THOMPSON RIVERS UNIVERSITY

MATH 3160
Differential Equations 2

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FINAL EXAM
SOLUTIONS

5 Dec. 2019   14:00–17:00

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
5		8
6		10
7		10
8		8
9		8
TOTAL:		84

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 10 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

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Problem 1: Consider the differential equation $y'' + k^2x^2y = 0$ in which k is a constant.

(a) Is $x = 0$ an ordinary point or a singular point for this equation? Explain.

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$y'' + P(x)y' + Q(x)y = 0$; coefficients $P(x) = 0$ and $Q(x) = k^2x^2$ are both analytic functions (polynomials) so *every* point is an ordinary point. So, in particular, $x = 0$ is an ordinary point.

(b) Derive a set of recurrence relations for the coefficients a_n in the power series $y = \sum_{n=0}^{\infty} a_n x^n$ such that y is a solution of the given equation.

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$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$\text{(sub into DE)} \implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n k^2 x^{n+2} = 0$$

$$\text{(re-index)} \implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=4}^{\infty} a_{n-4} k^2 x^{n-2} = 0$$

$$\text{(pull out leading terms)} \implies 2a_2 + 6a_3 x + \sum_{n=4}^{\infty} \left[n(n-1)a_n + a_{n-4}k^2\right] x^{n-2} = 0$$

$$\text{(equate all coeffs. to 0)} \implies \boxed{\begin{cases} a_2 = 0 \\ a_3 = 0 \\ a_n = -\dfrac{k^2 a_{n-4}}{n(n-1)} \quad (n = 4, 5, \ldots) \end{cases}}$$

(c) The general solution of this equation can be expressed as a linear combination $y = c_1y_1 + c_2y_2$. Solve your recurrence relation find the solutions y_1 and y_2, expressed as power series about $x = 0$.

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$$a_2 = a_3 = a_6 = a_7 = \cdots = c_{2m+2} = c_{2m+3} = 0 \quad (m = 0, 1, 2, \ldots)$$

$$a_4 = -\frac{k^2 a_0}{4 \cdot 3}, \quad a_8 = -\frac{k^2 a_4}{8 \cdot 7} = \frac{k^4 a_0}{8 \cdot 7 \cdot 4 \cdot 3}, \quad a_{12} = -\frac{k^2 a_8}{12 \cdot 11} = -\frac{k^6 a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$$

$$a_5 = -\frac{k^2 a_1}{5 \cdot 4}, \quad a_9 = -\frac{k^2 a_5}{9 \cdot 8} = \frac{k^4 a_1}{9 \cdot 8 \cdot 5 \cdot 4}, \quad a_{13} = -\frac{k^2 a_9}{13 \cdot 12} = -\frac{k^6 a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$$

$$y(x) = a_0 \underbrace{\left[1 - \frac{k^2x^4}{4 \cdot 3} + \frac{k^4x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{k^6x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \cdots\right]}_{y_0(x)} + a_1 \underbrace{\left[x - \frac{k^2x^5}{5 \cdot 4} + \frac{k^4x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{k^6x^{13}}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \cdots\right]}_{y_1(x)}$$

Two linearly independent solutions are the functions y_0, y_1 above.

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Problem 2: Consider the differential equation $3x^2y'' + 2xy' + x^2y = 0$.

(a) Is $x = 0$ an ordinary point or a singular point for this equation? Explain.

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$y'' + P(x)y' + Q(x)y = 0$ with coefficients $P(x) = \dfrac{2}{3x}$ and $Q(x) = \dfrac{1}{3}$.

We see that $x = 0$ is a singular point since P is singular (not analytic) at $x = 0$.

We have that $xP(x) = \frac{2}{3}$ and $x^2Q(x) = \frac{1}{3}x^2$ are both analytic functions (polynomials) so $x = 0$ is a *regular* singular point.

(b) Find values of r such that a solution of this equation can we written as $y = x^r \sum_{n=0}^{\infty} a_n x^n$.

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$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

$$\text{(sub into DE)} \implies \sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\text{(re-index)} \implies \sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Pull out leading ($n = 0$, $n = 1$) terms:

$$\implies \big[3r(r-1) + 2r\big]a_0 x^r + \big[3(r+1)r + 2(1+r)\big]a_1 x^{1+r} + \sum_{n=2}^{\infty}\Big[3(n+r)(n+r-1)a_n + 2(n+r)a_n + a_{n-2}\Big]x^{n+r} = 0.$$

Leading term gives the "indicial equation":

$$0 = 3r(r-1) + 2r = r(3r-1) \implies \boxed{r = 0 \text{ or } r = \frac{1}{3}}$$

(c) For the largest of your values of r, find the first three non-zero terms in the corresponding series solution of the differential equation.

