THOMPSON RIVERS UNIVERSITY

MATH 316
Differential Equations II

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FINAL EXAM
SOLUTIONS

18 April 2008 14:00–17:00

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
TOTAL:		80

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 9 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

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Problem 1: Consider the following differential equation for $y(x)$:

$$(1+x^2)y'' + 6xy' + 6y = 0$$

(a) Explain why $x = 0$ is an ordinary point (i.e. not a singular point) for this equation.

We have $y'' + P(x)y' + Q(x)y = 0$ where $P(x) = 6x/(1+x^2)$ and $Q(x) = 6/(1+x^2)$ are both analytic at $x = 0$, hence $x = 0$ is an ordinary point.

(b) Find the general solution of this equation in terms of power series.

Assume:

$$y = \sum_{n=0}^{\infty} c_n x^n \implies y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Subbing into the DE:

$$\sum_{2} n(n-1)c_n x^{n-2} + \sum_{2} n(n-1)c_n x^n + 6\sum_{1} n c_n x^n + 6\sum_{0} c_n x^n = 0$$

and re-indexing:

$$\sum_{0} (n+2)(n+1)c_{n+2} x^n + \sum_{2} n(n-1)c_n x^n + 6\sum_{1} n c_n x^n + 6\sum_{0} c_n x^n = 0$$

$$\implies 2c_2 + 6c_3 x + 6c_1 x + 6c_0 + 6c_1 x + \sum_{2}\Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n + 6nc_n + 6c_n\Big]x^n = 0$$

gives the recurrence relations:

$$\begin{cases} 2c_2 + 6c_0 = 0 \\ 6c_3 + 12c_1 = 0 \\ (n+2)(n+1)c_{n+2} + (n+2)(n+3)c_n = 0 \end{cases} \implies c_{n+2} = -\frac{n+3}{n+1}c_n; \;\; n = 2, 3, \ldots$$

so that

$$c_2 = -3c_0 \qquad c_3 = -2c_1$$
$$c_4 = -\frac{5}{3}c_2 = 5c_0 \qquad c_5 = -\frac{6}{4}c_3 = 3c_1$$
$$c_6 = -\frac{7}{5}c_4 = -7c_0 \qquad c_7 = -\frac{8}{6}c_5 = -4c_1$$
$$c_8 = -\frac{9}{7}c_6 = 9c_0 \qquad c_9 = -\frac{10}{8}c_7 = 5c_1$$
$$\ldots \qquad \ldots$$
$$c_{2k} = (-1)^n(2k+1)c_0 \qquad c_{2k+1} = (-1)^k(k+1)c_1$$

and the general solution is

$$y(x) = c_0\sum_{k=0}^{\infty}(-1)^k(2k+1)x^{2k} + c_1\sum_{k=0}^{\infty}(-1)^k(k+1)x^{2k+1}$$

(c) Give (and justify) a lower bound on the radius of convergence for the series solution(s) in part (b).

The only singular points (i.e. values of x at which $P(x)$ or $Q(x)$ fail to be analytic) are $x = \pm i$. The radius of convergence for the series in (a) is therefore at least $|i - 0| = 1$.

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Problem 2: Consider the following differential equation for $y(x)$:

$$3x^2y'' + x(1+x)y' - y = 0$$

(a) Explain why $x = 0$ is a regular singular point (i.e. not an ordinary point or irregular singular point) for this equation.

We have $y'' + P(x)y' + Q(x)y = 0$ where $P(x) = (1+x)/(2x)$ and $Q(x) = -1/(3x^2)$. Both P and Q are non-analytic at $x = 0$, so $x = 0$ is a singular point. Since $xP(x) = (1+x)/2$ and $x^2Q(x) = -1/3$ are both analytic at $x = 0$, $x = 0$ is a *regular* singular point.

(b) Find the value(s) of r such that this equation has a solution of the form $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n$. (Do not attempt to determine the coefficients c_n.) Are there two linearly independent solutions of this form?

Assume:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \implies y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \implies y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

Subbing into the DE:

$$\sum_0 3(n+r)(n+r-1)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r+1} - \sum_0 c_n x^{n+r} = 0$$

and re-indexing:

$$\sum_0 3(n+r)(n+r-1)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r} + \sum_1 (n+r-1)c_{n-1} x^{n+r} - \sum_0 c_n x^{n+r} = 0$$

$$\implies \Big[3r(r-1) + r - 1\Big]c_0 x^r + \sum_1 \Big[3(n+r)(n+r-1)c_n + (n+r)c_n + n + r - 1)c_{n-1} - c_n\Big]x^{n+r} = 0$$

gives the indicial equation

$$3r(r-1) + r - 1 = 0 \implies 3r^2 - 2r - 1 = 0$$

$$\implies \boxed{r = \frac{2 \pm 4}{-6} = -1, \ \frac{1}{3}}$$

Since the roots of the indicial equation do not differ by an integer, there will be two linearly independent solutions of the form $y(x) = x^r P(x)$ where P is a power series.

