MATH 3160: Quiz #6 – SOLUTIONS

/5 **Problem 1:** Consider the following initial value problem for a function $u(x,t)$, which describes one-dimensional diffusion on an interval with insulated boundaries, starting with an initial point-source distribution at the middle of the interval:

$$\begin{cases} u_t = u_{xx} \quad 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = \pi\delta(x - \frac{\pi}{2}) \end{cases}$$

(a) Solve for $u(x,t)$.

For these boundary conditions we have the general solution (derived in class):

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-n^2 t}.$$

Imposing the initial conditions gives

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) = \pi\delta(x - \tfrac{\pi}{2}) \quad \text{(a Fourier cosine series)}$$

$$\Longrightarrow A_n = \frac{2}{\pi}\int_0^{\pi} \pi\delta(x - \tfrac{\pi}{2})\cos(nx)\,dx = 2\cos\left(n\tfrac{\pi}{2}\right) \quad (n > 0)$$

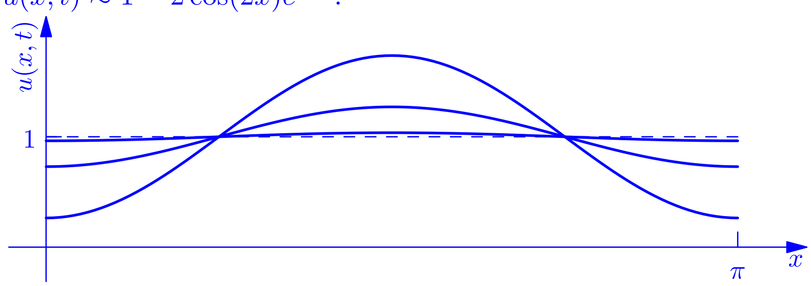
$$A_0 = \frac{2}{\pi}\int_0^{\pi} \pi\delta(x - \tfrac{\pi}{2})\,dx = 2.$$

$$\Longrightarrow u(x,t) = 1 + 2\sum_{n=1}^{\infty} \cos\left(n\tfrac{\pi}{2}\right)\cos(nx) e^{-n^2 t}$$

$$= 1 - 2\cos(2x)e^{-2^2 t} + 2\cos(4x)e^{-4^2 t} - 2\cos(6x)e^{-6^2 t} + \cdots$$

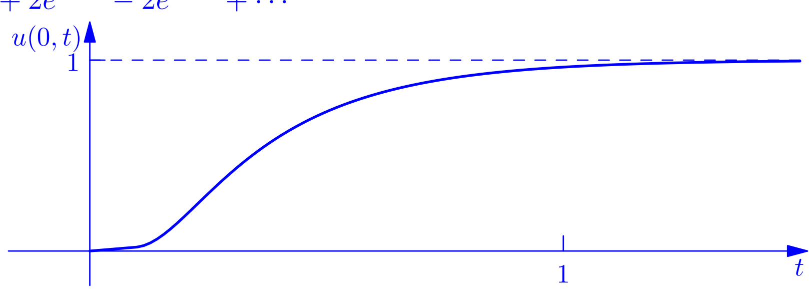
(b) Sketch the graph of $u(x,t)$ (as a function of x only) for several values of $t \gg 1$.

For $t \gg 1$ we have $u(x,t) \approx 1 - 2\cos(2x)e^{-4t}$.



(c) Sketch the graph of $u(0,t)$ (as a function of t).

$u(0,t) = 1 - 2e^{-4t} + 2e^{-16t} - 2e^{-36t} + \cdots$



/5 **Problem 2:** Find the general solution of the wave equation with one fixed and one free boundary:

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L \\ u(0,t) = 0 = u_x(L,t) = 0 \end{cases}$$

Separating variables according to $u(x,t) = X(x)T(t)$ gives (as before)

$$XT'' = c^2 X'' T \implies \frac{T''}{c^2 T} = \frac{X''}{X} = k \text{ (const)}.$$

Also as before, it can be shown that $k \geq 0$ yields only the trivial solution $u = 0$, but $k = -\lambda^2 < 0$ gives

$$X'' + \lambda^2 X = 0 \implies X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

The boundary conditions imply:

$$\begin{aligned} 0 &= X(0) = A \implies X(x) = B\sin(\lambda x) \\ 0 &= X'(L) = \lambda B\cos(\lambda L) \implies \lambda L = \frac{\pi}{2} + n\pi \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

Thus we obtain non-trivial solutions for $\lambda = (n + \frac{1}{2})\pi/L$. The corresponding DE for $T(t)$ gives

$$T'' + \lambda^2 c^2 T = 0 \implies T(t) = a\cos(\lambda ct) + b\sin(\lambda ct).$$

By linearity of the initial value problem, the general solution is a linear combination:

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos\left((n + \tfrac{1}{2})\pi ct/L\right) + B_n \sin\left((n + \tfrac{1}{2})\pi ct/L\right)\right] \sin\left((n + \tfrac{1}{2})\pi x/L\right)}$$