THOMPSON RIVERS UNIVERSITY

# MATH 3160
# Differential Equations 2

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# MIDTERM EXAM #2
# SOLUTIONS

20 Nov 2019 9:30–11:20

**Instructions:**

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

| PROBLEM | GRADE | OUT OF |
|---|---|---|
| 1 | | 9 |
| 2 | | 9 |
| 3 | | 9 |
| 4 | | 9 |
| TOTAL: | | 36 |

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**Problem 1:** Consider the function $f(x) = 2x$ for $-5 \le x \le 5$.

(a) Let $g(x) = \dfrac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\dfrac{n\pi x}{5}\right) + B_n \sin\left(\dfrac{n\pi x}{5}\right)$ be a Fourier series representation of $f(x)$. Without calculating the coefficients $A_n$, $B_n$, sketch the graph of $g(x)$ on the interval $[-10, 10]$.

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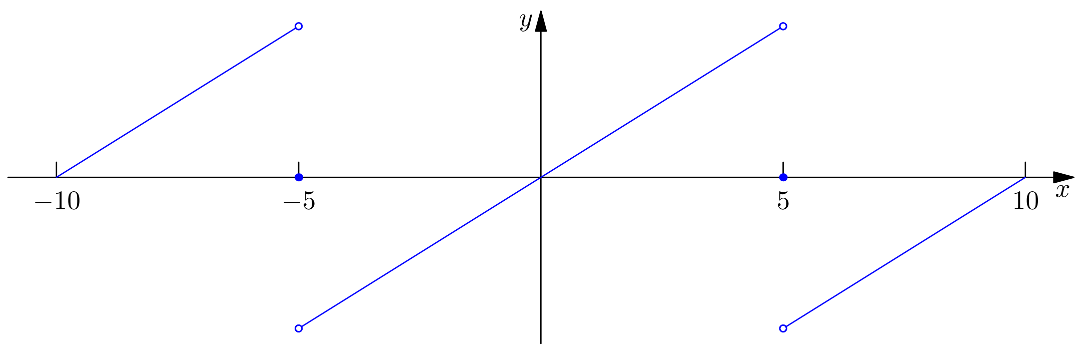


(b) Find a sine series representation of the function $f(x)$ restricted to the interval $[0, 5]$.

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$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{5}\right)$$

where

$$\begin{aligned}
B_n &= \frac{2}{5}\int_0^5 f(x)\sin\frac{n\pi x}{5}\,dx \\
&= \frac{4}{5}\int_0^5 x\sin\frac{n\pi x}{5}\,dx \qquad \left\{\begin{array}{ll} u = x & dv = \sin\dfrac{n\pi x}{5}\,dx \\ du = dx & v = -\dfrac{5}{n\pi}\cos\dfrac{n\pi x}{5} \end{array}\right. \\
&= \frac{4}{5}\left[-\frac{5x}{n\pi}\cos\frac{n\pi x}{5}\bigg|_0^5 + \frac{5}{n\pi}\int_0^5 \cos\frac{n\pi x}{5}\,dx\right] \\
&= \frac{4}{5}\left[-\frac{25}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n} + \left(\frac{5}{n\pi}\right)^2 \underbrace{\sin\frac{n\pi x}{5}\bigg|_0^5}_{0}\right] \\
&= \boxed{\frac{20}{n\pi}(-1)^{n+1}}
\end{aligned}$$

(c) Sketch the graph of the sine series from part (b), on the interval $[-10, 10]$.