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Problem 3: Solve the initial value problem

$$y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \qquad y(0) = 0, \qquad y'(0) = 1.$$

Simplify your solution as much as possible, and sketch the graph of the solution $y(t)$ on the interval $[0, 6\pi]$.

Laplace transform:

$$(s^2 Y(s) - 1) + Y(s) = \sum_{k=1}^{\infty} e^{-2k\pi s} \implies (1 + s^2)Y(s) = 1 + \sum_{k=1}^{\infty} e^{-2k\pi s}$$

$$\implies Y(s) = \frac{1}{1+s^2} + \sum_{k=1}^{\infty} e^{-2k\pi s}\frac{1}{1+s^2}$$

Therefore

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \sin t + \sum_{k=1}^{\infty} u(t - 2k\pi)\sin(t - 2k\pi) \\ &= \sin t + \sum_{k=1}^{\infty} u(t - 2k\pi)\sin t \\ &= (m+1)\sin t; \quad t \in [m2\pi, (m+1)2\pi]; \quad m = 0, 1, 2, \ldots \end{aligned}$$

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Problem 4: Find the inverse Laplace transform of

$$F(s) = \frac{s+2}{(s-3)(s^2+2s+5)}$$

Write

$$\begin{aligned} F(s) &= \frac{A}{s-3} + \frac{Bs+C}{s^2+2s+5} \\ &= \frac{A(s^2+2s+5)+(Bs+C)(s-3)}{(s-3)(s^2+2s+5)} \\ &= \frac{(A+B)s^2+(2A-3B+C)s+(5A-3C)}{(s-3)(s^2+2s+5)} \implies \begin{cases} A+B=0 \\ 2A-3B+C=1 \\ 5A-3C=2 \end{cases} \end{aligned}$$

$$s \to 3 \implies 20A = 5 \implies A = \frac{1}{4} \implies B = -\frac{1}{4} \implies C = -\frac{1}{4}$$

so that

$$\begin{aligned} F(s) &= \frac{1}{4}\cdot\frac{1}{s-3} - \frac{1}{4}\cdot\frac{s+1}{s^2+2s+5} \\ &= \frac{1}{4}\cdot\frac{1}{s-3} - \frac{1}{4}\cdot\frac{(s+1)}{(s+1)^2+2^2} \end{aligned}$$

which gives

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{4}e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\ &= \boxed{\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}\cos 2t} \end{aligned}$$

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Problem 5: Consider the following Sturm-Liouville problem on $[0, 3]$:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y'(3) = 0$$

(a) Find the eigenvalues and eigenfunctions for this problem.

case $\lambda = -\alpha^2 < 0$:

$$y(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_1 = c_2 = 0 \implies \text{only the trivial solution}$$

case $\lambda = 0$:

$$y(x) = c_1 + c_2 x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_2 = 0 \implies y(x) = c_1$$

case $\lambda = \alpha^2 > 0$:

$$y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_2 = 0;\ \sin 3\alpha = 0 \implies 3\alpha = n\pi$$

So the eigenvalues are

$$\boxed{\lambda_n = \left(\frac{n\pi}{3}\right)^2; \quad n = 0, 1, 2, \ldots}$$

with corresponding eigenfunctions

$$\boxed{y_n(x) = \cos\left(\frac{n\pi x}{3}\right)}$$

(b) Is $\lambda = 0$ an eigenvalue for this problem? Justify your answer.

Yes. $\lambda = 0$ gives non-trivial solutions $y(x) = c$; the corresponding eigenfunctions are the constant functions.

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Problem 6: Consider the following piecewise continuous function on $[-2, 2]$.

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ \frac{1}{2}, & 0 < x < 2. \end{cases}$$

(a) Find the Fourier series representation of this function.