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$$3(n+r)(n+r-1)a_n + 2(n+r)a_n + a_{n-2} = 0 \implies a_n = \frac{-a_{n-2}}{(n+r)\big[3(n+r-1)+2\big]}$$

<u>case $r = \frac{1}{3}$:</u>

$$a_n = \frac{-a_{n-2}}{(n+\frac{1}{3})\big[3(n+\frac{1}{3}-1)+2\big]} = \frac{-a_{n-2}}{(n+\frac{1}{3})(3n)} = \frac{-a_{n-2}}{n(3n+1)}$$

The coefficient of x^{1+r} becomes $4a_1$ so $a_1 = 0$, hence $0 = a_1 = a_3 = a_5 = \cdots$ and

$$a_2 = -\frac{a_0}{2\cdot 7}, \qquad a_4 = -\frac{a_2}{4\cdot 13} = \frac{a_0}{2\cdot 4\cdot 7\cdot 13}$$

$$\implies y(x) = x^{1/3}\big[a_0 + a_2x^2 + a_4x^4 + \cdots\big]$$

$$= \boxed{a_0 x^{1/3}\left[1 - \frac{x^2}{2\cdot 7} + \frac{x^4}{2\cdot 4\cdot 7\cdot 13} + \cdots\right]}$$

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Problem 3: Let $\mathcal{L}$ denote the Laplace transform. Evaluate (and simplify) the following:

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(a) $F(s) = \mathcal{L}\{t^{-1/2}\} \quad (s > 0)$

(Hint: It might help to substitute $u = \sqrt{st}$. Also $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.)

$$\begin{aligned}
F(s) = \mathcal{L}\{t^{-1/2}\} &= \int_0^\infty e^{-st}\,dt \qquad (u = \sqrt{st} = s^{1/2}t^{1/2},\ du = \tfrac{1}{2}t^{-1/2}s^{1/2}\,dt) \\
&= \int_0^\infty e^{-u^2}\frac{2\,du}{\sqrt{s}} \\
&= \frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}\,du \\
&= \frac{2}{\sqrt{s}}\frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\frac{\pi}{s}}}
\end{aligned}$$

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(b) $f(t) = \mathcal{L}^{-1}\left\{\frac{7s^2+3s+16}{(s+1)(s^2+4)}\right\}$

Partial fractions expansion:

$$\frac{7s^2+3s+16}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4)+(Bs+C)(s+1)}{(s+1)(s^2+4)}$$

At $s = -1$ in the numerator this gives

$$20 = 5A \implies A = 4.$$

Matching coefficients of s^2 gives

$$7 = A + B \implies B = 3.$$

Matching coefficients of s^0 gives

$$16 = 4A + C \implies C = 0.$$

Thus:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{4}{s^2+1} + \frac{3s}{s^2+4}\right\} \\
&= 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\
&= \boxed{4e^{-t} + 3\cos(2t)}
\end{aligned}$$

/10 **Problem 4:** Let $\delta(t)$ denote the Dirac delta function. Find the solution $y(t)$ of the following initial value problem:

$$y^{(4)} - y = \delta(t-1), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Laplace transform both sides:

$$(s^4 Y - 0) - Y = e^{-s} \implies Y(s) = \frac{e^{-s}}{s^4 - 1}$$

$$\implies y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4-1}e^{-s}\right\} = f(t-1)u(t-1)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\}.$$

Partial fractions expansion:

$$\begin{aligned}\frac{1}{s^4-1} &= \frac{1}{(s^2+1)(s^2-1)} = \frac{1}{(s^2+1)(s+1)(s-1)} \\ &= \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s-1} = \frac{(As+B)(s^2-1) + C(s^2+1)(s-1) + D(s^2+1)(s+1)}{s^4-1}.\end{aligned}$$

Some convenient values of s in the numerator give:

$$\begin{aligned} s = 1: &\quad 1 = 4D \implies D = \tfrac{1}{4} \\ s = -1: &\quad 1 = -4C \implies C = -\tfrac{1}{4}.\end{aligned}$$

