THOMPSON RIVERS UNIVERSITY

MATH 2120
Linear Algebra 1

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MIDTERM EXAM #1
SOLUTIONS

11 Feb 2016 11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 6 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		7
3		6
4		5
5		4
6		5
TOTAL:		33

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Problem 1: Find the general solution of the following linear system:

$$\begin{aligned} x + 2y - 3z &= -5 \\ 2x + 4y - 6z + w &= -8 \\ 6x + 13y - 17z + 4w &= -21 \end{aligned}$$

By Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{bmatrix} \xrightarrow[R_3-6R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2-4R_3} \begin{bmatrix} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & -5 & 0 & -7 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{(RREF)}$$

$$\Longrightarrow \boxed{\begin{cases} x = -7 + 5z \\ y = 1 - z \\ z \in \mathbb{R} \text{ is free} \\ w = 2. \end{cases}} \qquad \text{or in parametric form:} \qquad \boxed{\begin{cases} x = -7 + 5t \\ y = 1 - t \\ z = t \in \mathbb{R} \\ w = 2 \end{cases}}$$

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Problem 2: Consider the following linear system in x, y, z and w:

$$\begin{aligned} x+y+w &= b \\ 2x+3y+z+5w &= 6 \\ z+w &= 4 \\ 2y+2z+aw &= 1 \end{aligned}$$

For what value(s) of the constants a and b is the system

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(i) inconsistent?

By Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & b \\ 2 & 3 & 1 & 5 & 6 \\ 0 & 2 & 2 & a & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 2 & 2 & a & 1 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 0 & a-6 & -11+4b \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3\leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & a-6 & -11+4b \end{bmatrix}$$

The system is inconsistent if (and only if)

$$\begin{cases} a-6=0 \\ -11+4b\neq 0 \end{cases} \implies \boxed{a=6 \text{ and } b\neq \frac{11}{4}}$$

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(ii) consistent with a unique solution?

The system is consistent with a unique solution if (and only if)

$$a-6\neq 0 \implies \boxed{a\neq 6,\ b\in\mathbb{R}}$$

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(iii) consistent with infinitely many solutions?

The system has infinitely many solutions if there is a free variable, i.e. if

$$\begin{cases} a-6=0 \\ -11+4b=0 \end{cases} \implies \boxed{a=6,\ b=\frac{11}{4}}$$

/6 **Problem 3:** Smith and Jones are the only competing suppliers of communication services in their community. At present they each have a 50% share of the market. However, Smith has recently upgraded his service, and a survey indicates that from one month to the next, 90% of Smith's customers remain loyal, while 10% switch to Jones. On the other hand, 70% of Jones's customers remain loyal and 30% switch to Smith.

(i) After 3 months how large are their market shares?

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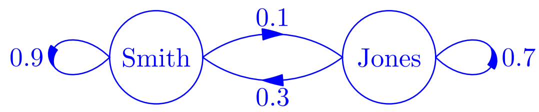
The following diagram summarizes this Markov process:



With state vector $\mathbf{x} = (x_S, x_J)$ having components gives the market shares of Smith and Jones, respectively, the transition matrix is

$$A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}.$$

After three years the market shares are given by

$$\begin{aligned} \mathbf{x}^{(3)} = A^3\mathbf{x}^{(0)} &= \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^3 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}^2 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.66 \\ 0.34 \end{bmatrix} = \begin{bmatrix} 0.696 \\ 0.304 \end{bmatrix} \end{aligned}$$

So Smith has 69.6% of the market while Jones has 30.4%.

(ii) If this goes on for a long time, how large will Smith's market share be?

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The equilibrium state vector is characterized by

$$A\mathbf{x} = \mathbf{x} \implies (A - I)\mathbf{x} = \mathbf{0}$$

which gives a linear system with augmented matrix

$$\left[\begin{array}{cc|c} -0.1 & 0.3 & 0 \\ 0.1 & -0.3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and so we have the general solution

$$\begin{cases} x_S = 3t \\ x_J = t \in \mathbb{R}. \end{cases}$$

Since market shares must sum to 1 we have

$$1 = x_S + x_J = 3t + t \implies t = \frac{1}{4} \implies \boxed{x_S = \frac{3}{4} = 0.75}$$

/5 **Problem 4:** Suppose A and B invertible matrices and c is a scalar. Solve the following equation for the matrix X, and simplify your answer as much as (but no more than!) possible:

$$AX + cX = A + B$$

$$\begin{aligned} AX + cX = A + B &\implies AX + cIX = A + B \\ &\implies (A + cI)X = A + B \\ &\implies \boxed{X = (A + cI)^{-1}(A + B)} \end{aligned}$$

which does not simplify further. Alternatively,

$$\begin{aligned} AX + cX = A + B &\implies A^{-1}(AX + cX) = A^{-1}(A + B) \\ &\implies X + cA^{-1}X = I + A^{-1}B \\ &\implies (I + cA^{-1})X = I + A^{-1}B \\ &\implies \boxed{X = (I + cA^{-1})^{-1}(I + A^{-1}B)} \end{aligned}$$

/4 **Problem 5:** Let $A = \begin{bmatrix} 3 & 5 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find a 2×2 matrix X such that $AX = B$.

$$AX = B \implies A^{-1}AX = A^{-1}B \implies X = A^{-1}B$$

Since

$$A^{-1} = \frac{1}{(3)(5) - (5)(4)} \begin{bmatrix} 5 & -5 \\ -4 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -5 & 5 \\ 4 & -3 \end{bmatrix}$$

we have

$$\begin{aligned} X = A^{-1}B &= \frac{1}{5} \begin{bmatrix} -5 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 10 & 10 \\ -5 & -4 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 2 & 2 \\ -1 & -4/5 \end{bmatrix}} \end{aligned}$$

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Problem 6: Determine all the values of p such that the following matrix is invertible.

$$A = \begin{bmatrix} 2 & 3 & 1 & p \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 7 & 6 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

By cofactor expansion on the first column of A:

$$\det A = (2)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 7 & 6 \\ 0 & 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} 3 & 1 & p \\ 1 & 2 & 1 \\ 1 & 7 & 6 \end{vmatrix}$$

$$= (2)(-1)\begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} - \left[(3)\begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} - (1)\begin{vmatrix} 1 & p \\ 7 & 6 \end{vmatrix} + (1)\begin{vmatrix} 1 & p \\ 2 & 1 \end{vmatrix}\right]$$

$$= (-2)(5) - \big[(3)(5) - (6 - 7p) + (1 - 2p)\big]$$

$$= -20 - 5p.$$

Therefore A is invertible provided that

$$0 \neq \det A = -20 - 5p \implies \boxed{p \in \mathbb{R},\ p \neq -4}$$