MATH 3000: Quiz #2 – SOLUTIONS

/5 **Problem 1:** Find all four complex roots of the equation $z^4 + 16 = 0$.

$$z^4 = -16 = 16e^{i(\pi + n2\pi)} \quad (n \in \mathbb{Z})$$

$$\implies z = \left(16e^{i(\pi+n2\pi)}\right)^{1/4}$$

$$= 16^{1/4} e^{i(\pi+n2\pi)/4}$$

$$= 2e^{i(\pi/4+n\pi/2)}$$

$$n = 0: \quad z = 2e^{i\pi/4} = 2(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}) = \boxed{\sqrt{2} + i\sqrt{2}}$$

$$n = 1: \quad z = 2e^{i3\pi/4} = \boxed{-\sqrt{2} + i\sqrt{2}}$$

$$n = 2: \quad z = 2e^{i5\pi/4} = \boxed{-\sqrt{2} - i\sqrt{2}}$$

$$n = 3: \quad z = 2e^{i7\pi/4} = \boxed{\sqrt{2} - i\sqrt{2}}$$

Or more compactly:

$$\boxed{z = \sqrt{2}(\pm 1 \pm i)}$$

/5 **Problem 2:** Consider the complex function $f(z) = z^2$. Find $f(S)$ where S is the hyberbola

$$S = \{x + iy \in \mathbb{C} : \ xy = 1\}.$$

If $xy = 1$ then

$$w = f(z) = f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2\underbrace{xy}_{1}i = x^2 - \frac{1}{x^2} + 2i.$$

So $f(S)$ is given parametrically as

$$f(S) = \{x^2 - \frac{1}{x^2} + 2i : \ x \in \mathbb{R}\}$$

This is clearly on the line $\text{Re}(w) = 2$. Since $\text{Range}\{x^2 - \frac{1}{x^2}\} = (-\infty, \infty)$, $f(S)$ is in fact the whole line:

$$\boxed{f(S) = \{w \in \mathbb{C} : \ \text{Im}(w) = 2\}}$$