THOMPSON RIVERS UNIVERSITY

MATH 130
Linear Algebra for Engineers

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MIDTERM EXAM #2
SOLUTIONS

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Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		6
3		6
4		8
5		6
TOTAL:		32

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Problem 1: Let a, b be given constants and let

$$A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 2 & 0 & 0 & a \\ 3 & 0 & 0 & b \\ 0 & a & b & 0 \end{bmatrix}.$$

(a) Evaluate $\det A$.

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$$\det A = 0 - 2\begin{vmatrix} 2 & 3 & 0 \\ 0 & 0 & b \\ a & b & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 & 0 \\ 0 & 0 & a \\ a & b & 0 \end{vmatrix} - 0$$

$$= -2(-b)\begin{vmatrix} 2 & 3 \\ a & b \end{vmatrix} + 3(-a)\begin{vmatrix} 2 & 3 \\ a & b \end{vmatrix}$$

$$= 2b(2b - 3a) - 3a(2b - 3a)$$

$$= \boxed{(2b - 3a)^2}$$

(b) Find conditions on a and b such that A is *not* invertible.

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A is not invertible if (and only if) $\det A = 0$:

$$\implies \det A = (2b - 3a)^2 = 0 \implies \boxed{b = \frac{3}{2}a \text{ where } a \text{ is arbitrary}}$$

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Problem 2: Let $\mathbf{a} = (1, 1, -1)$, $\mathbf{b} = (2, 1, k)$ and $\mathbf{c} = (2, -k, 2)$ be given vectors in $\mathbb{R}^3$.

(a) For what value(s) of k are $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ linear independent? Justify your answer.

Consider the vector equation $c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{c} = \mathbf{0}$. Form the augmented matrix and reduce:

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & -k & 0 \\ -1 & k & 2 & 0 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -1 & -(k+2) & 0 \\ 0 & k+2 & 4 & 0 \end{bmatrix} \xrightarrow{R_3+(k+2)R_2} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -1 & -(k+2) & 0 \\ 0 & 0 & 4-(k+2)^2 & 0 \end{bmatrix}$$

Therefore we get only the trivial solution (hence $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linear independent) only if

$$4 - (k+2)^2 \neq 0 \implies k + 2 \neq \pm 2 \implies \boxed{k \in \mathbb{R},\ k \neq 0, -4}$$

(Alternatively, calculate $\det A = k(k+4) \neq 0$ if (and only if) $k \neq 0, -4$.)

(b) For what value(s) of k do $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ form a basis for $\mathbb{R}^3$? Justify your answer.

A basis for $\mathbb{R}^3$ needs 3 linearly independent vectors, so $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ form a basis if (and only if) $k \neq 0, -4$ as above.

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Problem 3: Let $\mathcal{B} = \{(3, 6), (2, 1)\}$ be a basis for $\mathbb{R}^2$.

(a) Let $\mathbf{x} = (0, -9)$. Find $\mathbf{x}_{\mathcal{B}}$ (i.e. find the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$).

$\mathbf{x}_{\mathcal{B}} = (c_1, c_2)$ where

$$c_1 \begin{bmatrix} 3 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \end{bmatrix}.$$

Solve this system by row reduction and back-substitution:

$$\begin{bmatrix} 3 & 2 & 0 \\ 6 & 1 & -9 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & -3 & -9 \end{bmatrix} \implies \begin{cases} c_2 = 3 \\ c_1 = (0 - 2c_2)/3 = -2 \end{cases} \implies \boxed{\mathbf{x}_{\mathcal{B}} = (-2, 3)}$$

(b) If $\mathbf{x}_{\mathcal{B}} = (5, 2)$ find $\mathbf{x}$.

$$\mathbf{x} = (5) \begin{bmatrix} 3 \\ 6 \end{bmatrix} + (2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 19 \\ 32 \end{bmatrix}}$$

/8 **Problem 4:** Let $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

(a) Find the characteristic polynomial $p(\lambda)$ associated with A.

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$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} \\ &= (5-\lambda)^2 - 16 \\ &= 5 - 10\lambda + \lambda^2 - 16 \\ &= \boxed{\lambda^2 - 10\lambda + 9} \end{aligned}$$

(b) Determine the eigenvalues of A and their algebraic multiplicity.

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$$p(\lambda) = 0 = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9)$$

$$\Longrightarrow \lambda = 1, 9 \text{ (both with algebraic multicplicity 1)}$$

(c) Determine the eigenvectors of A.

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Eigenvectors $\mathbf{v} = (v_1, v_2)$ are solutions of $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Case $\lambda_1 = 1$:

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} v_1 = -v_2 = -t \\ v_2 = t \in \mathbb{R} \end{cases} \Longrightarrow \boxed{\mathbf{v}_1 = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}}$$

Case $\lambda_1 = 9$:

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} v_1 = v_2 = t \\ v_2 = t \in \mathbb{R} \end{cases} \Longrightarrow \boxed{\mathbf{v}_2 = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}}$$

/6 **Problem 5:** The matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Verify that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are eigenvectors of A, and find their associated eigenvalues.

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By direct calculation and inspection:

$$A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (-1)\mathbf{v}_1 \implies \text{eigenvalue } \lambda_1 = -1$$

$$A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (1)\mathbf{v}_2 \implies \text{eigenvalue } \lambda_2 = 1$$

$$A\mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = (3)\mathbf{v}_3 \implies \text{eigenvalue } \lambda_3 = 3$$

(b) Is A diagonalizable? Justify your answer.

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Yes. A has distinct eigenvalues, so the eigenvectors are linearly independent. Therefore A is diagonalizable.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

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$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(invertible since its columns are linearly independent)

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$