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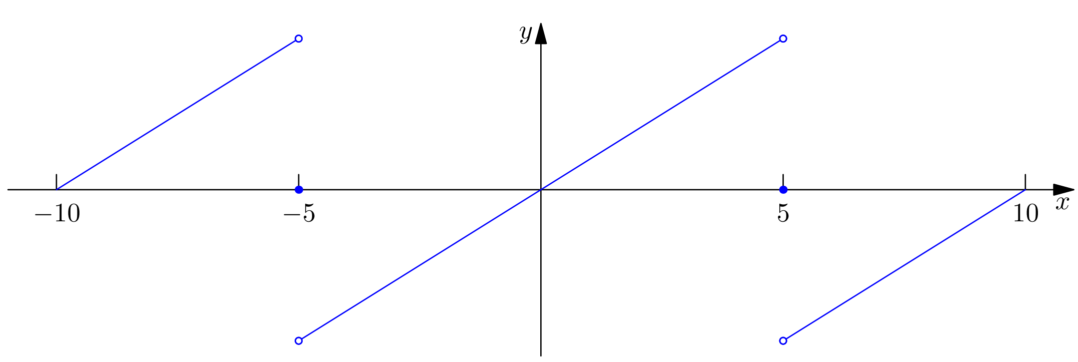

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**Problem 2:** Find the solution $u(x,t)$ of the following initial boundary value problem:

$$\begin{cases} u_{tt} = 9u_{xx}, & 0 < x < \pi, \quad t \geq 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \sin(4x) + 7\sin(5x) \end{cases}$$

This is the wave equation with $c = 3$. For these boundary conditions we already have the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(3nt) + B_n \sin(3nt)] \sin(nx) \qquad (A_n, B_n \in \mathbb{R}).$$

Imposing the initial conditions gives

$$u(x,0) = 0 = \sum_{n=1}^{\infty} A_n \sin(nx) \implies A_n = 0 \quad \forall n.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(3nt) \sin(nx)$$

and imposing the other initial condition gives

$$u_t(x,0) = \sin(4x) + 7\sin(5x) = \sum_{n=1}^{\infty} 3nB_n \sin(nx)$$

This is a sine series, where we can simply match coefficients by inspection:

$$1 = 3(4)B_4 \implies B_4 = \frac{1}{12},$$

$$7 = 3(5)B_5 \implies B_5 = \frac{7}{15},$$

$$\text{all other } B_n = 0.$$

$$\implies \boxed{u(x,t) = \frac{1}{12}\sin(12t)\sin(4x) + \frac{7}{15}\sin(15t)\sin(5x)}$$

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**Problem 3:** Find the solution $u(x,y)$ of the following boundary value problem:

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0, & 0 < x < \pi, \quad 0 < y < \pi \\ u(0,y) = u(\pi,y) = 0, & 0 < y < \pi \\ u(x,0) = f(x), & 0 < x < \pi \\ u(x,\pi) = 0, & 0 < x < \pi. \end{cases}$$

The function $f(x)$ is arbitrary; express you answer in terms of integrals involving $f(x)$.

Separate variables:

$$u(x,y) = X(x)Y(y) \implies X''Y + XY'' = 0 \implies \frac{Y''}{Y} = -\frac{X''}{X} = k \quad \text{(const.)}$$

As many times before, for the given boundary conditions the case $k \le 0$ gives only the trivial solution, so we assume $k = \lambda^2 > 0$ and obtain

$$X'' + \lambda^2 X = 0 \implies X(x) = A\cos(\lambda x) + B\sin(\lambda x) \qquad (A, B \in \mathbb{R}).$$

Imposing the boundary conditions $u(0,y) = u(\pi,y) = 0$ gives

$$\begin{aligned} & X(0) = 0 = A \\ \implies & X(\pi) = 0 = B\sin(\lambda\pi) \implies \lambda = n \ (= 1, 2, \ldots) \\ \implies & X(x) = B\sin(nx) \qquad (n = 1, 2, \ldots). \end{aligned}$$

Then the DE for $Y(y)$ gives

$$Y'' - n^2 Y = 0 \implies Y = ae^{ny} + be^{-ny} \qquad (a, b \in \mathbb{R}).$$

