sqrt{x^2+y^2} = \sqrt{x^2+L^2}$.

b) Since $\ddot{y}=0$, $F_y = m\ddot{y}$ says $F_y = 0$, i.e. force due to wire must counteract the y-component of the spring force:

$$F_{wire} = k(R-L)\cos\theta$$

$$= k(R-L)\cdot\frac{L}{R} \quad \text{(downward)}$$

as before.