THOMPSON RIVERS UNIVERSITY

MATH 3000
Complex Variables

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MIDTERM EXAM #1
SOLUTIONS

26 Oct 2015 14:30–16:20

PROBLEM	GRADE	OUT OF
1		15
2		4
3		5
4		5
5		5
6		5
7		4
8		6
9		5
TOTAL:		54

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 7 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

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Problem 1: Evaluate the following. Express your answers in Cartesian form $(a + ib)$ and simplify as much as possible.

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(a) $\dfrac{(4-i)(1-3i)}{-1+2i}$

$$\frac{(4-i)(1-3i)}{-1+2i} = \frac{(4-i)(1-3i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{-27+11i}{5} = \boxed{-\frac{27}{5} + i\frac{11}{5}}$$

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(b) $\text{Im}\left(\dfrac{1}{x-iy}\right)$ (assume x, y are real)

$$\text{Im}\left(\frac{1}{x-iy}\right) = \text{Im}\left(\frac{x+iy}{(x-iy)(x+iy)}\right) = \text{Im}\left(\frac{x-iy}{x^2+y^2}\right) = \boxed{\frac{x}{x^2+y^2}}$$

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(c) $(-1+i)^7$

$$(-1+i)^7 = \left(\sqrt{2}e^{i3\pi/4}\right)^7 = 2^{7/2}e^{i21\pi/4} = 2^{7/2}e^{i5\pi/4} = 2^{7/2}\left[\cos(\tfrac{5\pi}{4}) + i\sin(\tfrac{5\pi}{4})\right]$$
$$= 2^{7/2}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = \boxed{-8-8i}$$

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(d) $\text{Log}(-1-i)$

$$\text{Log}(-1-i) = \log|-1-i| + i\text{Arg}(-1-i) = \log\sqrt{2} + i(-\frac{3\pi}{4}) = \boxed{\frac{1}{2}\log 2 - i\frac{3\pi}{4}}$$

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(e) 3^i

$$3^i = \left(e^{\ln 3}\right)^i = e^{i\ln 3} = \boxed{\cos(\ln 3) + i\sin(\ln 3)}$$

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Problem 2: Let z_1, z_2 be arbitrary complex numbers. Prove that $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number.

Let $z_1 = x + iy$, $z_2 = a + ib$. Then

$$\begin{aligned} z_1\overline{z_2} + \overline{z_1}z_2 &= (x+iy)(a-ib) + (x-iy)(a+ib) \\ &= (ax - ibx + iay + by) + (ax + ibx - iay + by) \\ &= 2ax + 2by \end{aligned}$$

is real.

Alternative solution:
We have

$$\begin{aligned} \operatorname{Im}(z_1\overline{z_2} + \overline{z_1}z_2) &= \frac{1}{2}\left(z_1\overline{z_2} + \overline{z_1}z_2 - \overline{z_1\overline{z_2} + \overline{z_1}z_2}\right) \\ &= \frac{1}{2}\left(z_1\overline{z_2} + \overline{z_1}z_2 - \overline{z_1}z_2 - z_1\overline{z_2}\right) = 0 \end{aligned}$$

hence $z_1\overline{z_2} + \overline{z_1}z_2$ is real.

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Problem 3: Find all the solutions of $z^{3/2} = 4\sqrt{2} + i4\sqrt{2}$.

In polar form the equation becomes

$$z^{3/2} = 4e^{i(\frac{\pi}{4} + k2\pi)} \quad (k \in \mathbb{Z})$$

$$\implies z = \left(4e^{i(\frac{\pi}{4}+k2\pi)}\right)^{2/3} = 8e^{i(\frac{\pi}{6}+k\frac{4\pi}{3})} \quad (k = 0, 1, 2)$$

$$\begin{aligned} k = 0: &\quad z_0 = 4e^{i\pi/6} = 4\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \boxed{2\sqrt{3} + 2i} \\ k = 1: &\quad z_1 = 4e^{i3\pi/2} = \boxed{-4i} \\ k = -1: &\quad z_2 = 4e^{-i7\pi/6} = \boxed{-2\sqrt{3} + 2i} \end{aligned}$$

Other k values just repeat these solutions.

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Problem 4: For $z = x + iy$ consider the function $f(z) = x^3 + iy^3$. On what subset of the complex numbers is this function differentiable? On what subset is it analytic?

We have $f(x+iy) = u(x,y) + iv(x,y)$ where $u = x^3$, $v = y^3$. The Cauchy-Riemann equations in this case read

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned} \iff \begin{aligned} 3x^2 &= 3y^2 \\ 0 &= 0 \end{aligned} \iff y = \pm x$$

Away the lines $y = \pm x$ the first of the Cauchy-Riemanns does not hold, so f is not differentiable. In fact we have that

$$\boxed{f \text{ is differentiable only on the lines } y = \pm x}$$

since the Cauchy-Riemann equations are satisfied on these lines and the various partials are all continuous.

But the lines $y = \pm x$ do not contain an open subset of $\mathbb{C}$, hence

$$\boxed{f \text{ is analytic nowhere.}}$$

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Problem 5: Consider a function $f(z)$ given by $f(x+iy) = +iv(x,y)$. Find a function v such that f is entire.

