THOMPSON RIVERS UNIVERSITY

MATH 3000
Complex Variables

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MIDTERM EXAM #2
SOLUTIONS

18 November 2013 14:30–16:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		6
3		5
4		10
TOTAL:		31

/10 **Problem 1:** Consider the function $f(z) = \dfrac{1}{z(z^2+1)^2}$.

/7 (a) Obtain a partial fractions expansion of $f(z)$.

We can factor $f(z)$ as

$$f(z) = \frac{1}{z(z+i)^2(z-i)^2}$$

so the partial fractions expansion will have the form

$$f(z) = \frac{A}{z} + \frac{B}{(z+i)^2} + \frac{C}{z+i} + \frac{D}{(z-i)^2} + \frac{E}{z-i}.$$

The coefficients are given by

$$A = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{1}{(z^2+1)^2} = 1$$

$$B = \lim_{z\to -i} (z+i)^2 f(z) = \lim_{z\to -i} \frac{1}{z(z-i)^2} = \frac{1}{(-i)(-2i)^2} = -\frac{i}{4}$$

$$C = \lim_{z\to -i} \frac{d}{dz}(z+i)^2 f(z) = \lim_{z\to -i} \frac{d}{dz} \frac{1}{z(z-i)^2} = \lim_{z\to -i} -\frac{(z-i)^2 + 2z(z-i)}{[z(z-i)^2]^2}$$
$$= -\frac{(-2i)^2 + 2(-i)(-2i)}{[-i(-2i)^2]^2} = -\frac{1}{2}$$

$$D = \lim_{z\to i} (z-i)^2 f(z) = \lim_{z\to i} \frac{1}{z(z+i)^2} = \frac{1}{i(2i)^2} = \frac{i}{4}$$

$$E = \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) = \lim_{z\to i} \frac{d}{dz} \frac{1}{z(z+i)^2} = \lim_{z\to i} -\frac{(z+i)^2 + 2z(z+i)}{[z(z+i)^2]^2}$$
$$= -\frac{(2i)^2 + 2i(2i)}{[i(2i)^2]^2} = -\frac{1}{2}$$

So finally,

$$\boxed{f(z) = \frac{1}{z} + \frac{-i/4}{(z+i)^2} + \frac{-1/2}{z+i} + \frac{i/4}{(z-i)^2} + \frac{-1/2}{z-i}}$$

/3 (b) Evaluate $\oint_C f(z)\,dz$ where C is the circle or radius $\frac{1}{2}$ centered at the origin.

$$\oint_C f(z)\,dz = \oint_C \frac{A}{z}\,dz + \underbrace{\oint_C \frac{B}{(z+i)^2} + \oint_C \frac{C}{z+i} + \oint_C \frac{D}{(z-i)^2} + \oint_C \frac{E}{z-i}}_{\text{all 0 by Cauchy's Thm}}$$
$$= 2\pi i \cdot A = \boxed{2\pi i}$$

/6 **Problem 2:** Solve for z: $\operatorname{Log}(z^2-1) = i\frac{\pi}{2}$

Let $z^2 - 1 = w = re^{i\theta}$, then

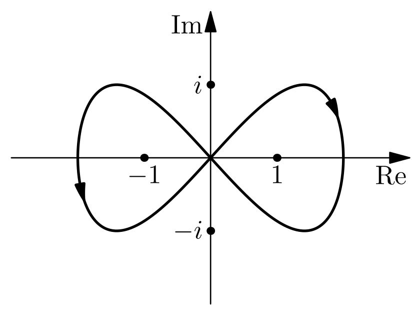
$$\operatorname{Log}(w) = \ln r + i\theta = i\frac{\pi}{2} \implies \begin{cases} \ln r = 0 \\ \theta = \frac{\pi}{2} \end{cases} \implies w = e^{i\pi/2} = i$$

so we have

$$z^2 - 1 = i \implies z^2 = 1 + i = \sqrt{2}e^{i\pi/4}$$

$$\implies z = \boxed{\pm 2^{1/4}e^{i\pi/8} = \pm 2^{1/4}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)}$$

/5 **Problem 3:** Let $\Gamma \subset \mathbb{C}$ be the figure-eight contour shown below. Evaluate: $\oint_\Gamma \frac{z}{z^2-2z+5}\,dz$.