We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$\frac{a_0}{2} = \frac{3}{4} \qquad \text{(average of } f(x)\text{, by inspection)}$$

and

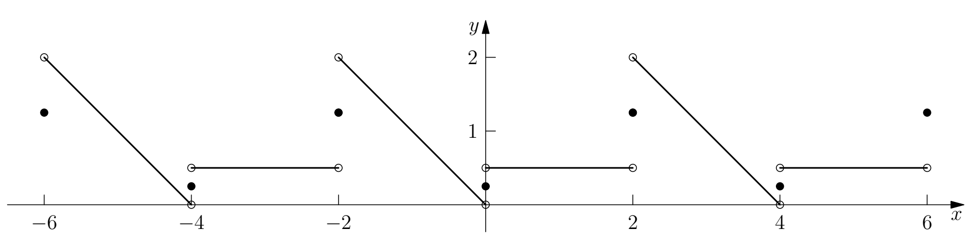
$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} (-x)\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} \frac{1}{2}\cos\left(\frac{n\pi x}{2}\right)dx \\
&= -\frac{1}{2}\left[\left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right) + \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[\frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{0}^{2} \\
&= -\frac{1}{2}\left[\left(\frac{2}{n\pi}\right)^2 - \left(\frac{2}{n\pi}\right)^2\cos(-n\pi)\right] \\
&= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} (-x)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} \frac{1}{2}\sin\left(\frac{n\pi x}{2}\right)dx \\
&= -\frac{1}{2}\left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right) - \frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[-\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{2} \\
&= -\frac{1}{2}\left[-\frac{4}{n\pi}\cos(-n\pi)\right] + \frac{1}{n2\pi}\left[1 - \cos(-n\pi)\right] \\
&= \frac{1}{n\pi}\left[3(-1)^n + 1\right]
\end{aligned}$$

so

$$f(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{3(-1)^n + 1}{n}\sin\left(\frac{n\pi x}{2}\right)$$

(b) Sketch the graph, on the interval $[-6, 6]$, of the function to which the Fourier series in part (a) converges.



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Problem 7: Solve the following initial boundary value problem for $u(x,t)$, which models heat flow in a one-dimensional object with one end insulated

$$\begin{aligned} &u_t = 16u_{xx}, && 0 < x < 2, && t > 0 \\ &u(0,t) = 0, && u_x(2,t) = 0, && t > 0 \\ &u(x,0) = x, && 0 \le x \le 2 \end{aligned}$$

Separation of variables $u(x,t) = X(x)T(t)$ gives

$$\begin{cases} X'' + \lambda X = 0 \\ T' = -16\lambda T \end{cases}$$

which give nontrivial solutions only for $\lambda = \alpha^2 > 0$:

$$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Boundary conditions $X(0) = 0,\ X'(2) = 0$ give $c_1 = 0$ and $\cos(2\alpha) = 0 \implies \alpha = (n+\frac{1}{2})\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$

We have $T(t) = Ae^{-16\lambda t} = Ae^{-4(n+\frac{1}{2})^2\pi^2 t}$ so the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-4(n+\frac{1}{2})^2\pi^2 t} \sin\left((n+\tfrac{1}{2})\frac{\pi x}{2}\right)$$

where

$$u(x,0) = x = \sum_{n=0}^{\infty} A_n \sin\left((n+\tfrac{1}{2})\frac{\pi x}{2}\right).$$

together with orthogonality gives

$$A_n = \frac{\displaystyle\int_0^2 x \sin\left((n+\tfrac{1}{2})\frac{\pi x}{2}\right) dx}{\displaystyle\int_0^2 \sin^2\left((n+\tfrac{1}{2})\frac{\pi x}{2}\right) dx}$$

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Problem 8: Solve the following initial boundary value problem for $u(x,t)$, which models vibration of a taut string.

$$u_{tt} = 9u_{xx}, \qquad 0 < x < 1, \qquad t > 0$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = \sin(\pi x) - 5\sin(3\pi x), \qquad u_t(x,0) = 0, \qquad 0 \le x \le 1$$

The standard solution for the wave equation gives

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n \cos(n3\pi t) + B_n \sin(n3\pi t) \Big] \sin(n\pi x)$$

where

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} \Big[B_n \cdot n3\pi \Big] \sin(n\pi x)$$

gives $B_n = 0$ for all $n = 1, 2, \ldots$ and

$$u(x,t) = \sin(\pi x) - 5\sin(3\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

gives $A_1 = 1$, $A_3 = -5$ and $A_n = 0$ for all other $n \neq 1, 3$.

Therefore the solution is

$$\boxed{u(x,t) = \cos(3\pi t)\sin(\pi x) - 5\cos(9\pi t)\sin(3\pi x)}$$