MATH 316: Quiz #1 – SOLUTIONS

/3 **Problem 1:** Find the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} \frac{n!x^n}{(2n)!}$

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}/(2n+2)!}{n!x^n/(2n)!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!} \left| \frac{x^{n+1}}{x^n} \right| = |x| \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n+2)} = 0 < 1$$

So the radius of convergence is $R = \infty$ (i.e. the series converges for all x).

/7 **Problem 2:** Find two power series solutions (about $x_0 = 0$) of the differential equation $y'' + x^2 y = 0$.

Since $x_0 = 0$ is an ordinary point for this DE we can assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \Longrightarrow \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \Longrightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute this into the DE:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

and re-index the sums:

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=2}^{\infty} c_{n-2}x^n = 0$$

$$\Longrightarrow \; 2c_2 + 6c_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + c_{n-2}]x^n = 0$$

Every power series coefficient must be zero, which gives $c_2 = c_3 = 0$ and the recursion relation

$$c_{n+2} = \frac{-c_{n-2}}{(n+2)(n+1)} \quad \text{or equivalently} \quad c_n = \frac{-c_{n-4}}{n(n-1)}$$

The recursion relation gives $c_2 = c_3 = c_6 = c_7 = c_{10} = c_{11} = \cdots = 0$ and

$$c_4 = \frac{-c_0}{4 \cdot 3} \qquad c_5 = \frac{-c_1}{5 \cdot 4}$$

$$c_8 = \frac{-c_4}{8 \cdot 7} = \frac{c_0}{8 \cdot 7 \cdot 4 \cdot 3} \qquad c_9 = \frac{-c_5}{9 \cdot 8} = \frac{c_1}{9 \cdot 8 \cdot 5 \cdot 4}$$

$$\vdots \qquad \vdots$$

$$c_{4m} = \frac{(-1)^m c_0}{(4m)(4m-1)\cdots 8 \cdot 7 \cdot 4 \cdot 3} \qquad c_{4m+1} = \frac{(-1)^m c_1}{(4m+1)(4m)\cdots 9 \cdot 8 \cdot 5 \cdot 4}$$

So the general solution is

$$y = c_0 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)}{(4m)(4m-1)\cdots 8 \cdot 7 \cdot 4 \cdot 3} x^{4m}}_{y_1(x)} + c_1 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(4m+1)(4m)\cdots 9 \cdot 8 \cdot 5 \cdot 4} x^{4m+1}}_{y_2(x)}$$

Clearly, each of $y_1(x)$ and $y_2(x)$ is a power series solution of the given DE.