Matching coefficients of s^3 gives

$$0 = A + C + D \implies A = 0.$$

Matching coefficients of s^2 gives

$$0 = B - C + D \implies B = -\tfrac{1}{2}.$$

Thus

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}}{s^2+1} + \frac{-\frac{1}{4}}{s+1} + \frac{\frac{1}{4}}{s-1}\right\} \\ &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= -\tfrac{1}{2}\sin t - \tfrac{1}{4}e^{-t} + \tfrac{1}{4}e^{t}\end{aligned}$$

and so

$$\begin{aligned} y(t) &= f(t-1)u(t-1) \\ &= \left[-\tfrac{1}{2}\sin(t-1) - \tfrac{1}{4}e^{-(t-1)} + \tfrac{1}{4}e^{t-1}\right]u(t-1) \\ &= \tfrac{1}{2}\left[\sinh(t-1) - \sin(t-1)\right]u(t-1)\end{aligned}$$

/8 **Problem 5:** Consider the function $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi. \end{cases}$

(a) Find the Fourier series for $f(x)$.

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f if has period $2p$ with $p = \pi$. Thus

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx \qquad \text{int. by parts: } \left\{ \begin{array}{ll} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx \end{array} \right. \\ &= \frac{1}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right] \\ &= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right] = \frac{(-1)^n - 1}{n^2\pi} \end{aligned}$$

except

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

and

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx \qquad \text{int. by parts: } \left\{ \begin{array}{ll} u = x & dv = \sin nx\,dx \\ du = dx & v = -\frac{1}{n}\cos nx \end{array} \right. \\ &= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right] \\ &= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right] = \frac{-(-1)^n}{n}. \end{aligned}$$

Thus

$$f(x) = \boxed{\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi}\cos nx - \frac{(-1)^n}{n}\sin nx}$$

(b) Sketch the graph of the function the series converges to over the interval $[-3\pi, 3\pi]$.

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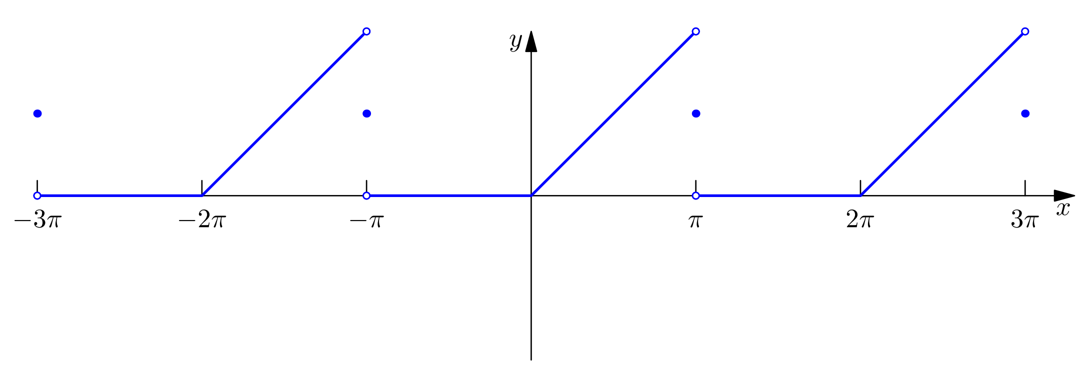

/10 **Problem 6:** Consider the following initial boundary value problem for a function $u(x,t)$:

$$\begin{cases} u_t = u_{xx}, & 0 < x < \pi \\ u_x(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x). \end{cases}$$

(a) Find the solution $u(x,t)$.

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$$u(x,t) = X(x)T(t) \implies XT' = X''T \implies \frac{T'}{T} = \frac{X''}{X} = \lambda \quad \text{(const.)}$$

$$\implies \begin{cases} X'' - \lambda X = 0, & X'(0) = X(\pi) = 0 \\ T' = \lambda T \end{cases}$$

With these boundary conditions, the cases $\lambda \geq 0$ give $X(x) = 0$ (trivial solution). The case $\lambda = -\alpha^2 < 0$ gives

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x).$$

$$X'(0) = 0 = A\cos(0) \implies A = 0$$

$$X(\pi) = 0 = B\cos(\alpha\pi) \implies \alpha\pi = \frac{\pi}{2} + n\pi \implies \alpha = n + \tfrac{1}{2} \quad (n = 0, 1, 2, \ldots)$$

$$\implies X_n(x) = \cos\big((n + \tfrac{1}{2})x\big) \quad (n = 0, 1, 2, \ldots).$$

$$T' = -\alpha^2 T = -(n+\tfrac{1}{2})^2 T \implies T_n(t) = Ce^{-(n+\frac{1}{2})^2 t}.$$