The poles of $f(z) = \frac{z}{z^2-2z+5}$ are given by

$$z^2 - 2z + 5 = 0 \implies z = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

Except at these poles, f is analytic.

Note that Γ is *not* a simple closed curve, so we can't apply the Residue Theorem directly. But we can look at Γ as a sum of two loops. Let Γ_1, Γ_2 be the loops on the left/right, respectively. We have

$$\oint_\Gamma f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz.$$

Since f is analytic in a domain containing each of Γ_1, Γ_2, both integrals are zero (by the Residue Theorem).

Alternatively, note that there is a simply connected domain D, containing Γ, in which f is analytic (since the poles of f lie some distance from Γ). So the given integral is zero by Cauchy's Theorem: within D, Γ can be continuously deformed to a point.

/10 **Problem 4:** Evaluate: $\displaystyle\int_0^\infty \frac{x\sin x}{(x^2+1)^2}\,dx$

Because the integrand is an even function we have

$$\int_0^\infty \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{1}{2}\text{Im}\underbrace{\int_{-\infty}^\infty \frac{xe^{ix}}{(x^2+1)^2}\,dx}_{I}.$$

Now let $f(z) = \dfrac{ze^{iz}}{(z^2+1)^2}$ and consider

$$\int_{-R}^R f(x)\,dx + \int_C f(z)\,dz = 2\pi i \sum_j \text{Res}(f; z_j) \tag{1}$$

where C is semi-circle of radius R, centered at the origin, in the upper half-plane; the sum is taken over poles z_j of f inside this semi-circle. On C we have

$$\begin{aligned}|f(z)| = \frac{|z||e^{i(x+iy)}|}{|(z^2+1)^2|} = \frac{|z||e^{ix}||e^{-y}|}{|(z^2+1)^2|} &\le \frac{|z|}{|(z^2+1)^2|} \quad (\text{since } |e^{ix}| = 1 \text{ and } e^{-y} \le 1)\\ &\le \frac{|z|}{|z|^4/2} \quad (\text{since } |z^4+1| \ge |z|^4/2 \text{ for all } |z| \text{ suff. large})\\ &= \frac{2}{R^3}\end{aligned}$$

so that

$$\left|\int_C f(z)\,dz\right| \le \frac{2}{R^3}\cdot\frac{\pi}{R} = \frac{2\pi}{R^2} \to 0 \text{ as } R \to \infty.$$

Also, the poles of $f(z)$ are given by $z^2+1 = 0 \implies z = \pm i$. Only $z = i$ is enclosed by C. Thus, letting $R \to \infty$ in (1) gives

$$\begin{aligned}I = \int_{-\infty}^\infty f(z)\,dz &= 2\pi i \cdot \text{Res}(f; i)\\ &= 2\pi i \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) \quad (\text{since } i \text{ is a 2nd-order pole})\\ &= 2\pi i \lim_{z\to i} \frac{d}{dz}\frac{ze^{iz}}{(z+i)^2}\\ &= 2\pi i \lim_{z\to i} \frac{(e^{iz}+ize^{iz})(z+i)^2 - ze^{iz}\cdot 2(z+i)}{(z+i)^4}\\ &= 2\pi i \lim_{z\to i} \frac{(e^{iz}+ize^{iz})(z+i) - 2ze^{iz}}{(z+i)^3}\\ &= 2\pi i \lim_{z\to i} e^{iz}\frac{(1+iz)(z+i)-2z}{(z+i)^3} = 2\pi i\cdot e^{-1}\cdot\frac{1}{4} = \frac{\pi i}{2e}\end{aligned}$$

and so

$$\int_0^\infty \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{1}{2}\text{Im}\left(\frac{\pi i}{2e}\right) = \boxed{\frac{\pi}{4e}}$$