MATH 3650 – Numerical Analysis

Assignment #4

Written solutions are due Oct. 20

SOLUTIONS

1. Approximate the integral $\int_1^{1.5} x^2 \ln x \, dx$ using the (non-composite) trapezoidal rule. Give a rigorous error bound on this approximation.

With $f(x) = x^2 \ln x$ and $h = (1.5 - 1) = 0.5$ the Trapezoid Rule gives

$$\begin{aligned}\int_1^{1.5} x^2 \ln x \, dx &\approx \frac{h}{2}[f(1) + f(1.5)] \\ &= \frac{0.5}{2}[1^2 \ln 1 + 1.5^2 \ln 1.5] \approx \boxed{0.228}\end{aligned}$$

The absolute error is

$$|E| = \frac{h^3}{12}|f''(\xi)|$$

for some $\xi \in [1, 1.5]$. Note that

$$|f''(x)| = 3 + 2\ln x$$

is increasing $\forall x > 1$ so

$$|f''(\xi)| \leq |f''(1.5)| = 3 + 2\ln 1.5.$$

Thus

$$|E| = \frac{0.5^3}{12}|f''(\xi)| \leq \frac{0.5^3}{12}(3 + 2\ln 1.5) \approx \boxed{0.040}.$$

We can conclude:

$$\boxed{\int_1^{1.5} x^2 \ln x \, dx = 0.228 \pm 0.040}$$

2. Approximate the integral $\int_0^{0.5} \frac{2}{x-4} \, dx$ using the (non-composite) Simpson's rule. Give a rigorous error bound on this approximation.

With $f(x) = 2/(x - 4)$ and $h = 0.5/2 = 0.25$ Simpson's Rule gives

$$\begin{aligned}\int_0^{0.5} f(x) \, dx &\approx \frac{h}{3}[f(0) + 4f(0.25) + f(0.5)] \\ &= \frac{0.25}{3}\left[\frac{2}{0-4} + 4 \cdot \frac{2}{0.25-4} + \frac{2}{0.5-4}\right] \approx \boxed{-0.26706349}\end{aligned}$$

The absolute error is

$$|E| = \frac{h^5}{90}|f^{(4)}(\xi)|$$

for some $\xi \in [0, 0.5]$. Note that

$$|f^{(4)}(x)| = \frac{48}{|x-4|^5}$$

is increasing on $[0, 0.5]$ so

$$|f^{(4)}(\xi)| \le |f^{(4)}(0.5)| = 48/|0.5-4|^5 = \frac{48}{(3.5)^5}.$$

Thus

$$|E| = \frac{0.25^5}{90}|f^{(4)}(\xi)| \le \frac{0.25^5}{90} \cdot \frac{48}{(3.5)^5} \approx \boxed{10^{-6}}.$$

We can conclude:

$$\boxed{\int_0^{0.5} \frac{2}{x-4}\,dx = -0.267063 \pm 0.000001}$$

3. Determine the values of n and h required to approximate $\int_0^2 e^{2x} \sin 3x \, dx$ to within 10^{-4} using the

   (a) composite trapezoid rule.

   The absolute error is

   $$|E| = \frac{(b-a)}{12} h^2 |f''(\xi)| = \frac{2}{12} h^2 |f''(\xi)|$$

   for some $\xi \in [0, 2]$. Note that

   $$\begin{aligned} |f''(x)| &= |12e^{2x} \cos 3x - 5e^{2x} \sin 3x| \\ &\le |12e^{2x} \cos 3x| + |5e^{2x} \sin 3x| \qquad \text{(triangle inequality)} \\ &= 12e^{2x} |\cos 3x| + 5e^{2x} |\sin 3x| \\ &\le 12e^{2x} + 5e^{2x} = 17e^{2x} \end{aligned}$$

   which is increasing on $[0, 2]$ so

   $$|f''(\xi)| \le 17e^{2\cdot 2} = 17e^4.$$

   Thus

   $$|E| = \frac{1}{6} h^2 |f''(\xi)| \le \frac{1}{6} h^2 \cdot 17e^4 \le 10^{-4}$$

   $$\implies h \le \left[\frac{(6)(10^{-4})}{17e^4}\right]^{1/2} \approx 8.04 \times 10^{-4}$$

   $$\implies n \ge \frac{2}{h} = \frac{2}{8.04 \times 10^{-4}} \approx 2487.5$$

   $$\implies \boxed{h \le 8.04 \times 10^{-4}, \quad n \ge 2488}$$

(b) composite Simpson's rule.

The absolute error is

$$|E| = \frac{(b-a)}{180}h^4|f^{(4)}(\xi)| = \frac{2}{180}h^4|f^{(4)}(\xi)|$$

for some $\xi \in [0,2]$. Note that

$$\begin{aligned}|f^{(4)}(x)| &= |-119e^{2x}\sin 3x - 120e^{2x}\cos 3x| \\ &\le |119e^{2x}\sin 3x| + |120e^{2x}\cos 3x| \qquad \text{(triangle inequality)} \\ &= 119e^{2x}|\sin 3x| + 120e^{2x}|\cos 3x| \\ &\le 119e^{2x} + 120e^{2x} = 239e^{2x}\end{aligned}$$

which is increasing on $[0,2]$ so

$$|f^{(4)}(\xi)| \le 239e^{2\cdot 2} = 239e^4.$$

Thus

$$|E| = \frac{1}{90}h^4|f^{(4)}(\xi)| \le \frac{1}{90}h^4 \cdot 239e^4 \le 10^{-4}$$

$$\implies h \le \left[\frac{(90)(10^{-4})}{239e^4}\right]^{1/4} \approx 0.0288$$

$$\implies n \ge \frac{2}{h} = \frac{2}{0.0288} \approx 69.4$$

$$\implies \boxed{h \le 0.0288, \quad n \ge 70}$$

4. Suppose the quadrature formula

$$\int_{-1}^{1} f(x)\,dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact whenever $f(x)$ is a polynomial of degree less than or equal to 2 (i.e. the formula has degree of precision 2). Determine c_0, c_1 and c_2.

