THOMPSON RIVERS UNIVERSITY

MATH 3000
Complex Variables

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

23 Nov 2015   14:30–16:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 6 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		14
2		5
3		3
4		7
5		8
6		8
TOTAL:		45

/14

Problem 1: Evaluate the following.

/4 (a) $\displaystyle\int_\Gamma (1+\bar{z})\,dz$ where Γ is the semi-circle $z = 1 + e^{i\theta}$ $(0 \le \theta \le \pi)$.

$$\begin{aligned} z &= 1 + e^{i\theta} \\ dz &= ie^{i\theta}\,d\theta \end{aligned} \implies \int_\Gamma (1+\bar{z})\,dz = \int_0^\pi (2+e^{-i\theta})ie^{i\theta}\,d\theta$$

$$= \int_0^\pi (2ie^{i\theta} + i)\,d\theta$$

$$= 2e^{i\theta} + i\theta\Big|_0^\pi$$

$$= (2e^{i\pi} + i\pi) - (2) = -2 + i\pi - 2 = \boxed{i\pi - 4}$$

/3 (b) $\text{Res}(f;0)$ where $f(z) = \dfrac{1}{z+z^2}$.

f has a simple pole at $z=0$ so

$$\text{Res}(f;0)] = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{1}{1+z} = \boxed{1}$$

/3 (c) $\displaystyle\int_C \frac{dz}{z^2}$ where C is the positively-oriented unit circle.

Since $f(z) = z^{-2}$ has an antiderivative $F(z) = -z^{-1}$ throughout a domain D that contains Γ (in fact $D = \mathbb{C}\backslash\{0\}$) we have

$$\int_C f(z)\,dz = F(z)\Big|_1^1 = \boxed{0}.$$

Alternatively,

$$\int_C f(z)\,dz = 2\pi i \text{Res}(0) = 2\pi i \lim_{z\to 0}\frac{d}{dz}z^2 f(z) = 2\pi i \lim_{z\to 0}\frac{d}{dz}1 = 2\pi i \lim_{z\to 0} 0 = 0.$$

/4 (d) The residues at all the poles of $f(z) = \dfrac{e^z}{z^2+\pi^2}$.

Poles are roots of $z^2 + \pi^2 = 0 \implies z = \pm i\pi$. These are simple poles. Note that

$$f(z) = \frac{e^z}{(z+i\pi)(z-i\pi)}$$

so that

$$\text{Res}(i\pi) = \lim_{z\to i\pi}(z-i\pi)f(z) = \lim_{z\to i\pi}\frac{e^z}{z+i\pi} = \frac{e^{i\pi}}{i2\pi} = \frac{-1}{i2\pi} = \boxed{\frac{i}{2\pi}}$$

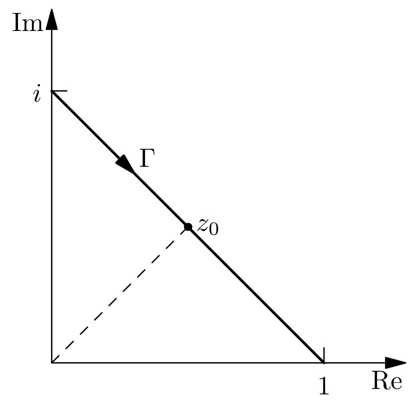
$$\text{Res}(-i\pi) = \lim_{z\to -i\pi}(z+i\pi)f(z) = \lim_{z\to -i\pi}\frac{e^z}{z-i\pi} = \frac{e^{-i\pi}}{-i2\pi} = \frac{-1}{-i2\pi} = \boxed{-\frac{i}{2\pi}}$$

/5

Problem 2: Let Γ be the line segment from $z = i$ to $z = 1$. Without evaluating the integral, show that

$$\left| \int_\Gamma \frac{dz}{z^4} \right| \le 4\sqrt{2}.$$

(Hint: of all the points on that line segment, the midpoint is closest to the origin.)



The midpoint of Γ is

$$z_0 = \frac{1+i}{2}$$

so on Γ we have

$$\left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} \le \frac{1}{|z_0|^4} = \left|\frac{1}{z_0}\right|^4 = \left|\frac{2}{1+i}\right|^4 = \frac{2^4}{|1+i|^4} = \frac{2^4}{2^2} = 4 = M.$$

Also, the length of Γ is

$$L = \sqrt{2}.$$

Therefore

$$\left| \int_\Gamma \frac{dz}{z^4} \right| \le M \cdot L = 4\sqrt{2}.$$

/3

Problem 3: Explain, without evaluating the integral, why $\int_C \tan z \, dz = 0$ if C is the circle $|z| = 1$.

