MATH 3000: Quiz #3 – SOLUTIONS

/5 **Problem 1:** Suppose a function f is analytic in a domain D, and that $\operatorname{Re} f = \operatorname{Im} f$ everywhere in D. Show that f must be constant in D.

Writing

$$f(x+iy) = u(x,y) + iv(x,y),$$

the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

must hold everywhere in D since f is analytic. But

$$\operatorname{Re} f = \operatorname{Im} f \implies u = v \implies \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

Adding these two equations gives

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$$

everywhere in D. This implies $u(x,y) = C$ is constant, hence $f(z) = C + iC$ is constant in D.

/5 **Problem 2:** Let $f(z) = \operatorname{Re} z + \operatorname{Im} z$. Use the definition of $f'(z)$ to show that f is nowhere differentiable. (It might help to write $f(x+iy) = x + y$.)

If $f'(x+iy)$ exists then both

$$\begin{aligned} f'(x+iy) &= \lim_{\Delta x \to 0} \frac{f(x+iy+\Delta x) - f(x+iy)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x+y) - (x+y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \end{aligned}$$

and

$$\begin{aligned} f'(x+iy) &= \lim_{\Delta y \to 0} \frac{f(x+iy+i\Delta y) - f(x+iy)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{(x+y+\Delta y) - (x+y)}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{i\Delta y} = -i. \end{aligned}$$

These are inconsistent, therefore $f'(z)$ does not exist for any z.