Thompson Rivers University

MATH 316
Differential Equations II

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MIDTERM EXAM #2
SOLUTIONS

27 March 2008 16:30–18:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
TOTAL:		40

/10 **Problem 1:** Consider the following Sturm-Liouville problem.

$$y'' + \lambda y = 0$$
$$y(0) = 0$$
$$y'(1) = 0$$

/7 (a) Find the eigenvalues λ_n and the corresponding eigenfunctions y_n for this problem.

3 cases: **i)** $\lambda = -\alpha^2 < 0 \implies y = Ae^{\alpha x} + Be^{-\alpha x}$

$$\begin{cases} y(0) = 0 \implies A + B = 0 \\ y'(1) = 0 \implies \alpha A e^{\alpha} - \alpha B e^{-\alpha} = 0 \end{cases} \implies A = B = 0 \implies y(x) = 0$$

ii) $\lambda = 0 \implies y = Ax + B$

$$\begin{cases} y(0) = 0 \implies B = 0 \\ y'(1) = 0 \implies A = 0 \end{cases} \implies y(x) = 0$$

iii) $\lambda = \alpha^2 > 0 \implies y = A\cos\alpha x + B\sin\alpha x$

$$\begin{cases} y(0) = 0 \implies A = 0 \\ y'(1) = 0 \implies \alpha B\cos\alpha = 0 \end{cases} \implies \alpha = \frac{\pi}{2} + n\pi;\ n = 0, 1, 2, \ldots$$

Therefore the eigenvalues are

$$\boxed{\lambda_n = \alpha^2 = (n + \tfrac{1}{2})^2\pi^2; \quad n = 0, 1, 2, \ldots}$$

and the corresponding eigenfunctions

$$\boxed{y_n(x) = \sin\left((n + \tfrac{1}{2})\pi x\right)}$$

/3 (b) Is $\lambda = 0$ an eigenvalue for this problem? If so, find the corresponding eigenfunction; if not, explain why.

No. For $\lambda = 0$ to be an eigenvalue, there would have to be non-trivial solutions of the boundary-value problem

$$\begin{cases} y'' + (0)y = 0 \\ y(0) = y'(1) = 0. \end{cases}$$

As already shown above, this problem has only the trivial solution, hence $\lambda = 0$ is not an eigenvalue.

/10 **Problem 2:** Consider the Legendre polynomials $P_n(x)$. Rodrigues' formula provides the following explicit representation:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

(a) Find $P_0(x)$, $P_1(x)$ and $P_2(x)$.

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$$P_0(x) = 1$$
$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$$
$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}(4x^3-4x) = \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1)$$

(b) Show that the set of functions $\{P_0, P_1, P_2\}$ is orthogonal with respect to $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

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We have:

$$\langle P_0, P_1\rangle = \int_{-1}^{1}(1)\cdot(x)\,dx = \tfrac{1}{2}x^2\Big|_{-1}^{1} = \tfrac{1}{2} - \tfrac{1}{2} = 0$$
$$\langle P_0, P_2\rangle = \int_{-1}^{1}(1)\cdot\tfrac{1}{2}(3x^2-1)\,dx = \tfrac{1}{2}(x^3-x)\Big|_{-1}^{1} = 0 - 0 = 0$$
$$\langle P_1, P_2\rangle = \int_{-1}^{1}(x)\cdot\tfrac{1}{2}(3x^2-1)\,dx = \tfrac{1}{2}\int_{-1}^{1}3x^3 - x\,dx = \tfrac{1}{2}(\tfrac{3}{4}x^4 - \tfrac{1}{2}x^2)\Big|_{-1}^{1} = \tfrac{1}{8} - \tfrac{1}{8} = 0$$

Thus $\{P_0, P_1, P_2\}$ is an orthogonal set, since $\langle P_i, P_j\rangle = 0\ \ \forall i, j = 0, 1, 2,\ i \neq j$.

(c) The infinite set $\{P_0, P_1, P_2, \ldots\}$ is an orthogonal basis for the vector space of continuous functions on $[-1, 1]$. That is, for any given continuous $f(x)$ on $[-1, 1]$, we can expand $f(x)$ in a series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

Find the first three terms in this expansion for the function $f(x) = x^3$.

