MATH 2110: Quiz #3 – SOLUTIONS

/6 **Problem 1:** Find and classify the critical points of $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

Solve for the critical points:

$$\begin{aligned} 0 &= f_x = -4x^3 + 4x = 4x(1 - x^2) \implies x = 0, \pm 1 \\ 0 &= f_y = -2y \implies y = 0. \end{aligned}$$

So the critical points are $(0, 0)$, $(1, 0)$ and $(-1, 0)$.
We have:

$$\begin{aligned} f_{xx} &= -12x^2 + 4 \\ f_{yy} &= -2 \\ f_{xy} &= 0 \end{aligned}$$

At $(0, 0)$:

$$D = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = (4)(-2) - 0 = -8 < 0$$

so $f(x, y)$ has a **saddle point** at $(0, 0)$.

At $(1, 0)$:

$$D = f_{xx}(1,0)f_{yy}(1,0) - [f_{xy}(1,0)]^2 = (-8)(-2) - 0 = 16 > 0, \quad f_{xx}(0,0) = -8 < 0$$

so $f(x, y)$ has a **local max** at $(1, 0)$.

At $(-1, 0)$:

$$D = f_{xx}(-1,0)f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = (-8)(-2) - 0 = 16 > 0, \quad f_{xx}(0,0) = -8 < 0$$

so $f(x, y)$ has a **local max** at $(-1, 0)$.

/4 **Problem 2:** Find the equation of the tangent plane at $(1, 2, 1)$ to the surface $xy + yz + zx = 5$.

Let $f(x, y, z) = xy + yz + zx$. The surface is a "level surface" of f so $\nabla f(1, 2, 1)$ is perpendicular to the tangent plane. A vector normal to the tangent plane is therefore

$$\mathbf{n} = \nabla f(1, 2, 1) = (y + z, x + z, y + x)\Big|_{(1,2,1)} = (3, 2, 3).$$

The equation of the plane in "normal form" is then

$$\begin{aligned} [(x, y, z) - (1, 2, 1)] \cdot (3, 2, 3) = 0 &\implies 3(x - 1) + 2(y - 2) + 3(z - 1) = 0 \\ &\implies \boxed{3x + 2y + 3z = 10} \end{aligned}$$