MATH 3160: Quiz #1 – SOLUTIONS

/10 **Problem 1:** The differential equation $y'' + xy' - y = 0$ has the general solution $y(x) = c_0 y_0(x) + c_1 y_1(x)$ $(c_1, c_2 \in \mathbb{R})$. Find power series representations of the functions y_0, y_1.

Assume a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.

Substituting these into the DE gives

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0.$$

Re-index to obtain:

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\implies 2c_2 - c_0 + \sum_{n=1}^{\infty} \big[(n+2)(n+1) c_{n+2} + n c_n - c_n\big] x^n = 0.$$

This gives the following recurrence relations:

$$c_2 = \tfrac{1}{2} c_0, \quad c_{n+2} = -\frac{(n-1)c_n}{(n+2)(n+1)} \quad (n = 1, 2, \ldots).$$

$$\begin{aligned}
n = 1: &\quad c_3 = 0 \quad (\text{and so } c_3 = c_5 = c_7 = \cdots = 0) \\
n = 2: &\quad c_4 = -\frac{1}{4 \cdot 3} c_2 = -\frac{1}{4 \cdot 3 \cdot 2} c_0 = -\frac{1}{4!} c_0 \\
n = 4: &\quad c_6 = -\frac{3}{6 \cdot 5} c_4 = \frac{3}{6!} c_0 \\
n = 6: &\quad c_8 = -\frac{5}{8 \cdot 7} c_6 = -\frac{5 \cdot 3}{8!} c_0 \\
n = 8: &\quad c_{10} = -\frac{7}{10 \cdot 9} c_8 = \frac{7 \cdot 5 \cdot 3}{10!} c_0.
\end{aligned}$$

The even-order terms can be written compactly as

$$c_{2m} = \frac{(-1)^{m+1}(2m-3)!!}{(2m)!} \quad (m = 2, 3, \ldots).$$

Thus

$$\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \\
&= c_0 + c_1 x + \frac{1}{2} c_0 x^2 - \frac{1}{4!} c_0 x^4 + \frac{3}{6!} c_0 x^6 - \frac{5 \cdot 3}{8!} c_0 x^8 + \cdots \\
&= c_1 x + c_0 \left(1 + \frac{1}{2} x^2 - \frac{1}{4!} x^4 + \frac{3}{6!} x^6 - \frac{5 \cdot 3}{8!} x^8 + \cdots\right) \\
&= c_1 x + c_0 \left(1 + \frac{1}{2} x^2 + \sum_{m=2}^{\infty} \frac{(-1)^{m+1}(2m-3)!!}{(2m)!} x^{2m}\right) \\
&= c_0 y_0(x) + c_1 y_1(x)
\end{aligned}$$

where we have the two linearly independent solutions

$$\boxed{y_1(x) = x \quad \text{and} \quad y_0(x) = 1 + \frac{1}{2} x^2 - \sum_{m=2}^{\infty} \frac{(-1)^m (2m-3)!!}{(2m)!} x^{2m}}$$