THOMPSON RIVERS UNIVERSITY

# MATH 3160
# Differential Equations II

Instructor: Richard Taylor

# FINAL EXAM
# SOLUTIONS

20 April 2013 19:00–22:00

| PROBLEM | GRADE | OUT OF |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| TOTAL: | | 80 |

**Instructions:**

1. Read the whole exam before beginning.
2. Make sure you have all 9 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/10

**Problem 1:** Consider the following differential equation for $y(x)$:

$$(1+x^2)y'' + 6xy' + 6y = 0$$

(a) Explain why $x = 0$ is an ordinary point (i.e. not a singular point) for this equation.

We have $y'' + P(x)y' + Q(x)y = 0$ where $P(x) = 6x/(1+x^2)$ and $Q(x) = 6/(1+x^2)$ are both analytic at $x = 0$, hence $x = 0$ is an ordinary point.

(b) Find the general solution of this equation in terms of power series.

Assume:

$$y = \sum_{n=0}^{\infty} c_n x^n \implies y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \implies y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Subbing into the DE:

$$\sum_{2} n(n-1)c_n x^{n-2} + \sum_{2} n(n-1)c_n x^n + 6\sum_{1} nc_n x^n + 6\sum_{0} c_n x^n = 0$$

and re-indexing:

$$\sum_{0} (n+2)(n+1)c_{n+2} x^n + \sum_{2} n(n-1)c_n x^n + 6\sum_{1} nc_n x^n + 6\sum_{0} c_n x^n = 0$$

$$\implies 2c_2 + 6c_3 x + 6c_1 x + 6c_0 + 6c_1 x + \sum_{2} \Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n + 6nc_n + 6c_n\Big] x^n = 0$$

gives the recurrence relations:

$$\begin{cases} 2c_2 + 6c_0 = 0 \\ 6c_3 + 12c_1 = 0 \\ (n+2)(n+1)c_{n+2} + (n+2)(n+3)c_n = 0 \end{cases} \implies c_{n+2} = -\frac{n+3}{n+1}c_n; \quad n = 2, 3, \ldots$$

so that

$$\begin{aligned} c_2 &= -3c_0 & c_3 &= -2c_1 \\ c_4 &= -\frac{5}{3}c_2 = 5c_0 & c_5 &= -\frac{6}{4}c_3 = 3c_1 \\ c_6 &= -\frac{7}{5}c_4 = -7c_0 & c_7 &= -\frac{8}{6}c_5 = -4c_1 \\ c_8 &= -\frac{9}{7}c_6 = 9c_0 & c_9 &= -\frac{10}{8}c_7 = 5c_1 \\ &\ldots & &\ldots \\ c_{2k} &= (-1)^n(2k+1)c_0 & c_{2k+1} &= (-1)^k(k+1)c_1 \end{aligned}$$

and the general solution is

$$y(x) = c_0 \sum_{k=0}^{\infty} (-1)^k (2k+1)x^{2k} + c_1 \sum_{k=0}^{\infty} (-1)^k (k+1)x^{2k+1}$$

(c) Give (and justify) a lower bound on the radius of convergence for the series solution(s) in part (b).

The only singular points (i.e. values of $x$ at which $P(x)$ or $Q(x)$ fail to be analytic) are $x = \pm i$. The radius of convergence for the series in (a) is therefore at least $|i - 0| = 1$.

/10

**Problem 2:** Consider the following differential equation for $y(x)$:

$$3x^2y'' + x(1+x)y' - y = 0$$

(a) Explain why $x = 0$ is a regular singular point (i.e. not an ordinary point or irregular singular point) for this equation.

We have $y'' + P(x)y' + Q(x)y = 0$ where $P(x) = (1+x)/(2x)$ and $Q(x) = -1/(3x^2)$. Both $P$ and $Q$ are non-analytic at $x = 0$, so $x = 0$ is a singular point. Since $xP(x) = (1+x)/2$ and $x^2Q(x) = -1/3$ are both analytic at $x = 0$, $x = 0$ is a *regular* singular point.

(b) Find the value(s) of $r$ such that this equation has a solution of the form $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n$. (Do not attempt to determine the coefficients $c_n$.) Are there two linearly independent solutions of this form?

Assume:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \implies y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \implies y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

Subbing into the DE:

$$\sum_0 3(n+r)(n+r-1)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r+1} - \sum_0 c_n x^{n+r} = 0$$

and re-indexing:

$$\sum_0 3(n+r)(n+r-1)c_n x^{n+r} + \sum_0 (n+r)c_n x^{n+r} + \sum_1 (n+r-1)c_{n-1} x^{n+r} - \sum_0 c_n x^{n+r} = 0$$

$$\implies \Big[3r(r-1) + r - 1\Big]c_0 x^r + \sum_1 \Big[3(n+r)(n+r-1)c_n + (n+r)c_n + n + r - 1)c_{n-1} - c_n\Big]x^{n+r} = 0$$

gives the indicial equation

$$3r(r-1) + r - 1 = 0 \implies 3r^2 - 2r - 1 = 0$$

$$\implies \boxed{r = \frac{2 \pm 4}{-6} = -1,\ \frac{1}{3}}$$

Since the roots of the indicial equation do not differ by an integer, there will be two linearly independent solutions of the form $y(x) = x^r P(x)$ where $P$ is a power series.

