THOMPSON RIVERS UNIVERSITY

MATH 3000
Complex Variables

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MIDTERM EXAM #1
SOLUTIONS

9 October 2013 15:30–16:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		5
3		5
4		5
5		6
6		8
TOTAL:		34

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Problem 1: Let L be the line in the complex plane that passes through the origin and through $z_0 = 3 + i$. Find and sketch the image of L under the transformation $f(z) = \dfrac{1}{z}$.

Parametrize L:

$$z = (3 + i)t, \quad t \in \mathbb{R}.$$

Then $f(L)$ is given parametrically by

$$w = f(z) = \frac{1}{z} = \frac{1}{(3 + i)t} = \frac{1}{t} \cdot \frac{3 - i}{10}, \quad t \in \mathbb{R}.$$

Let $s = 1/(10t)$; then

$$w = (3 - i)s, \quad s \in \mathbb{R} \setminus \{0\}.$$

This describes the line through the origin and $w_0 = 3 - i$, minus the point at the origin.

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Problem 2: Let $S \subset \mathbb{C}$ be the quarter-circle with radius 2 (centered at the origin) that lies within the first quadrant ($\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$). Find and sketch the image of S under the transformation $g(z) = \dfrac{1}{z^2}$.

Parametrize S:

$$z = 2e^{i\theta}, \quad \theta \in (0, \pi/2).$$

Then $g(S)$ is given parametrically by

$$w = g(z) = \frac{1}{z^2} = \frac{1}{(2e^{i\theta})^2} = \frac{1}{4}e^{-i2\theta}, \theta \in (0, \pi/2).$$

Let $\alpha = -2\theta$; then

$$w = \frac{1}{4}e^{i\alpha}, \quad \alpha \in (0, -\pi).$$

This describes the half-circle with radius 1/4 (centered at the origin) that lies within the lower half-plane.

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Problem 3: Show that $\overline{e^z} = e^{\bar{z}}$ for all $z \in \mathbb{C}$.

Let $z = x + iy$, then

$$(1): \quad \overline{e^z} = \overline{e^{x+iy}} = \overline{e^x e^{iy}} = \overline{e^x(\cos x + i\sin y)} = \overline{e^x} \cdot \overline{\cos x + i \sin y} = e^x(\cos x - i\sin y)$$

$$(2): \quad e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos y - i\sin y)$$

By inspection these agree for all z.

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Problem 4: Prove that $f(z) = \operatorname{Arg} z$ is nowhere analytic. (Recall that $\operatorname{Arg} z = \theta$ for $z = |z|e^{i\theta}$).

the easy way:
Suppose f is analytic in some domain D. Since $\operatorname{Im} f = 0$ is constant in D, he have that $\operatorname{Re} f$ is also constant in D (by a result from the homework). But clearly $\operatorname{Re} f(z) = \operatorname{Arg} z$ is *not* constant in any open set in $\mathbb{C}$ ... a contradiction. Hence f is nowhere analytic.

a harder way:
We have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\operatorname{Arg}(z + \Delta z) - \operatorname{Arg}(z)}{\Delta z}.$$

Write $z = |z|e^{i\theta}$ and consider $\Delta z \to 0$ along the line through z and the origin; i.e. take $\Delta z = \Delta r e^{i\theta}$. Then

$$\begin{aligned} f'(z) &= \lim_{\Delta r \to 0} \frac{\operatorname{Arg}(|z|e^{i\theta} + \Delta r e^{i\theta}) - \operatorname{Arg}(|z|e^{i\theta})}{\Delta r e^{i\theta}} \\ &= \lim_{\Delta r \to 0} \frac{\operatorname{Arg}((|z| + \Delta r)e^{i\theta}) - \operatorname{Arg}(|z|e^{i\theta})}{\Delta r e^{i\theta}} \\ &= \lim_{\Delta r \to 0} \frac{\theta - \theta}{\Delta r e^{i\theta}} = 0. \end{aligned}$$

Now suppose f is analytic in D. Then f is constant in D (since $f' = 0$). But clearly $f(z) = \operatorname{Arg} z$ is *not* constant in any open set in D ... a contradiction. Hence f is nowhere analytic.

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Problem 5: Consider the polynomial $P(z) = z^3 - 27$.

(a) Show that the sum of the roots of P is 0.

