# MATH 1300: Quiz #8 – SOLUTIONS

/10 **Problem 1:** Let $A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$. (a) Find the eigenvalues and associated eigenvectors of $A$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 8 = \lambda^2 - 1$$
$$= (\lambda+1)(\lambda-1) \implies \boxed{\lambda = \pm 1}$$

case $\lambda = 1$:

$$(A - (1)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 2 & -2 & 0 \\ 4 & -4 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} v_1 = t \\ v_2 = t \in \mathbb{R} \end{cases} \implies \mathbf{v} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left(\text{choosing } t = 1 \text{ gives } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

case $\lambda = -1$:

$$(A - (-1)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 4 & -2 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} v_1 = \frac{1}{2}t \\ v_2 = t \in \mathbb{R} \end{cases} \implies \mathbf{v} = \frac{1}{2}t\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \left(\text{choosing } t = 2 \text{ gives } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

So the eigenvalues are

$$\boxed{\lambda_1 = 1,\ \lambda_2 = -1}$$

with corresponding eigenvectors

$$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(b) Diagonalize $A$. That is, find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$.

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$$\boxed{D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$$

(c) Evaluate $A^{100}$.

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$$A^{100} = PD^{100}P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{100} P^{-1}$$
$$= P\begin{bmatrix} 1^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} P^{-1}$$
$$= P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = PP^{-1} = \boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$