The function

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

has poles at zeroes of $\cos z$, i.e. at $z = \frac{\pi}{2} + n\pi$ ($n \in \mathbb{Z}$). For each of these poles we have

$$|z| \ge \frac{\pi}{2} > 1$$

so all the poles are outside the unit circle. Since $\sin z$ and $\cos z$ are both analytic, we have that f is analytic everywhere in a simply connected domain (e.g. the circle $|z| < 1.1$) that contains Γ, so by Cauchy's Integral Theorem the integral is 0.

/7 **Problem 4:** Consider the integral $\displaystyle\int_C \frac{dz}{z^3(z+4)}$.

(a) Evaluate the integral when C is the circle $|z| = 2$.

/4

The function

$$f(z) = \frac{1}{z^3(z+4)}$$

has poles at $z = 0$ (order 3) and $z = -4$ (order 1). Only the pole at $z = 0$ is enclosed by C so

$$\int_C f\,dz = 2\pi i \text{Res}(0)$$

where, since the pole as zero has order 3,

$$\text{Res}(0) = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}z^3 f(z) = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{1}{z+4} = \frac{1}{2}\lim_{z\to 0}\frac{2}{(z+4)^3} = \frac{1}{64}$$

so that

$$\int_C f\,dz = 2\pi i\cdot\frac{1}{64} = \boxed{\frac{i\pi}{32}}$$

(b) Evaluate the integral when C is the circle $|z+2| = 3$.

/3

The given circle encloses both poles, so

$$\int_C f\,dz = 2\pi i[\text{Res}(0) + \text{Res}(-4)]$$

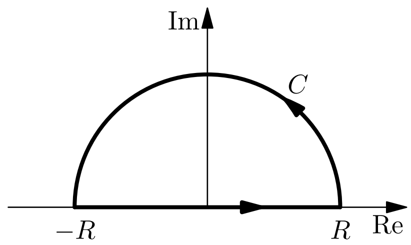
where, since $z = -4$ is a simple pole,

$$\text{Res}(-4) = \lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4}\frac{1}{z^3} = -\frac{1}{64}.$$

Therefore,

$$\int_C f\,dz = 2\pi i[\frac{1}{64} - \frac{1}{64}] = \boxed{0}$$

/8 **Problem 5:** Calculate the Cauchy principal value of $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$.



Let

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}.$$

Refer to the diagram above. Since the loop encloses only the (order 2) pole at $z = i$ we have

$$\begin{aligned}\left(\int_{[-R,R]} + \int_C\right) f\,dz &= 2\pi i \text{Res}(i) \\ &= 2\pi i \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) \\ &= 2\pi i \lim_{z\to i} \frac{d}{dz} \frac{1}{(z+i)^2} \\ &= 2\pi i \lim_{z\to i} \frac{-2}{(z+i)^3} = 2\pi i \cdot \frac{-2}{(2i)^3} = \frac{\pi}{2}.\end{aligned}$$

We have

$$\lim_{R\to\infty} \int_{[-R,R]} f\,dz = \text{p.v.} \int_{-\infty}^{\infty} f\,dx$$

and

$$\lim_{R\to\infty} \int_C f\,dz = 0$$

the "The Lemma" since $\deg(x^2+1)^2 \geq \deg(1) + 2$. Thus letting $R \to \infty$ in the equation above gives

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \boxed{\frac{\pi}{2}}$$

/8 **Problem 6:** Evaluate: $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

The substitution $z = e^{i\theta}$ (hence $d\theta = -i\dfrac{dz}{z}$) gives

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{-i\,dz/z}{5+\frac{4}{2i}(z-\frac{1}{z})} = \int_C f(z)\,dz$$

where C is the unit circle and (after a little algebra)

$$f(z) = \frac{i}{2iz^2-5z-2i}.$$

The function f has simple poles where

$$2iz^2-5z-2i=0 \implies z = \frac{5\pm\sqrt{25-16}}{4i} \implies \begin{array}{l} z_1=-i/2 \\ z_2=-2i \end{array}$$

so that

$$f(z) = \frac{i}{2i(z-z_1)(z-z_2)}.$$

Since only the pole at $z_1 = -i/2$ is inside C we have

$$\begin{aligned}\int_C f(z)\,dz = 2\pi i \mathrm{Res}(z_1) &= 2\pi i \lim_{z\to z_1}(z-z_1)f(z) \\ &= 2\pi i \lim_{z\to z_1} \frac{i}{2i(z-z_2)} \\ &= 2\pi i \cdot \frac{i}{2i(z_1-z_2)} \\ &= \frac{i\pi}{z_1-z_2} \\ &= \frac{i\pi}{-\frac{i}{2}+2i} = \boxed{\frac{2\pi}{3}}\end{aligned}$$