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the easy way:

$$z^3 - 27 = 0 \implies \left(\frac{z}{3}\right)^3 = 1$$

so $z/3$ is a cube-root of unity, i.e. the roots are $z_1 = 1$, $z_2 = 3\omega_3$, $z_2 = 3\omega_3^2$ and their sum is

$$z_1 + z_2 + z_3 = 3(\underbrace{1 + \omega_3 + \omega_3^2}_{0}) = 0 \quad \surd$$

by a theorem discussed in class re: sums of roots of unity.

by direct computation:
Just solve for the roots...

$$z^3 = 27 = 27e^{in2\pi} \quad (n = 0, 1, 2, \ldots) \implies z = (27e^{in2\pi})^{1/3} = 3e^{in2\pi/2}$$

$$\implies \begin{cases} z_1 = 3e^0 = 3 \\ z_2 = 3e^{i2\pi 3} = 3(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 3(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\ z_3 = 3e^{i4\pi 3} = 3(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = 3(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) \end{cases}$$

Now we can just compute:

$$z_1 + z_2 + z_3 = 3 + 3(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) + 3(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 0 \quad \surd$$

(b) Factor $P(z)$ completely.

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If you didn't already solve for the roots above, you'll need to do that now. The we have

$$\begin{aligned} P(z) &= (z - z_1)(z - z_2)(z - z_3) \\ &= (z - 3)\left(z + \frac{3}{2} - i\frac{3\sqrt{3}}{2}\right)\left(z + \frac{3}{2} + i\frac{3\sqrt{3}}{2}\right) \end{aligned}$$

/8 **Problem 6:** For $z \in \mathbb{C}$ a natural definition of the cos function is $g(z) = \cos z \equiv \dfrac{e^{iz} + e^{-iz}}{2}$.

(a) Show that if $\operatorname{Im} z = 0$ then $g(z)$ agrees with the usual (real-valued) cos function.

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For $z = x \in \mathbb{R}$ we have

$$\begin{aligned} g(z) = \frac{1}{2}(e^{ix} + e^{-ix}) &= \frac{1}{2}(\cos x + i \sin x + \cos(-x) + i\sin(-x)) \\ &= \frac{1}{2}(\cos x + i \sin x + \cos(x) - i\sin(-x)) = \frac{1}{2}(2\cos x) = \cos x \quad \checkmark \end{aligned}$$

(b) Prove that $g(z)$ is entire (i.e. analytic on $\mathbb{C}$).

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easy way:
We know e^z is everywhere differentiable, hence so are both e^{iz} and e^{-iz}, and thus also $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$.

hard(er) way:
Write

$$\begin{aligned} g(x+iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{(\cos x + i\sin x)e^{-y} + (\cos x - i \sin x)e^{y}}{2} \\ &= \underbrace{\frac{1}{2}(e^{y} + e^{-y})\cos x}_{u(x,y)} + i \cdot \underbrace{\frac{1}{2}(e^{-y} - e^{y})\sin x}_{v(x,y)} \end{aligned}$$

Now check that the Cauchy-Riemann equations hold everywhere:

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(e^{y} + e^{-y})\sin x = \frac{\partial v}{\partial y} \quad \checkmark \qquad\qquad \frac{\partial v}{\partial x} = \frac{1}{2}(e^{-y} - e^{y})\cos x = -\frac{\partial u}{\partial y} \quad \checkmark$$

Together with the (obvious) continuity of the partials these imply g is everywhere analytic.

(c) Find at least one solution of $\cos z = 2$.

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$$\begin{aligned} \cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = 2 &\implies e^{-iz} + e^{-iz} = 4 \\ &\implies (e^{iz})^2 - 4e^{iz} + 1 = 0 \quad \text{(a quadratic equation in } e^{iz}) \end{aligned}$$

which gives

$$e^{iz} = e^{i(x+iy)} = e^{-y}(\cos x + i \sin x) = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Matching both real and imaginary parts gives

$$\begin{cases} \sin x = 0 \implies x = n\pi \quad (n = 0, 1, 2, \ldots) \\ \cos x > 0 \implies x = m2\pi \quad (m = 0, 1, 2, \ldots) \end{cases}$$

Hence $\cos x = 1$ and

$$e^{-y} = 2 \pm \sqrt{3} \implies y = -\ln(2 \pm \sqrt{3}) = \pm \ln(2 + \sqrt{3})$$

$$\implies \boxed{z = \pm i \ln(2+\sqrt{3}) + n2\pi}$$