THOMPSON RIVERS UNIVERSITY

MATH 1300
Linear Algebra for Engineers

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

18 November 2011   12:30–13:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		4
2		4
3		4
4		4
5		11
6		5
TOTAL:		32

/4

Problem 1: Find a vector $\mathbf{x} \in \mathbb{R}^3$ that is orthogonal to *both* $\mathbf{v} = (1, 1, 1)$ and $\mathbf{w} = (2, 2, 1)$.

Let $\mathbf{x} = (x, y, z)$. Then

$$\begin{cases} \mathbf{x} \cdot \mathbf{v} = 0 = x + y + z \\ \mathbf{x} \cdot \mathbf{w} = 0 = 2x + 2y + z \end{cases}$$

Solving this linear system . . .

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow[-R_1]{R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus the general solution is

$$\begin{cases} z = 0 \\ y = t \in \mathbb{R} \\ x = -y = -t \end{cases} \implies \mathbf{x} = t(-1, 1, 0),\ t \in \mathbb{R}$$

We can take $t = 1$, for instance, to get

$$\boxed{\mathbf{x} = (-1, 1, 0)}$$

/4

Problem 2: For what value(s) of $c \in \mathbb{R}$ are the following vectors linearly independent?

$$\mathbf{v}_1 = (c, -1, -1) \qquad \mathbf{v}_2 = (-1, c, -1) \qquad \mathbf{v}_3 = (-1, -1, c)$$

We know the vectors are linearly independent if an only if the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$

is invertible, i.e. has non-zero determinant. So evaluate. . .

$$\begin{aligned} \det A = \begin{vmatrix} c & -1 & -1 \\ -1 & c & -1 \\ -1 & -1 & c \end{vmatrix} &= c(c^2 - 1) - (-1)(-c - 1) + (-1)(1 + c) \\ &= c(c - 1)(1 + c) - 2(1 + c) \\ &= (1 + c)\big[c(c - 1) - 2\big] \\ &= (1 + c)(c^2 - c - 2) \\ &= (1 + c)(c + 1)(c - 2) \end{aligned}$$

Thus $\det A \neq 0$ (and the given vectors are linearly independent) if and only if

$$\boxed{c \in \mathbb{R},\ c \neq -1, 2}$$

/4

Problem 3: Let $\mathbf{x} = (5, -8, -9)$. Is $\mathbf{x}$ in the plane spanned by $\mathbf{v} = (1, 2, 3)$ and $\mathbf{w} = (-1, 4, 5)$?

Check whether $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{x}$ is consistent:

$$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & -8 \\ 3 & 5 & -9 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 6 & -18 \\ 0 & 8 & -24 \end{bmatrix}$$
$$\xrightarrow[R_3/8]{R_2/6} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, therefore *yes*, $\mathbf{x} \in \text{span}\{\mathbf{v}, \mathbf{w}\}$.

/4

Problem 4: Calculate, to the nearest degree, the angle between the diagonal of a cube and the diagonal of one of its faces.

The diagonal of the unit cube can be represented by $\mathbf{v} = (1, 1, 1)$, while the diagonal of one of its sides can be represented by $\mathbf{w} = (0, 1, 1)$.

The angle between $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\cos\theta = \frac{\mathbf{w} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{6}}$$
$$\implies \theta = \boxed{\cos^{-1}\frac{2}{\sqrt{6}} \approx 35^\circ}$$

/11 **Problem 5:** Consider the matrix $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{bmatrix}$. (a) Find the characteristic polynomial for A.

/3

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 4 & -2 & 5-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)(5-\lambda) = \boxed{-(3-\lambda)(1+\lambda)(5-\lambda)}$$

(b) Verify that $\lambda = -1$ is an eigenvalue for A.

/1

Clearly $P(-1) = 0$, so $\lambda = -1$ is an eigenvalue of A.

(c) Show that $\lambda = 0$ is *not* an eigenvalue for A. Is A invertible?

/2

Since $P(0) = (3)(-1)(5) = -15 \neq 0$, $\lambda = 0$ is not an eigenvalue of A (hence A is invertible).

(d) Verify that $\mathbf{v} = (1, 4, 2)$ is an eigenvector for A, and determine the corresponding eigenvalue.

/2

$$A\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} = 3\mathbf{v} \implies \lambda = 3$$

(e) Find the eigenvector corresponding to the eigenvalue $\lambda = -1$.

/3

$$(A - (-1)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 4 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - 4R_2]{R_1 + R_2} \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} v_3 = t \in \mathbb{R} \\ v_2 = 0 \\ v_1 = -\frac{3}{2}v_3 = -\frac{3}{2}t \end{cases} \qquad t = 2 \implies \boxed{\mathbf{v} = (-3, 0, 2)}$$

/5

Problem 6: Find an invertible matrix P and a diagonal matrix D such that

$$A^m = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}^m = PD^mP^{-1}$$

for any integer m. (You do not need to evaluate A^m or P^{-1}.) Without calculating P^{-1}, how do you know P is invertible?

We need the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$$

Determine eigenvalues first...

$$\begin{aligned} 0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{vmatrix} &= (1-\lambda)(5-\lambda) - (-3) \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 2)(\lambda - 4) \end{aligned}$$

$$\Longrightarrow \lambda = 2,\ 4$$

New determine the corresponding eigenvectors...

Case $\lambda_1 = 2$:

$$(A - 2I)\mathbf{v}_1 = \mathbf{0} \Longrightarrow \begin{bmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathbf{v}_1 = t\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ t \in \mathbb{R}$$

Case $\lambda_1 = 4$:

$$(A - 24)\mathbf{v}_2 = \mathbf{0} \Longrightarrow \begin{bmatrix} -3 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathbf{v}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ t \in \mathbb{R}$$

So

$$\boxed{P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}$$

We know P is invertible because its columns are not scalar multiples of each other, hence they're linearly independent (also a consequence of distinct eigenvalues).