By linearity, the general solution is an arbitrary linear combination

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2 t} \cos\big((n+\tfrac{1}{2})x\big).}$$

Imposing the initial conditions gives

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\big((n+\tfrac{1}{2})x\big).$$

The functions $X_n(x)$ are solutions of a regular Sturm-Liouville problem and so are orthogonal (with respect to the usual L^2 inner product). Thus

$$\boxed{A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\int_0^\pi f(x)\cos\big((n+\frac{1}{2})x\big)\,dx}{\int_0^\pi \cos^2\big((n+\frac{1}{2})x\big)\,dx} = \frac{2}{\pi}\int_0^\pi f(x)\cos\big((n+\tfrac{1}{2})x\big)\,dx}$$

(b) Consider the particular case where $f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$ For several values of $t > 0$, sketch the graph of the solution (as a function of x only)

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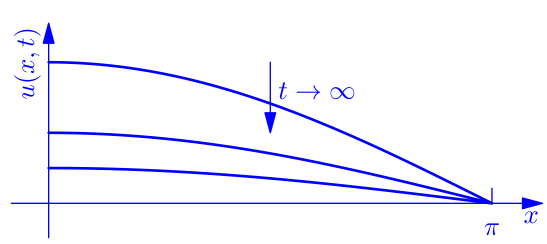
For $t \gg \frac{4}{9}$ the first term in the series dominates and so $u(x,t) \sim A_0 e^{-t/4}\cos(x/2)$.



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Problem 7: Find the solution $u(x,y)$ of the following boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \\ u(x,0) = \sin(2\pi x) \\ u(x,2) = 0 = u(0,y) = u(1,y). \end{cases}$$

$$u(x,y) = X(x)Y(y) \implies X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \quad \text{(const.)}$$

$$\implies \begin{cases} X'' - \lambda X = 0, \quad X(0) = X(1) = 0 \\ Y'' + \lambda^2 Y = 0, \quad Y(2) = 0. \end{cases}$$

The cases $\lambda \geq 0$ gives $X(x)$ (trivial solution).
The cases $\lambda = -\alpha^2 < 0$ gives

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x).$$

$$X(0) = 0 = A\cos 0 \implies A = 0$$
$$X(1) = 0 = B\sin\alpha \implies \alpha = n\pi \quad (n = 1, 2, \ldots)$$
$$\implies X_n(x) = \sin(n\pi x) \quad (n = 1, 2, \ldots).$$

$$Y'' - \lambda Y = 0 \implies Y'' - (n\pi)^2 Y = 0 \implies Y(y) = Ce^{n\pi y} + De^{-n\pi y}.$$
$$Y(2) = 0 = Ce^{n2\pi} + De^{-n2\pi} \implies D = -Ce^{n4\pi}$$
$$\implies Y_n(y) = e^{n\pi y} + e^{n4\pi}e^{-n\pi y} = e^{n\pi y} + e^{n\pi(4-y)}.$$

By linearity, the general solution is an arbitrary linear combination

$$u(x,y) = \sum_{n=0}^{\infty} A_n \left[e^{n\pi y} - e^{n\pi(4-y)}\right] \sin(n\pi x).$$

Imposing the final boundary condition gives

$$u(x,0) = \sin(2\pi x) = \sum_{n=0}^{\infty} A_n \left[1 - e^{n4\pi}\right] \sin nx$$

$$\implies 1 = A_2 \left[1 - e^{2\cdot 4\pi}\right] \implies A_2 = \left[1 - e^{2\cdot 4\pi}\right]^{-1}, \qquad \text{all other } A_n = 0.$$

$$\implies \boxed{u(x,y) = \frac{e^{2\pi y} - e^{2\pi(4-y)}}{1 - e^{8\pi}} \sin(2\pi x)}$$

/8 **Problem 8:** Consider the following Sturm-Liouville problem for a function $\phi(x)$:

$$\begin{cases} -\phi'' = \lambda\phi, & 0 < x < 1 \\ \phi(0) = \phi'(1) = 0 \end{cases}$$

(a) Find the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ $(n = 1, 2, \ldots)$.

