THOMPSON RIVERS UNIVERSITY

MATH 1300
Linear Algebra for Engineers

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MIDTERM EXAM #1
SOLUTIONS

12 October 2012 12:30–13:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		6
3		5
4		5
5		7
6		5
TOTAL:		33

/5

Problem 1: Consider the following system of linear equations, in which $k \in \mathbb{R}$ is a constant.

$$\begin{cases} x + 2y = 1 \\ 2x + (k^2 - 5)y = k - 1 \end{cases}$$

For what value(s) of k does this system have:

(a) no solution?

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$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & k^2 - 5 & k - 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & k^2 - 9 & k - 3 \end{bmatrix}$$

There will be no solution provided that

$$\begin{cases} k^2 - 9 = 0 \\ k - 3 \neq 0 \end{cases} \implies \begin{cases} k = \pm 3 \\ k \neq 3 \end{cases} \implies \boxed{k = -3}$$

(b) a unique solution?

/1

There will be a unique solution provided that

$$k^2 - 9 \neq= 0 \implies \boxed{k \neq \pm 3,\ k \in \mathbb{R}}$$

(c) an infinite number of solutions?

/1

There will be infinitely many solutions provided that

$$\begin{cases} k^2 - 9 = 0 \\ k - 3 = 0 \end{cases} \implies \begin{cases} k = \pm 3 \\ k = 3 \end{cases} \implies \boxed{k = 3}$$

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Problem 2: Find all solutions of the following linear system: $\begin{cases} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{cases}$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow[R_3+R_1;\ R_4-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4-R_2;\ \frac{1}{3}R_2]{R_3-\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\boxed{\begin{cases} x = -1 + t \\ y = 2s \\ z = s \in \mathbb{R} \\ w = t \in \mathbb{R} \end{cases}}$$

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Problem 3: Solve for X and simplify as much as (but no more than) possible. A, B, X are invertible matrices and c is scalar.

$$X^{-1}A - cX^{-1} = B^{-1}A$$

Several approches will work. Here is one:

$$\begin{aligned} X^{-1}A - cX^{-1} = B^{-1}A &\implies X^{-1}(A - cI) = B^{-1}A \\ &\implies (A - cI)^{-1}X = A^{-1}B \\ &\implies X = (A - cI)A^{-1}B \\ &\implies \boxed{X = B - cA^{-1}B} \end{aligned}$$

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Problem 4: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find a matrix X such that $AXA^{-1} = B$.

Solving algebraically gives $X = A^{-1}BA$:

$$\begin{aligned} \implies X &= \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \\ &= \frac{1}{14 - 12} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}} \end{aligned}$$

/7 **Problem 5:** Suppose that on any given day it must be either "sunny" or "rainy". 80% of "sunny" days are followed by another "sunny" day; 40% of "rainy" days are followed by another "rainy" day.

(a) This weather system can be modeled by a Markov chain. Write the corresponding transition matrix.

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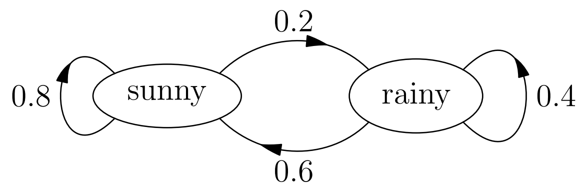
The transitions can be pictured as follows:



The transition matrix is

$$A = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

(b) The probability of rain on Thursday is 90%. What is the probability that Saturday will be "sunny"?

/2

The initial state is $\mathbf{x}^0 = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$ (probabilities of sun and rain, respectively).

Thus after two days the system state is

$$\mathbf{x}^2 = A^2\mathbf{x}^0 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}^2 \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.724 \\ 0.276 \end{bmatrix}$$

giving a $\boxed{72.4\%}$ probability that Saturday is "sunny".

(c) In the long run, what is the probability that any given day will be "rainy"?

/3

Equilibrium corresponds to a system state $\mathbf{x}$ such that

$$\mathbf{x} = A\mathbf{x} \implies (A - I)\mathbf{x} = \mathbf{0}$$

$$\to \begin{bmatrix} -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies x_1 = 3x_2$$

This gives a "rainy" probability of

$$\frac{x_2}{3x_2 + x_2} = \frac{1}{4} = \boxed{25\%}$$

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/2 **Problem 6:** Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$. (a) Show that $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$.

Simply check that both AA^{-1} and $A^{-1}A$ give the identity matrix I.

/3 (b) Use the inverse of the coefficient matrix to solve the following system: $\begin{cases} x + y + z &= 2 \\ 3x + 5y + 4z &= -1 \\ 3x + 6y + 5z &= 1 \end{cases}$

Writing the system in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 11 \end{bmatrix}$$

so

$$\boxed{\begin{cases} x = 0 \\ y = -9 \\ z = 11 \end{cases}}$$