MATH 316: Quiz #3 – SOLUTIONS

/5 **Problem 1:** Use the Laplace Transform to solve the following initial-value problem:

$$y'' + y = \delta(t-2\pi) + \delta(t-4\pi), \quad y(0)=1,\ y'(0)=0.$$

$$\xrightarrow{\mathcal{L}} (s^2Y - s - 0) + Y = e^{-2\pi s} + e^{-4\pi s}$$

$$\Longrightarrow Y(s) = \frac{s}{s^2+1} + e^{-2\pi s}\underbrace{\frac{1}{s^2+1}}_{W(s)} + e^{-4\pi s}\underbrace{\frac{1}{s^2+1}}_{W(s)}$$

$$\Longrightarrow y(t) = \cos t + u(t-2\pi)w(t-2\pi) + u(t-4\pi)w(t-4\pi)$$

$$\text{where } w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

$$\Longrightarrow \ y(t) = \cos t + u(t-2\pi) sin(t-2\pi) + u(t-4\pi)\sin(t-4\pi)$$

$$= \boxed{\cos t + \big[u(t-2\pi) + u(t-4\pi)\big]\sin(t)}$$

/5 **Problem 2:** Use the Laplace Transform to solve the initial-value problem

$$y' + 2y = f(t), \quad y(0) = 0$$

where $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & t \ge 1. \end{cases}$

We can write $f(t) = t - u(t-1)t = t - u(t-1)\big[(t-1)+1\big] \ \Longrightarrow\ F(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$.
Then transforming the DE gives

$$sY + 2Y = F(s) \ \Longrightarrow Y(s) = \underbrace{\frac{1}{s^2(s+2)}}_{G(s)} - e^{-s}\Big(\underbrace{\frac{1}{s^2(s+2)}}_{G(s)} + \underbrace{\frac{1}{s(s+2)}}_{W(s)}\Big)$$

$$\Longrightarrow y(t) = g(t) + u(t-1)g(t-1) + u(t-1)w(t-1)$$

where

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-(1/4)}{s} + \frac{(1/2)}{s^2} + \frac{(1/4)}{s+2}\right\} = -\tfrac14 + \tfrac12 t + \tfrac14 e^{-2t}$$

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{(1/2)}{s} + \frac{-(1/2)}{s+2}\right\} = \tfrac12 - \tfrac12 e^{-2t}.$$

$$\Longrightarrow\ y(t) = -\tfrac14 + \tfrac12 t + \tfrac14 e^{-2t} + u(t-1)\left[-\tfrac14 + \tfrac12(t-1) + \tfrac14 e^{-2(t-1)}\right] + u(t-1)\left[\tfrac12 - \tfrac12 e^{-2(t-1)}\right]$$

$$= \boxed{-\tfrac14 + \tfrac12 t + \tfrac14 e^{-2t} + u(t-1)\left[-\tfrac14 + \tfrac12 t - \tfrac14 e^{-2(t-1)}\right]}$$