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$$\phi'' + \lambda\phi = 0$$

Case $\lambda = -\alpha^2 < 0$:

$$\phi'' - \alpha^2\phi = 0 \implies \phi(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

$$\begin{cases} \phi(0) = 0 = A + B \\ \phi'(1) = 0 = \alpha A e^{\alpha} - \alpha B e^{-\alpha} \end{cases} \implies A = B = 0 \implies \phi(x) = 0 \quad \text{(trivial solution).}$$

Case $\lambda = 0$:

$$\phi'' = 0 \implies \phi(x) = Ax + B.$$

$$\begin{cases} \phi(0) = 0 = B \\ \phi'(1) = 0 = A \end{cases} \implies A = B = 0 \implies \phi(x) = 0 \quad \text{(trivial solution).}$$

Case $\lambda = \alpha^2 > 0$:

$$\phi'' + \alpha^2\phi = 0 \implies \phi(x) = A\cos\alpha x + B\sin\alpha x.$$

$$\phi(0) = 0 = A\cos 0 \implies A = 0$$
$$\phi'(1) = 0 = \alpha B\cos\alpha \implies \alpha = (n + \tfrac{1}{2})\pi \quad (n = 0, 1, 2, \ldots)$$

Therefore,

$$\boxed{\begin{array}{ll} \text{eigenfunctions:} & \phi_n(x) = \sin\left((n + \tfrac{1}{2})\pi x\right) \\ \text{eigenvalues:} & \lambda_n = (n + \tfrac{1}{2})^2\pi^2 \end{array}}$$

(b) Give an orthogonality relation for the eigenfunctions.

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In self-adjoint form the DE is

$$\frac{d}{dx}[\phi'(x)] + \lambda\phi = 0$$

which is a regular Sturm-Liouville problem (with $p(x) = r(x) = 1$ in our standard notation). Therefore the eigenfunctions are orthogonal, i.e.

$$\boxed{\langle \phi_m, \phi_n \rangle = 0 \text{ if } m \neq n}$$

with

$$\boxed{\langle \phi_m, \phi_n \rangle = \int_0^1 \phi_m(x)\phi_n(x)\, dx}$$

(c) For a given function $f(x)$ on $[0, 1]$, suppose we wish to express f as a series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

(a "generalized Fourier series") where the $\phi_n(x)$ are the eigenfunctions of this Sturm-Liouville problem. Give a formula for the coefficients c_n in this series.

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By orthogonal project of f onto ϕ_n:

$$c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^1 f(x)\sin\left((n + \frac{1}{2})\pi x\right) dx}{\int_0^1 \sin^2\left((n + \frac{1}{2})\pi x\right) dx} = \boxed{2\int_0^1 f(x)\sin\left((n + \tfrac{1}{2})\pi x\right) dx}$$

/8 **Problem 9:** The following initial boundary value problem models the time evolution of the temperature $u(r,\theta,t)$ in a circular plate with insulated boundary:

$$\begin{cases} u_t = \nabla^2 u, \quad 0 \le r < 1 \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=1} = 0 \\ u(r,\theta,0) = f(r). \end{cases}$$

It is safe to assume that $u(r,\theta,t) = u(r,t)$ is independent of θ.

(a) Formulate the eigenvalue problem that arises in the solution of this problem by separation of variables. (You do **not** need to find the solution $u(r,t)$.)

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$$u(r,t) = R(r)T(t) \implies RT' = \left(R'' + \frac{1}{r}R'\right)T \implies \frac{T'}{T} = \frac{R'' + \frac{1}{r}R'}{R} = -\lambda \quad \text{(const.)}$$

$$\implies rR'' + R' + \lambda r R = 0$$

In self-adjoint form:

$$\implies \begin{cases} \dfrac{d}{dr}(rR') + \lambda r R = 0 \\ R'(1) = 0 \\ R(0) \text{ is finite. (Or better: } R'(0) = 0 \text{ by symmetry).} \end{cases}$$

(b) What are the eigenvalues?

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With $\lambda = \alpha^2$ the DE is a Bessel equation of order 0, with general solution

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r).$$

The boundary condition at $r = 0$ requires $c_2 = 0$. The boundary condition at $r = 1$ gives

$$R'(1) = 0 = \alpha c_1 J_0'(\alpha) \implies \alpha \text{ is a root of } J_0'.$$

Thus the eigenvalues are $\lambda_n = \alpha_n^2$ where α_n is the nth zero of J_0' (of which there are infinitely many).