THOMPSON RIVERS UNIVERSITY

MATH 1300
Linear Algebra for Engineers

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

21 November 2012 12:30–13:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		4
2		6
3		6
4		5
5		4
6		7
TOTAL:		32

/4 **Problem 1:** Find the value(s) of a and b such that $A = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}$ is *not* invertible.

$$\begin{aligned} 0 = \det A &= a\begin{vmatrix} a & b \\ b & a \end{vmatrix} - b\begin{vmatrix} b & b \\ 0 & a \end{vmatrix} + 0 \\ &= a(a^2 - b^2) - b(ab - 0) \\ &= a(a^2 - 2b^2) \end{aligned}$$

$$\implies \boxed{a = 0 \text{ or } a = \pm\sqrt{2}b}$$

/6 **Problem 2:** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$.

For each of the following sets of vectors: (i) Is $\mathcal{B}$ linear independent? (ii) Is $\mathcal{B}$ a basis for $\mathbb{R}^2$? (Give a brief justification for each answer.)

(a) $\mathcal{B} = \{\mathbf{v}_1,\ \mathbf{v}_2\}$

(i) Yes: $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$

(ii) Yes: any 2 linearly independent vectors are a basis for $\mathbb{R}^2$

(b) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

(i) No: in $\mathbb{R}^2$ a linearly independent set can contain at most 2 vectors.

(ii) No: $\mathcal{B}$ is not linearly independent so it can't be a basis.

(c) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_3\}$

(i) No: by inspection $\mathbf{v}_2 = -3\mathbf{v}_1$

(ii) No: $\mathcal{B}$ is not linearly independent so it can't be a basis.

/6 **Problem 3:** Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}-2\\2\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}3\\0\\-4\end{bmatrix}$.

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

Consider the linear system

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

which we can write in equivalent matrix form as

$$\underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}}_{A}\mathbf{c} = \mathbf{0}.$$

We have

$$\det A = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{vmatrix} = (1)(2)(-4) = -8 \neq 0$$

so A is invertible, hence the linear system above has only the trivial solution $\mathbf{c} = A^{-1}\mathbf{0} = \mathbf{0}$. Therefore $\mathcal{B}$ is a linearly independent set.

Since $\mathcal{B}$ contains 3 linearly independent vectors it is a basis for $\mathbb{R}^3$.

(b) Let $\mathbf{w} = \begin{bmatrix}a\\b\\c\end{bmatrix}$. Find $\mathbf{w}_{\mathcal{B}}$ (the coordinates of $\mathbf{w}$ relative to $\mathcal{B}$).

We need to find $\mathbf{w}_{\mathcal{B}} = (c_1, c_2, c_3)$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}.$$

This is a linear system with augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & -4 & c \end{bmatrix}$$

which is already in row echelon form, so we can solve by back-substitution:

$$\begin{aligned} c_3 &= -\frac{1}{4}c \\ c_2 &= \frac{1}{2}b \\ c_1 &= a + 2c_2 - 3c_3 = a + b + \frac{3}{4}c. \end{aligned}$$

Therefore

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} a + b + \frac{3}{4}c \\ \frac{1}{2}b \\ -\frac{1}{4}c \end{bmatrix}$$

/5 **Problem 4:** Find all the complex roots of $z^3 = -i$.

$$\begin{aligned} z^3 = -i = e^{i(\frac{3\pi}{2} + n2\pi)} \implies z &= e^{i(\frac{3\pi}{2} + n2\pi)/3} \quad (n = 0, 1, 2, \ldots) \\ &= e^{i(\frac{\pi}{2} + n\frac{2\pi}{3})} \end{aligned}$$

$$n = 0: \quad z = e^{i\frac{\pi}{2}} = i$$

$$n = 1: \quad z = e^{i\frac{7\pi}{6}} = \cos\tfrac{7\pi}{6} + i\sin\tfrac{7\pi}{6} = -\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i$$

$$n = 2: \quad z = e^{i\frac{11\pi}{6}} = \cos\tfrac{11\pi}{6} + i\sin\tfrac{11\pi}{6} = \tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i$$

$$\implies \boxed{z = i,\ \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i}$$

/4 **Problem 5:** Find the inverse of $A = \begin{bmatrix} i & 1-i \\ 1+i & -i \end{bmatrix}$.

$$A^{-1} = \frac{1}{(i)(-i)-(1-i)(1+i)} \begin{bmatrix} -i & -1+i \\ -1-i & i \end{bmatrix}$$

$$= \frac{1}{1-2} \begin{bmatrix} -i & -1+i \\ -1-i & i \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} i & 1-i \\ 1+i & -i \end{bmatrix} = A}$$

/7 **Problem 6:** Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Determine the eigenvalues and corresponding eigenvectors of A, and state the algebraic and geometric multiplicity of each eigenvalue.

Eigenvalues...

$$0 = \det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda)(1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)(\lambda^2-1)$$

$$= -(1-\lambda)^2(1+\lambda) \implies \lambda = \pm 1$$

Eigenvectors...

case $\lambda = 1$:

$$(A-(1)I)\mathbf{v} = \mathbf{0} \iff \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} v_1 = s \\ v_2 = s \\ v_3 = t \end{cases} \implies \mathbf{v} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{so } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ are eigenvectors.}$$

case $\lambda = -1$:

$$(A-(-1)I)\mathbf{v} = \mathbf{0} \iff \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} v_1 = -t \\ v_2 = t \\ v_3 = 0 \end{cases} \implies \mathbf{v} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{so } \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ is an eigenvector.}$$

Summary:

$$\lambda_1 = \lambda_2 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{alg. multiplicity} = \text{geom. multiplicity} = 2)$$

$$\lambda_3 = -1, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad (\text{alg. multiplicity} = \text{geom. multiplicity} = 1)$$

(b) Find (if possible) a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$