MATH 316: Quiz #2 – SOLUTIONS

/2 **Problem 1:** Find the Laplace transform of $f(t) = t^2 - e^{-9t} + 5$.

$$F(s) = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$$

/8 **Problem 2:** Consider the differential equation $2xy'' + y' + xy = 0$.

(a) Show that this equation has a regular singular point at $x = 0$.

$$2xy'' + y' + xy = 0 \implies y'' + \underbrace{\frac{1}{2x}}_{P(x)} y' + \underbrace{\frac{1}{2}}_{Q(x)} y = 0$$

$P(x)$ is not analytic at $x = 0$ so $x = 0$ is a singular point, but both $xP(x) = 1/2$ and $x^2Q(x) = x^2/2$ *are* analytic at $x = 0$, so $x = 0$ is a *regular* singular point.

(b) Find just one solution, expressed as a series about $x = 0$.

Method of Frobenius: assume

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \implies y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \implies y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$$

and substitute these into the DE:

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}}_{\sum_{n=2}^{\infty} c_{n-2}x^{n+r-1}} = 0$$

$$c_0\big[2r(r-1)+r\big]x^{r-1} + c_1\big[2(1+r)r+(n+r)\big]x^r + \sum_{n=2}^{\infty}\big[2c_n(n+r)(n+r-1)+c_n(n+r)+c_{n-2}\big]x^{n+r-1} = 0$$

The coefficient of the leading term (x^{r-1}) gives the indicial equation:

$$0 = 2r(r-1) + r = r(2r-1) \implies r = 0 \text{ or } \frac{1}{2}$$

With $r = \frac{1}{2}$ the other coefficients give $c_1 = 0$ together with the recurrence relation:

$$2c_n(n+\tfrac{1}{2})(n+\tfrac{1}{2}-1) + c_n(n+\tfrac{1}{2}) + c_{n-2} = 0 \implies c_n = \frac{-c_{n-2}}{n(2n+1)}.$$

Thus $c_1 = c_3 = c_5 = \cdots = 0$ and

$$c_2 = \frac{-c_0}{2\cdot 5} \implies c_4 = \frac{-c_2}{4\cdot 9} = \frac{c_0}{(2\cdot 4)(5\cdot 9)} \implies c_6 = \frac{-c_4}{6\cdot 13} = \frac{-c_0}{(2\cdot 4\cdot 6)(5\cdot 9\cdot 13)} \implies \cdots$$

$$\implies c_{2m} = \frac{(-1)^m c_0}{(2\cdot 4\cdots(2m))(5\cdot 9\cdots(4m+1))} = \frac{(-1)^m c_0}{2^m m!(5\cdot 9\cdots(4m+1))}$$

$$\implies \boxed{y(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!(5\cdot 9\cdots(4m+1))} x^{2m+1/2}}$$