We have $f(x+iy) = u(x,y) + iv(x,y)$ where $u = xy^3 - x^3y$. Since f is entire, the Cauchy-Riemann equations must hold:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned} \iff \begin{aligned} \frac{\partial v}{\partial y} &= y^3 - 3x^2y \\ \frac{\partial v}{\partial x} &= -3xy^2 + x^3 \end{aligned}$$

Taking an anti-derivative with respect to y in the first equation gives

$$v = \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + C(x)$$

where $C(x)$ is an arbitrary function of x only. The second equation then gives

$$\begin{aligned} -3xy^2 + C'(x) = -3xy^2 + x^3 &\implies C'(x) = x^3 \\ &\implies C(x) = \frac{1}{4}x^4 \\ &\implies \boxed{v(x,y) = \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}x^4} \end{aligned}$$

Now f is entire since u, v satisfy the Cauchy-Riemann equations and have continuous partials everywhere.

/5 **Problem 6:** Prove that $f(z) = \dfrac{z}{\bar{z}+1}$ is differentiable at $z = 0$.

Note that

$$f(0) = \frac{0}{\bar{0}+1} = 0.$$

The definition of $f'(0)$ gives

$$\begin{aligned} f'(0) &= \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\dfrac{\Delta z}{\overline{\Delta z}+1} - 0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\overline{\Delta z}+1} \\ &= \frac{1}{0+1} = 1 \quad \text{(by continuity of } \bar{z} \text{ and } 1/(z+1)\text{).} \end{aligned}$$

Since this limit exists $f(z)$ is differentiable at $z = 0$.

/4 **Problem 7:** The function $f(z) = \dfrac{2z^3+3}{z^3(z+1)}$ has the partial fractions expansion

$$f(z) = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1}$$

where A, B, C, D are (complex) constants. Calculate B.

$$\begin{aligned} B &= \lim_{z \to 0} \frac{d}{dz} z^3 f(z) \\ &= \lim_{z \to 0} \frac{d}{dz} \frac{2z^3+3}{z+1} \\ &= \lim_{z \to 0} \frac{(6z)(z+1) - (2z^3+3)(1)}{(z+1)^2} \qquad \text{(quotient rule)} \\ &= \lim_{z \to 0} \frac{(6 \cdot 0)(0+1) - (2 \cdot 0^3+3)(1)}{(0+1)^2} \qquad \text{(by continuity at 0)} \\ &= \boxed{-3} \end{aligned}$$

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Problem 8: Sketch the following sets in the complex plane:

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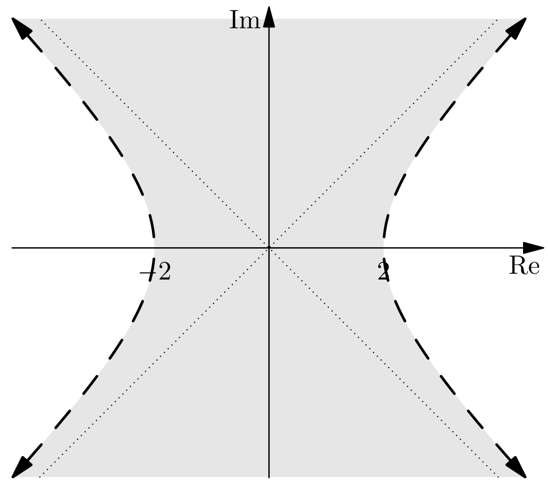
(a) $\mathrm{Re}(z^2) < 4$

$$\mathrm{Re}\big((x+iy)^2\big) = \mathrm{Re}(x^2 - y^2 + 2xyi) = x^2 - y^2 > 4$$

The equality $x^2 - y^2 = 4$ gives a pair of hyperbolas:



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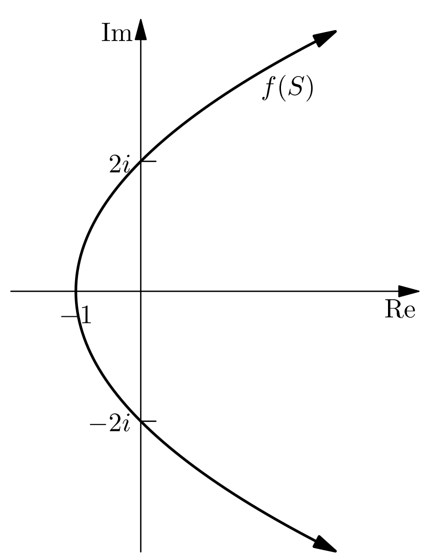
(b) $f(S)$ where $f(z) = z^2$ and $S = \{z \in \mathbb{C} : \mathrm{Im}(z) = 1\}$

We can parametrize S as $z(t) = t + i$; $t \in \mathbb{R}$. Then on $f(S)$ we have

$$w = f(z) = (t+i)^2 = \underbrace{(t^2-1)}_{u} + i\underbrace{(2t)}_{v} \iff u = \left(\frac{v}{2}\right)^2 - 1$$

so $f(S)$ is a parabola:



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Problem 9: Derive a trigonometric identity expressing $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

$$\begin{aligned}\cos 4\theta &= \mathrm{Re}[e^{i4\theta}] \\ &= \mathrm{Re}[(e^{i\theta})^4] \\ &= \mathrm{Re}[(\cos\theta + i\sin\theta)^4] \\ &= \mathrm{Re}[\cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4] \\ &= \boxed{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}\end{aligned}$$

Note that taking the imaginary part gives the complementary identity:

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$