THOMPSON RIVERS UNIVERSITY

MATH 130
Linear Algebra for Engineers

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FINAL EXAM
SOLUTIONS

10 December 2009 14:00–17:00

PROBLEM	GRADE	OUT OF
1		5
2		5
3		5
4		8
5		7
6		4
7		8
8		4
9		9
10		4
11		4
12		10
TOTAL:		73

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 9 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/5

Problem 1: Find all solutions of the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 38 \\ 2x_1 + x_2 + 3x_3 + 2x_4 &= 53 \\ 2x_1 + 4x_2 + x_3 + 4x_4 &= 78 \end{aligned}$$

Ans:

$$\begin{cases} x_1 = 26 - 8t \\ x_2 = 7 + 2t \\ x_3 = -2 + 4t \\ x_4 = t \end{cases} \qquad t \in \mathbb{R}$$

/5

Problem 2: Calculate the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$.

Ans:

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

/5 **Problem 3:** Let p and q be given constants and consider the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= q \\ x_2 + px_3 &= 1 \\ -x_1 + x_2 + px_3 &= 5. \end{aligned}$$

Find the values of p and q (if any) such that this system has

/3 (a) no solution.

Ans: $p = 1$, $q \neq -2$

/1 (b) a unique solution.

Ans: $p \neq 1$, q arbitrary

/1 (c) infinitely many solutions.

Ans: $p = 1$, $q = -2$

/8 **Problem 4:** Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} k & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(a) Find the value(s) of k such that A and B are inverses of each other.

/3

Ans: $k = 7$

(b) Solve the following linear system using the inverse of the coefficient matrix.

/2

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 12 \\ x_1 + 4x_2 + 3x_3 &= -10 \\ x_1 + 3x_2 + 4x_3 &= 16 \end{aligned}$$

Ans: $(x_1, x_2, x_3) = (66, -22, 4)$.

(c) Let k have the value you found in part (a). Find the matrix X such that $AXB = A + B$.

/3

Ans: $A + B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{bmatrix}$.

/7 **Problem 5:** Consider the matrix

$$A = \begin{bmatrix} -3 & 2 & 9 & 1 \\ 4 & 1 & -7 & 6 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & -6 & -2 \end{bmatrix}$$

(a) Evaluate $\det A$ in the most computationally efficient manner.

/3

Ans: $\det A = 44$

(b) Is A invertible? Justify your answer.

/1

Yes, because $\det A \neq 0$.

(c) List 3 other properties of A that are consequences of your result in part (a).

/3

- the columns of A are linearly independent vectors
- the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution
- $\lambda = 0$ is not an eigenvalue of A

/4 **Problem 6:** Let $T(\mathbf{x})$ be the linear transformation of $\mathbb{R}^2$ that carries out a reflection across the x-axis followed by a clockwise rotation by 30°. Find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Ans: $A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

/8

Problem 7: Consider the sets $\mathcal{B} = \{(1,2),\ (3,4)\}$ and $\mathcal{C} = \{(-1,1),\ (-3,-4)\}$ in $\mathbb{R}^2$.

(a) Show that both $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^2$.

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(b) Let $\mathbf{x}_{\mathcal{B}} = (4,-5)$ be the coordinates of a given vector $\mathbf{x}$ relative to the basis $\mathcal{B}$. Find $\mathbf{x}_{\mathcal{C}}$ (that is, find the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{C}$).

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Ans: $\mathbf{x}_{\mathcal{C}} = (8/7, 23/7)$

/4

Problem 8: Let $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Show that the vectors

$$\{\mathbf{u},\ \mathbf{u}+\mathbf{v},\ \mathbf{v}+\mathbf{w}\}$$

are also linearly independent.

/9 **Problem 9:** Consider the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

(a) Is A diagonalizable? Justify your answer.

/4

Ans: Eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4$ are distinct so the eigenvectors (which are $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (-1,1)$) are linearly independent, hence A is diagonalizable.

(b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Verify your answer by carrying out the matrix multiplication.

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Ans: $D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) Evaluate the matrix A^{10}.

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Ans: $A^{10} = \frac{1}{2} \begin{bmatrix} 4^{10} & -4^{10} \\ -4^{10} & 4^{10} \end{bmatrix}$

/4

Problem 10: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 4 \end{bmatrix}.$$

Is $\lambda = 2$ an eigenvalue of A? (*Hint:* Do not attempt to calculate all the eigenvalues.)

Ans: No, since $\det(A - 2I) = 6 \neq 0$

/4

Problem 11: (a) Write the imaginary number i in polar form.

$$i = e^{i\pi/2}$$

(b) Use your answer to part (a) to evaluate $\sqrt{i}$. Write your answer in Cartesian form. (*Hint*: use the relationship $\sqrt{e^{i\theta}} = (e^{i\theta})^{1/2} = e^{i(\theta/2)}$.)

$$\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2}(1+i)$$

/10 **Problem 12:** Consider the following system of linear differential equations:

$$y_1' = -y_2$$
$$y_2' = -4y_1$$

/5 (a) Find all solutions of this system.

$$y_1(t) = c_1 e^{-2t} + c_2 e^{2t}$$
$$y_2(t) = 2c_1 e^{-2t} - 2c_2 e^{2t}$$

/2 (b) Find the particular solution with initial conditions $y_1(0) = 4$, $y_2(0) = -2$.

$$y_1(t) = \tfrac{5}{2}e^{2t} + \tfrac{3}{2}e^{-2t}$$
$$y_2(t) = -5e^{2t} + 3e^{-2t}$$

/3 (c) Evaluate the matrix e^{3A} where $A = \begin{bmatrix} 0 & -1 \\ -4 & 0 \end{bmatrix}$.

Ans:

$$e^{3A} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{-6} & 0 \\ 0 & e^{6} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} (-\tfrac{1}{4}) = \begin{bmatrix} \frac{1}{2}(e^{-6} + e^{6}) & \frac{1}{4}(e^{-6} - e^{6}) \\ e^{-6} - e^{6} & \frac{1}{2}(e^{-6} + e^{6}) \end{bmatrix}$$