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As for any orthogonal basis, for any $m = 0, 1, 2, \ldots$ we have

$$x^3 = \sum c_n P_n(x) \implies \langle x^3, P_m\rangle = \left\langle \sum c_n P_n, P_m\right\rangle = \sum c_n\langle P_n, P_m\rangle = c_m\langle P_m, P_m\rangle$$
$$\implies c_m = \frac{\langle x^3, P_m\rangle}{\langle P_m, P_m\rangle}$$

so

$$c_0 = \frac{\langle x^3, P_0\rangle}{\langle P_0, P_0\rangle} = \frac{\int_{-1}^{1}(x^3)\cdot(1)\,dx}{\int_{-1}^{1}(1)^2\,dx} = 0 \quad \text{(by oddness)}$$

$$c_1 = \frac{\langle x^3, P_1\rangle}{\langle P_1, P_1\rangle} = \frac{\int_{-1}^{1}(x^3)\cdot(x)\,dx}{\int_{-1}^{1}(x)^2\,dx} = \frac{\frac{1}{5}x^5\Big|_{-1}^{1}}{\frac{1}{3}x^3\Big|_{-1}^{1}} = \frac{2/5}{2/3} = \frac{3}{5}$$

$$c_2 = \frac{\langle x^3, P_2\rangle}{\langle P_2, P_2\rangle} = \frac{\int_{-1}^{1}(x^3)\cdot\frac{1}{2}(3x^2-1)\,dx}{\int_{-1}^{1}\left(\frac{1}{2}(3x^2-1)\right)^2 dx} = \frac{\frac{1}{2}\int_{-1}^{1}(3x^5-x^3)\,dx}{\int_{-1}^{1}\frac{1}{4}(3x^2-1)^2\,dx} = 0 \quad \text{(by oddness)}$$

(You can easily check that in fact $x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$.)

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Problem 3: Find the Fourier series for the function

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ x^2 & \text{if } 0 \le x < 1 \end{cases}$$

and sketch the graph, on the interval $[-3, 3]$, of the function to which this Fourier series converges.

We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

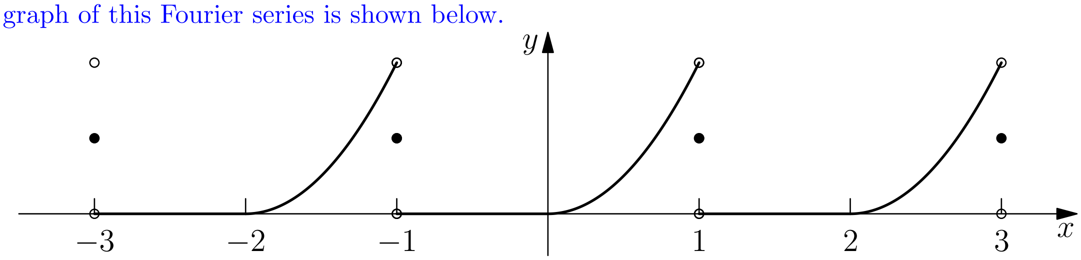
$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\,\cos n\pi x\,dx = \int_0^1 x^2 \cos n\pi x\,dx \\ &= \frac{x^2}{n\pi}\sin n\pi x\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x \sin n\pi x\,dx \\ &= 0 - 0 + \frac{x}{n^2\pi^2}\cos n\pi x\Big|_0^1 - \frac{2}{n^2\pi^2}\int_0^1 \cos n\pi x\,dx \\ &= \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{2}{n^3\pi^3}\sin n\pi x\Big|_0^1 \\ &= \frac{(-1)^n}{n^2\pi^2} \end{aligned}$$

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\,\sin n\pi x\,dx = \int_0^1 x^2 \sin n\pi x\,dx \\ &= -\frac{x^2}{n\pi}\cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x \cos n\pi x\,dx \\ &= -\frac{(-1)^n}{n\pi} + \frac{2x}{n^2\pi^2}\sin n\pi x\Big|_0^1 - \frac{2}{n^2\pi^2}\int_0^1 \sin n\pi x\,dx \\ &= -\frac{(-1)^n}{n\pi} + 0 - 0 + \frac{2}{n^3\pi^3}\cos n\pi x\Big|_0^1 \\ &= -\frac{(-1)^n}{n\pi} + \frac{2}{n^3\pi^3}\left[(-1)^n - 1\right] \end{aligned}$$

so the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty}\left(\frac{2}{n^3\pi^3}\left[(-1)^n - 1\right] - \frac{(-1)^n}{n\pi}\right)\sin n\pi x$$

The graph of this Fourier series is shown below.



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Problem 4: Use separation of variables to find the general solution $u(x,t)$ of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
$$u(0,t) = u(1,t) = 0.$$

$$u(x,t) = X(x)T(t) \implies X''T = XT'$$
$$\implies \frac{X''}{X} = \frac{T'}{T} = \lambda$$

As we have seen many times before, the Sturm-Liouville problem for $X(x)$ gives:

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \implies \begin{cases} \lambda_n = -n^2\pi^2; \;\; n = 1, 2, \dots \\ X(x) = A_n \sin n\pi x \end{cases}$$

The DE for $T(t)$ gives:

$$T' = \lambda_n T \implies T(t) = Ce^{\lambda_n t} = Ce^{-n^2\pi^2 t}$$

Putting these together we have a family of linearly independent solutions:

$$u(x,t) = X(x)T(t) = B_n e^{-n^2\pi^2 t} \sin n\pi x$$

so the most general solution is a superposition:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin n\pi x$$

where $B_1, B_2, \dots \in \mathbb{R}$ are arbitrary constants.