THOMPSON RIVERS UNIVERSITY

MATH 130
Linear Algebra for Engineers

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MIDTERM EXAM #1
SOLUTIONS

9 October 2009 12:30–13:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		5
3		7
4		7
TOTAL:		25

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Problem 1: Consider the following system of linear equations, in which k is a constant.

$$\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= -2 \end{aligned}$$

For what value(s) of k does this system have:

(a) no solution?

$$\begin{bmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 - kR_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & -2-k \end{bmatrix}$$

There will be no solution provided that:

$$\begin{cases} 2 - k - k^2 = 0 \\ -2 - k \neq 0 \end{cases} \implies \begin{cases} (2+k)(1-k) = 0 \\ k \neq -2 \end{cases} \implies \boxed{k = 1}$$

(b) a unique solution?

There will be a unique solution provided that:

$$2 - k - k^2 \neq 0 \implies (2+k)(1-k) \neq 0 \implies \boxed{k \in \mathbb{R}, k \neq 1, k \neq -2}$$

(c) an infinite number of solutions?

There will be an infinite number of solutions provided that:

$$\begin{cases} 2 - k - k^2 = 0 \\ -2 - k = 0 \end{cases} \implies \begin{cases} (2+k)(1-k) = 0 \\ k = -2 \end{cases} \implies \boxed{k = -2}$$

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Problem 2: Find all solutions of the following linear system:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= -1 \\ x_1 + x_2 - x_4 &= 0 \\ 2x_1 + 2x_2 - 2x_3 + 5x_4 &= 1 \end{aligned}$$

Calculate the reduced row echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 2 & -2 & 5 & 1 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2+2R_3]{R_1-R_3} \begin{bmatrix} 1 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now read off the general solution:

$$\begin{cases} x_1 = 1 - x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \\ x_4 = 1 \end{cases}$$

or equivalently:

$$\begin{cases} x_1 = 1 - t \\ x_2 = t \\ x_3 = 3 \\ x_4 = 1 \end{cases} \qquad t \in \mathbb{R}$$

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Problem 3: (a) Use the Gauss-Jordan method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$\left[\, A \,|\, I \,\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + 2R_3]{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = \left[\, I \,|\, A^{-1} \,\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}}$$

(b) Use your answerto part (a) to find the solution of the linear system

$$\begin{aligned} x + 2z &= a \\ x + y &= b \\ x + y + z &= c \end{aligned}$$

where a, b and c are constants.

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$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b - 2c \\ -a - b + 2c \\ -b + c \end{bmatrix}$$

$$\boxed{\begin{cases} x = a + 2b - 2c \\ y = -a - b + 2c \\ z = -b + c \end{cases}}$$

/7 **Problem 4:** Consider the matrices $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 4 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -9 & -1 & 3 \\ 16 & 2 & -5 \\ 6 & 1 & -2 \end{bmatrix}$.

(a) Show that A and B are inverses of each other.

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Check by direct calculation:

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(b) Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find the matrix X that satisfies the equation

$$AXB = C.$$

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$$\begin{aligned} AXB = C &\implies AXBB^{-1} = CB^{-1} \\ &\implies AX = CB^{-1} \\ &\implies A^{-1}AX = A^{-1}CB^{-1} \\ &\implies X = A^{-1}CB^{-1} \\ &\implies X = BCA \quad \text{(by part (a))} \end{aligned}$$

$$\begin{aligned} \implies X &= \begin{bmatrix} -9 & -1 & 3 \\ 16 & 2 & -5 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 4 & 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -9 & -1 & 3 \\ 16 & 2 & -5 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 7 & 4 & 0 \\ 6 & 3 & 1 \\ 4 & 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -57 & -30 & -7 \\ 104 & 55 & 12 \\ 40 & 21 & 5 \end{bmatrix} \end{aligned}$$