MATH 316: Quiz #5 – SOLUTIONS

/5 **Problem 1:** The following boundary value problem models the displacement of a vibrating string with frictional damping. Use separation of variables to find the general solution. (Assume $0 < \beta < 1$.)

$$u_{xx} = u_{tt} + 2\beta u_t, \quad t \geq 0, \quad x \in [0, \pi]. \qquad u(0) = u(\pi) = 0.$$

$$u = X(x)T(t) \implies X''T = T''X + 2\beta XT' \implies \frac{X''}{X} = \frac{T''}{T} + 2\beta\frac{T'}{T} = k$$

$$\implies \begin{cases} X'' - kX = 0 \\ T'' + 2\beta T' - kT = 0 \end{cases}$$

As before, only $k = -n^2 < 0$ gives non-trivial solutions $\implies X(x) = A\sin(nx)$, $n = 1, 2, \ldots$

The characteristic polynomial for the second ODE gives $r^2 + 2\beta r + n^2 = 0 \implies r = -\beta \pm i\sqrt{n^2 - \beta^2}$

$$\implies T(t) = Ce^{-\beta t}\cos(\sqrt{n^2 - \beta^2}\, t) + De^{-\beta t}\sin(\sqrt{n^2 - \beta^2}\, t)$$

Putting all of this together using superposition gives the general solution:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t}\sin(nx)\left[A_n\cos\left(\sqrt{n^2 - \beta^2}\, t\right) + B_n e^{-\beta t}\sin\left(\sqrt{n^2 - \beta^2}\, t\right)\right]$$

/5 **Problem 2:** Find the Fourier series for the following function. Sketch a graph of this series for $-3 \leq x \leq 3$.

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \leq x < 1. \end{cases}$$

We have

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x + B_n \sin n\pi x$$

where

$$\begin{aligned} A_n &= \int_{-1}^{1} f(x)\cos n\pi x\, dx \\ &= \int_0^1 x\cos n\pi x\, dx \\ &= \frac{x}{n\pi}\sin n\pi x\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x\, dx \\ &= \frac{1}{n^2\pi^2}\cos n\pi x\Big|_0^1 \\ &= \frac{\cos n\pi - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2} \end{aligned}$$

$$\begin{aligned} B_n &= \int_{-1}^{1} f(x)\sin n\pi x\, dx \\ &= \int_0^1 x\sin n\pi x\, dx \\ &= -\frac{x}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\, dx \\ &= -\frac{(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}\sin n\pi x\Big|_0^1 \\ &= \frac{(-1)^{n+1}}{n\pi} \end{aligned}$$

except $A_0 = 1/2$.

$$\implies f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x$$