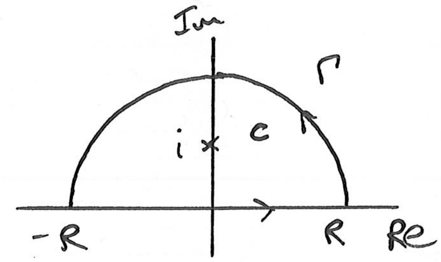
Name: SOLUTIONS Student #: ______ /10

MATH 3000: Quiz #5



/5 **Problem 1:** Evaluate p.v. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$.

poles $z = \pm i$ (order 2)

$$2\pi \mathrm{Res}(i) = \int_\Gamma f\,dz = \int_{-R}^{R} f(x)\,dx + \int_C f\,dz$$

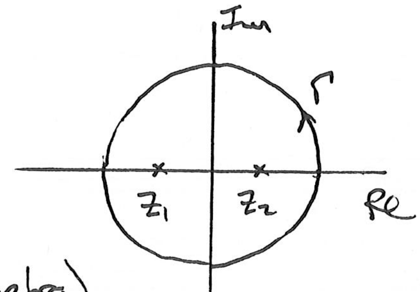
letting $R \to \infty$: $\quad \text{p.v.} \int_{-\infty}^{\infty} f\,dx \quad + \quad 0$ (by "The Lemma" since $\deg(x^2+1)^2 \geq \deg(x^2)+2$)

$$\text{So p.v.} \int_{-\infty}^{\infty} f\,dx = 2\pi i\,\mathrm{Res}(i) = 2\pi i \lim_{z\to i} \overset{d/dz}{\vee}(z-i)^2 f(z)$$

$$= 2\pi i \lim_{z\to\infty} \frac{d}{dz}\frac{z^2}{(z+i)^2} = 2\pi i \lim_{z\to i} \frac{2z(z+i)^2 - z^2\cdot 2(z+i)}{(z+i)^4}$$

$$= 2\pi i \cdot \frac{2i(2i)^2 - i^2\cdot 2(2i)}{(2i)^4}$$

$$= 2\pi i \cdot \frac{-8i+4i}{16} = \boxed{\frac{\pi}{2}}$$



/5 **Problem 2:** Evaluate $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$.

$z = e^{i\theta}$, $d\theta = -i\frac{dz}{z}$

$$= \int_\Gamma \frac{-i\,dz/z}{1+\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2} = \int_\Gamma \frac{4iz}{z^4-6z^2+1}\,dz \quad \text{(after some algebra)}$$

poles: $z^4 - 6z^2 + 1 = 0 \to z = \pm\sqrt{3 \pm 2^{3/2}} \to$ only $z_1, z_2 = \pm\sqrt{3-2^{3/2}}$ are inside Γ.

$$\to \int_\Gamma f\,dz = 2\pi i\,[\mathrm{Res}(z_1) + \mathrm{Res}(z_2)]$$

$$= 2\pi i\left(\lim_{z\to z_1} \frac{4iz}{(z-z_2)(z-z_3)(z-z_4)} + \lim_{z\to z_2}\frac{4iz}{(z-z_1)(z-z_3)(z-z_4)}\right)$$

$$= 2\pi i\left(\frac{4iz_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{4iz_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}\right)$$

$= \ldots$ (more tedious than I intended!)