MATH 316: Quiz #6 (take-home) – SOLUTIONS

/10 **Problem 1:** Solve the following initial boundary value problem for $u(x,t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{5}\dfrac{\partial^2 u}{\partial x^2} & 0 < x < 10, \quad t > 0 \\ u(0,t) = u(10,t) = 0 \\ u(x,0) = 4x \end{cases}$$

Separation of variables:

$$u(x,t) = X(x)T(t) \implies XT' = \frac{1}{5}X''T \implies 5\frac{T'}{T} = \frac{X''}{X} = k \implies \begin{cases} X'' - kX = 0 \\ T' = \frac{k}{5}T \end{cases}$$

As before, only $k = -\lambda^2 < 0$ gives nontrivial solutions:

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

The boundary conditions $X(0) = X(10) = 0$ give:

$$\begin{aligned} 0 =& X(0) = A \implies X(x) = B\sin\lambda x \\ 0 =& X(10) = B\sin(10\lambda) \implies 10\lambda = n\pi \implies \lambda = \frac{n\pi}{10} \quad n = 1, 2, \ldots \end{aligned}$$

The ODE for T gives:

$$T(t) = Ae^{(k/5)t} = e^{-(n\pi/10)^2/(5t)} = e^{-n^2\pi^2 t/500}$$

By superposition the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{10}\right) e^{-n^2\pi^2 t/500}$$

Imposing initial conditions...

$$u(x,0) = 4x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{10}\right) \qquad \text{(a sine series)}$$

$$\begin{aligned} \implies A_n &= \frac{2}{10}\int_0^{10} 4x\sin\left(\frac{n\pi x}{10}\right) \\ &= \frac{4}{5}\int_0^{10} x\sin\left(\frac{n\pi x}{10}\right)\,dx \\ &= \frac{8}{n\pi}\int_0^{n\pi} u\sin u\,du \\ &= \frac{8}{n\pi}(\sin u - u\cos u)\Big|_0^{n\pi} = -\frac{8}{n\pi}\cos n\pi = \frac{8(-1)^{n+1}}{n\pi} \end{aligned}$$

So

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{10}\right) e^{-n^2\pi^2 t/500}}$$