/10

**Problem 3:** Solve the initial value problem

$$y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \qquad y(0) = 0, \qquad y'(0) = 1.$$

Simplify your solution as much as possible, and sketch the graph of the solution $y(t)$ on the interval $[0, 6\pi]$.

Laplace transform:

$$\begin{aligned}
(s^2 Y(s) - 1) + Y(s) = \sum_{k=1}^{\infty} e^{-2k\pi s} &\implies (1 + s^2) Y(s) = 1 + \sum_{k=1}^{\infty} e^{-2k\pi s} \\
&\implies Y(s) = \frac{1}{1+s^2} + \sum_{k=1}^{\infty} e^{-2k\pi s} \frac{1}{1+s^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \sin t + \sum_{k=1}^{\infty} u(t - 2k\pi) \sin(t - 2k\pi) \\
&= \sin t + \sum_{k=1}^{\infty} u(t - 2k\pi) \sin t \\
&= (m+1) \sin t; \quad t \in [m2\pi, (m+1)2\pi]; \quad m = 0, 1, 2, \ldots
\end{aligned}$$

/10

**Problem 4:** Find the inverse Laplace transform of

$$F(s) = \frac{s+2}{(s-3)(s^2+2s+5)}$$

Write

$$\begin{aligned} F(s) &= \frac{A}{s-3} + \frac{Bs+C}{s^2+2s+5} \\ &= \frac{A(s^2+2s+5)+(Bs+C)(s-3)}{(s-3)(s^2+2s+5)} \\ &= \frac{(A+B)s^2+(2A-3B+C)s+(5A-3C)}{(s-3)(s^2+2s+5)} \implies \begin{cases} A+B=0 \\ 2A-3B+C=1 \\ 5A-3C=2 \end{cases} \end{aligned}$$

$$s \to 3 \implies 20A = 5 \implies A = \frac{1}{4} \implies B = -\frac{1}{4} \implies C = -\frac{1}{4}$$

so that

$$\begin{aligned} F(s) &= \frac{1}{4}\cdot\frac{1}{s-3} - \frac{1}{4}\cdot\frac{s+1}{s^2+2s+5} \\ &= \frac{1}{4}\cdot\frac{1}{s-3} - \frac{1}{4}\cdot\frac{(s+1)}{(s+1)^2+2^2} \end{aligned}$$

which gives

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{4}e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\ &= \boxed{\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}\cos 2t} \end{aligned}$$

/10

**Problem 5:** Consider the following Sturm-Liouville problem on $[0, 3]$:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y'(3) = 0$$

(a) Find the eigenvalues and eigenfunctions for this problem.

<u>case $\lambda = -\alpha^2 < 0$:</u>

$$y(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_1 = c_2 = 0 \implies \text{only the trivial solution}$$

<u>case $\lambda = 0$:</u>

$$y(x) = c_1 + c_2 x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_2 = 0 \implies y(x) = c_1$$

<u>case $\lambda = \alpha^2 > 0$:</u>

$$y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$\begin{cases} y'(0) = 0 \\ y'(3) = 0 \end{cases} \implies c_2 = 0;\ \sin 3\alpha = 0 \implies 3\alpha = n\pi$$

So the eigenvalues are

$$\boxed{\lambda_n = \left(\frac{n\pi}{3}\right)^2; \quad n = 0, 1, 2, \ldots}$$

with corresponding eigenfunctions

$$\boxed{y_n(x) = \cos\left(\frac{n\pi x}{3}\right)}$$

(b) Is $\lambda = 0$ an eigenvalue for this problem? Justify your answer.

Yes. $\lambda = 0$ gives non-trivial solutions $y(x) = c$; the corresponding eigenfunctions are the constant functions.

/10

**Problem 6:** Consider the following piecewise continuous function on $[-2, 2]$.

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ \frac{1}{2}, & 0 < x < 2. \end{cases}$$

(a) Find the Fourier series representation of this function.