For $f(x) = A + Bx + Cx^2$ we require:

$$\begin{aligned}\int_{-1}^{1}(A + Bx + Cx^2)\,dx &= 2A + \frac{2}{3}C \quad \text{(exact value)} \\ &= c_0(A - B + C) + c_1 A + c_2(A + B + C) \quad \text{(quadrature rule)}\end{aligned}$$

To make this equation an identity (i.e., true for all values of A, B, C) we match coefficients of A, B, C to get 3 (linear) equations in 3 unknowns:

$$\begin{cases} A: & 2 = c_0 + c_1 + c_2 \\ B: & 0 = -c_0 + c_2 \\ C: & \frac{2}{3} = c_0 + c_2 \end{cases}$$

Solving this system (e.g. using linear algebra methods, or Wolfram Alpha, or...) gives

$$\boxed{c_0 = \frac{1}{3}, \quad c_1 = \frac{4}{3}, \quad c_2 = \frac{1}{3}}$$

Notice: this reproduces Simpson's Rule! So the degree of precision is actually 3, not 2.

5. Consider the quadrature formula

$$\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1).$$

(a) What is the largest integer n such that this formula has degree of precision n?

With 3 free parameters in the quadrature rule, we expect it can give the exact answer only for a 3-parameter family of functions (e.g. the family of degree-2 polynomials).

(b) Determine c_0, c_1 and x_1 in this case.

For $f(x) = A + Bx + Cx^2$ we require:

$$\begin{aligned} \int_0^1 (A + Bx + Cx^2)\,dx &= A + \frac{1}{2}B + \frac{1}{3}C \quad \text{(exact value)} \\ &= c_0 A + c_1(A + Bx_1 + Cx_1^2) \quad \text{(quadrature rule)} \end{aligned}$$

To make this equation an identity, we match coefficients of A, B, C to get 3 (nonlinear) equations in 3 unknowns:

$$\begin{cases} A: & 1 = c_0 + c_1 \\ B: & \frac{1}{2} = c_1 x_1 \\ C: & \frac{1}{3} = c_1 x_1^2 \end{cases}$$

The last 2 equations give

$$\frac{1}{c_1} = 2x_1 = 3x_1^2 \implies x_1(2 - 3x_1) = 0 \implies x_1 = \frac{2}{3} \implies c_1 = \frac{1}{2x_1} = \frac{3}{4}.$$

(the other root $x_1 = 0$ is inconsistent with both equations). Subsituting this into the first equation gives

$$\boxed{c_1 = \frac{3}{4}, \quad c_0 = \frac{1}{4}, \quad x_1 = \frac{2}{3}}$$

Note that if we apply this quadrature rule to $f(x) = x^3$ we get

$$\tfrac{1}{4}f(0) + \tfrac{3}{4}f(\tfrac{2}{3}) = 0 + \frac{3}{4}\left(\frac{2}{3}\right)^3 = \frac{2}{9} \neq \int_0^1 f(x)\,dx = \frac{1}{4}.$$

Thus the degree of precision is indeed 2 (not 3).

(c) Use your result to approximate the integral $\int_1^2 x \ln x \, dx$. Determine the absolute error by evaluating the integral exactly.

We need to make a substitution to shift the interval:

$$u = x - 1 \implies \int_1^2 x \ln x \, dx = \int_0^1 \underbrace{(u+1)\ln(u+1)}_{f(u)}\, du$$
$$\approx \tfrac{1}{4}f(0) + \tfrac{3}{4}f\left(\tfrac{2}{3}\right)$$
$$= \tfrac{1}{4}(1)\ln(1) + \tfrac{3}{4}(\tfrac{2}{3}+1)\ln(\tfrac{2}{3}+1) \approx \boxed{0.6385}$$

The exact value is

$$\int_1^2 x \ln x \, dx = \boxed{2\ln 2 - \frac{3}{4} \approx 0.6363}$$

so the absolute error is

$$\boxed{|E| = |0.6385 - 0.6363| \approx 0.0022}$$

6. Determine constants a, b, c and d so that the quadrature formula

$$\int_{-1}^1 f(x)\,dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

has degree of precision 3.

For $f(x) = A + Bx + Cx^2 + Dx^3$ we require:

$$\int_{-1}^1 (A + Bx + Cx^2 + Dx^3)\,dx = 2A + \frac{2}{3}C \quad \text{(exact value)}$$
$$= a(A - B + C - D) + b(A + B + C + D)$$
$$+ c(B - 2C + 3D) + d(B + 2C + 3D) \quad \text{(quadrature rule)}$$

To make this equation an identity, we match coefficients of A, B, C, D to get 4 (linear) equations in 4 unknowns:

$$\begin{cases} A: & 2 = a + b \\ B: & 0 = -a + b + c + d \\ C: & \frac{2}{3} = a + b - 2c + 2d \\ D: & 0 = -a + b + 3c + 3d \end{cases}$$

Solving this linear system gives

$$\boxed{a = 1, \quad b = 1, \quad c = \frac{1}{3}, \quad d = -\frac{1}{3}}$$