Imposing the boundary condition $u(x,\pi) = 0$ gives

$$Y(\pi) = 0 = ae^{n\pi} + be^{-n\pi} \implies b = -ae^{n2\pi} \implies Y(y) = a\left[e^{ny} - e^{n2\pi}e^{-ny}\right]$$

so we get solutions

$$u(x,y) = B\left[e^{ny} - e^{n2\pi}e^{-ny}\right]\sin(nx) \qquad (n = 1, 2, \ldots)$$

An arbitrary linear combination of these gives the general solution:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \left(e^{ny} - e^{n2\pi}e^{-ny}\right)\sin(nx).$$

Imposing the final boundary condition gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \left(1 - e^{n2\pi}\right)\sin(nx) \qquad \text{(a sine series for } f(x))$$

$$\implies B_n\left(1 - e^{n2\pi}\right) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

So finally we obtain

$$\boxed{\begin{aligned} & u(x,y) = \sum_{n=1}^{\infty} B_n \left(e^{ny} - e^{n2\pi}e^{-ny}\right)\sin(nx) \\ & \text{with } B_n = \tfrac{2}{\pi}\left(1 - e^{n2\pi}\right)^{-1}\int_0^{\pi} f(x)\sin(nx)\,dx. \end{aligned}}$$

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**Problem 4:** One-dimensional diffusion of a substance that is also being depleted (e.g. by chemical reaction or radioactive decay) can be modeled by the boundary value problem

$$\begin{cases} u_t = u_{xx} - ku, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0 \end{cases}$$

where $u(x,t)$ is the concentration of the substance and $k > 0$ is a constant that determines the rate of decay.

(a) Find the general solution $u(x,t)$ for this problem.

Separate variables:

$$u(x,t) = X(x)T(t) \implies XT' = X''T - kXT \implies \frac{T'}{T} = \frac{X''}{X} - k = c \text{ (const.)}$$

$$\implies \begin{cases} X'' - (k+c)X = 0 \\ T' = cT. \end{cases}$$

As many times before, for these boundary conditions only the case $k + c \equiv -\lambda^2 < 0$ yields non-trivial solutions:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \qquad (A, B \in \mathbb{R}).$$

The boundary conditions require:

$$\begin{aligned} &X(0) = A = 0 \\ \implies &X(L) = B\sin(\lambda L) \implies \lambda L = n\pi \implies \lambda = \frac{n\pi}{L} \qquad (n = 1, 2, \ldots) \\ \implies &X(x) = B\sin\frac{n\pi x}{L}. \end{aligned}$$

The DE for $T(t)$ gives

$$T' = cT \implies T(t) = Ae^{ct} \qquad (A \in \mathbb{R})$$

where

$$c = -\lambda^2 - k = -\left(\frac{n\pi}{L}\right)^2 - k.$$

Thus we obtain the solutions

$$u(x,t) = XT = Be^{-((\frac{n\pi}{L})^2 + k)t}\sin\frac{n\pi x}{L} \qquad (n = 1, 2, \ldots).$$

We get the general solution by forming an arbitrary linear combination of these:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-((\frac{n\pi}{L})^2 + k)t} = e^{-kt}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 t}}$$

(b) What happens to the concentration $u(x,t)$ as $t \to \infty$?

Even without the general solution we could have discovered that

$$T' = cT \implies T = Ae^{ct} \qquad (A \in \mathbb{R})$$

where $c = -\lambda^2 - k < 0$. Thus $T(t) \to 0$ (exponentially) as $t \to \infty$, hence $u(x,t) \to 0$.

We can be more precise about this. For $t \gg \frac{L^2}{4\pi^2}$ the leading ($n = 1$) term dominates so that

$$u(x,t) \approx A_1 e^{-kt} e^{-(\frac{\pi}{L})^2 t}\sin\frac{\pi x}{L} \sim \sin\frac{\pi x}{L}$$

so the approach to equilibrium ($u = 0$) looks like this: