Thompson Rivers University

MATH 316
Differential Equations II

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MIDTERM EXAM #1
SOLUTIONS

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PROBLEM	GRADE	OUT OF
1		10
2		4
3		6
4		10
5		8
6		8
TOTAL:		46

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

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Problem 1: Consider the differential equation

$$y'' - x^2 y' - xy = 0.$$

(a) For this equation, is $x = 0$ an ordinary point or a singular point? Explain.

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We have $y'' - P(x)y' - Q(x)y = 0$, where $P(x) = x^2$ and $Q(x) = x$ are both analytic functions at $x = 0$. So $x = 0$ is an *ordinary point*.

(b) Find the general solution of this equation, represented as a power series about $x = 0$. (Your answer should include a general formula for the n'th coefficient in the series).

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$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Subbing into the DE:

$$\sum_2 n(n-1)c_n x^{n-2} - \sum_1 n c_n x^{n+1} - \sum_0 c_n x^{n+1} = 0$$

$$\text{re-index: } \Longrightarrow \sum_0 (n+2)(n+1)c_{n+2}x^n - \sum_2 (n-1)c_{n-1}x^n - \sum_1 c_{n-1}x^n = 0$$

$$\Longrightarrow 2c_2 + (3\cdot 2)c_3 x - c_0 x + \sum_2 \big[(n+2)(n+1)c_{n+2} - (n-1)c_{n-1} - c_{n-1}\big]x^n = 0$$

gives the recurrence relations:

$$\begin{cases} c_2 = 0 \\ c_3 = \dfrac{1}{3\cdot 2}c_0 \\ c_{n+2} = \dfrac{n}{(n+2)(n+1)}c_{n-1} \quad (n = 2, 3, 4, \dots) \end{cases}$$

so that

$$c_2 = c_5 = c_8 = \cdots = c_{3k+2} = 0 \quad \forall k = 0, 1, 2, \dots$$
$$c_3 = \frac{1}{3\cdot 2}c_0$$
$$c_4 = \frac{2}{4\cdot 3}c_1$$
$$c_6 = \frac{4}{6\cdot 5}c_3 = \frac{4}{6\cdot 5\cdot 3\cdot 2}c_0$$
$$c_7 = \frac{5}{7\cdot 6}c_4 = \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}c_1$$

In general the coefficients are

$$c_{3k} = \frac{4\cdot 7\cdot 10\cdots(3k-2)}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)(3k)}c_0 \qquad c_{3k+1} = \frac{2\cdot 5\cdot 8\cdots(3k-1)}{3\cdot 4\cdot 6\cdot 7\cdots(3k)(3k+1)}c_1 \qquad c_{3k+2} = 0$$

so the general solution is

$$y(x) = c_0\left[1 + \sum_{k=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3k-2)}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)(3k)}x^{3k}\right] + c_1 \sum_{k=0}^{\infty} \frac{2\cdot 5\cdot 8\cdots(3k-1)}{3\cdot 4\cdot 6\cdot 7\cdots(3k)(3k+1)}x^{3k+1}$$

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Problem 2: Classify the singular points of the equation

$$(x^2-1)^2y'' + (x+1)y' - y = 0.$$

$$y'' + P(x)y' + Q(x)y = 0$$

where

$$P(x) = \frac{x+1}{(x^2-1)^2} = \frac{x+1}{(x+1)^2(x-1)^2} = \frac{1}{(x+1)(x-1)^2}$$
$$Q(x) = -\frac{1}{(x+1)^2(x-1)^2}$$

Both P and Q are analytic functions except at $x = \pm 1$, which are the singular points.

Case $x = 1$:

$(x-1)P(x) = \dfrac{1}{(x+1)(x-1)}$ is *not* analytic at $x = 1$, so $x = 1$ is an *irregular* singular point.

Case $x = -1$:

$$(x+1)P(x) = \frac{1}{(x-1)^2} \quad \text{and} \quad (x+1)^2Q(x) = -\frac{1}{(x-1)^2}$$

are both analytic at $x = -1$, so $x = -1$ is a *regular* singular point.

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Problem 3: Use the *definition* of the Laplace transform $\mathcal{L}$ to find $\mathcal{L}\{f\}$ where

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 5 \\ 0 & \text{if } 5 \le t < 10 \\ e^{4t} & \text{if } t \ge 10 \end{cases}$$

$$\begin{aligned}
\mathcal{L}\{f\} &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \int_0^5 2e^{-st}\,dt + \int_{10}^\infty e^{4t}e^{-st}\,dt \\
&= \left[-\frac{2}{s}e^{-st}\right]_{t=0}^{5} + \left[-\frac{1}{4-s}e^{(4-s)t}\right]_{t=10}^{\infty} \\
&= \boxed{\frac{2}{s} - \frac{2}{s}e^{-5s} + \frac{1}{4-s}e^{10(4-s)}}
\end{aligned}$$

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Problem 4: Consider the differential equation

$$(x+2)x^2y'' - xy' + (1+x)y = 0.$$

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(a) Show that $x = 0$ is a regular singular point for this equation.

$$y'' - \underbrace{\frac{x}{(x+2)x^2}}_{P(x)} y' + \underbrace{\frac{1+x}{(x+2)x^2}}_{Q(x)} y = 0$$

$P(x)$, $Q(x)$ are not analytic at $x = 0$, so this is a singular point. We have $xP(x) = 1/(x+2)$ and $x^2Q(x) = (1+x)/(x+2)$ both analytic at $x = 0$, so $x = 0$ is a *regular* singular point.

(b) Find the general solution, represented as a series about $x = 0$. (Find the *first four* terms in each of the two linearly independent solutions; *do not* attempt to find a general formula for the n'th term.)

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$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

Subbing into the DE:

$$\sum_0 (n+r)(n+r-1)c_n x^{n+r+1} + \sum_0 2(n+r)(n+r-1)c_n x^{n+r} - \sum_0 (n+r)c_n x^{n+r} + \sum_0 c_n x^{n+r} + \sum_0 c_n x^{n+r+1} = 0$$

$$\implies \sum_1 (n+r-1)(n+r-2)c_{n-1}x^{n+r} + \sum_0 2(n+r)(n+r-1)c_n x^{n+r} - \sum_0 (n+r)c_n x^{n+r} + \sum_0 c_n x^{n+r} + \sum_1 c_{n-1}x^{n+r} = 0$$

$$\implies c_0\Big[2r(r-1) - r + 1\Big]x^r + x^r \sum_1 \Big[(n+r-1)(n+r-2)c_{n-1} + 2(n+r)(n+r-1)c_n - (n+r)c_n + c_n + c_{n-1}\Big]x^n = 0$$

The indicial equation gives

$$2r^2 - 3r + 1 = 0 \implies \boxed{r = 1,\ \frac{1}{2}.}$$

Case $r = 1$:

$$n(n-1)c_{n-1} + 2(n+1)nc_n - (n+1)c_n + c_n + c_{n-1} = 0 \implies c_n = -\frac{n(n-1)+1}{2(n+1)n - n}c_{n-1} = -\frac{n^2-n+1}{n(2n+1)}c_{n-1}$$

$$\therefore\ c_1 = -\frac{1}{3}c_0, \quad c_2 = -\frac{3}{10}c_1 = \frac{1}{10}c_0, \quad c_3 = -\frac{7}{21}c_2 = -\frac{1}{30}c_0$$

$$\therefore\ \boxed{y_1(x) = x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \cdots}$$

Case $r = \frac{1}{2}$:

$$(n-\tfrac{1}{2})(n-\tfrac{3}{2})c_{n-1} + 2(n+\tfrac{1}{2})(n-\tfrac{1}{2})c_n - (n+\tfrac{1}{2})c_n + c_n + c_{n-1} = 0 \implies c_n = -\frac{(n-\frac{1}{2})(n-\frac{3}{2})+1}{(2n+1)(n-\frac{1}{2}) - n + \frac{1}{2}}c_{n-1}$$

$$\therefore\ c_1 = -\frac{3}{4}c_0, \quad c_2 = -\frac{7}{24}c_1 = \frac{7}{32}c_0, \quad c_3 = -\frac{16}{90}c_2 = -\frac{7}{180}c_0$$

$$\therefore\ \boxed{y_2(x) = x^{1/2} - \frac{3}{4}x^{3/2} + \frac{7}{24}x^{5/2} - \frac{7}{180}x^{7/2} + \cdots}$$

The general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

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Problem 5: Use the method of Laplace transforms to solve the following initial value problem:

$$y''(t) + y'(t) = t^2$$
$$y(0) = -1$$
$$y'(0) = 0$$

Version 1 ($y'' + y = t^2$)**:**

$$(s^2 Y + s) + Y = \frac{2}{s^3} \implies (s^2+1)Y(s) = \frac{2}{s^3} - s$$
$$\implies Y(s) = \frac{2}{s^3(s^2+1)} - \frac{s}{s^2+1}$$

Partial fractions:

$$\frac{2}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Dx+E}{s^2+1}$$
$$= \frac{As^2(s^2+1) + Bs(s^2+1) + C(s^2+1) + (Dx+E)s^3}{s^3(s^2+1)}$$

$$s = 0 \implies C = 2 \implies A = -2 \implies D = 2; \quad B = 0 \implies E = 0$$

$$\therefore \ y(t) = \mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{2}{s^3} + \frac{s}{s^2+1}\right\}$$

$$= \boxed{-2 + t^2 + \cos t}$$

Version 2 ($y'' + y' = t^2$)**:**

$$(s^2 Y + s) + (sY + 1) = \frac{2}{s^3} \implies s(s+1)Y(s) = \frac{2}{s^3} - (s+1)$$
$$\implies Y(s) = \frac{2}{s^4(s+1)} - \frac{1}{s}$$

Partial fractions:

$$\frac{2}{s^4(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s+1}$$
$$= \frac{As^3(s+1) + Bs^2(s+1) + Cs(s+1) + D(s+1) + Es^4}{s^4(s+1)}$$

$$s = -1 \implies E = 2 \implies A = -2; \qquad s = 0 \implies D = 2 \implies C = -2 \implies B = 2$$

$$\therefore \ y(t) = \mathcal{L}^{-1}\left\{-\frac{3}{s} + \frac{2}{s^2} - \frac{2}{s^3} + \frac{2}{s^4} + \frac{2}{s+1}\right\}$$

$$= \boxed{-3 + 2t - t^2 + \tfrac{1}{3}t^3 + 2e^{-t}}$$

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Problem 6: Let

$$g(t) = \begin{cases} 2, & t < 1 \\ 0, & t \geq 1. \end{cases}$$

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(a) Express $g(t)$ in terms of unit step functions, and find its Laplace transform.

$$\boxed{g(t) = 2 - 2u(t-1)} \implies \boxed{G(s) = \mathcal{L}\{g\} = \frac{2}{s} - \frac{2}{s}e^{-s}}$$

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(b) Solve the following initial value problem, with $g(t)$ as given above.

$$y' + 3y = g(t) + \delta(t-2)$$
$$y(0) = y'(0) = 0.$$

Laplace transform:

$$sY + 3Y = \frac{2}{s} - \frac{2}{s}e^{-s} + e^{-2s} \implies Y(s) = \underbrace{\frac{2}{s(s+3)}}_{F(s)} - \underbrace{\frac{2}{s(s+3)}}_{F(s)} e^{-s} + \underbrace{\frac{1}{s+3}}_{H(s)} e^{-2s}$$

$$\therefore \quad y(t) = f(t) - f(t-1)u(t-1) + h(t-2)u(t-2)$$

where we have

$$F(s) = \frac{2}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{A(s+3) + Bs}{s(s+3)} \implies A = \frac{2}{3}, \; B = -\frac{2}{3}$$

so that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{2/3}{s} - \frac{2/3}{s+3}\right\} = \frac{2}{3} - \frac{2}{3}e^{-3t} = \tfrac{2}{3}\left(1 - e^{-3t}\right)$$

$$\text{and} \quad h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}.$$

So finally,

$$\boxed{y(t) = \tfrac{2}{3}\left(1 - e^{-3t}\right) - \tfrac{2}{3}\left(1 - e^{-3(t-1)}\right)u(t-1) + e^{-3(t-2)}u(t-2)}$$