THOMPSON RIVERS UNIVERSITY

MATH 3160
Differential Equations II

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MIDTERM EXAM #2
SOLUTIONS

16 November 2011 10:30–12:20

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
5		10
TOTAL:		50

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

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Problem 1: Find the general solution $u(x,t)$ of the "telegraph equation": ($\alpha \in \mathbb{R}$ is an arbitrary parameter)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} + u = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

$$u(x,t) = X(x)T(t) \implies XT'' + XT' + XT = \alpha^2 X''T \implies \frac{T'' + T' + T}{\alpha^2 T} = \frac{X''}{X} = k$$

$$\implies \begin{cases} T'' + T' + (1 - \alpha^2 k)T = 0 \\ X'' - kX = 0 \end{cases}$$

As many times before, only $k = -\lambda^2 < 0$ yields nontrivial solutions with $X(0) = X(\pi) = 0$:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$0 = X(0) = A \implies A = 0$$
$$0 = X(\pi) = B\sin(\lambda\pi) \implies \lambda = n \ (n = 1, 2, \ldots)$$

$$\implies X(x) = B\sin(nx)$$

The ODE for T has characteristic polynomial $r^2 + r + (1 + \alpha^n n^2)$ with roots

$$r = \frac{-1 \pm \sqrt{1 - 4(1 + \alpha^2 n^2)}}{2} = -\frac{1}{2} \pm i \underbrace{\sqrt{\alpha^2 n^2 + \frac{3}{4}}}_{\omega_n}$$

$$\implies T(t) = Ce^{-t/2}\cos(\omega_n t) + De^{-t/2}\sin(\omega_n t)$$

Then superposition gives the general solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin(nx) e^{-t/2} \big[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\big]}$$

$$\text{where } \omega_n = \sqrt{\alpha^2 n^2 + \frac{3}{4}}$$

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Problem 2: Solve the following heat conduction problem for $u(x,t)$.

$$\begin{cases} u_t = 9u_{xx}, & 0 < x < 4, \quad t > 0 \\ u_x(0,t) = u_x(4,t) = 0, & t > 0 \\ u(x,0) = x^2, & 0 \le x \le 4 \end{cases}$$

$$u(x,t) = X(x)T(T) \implies XT' = 9X''T \implies \frac{T'}{9T} = \frac{X''}{X} = k$$

$$\implies \begin{cases} T' = 9kT \\ X'' - kX = 0 \end{cases}$$

To get nontrivial solutions with $X'(0) = X'(4) = 0$ requires $k = -\lambda^2 \le 0$:

$$X(x) = A\cos\lambda x + B\sin\lambda x$$
$$0 = X'(0) = \lambda B \implies B = 0$$
$$0 = X'(4) = \lambda A \sin 4\lambda \implies 4\lambda = n\pi \implies \lambda = \frac{n\pi}{4} \ (n = 0, 1, 2, \ldots)$$
$$\implies X(x) = A\cos\left(\frac{n\pi x}{4}\right)$$

The ODE for T then yields

$$\implies T(t) = Be^{9kt} = Be^{-9n^2\pi^2 t/16}$$

and superposition gives the general solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right) e^{-9n^2\pi^2 t/16}$$

Imposing initial conditions . . .

$$u(x,0) = x^2 = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right) \quad \text{(a cosine series for } x^2\text{)}$$

$$\begin{aligned}
\implies A_n &= \frac{2}{4}\int_0^4 x^2 \cos\left(\frac{n\pi x}{4}\right) dx \\
&= \frac{1}{2}\Bigg[\underbrace{x^2 \frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\Big|_0^4}_{0} - \int_0^4 2x \cdot \frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right) dx\Bigg] \\
&= -\frac{4}{n\pi}\int_0^4 x\sin\left(\frac{n\pi x}{4}\right) dx \\
&= -\frac{4}{n\pi}\left[-x\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\Big|_0^4 + \underbrace{\int_0^4 \frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right) dx}_{0}\right] = \frac{64(-1)^n}{n^2\pi^2}
\end{aligned}$$

$$\text{except } A_0 = \frac{1}{4}\int_0^4 x^2\,dx = \frac{1}{4}\cdot\frac{1}{3}x^3\Big|_0^4 = \frac{16}{3}$$

$$\implies \boxed{u(x,t) = \frac{16}{3} + \sum_{n=1}^{\infty} \frac{64(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right) e^{-9n^2\pi^2 t/16}}$$

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Problem 3: Solve the following wave equation for $u(x,t)$.

$$\begin{cases} u_{tt} = 9u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin 4x + 7\sin 5x, & 0 < x < \pi \\ u_t(x,0) = 3\sin 8x, & 0 < x < \pi \end{cases}$$

With these boundary conditions the general solution of the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\big[A_n\cos(3nt) + B_n\sin(3nt)\big]$$

Imposing initial conditions…

$$u(x,0) = \sin 4x + 7\sin 5x = \sum_{n=1}^{\infty} A_n \sin(nx)$$

$$\implies A_4 = 1,\ A_5 = 7, \ \text{ all other } A_n = 0$$

$$\implies u(x,t) = \sin 4x\cos 12t + 7\sin 5x\cos 15t + \sum_{n=1}^{\infty} B_n\sin(nx)\sin(3nt)$$

$$u_t(x,0) = 3\sin 8x = \sum_{n=1}^{\infty} 3nB_n\sin(nx)$$

$$B_8 = \frac{1}{8}, \ \text{ all other } B_n = 0$$

$$\implies \boxed{u(x,t) = \sin 4x\cos 12t + 7\sin 5x\cos 15t + \frac{1}{8}\sin 8x\sin 24t}$$

/10 **Problem 4:** Consider the function $f(x) = \begin{cases} 1, & 0 < x \le \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$ defined for $x \in [0, \pi]$.

(a) Find the Fourier series for $f(x)$. Sketch the graph of the function to which this series converges on $[-2\pi, 2\pi]$.

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$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos 2nx + B_n \sin 2nx \quad (\pi\text{-periodic})$$

$$A_n = \frac{1}{\pi/2}\int_0^{\pi} f(x)\cos(2nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos(2nx)\,dx = \frac{2}{\pi}\cdot\frac{1}{2n}\sin(2nx)\Big|_0^{\pi/2} = 0, \text{ except } A_0 = 1$$

$$B_n = \frac{1}{\pi/2}\int_0^{\pi} f(x)\sin(2nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin(2nx)\,dx = -\frac{2}{\pi}\cdot\frac{1}{2n}\cos(2nx)\Big|_0^{\pi/2} = -\frac{\cos(n\pi)-1}{n\pi} = \frac{1-(-1)^n}{n\pi}$$

$$\implies \boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin 2nx}$$

(b) Find the sine series for $f(x)$. Sketch the graph of the function to which this series converges on $[-2\pi, 2\pi]$.

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$$f(x) = \sum_{n=1}^{\infty} B_n \sin nx \quad (2\pi\text{-periodic})$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\,dx = -\frac{2}{\pi}\cdot\frac{1}{n}\cos(nx)\Big|_0^{\pi/2} = \frac{2(1-\cos(n\pi/2))}{n\pi}$$

$$\implies \boxed{f(x) = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi/2))}{n\pi}\sin nx}$$

(c) Find the cosine series for $f(x)$. Sketch the graph of the function to which this series converges on $[-2\pi, 2\pi]$.

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$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx \quad (2\pi\text{-periodic})$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos(nx)\,dx = \frac{2}{\pi}\cdot\frac{1}{n}\sin(nx)\Big|_0^{\pi/2} = \frac{2\sin(n\pi/2)}{n\pi}$$

$$\implies A_{2m} = 0,\quad A_{2m+1} = \frac{2(-1)^m}{(2m+1)\pi}, \text{ except } A_0 = 1$$

$$\implies \boxed{f(x) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2(-1)^m}{(2m+1)\pi}\cos(2m+1)x}$$

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Problem 5: Solve the following boundary value problem (Laplace's equation) for $u(x, y)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < \pi, \quad 0 < y < 1 \\ \dfrac{\partial u}{\partial x}(0, y) = \dfrac{\partial u}{\partial x}(\pi, y) = 0, & 0 \le y \le 1 \\ u(x, 0) = \cos x - \cos 3x, & 0 \le x \le \pi \\ u(x, 1) = 0 & 0 \le x \le \pi \end{cases}$$

$$u(x, y) = X(x)Y(y) \implies X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} = k$$

$$\implies \begin{cases} X'' - kX = 0 \\ Y'' + kY = 0 \end{cases}$$

To get nontrivial solutions with $X'(0) = X'(\pi) = 0$ requires $k = -\lambda^2 < 0$:

$$X(x) = A\cos\lambda x + B\sin\lambda x$$
$$0 = X'(0) = \lambda B \implies B = 0$$
$$0 = X'(\pi) = \lambda A \sin\lambda\pi \implies \lambda\pi = n\pi \implies \lambda = n \ (n = 0, 1, 2, \ldots)$$

$$\implies X(x) = A\cos(nx)$$

The ODE for Y then yields

$$\implies Y(y) = Be^{ny} + Ce^{-ny}$$

Enforcing the boundary condition on Y ...

$$0 = Y(1) = Be^{n} + Ce^{-n} \implies C = -Be^{2n} \implies Y(y) = B\left[e^{ny} - e^{n(2-y)}\right]$$

Then superposition gives:

$$u(x, y) = \sum_{n=0}^{\infty} A_n \cos(nx)\left[e^{ny} - e^{n(2-y)}\right]$$

Enforcing the final boundary condition ...

$$u(x, 0) = \cos x - \cos 3x = \sum_{n=0}^{\infty} A_n \cos(nx)\left[1 - e^{2n}\right]$$

$$\implies A_1(1 - e^2) = 1, \quad A_3(1 - e^6) = -1, \quad \text{all other } A_n = 0$$

$$\implies \boxed{u(x, y) = \cos x \frac{e^{y} - e^{2-y}}{1 - e^2} - \cos 3x \frac{e^{3y} - e^{3(2-y)}}{1 - e^6}}$$