We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$\frac{a_0}{2} = \frac{3}{4} \qquad \text{(average of } f(x)\text{, by inspection)}$$

and

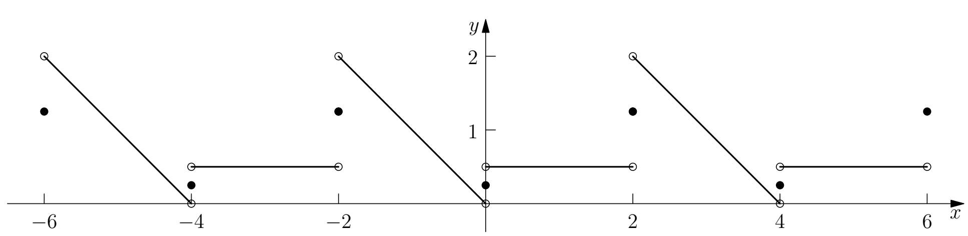
$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} (-x)\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} \frac{1}{2}\cos\left(\frac{n\pi x}{2}\right)dx \\
&= -\frac{1}{2}\Big[\left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right) + \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big]_{-2}^{0} + \frac{1}{2}\Big[\frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big]_{0}^{2} \\
&= -\frac{1}{2}\Big[\left(\frac{2}{n\pi}\right)^2 - \left(\frac{2}{n\pi}\right)^2\cos(-n\pi)\Big] \\
&= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\int_{-2}^{0} (-x)\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} \frac{1}{2}\sin\left(\frac{n\pi x}{2}\right)dx \\
&= -\frac{1}{2}\Big[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right) - \frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big]_{-2}^{0} + \frac{1}{2}\Big[-\frac{1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big]_{0}^{2} \\
&= -\frac{1}{2}\Big[-\frac{4}{n\pi}\cos(-n\pi)\Big] + \frac{1}{n2\pi}\Big[1 - \cos(-n\pi)\Big] \\
&= \frac{1}{n\pi}\Big[3(-1)^n + 1\Big]
\end{aligned}$$

so

$$f(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{3(-1)^n + 1}{n}\sin\left(\frac{n\pi x}{2}\right)$$

(b) Sketch the graph, on the interval $[-6, 6]$, of the function to which the Fourier series in part (a) converges.



/10

**Problem 7:** Solve the following initial boundary value problem for $u(x,t)$, which models heat flow in a one-dimensional object with one end insulated

$$\begin{aligned} &u_t = 16u_{xx}, \qquad 0 < x < 2, \qquad t > 0 \\ &u(0,t) = 0, \qquad u_x(2,t) = 0, \qquad t > 0 \\ &u(x,0) = x, \qquad 0 \le x \le 2 \end{aligned}$$

Separation of variables $u(x,t) = X(x)T(t)$ gives

$$\begin{cases} X'' + \lambda X = 0 \\ T' = -16\lambda T \end{cases}$$

which give nontrivial solutions only for $\lambda = \alpha^2 > 0$:

$$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

Boundary conditions $X(0) = 0,\ X'(2) = 0$ give $c_1 = 0$ and $\cos(2\alpha) = 0 \implies \alpha = (n + \frac{1}{2})\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$

We have $T(t) = Ae^{-16\lambda t} = Ae^{-4(n+\frac{1}{2})^2\pi^2 t}$ so the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-4(n+\frac{1}{2})^2\pi^2 t} \sin\left((n + \tfrac{1}{2})\frac{\pi x}{2}\right)$$

where

$$u(x,0) = x = \sum_{n=0}^{\infty} A_n \sin\left((n + \tfrac{1}{2})\frac{\pi x}{2}\right).$$

together with orthogonality gives

$$A_n = \frac{\displaystyle\int_0^2 x \sin\left((n + \tfrac{1}{2})\frac{\pi x}{2}\right) dx}{\displaystyle\int_0^2 \sin^2\left((n + \tfrac{1}{2})\frac{\pi x}{2}\right) dx}$$

/10

**Problem 8:** Solve the following initial boundary value problem for $u(x,t)$, which models vibration of a taut string.

$$\begin{aligned}
&u_{tt} = 9u_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\
&u(0,t) = u(1,t) = 0, \qquad t > 0 \\
&u(x,0) = \sin(\pi x) - 5\sin(3\pi x), \qquad u_t(x,0) = 0, \qquad 0 \le x \le 1
\end{aligned}$$

The standard solution for the wave equation gives

$$u(x,t) = \sum_{n=1}^{\infty} \Big[A_n \cos(n3\pi t) + B_n \sin(n3\pi t)\Big] \sin(n\pi x)$$

where

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} \Big[B_n \cdot n3\pi\Big] \sin(n\pi x)$$

gives $B_n = 0$ for all $n = 1, 2, \ldots$ and

$$u(x,t) = \sin(\pi x) - 5\sin(3\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

gives $A_1 = 1$, $A_3 = -5$ and $A_n = 0$ for all other $n \neq 1, 3$.

Therefore the solution is

$$\boxed{u(x,t) = \cos(3\pi t)\sin(\pi x) - 5\cos(9\pi t)\sin(3\pi x)}$$