THOMPSON RIVERS UNIVERSITY

MATH 1300
Linear Algebra for Engineers

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MIDTERM EXAM #1
SOLUTIONS

19 October 2011   12:30–13:20

PROBLEM	GRADE	OUT OF
1		5
2		5
3		4
4		5
5		7
6		7
TOTAL:		33

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

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Problem 1: Consider the following system of linear equations, in which $k \in \mathbb{R}$ is a constant.

$$x + y + 7z = -7$$
$$2x + 3y + 17z = -16$$
$$x + 2y + (k^2 + 1)z = 3k$$

For what value(s) of k does this system have:

(a) no solution?

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$$\begin{bmatrix} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & k^2+1 & 3k \end{bmatrix} \xrightarrow[R_3-R1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & k^2-6 & 3k+7 \end{bmatrix}$$
$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & k^2-9 & 3k+9 \end{bmatrix}$$

No solution requires:

$$\begin{cases} k^2 - 9 = 0 \\ 3k + 9 \neq 0 \end{cases} \implies \begin{cases} k = \pm 3 \\ k \neq -3 \end{cases} \implies \boxed{k = 3}$$

(b) a unique solution?

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$$k^2 - 9 \neq 0 \implies \boxed{k \neq \pm 3,\ k \in \mathbb{R}}$$

(c) an infinite number of solutions?

/1

$$\begin{cases} k^2 - 9 = 0 \\ 3k + 9 = 0 \end{cases} \implies \begin{cases} k = \pm 3 \\ k = -3 \end{cases} \implies \boxed{k = -3}$$

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Problem 2: Find all solutions of the following linear system:

$$\begin{aligned} 2x + 2y + 4z &= 8 \\ w - y - 3z &= -3 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

$$\left[\begin{array}{rrrrr} 1 & 0 & -1 & -3 & -3 \\ -2 & 1 & 3 & -2 & 0 \\ 0 & 2 & 2 & 4 & 8 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{R_2+2R_1} \left[\begin{array}{rrrrr} 1 & 0 & -1 & -3 & -3 \\ 0 & 1 & 1 & -8 & -6 \\ 0 & 1 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[\frac{1}{10}R_3]{R_3-R_2} \left[\begin{array}{rrrrr} 1 & 0 & -1 & -3 & -3 \\ 0 & 1 & 1 & -8 & -6 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+8R_3]{R_1+3R_3} \left[\begin{array}{rrrrr} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Longrightarrow \boxed{\begin{cases} w = y \\ x = 2 - y \\ y \text{ is free} \\ z = 1 \end{cases}} \quad \text{or alternatively} \quad \boxed{\begin{cases} w = t \\ x = 2 - t \\ y = t \\ z = 1 \end{cases} \quad (t \in \mathbb{R})}$$

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Problem 3: Let A, B, C, D be invertible matrices. Simplify as much as possible:

$$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$$

$$\begin{aligned}(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1} &= B^{-1}A^{-1}AC^{-1}(C^{-1})^{-1}(D^{-1})^{-1}D^{-1} \\ &= B^{-1}\underbrace{A^{-1}A}_{I}\underbrace{C^{-1}C}_{I}\underbrace{DD^{-1}}_{I} \\ &= \boxed{B^{-1}}\end{aligned}$$

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Problem 4: Consider the matrix $A = \begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$.

(a) Evaluate $\det A$.

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$$\begin{aligned}\det A &= (c)\underbrace{\begin{vmatrix} c & c \\ 1 & c \end{vmatrix}}_{c^2-c} - (1)\underbrace{\begin{vmatrix} c & c \\ 1 & c \end{vmatrix}}_{c^2-c} + (1)\underbrace{\begin{vmatrix} c & c \\ c & c \end{vmatrix}}_{0} \\ &= c(c^2-c) - (c^2-c) \\ &= (c-1)(c^2-c) \\ &= \boxed{c(c-1)^2}\end{aligned}$$

(b) Find all values of c for which A is *not* invertible.

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$$A \text{ not invertible} \implies \det A = 0 \implies c(c-1)^2 = 0 \implies \boxed{c = 0,\ 1}$$

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Problem 5: Use a matrix inverse to solve the following linear systems:

$$(a) \quad \begin{cases} 2x + 5y = & 9 \\ 3x + 8y = & 14 \end{cases} \qquad\qquad (b) \quad \begin{cases} 2x + 5y = & 3 \\ 3x + 8y = & 5 \end{cases}$$

coefficient matrix: $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \implies A^{-1} = \dfrac{1}{(2)(8) - (5)(3)} \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

a) $$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\implies \boxed{\begin{cases} x = 2 \\ y = 1 \end{cases}}$$

b) $$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\implies \boxed{\begin{cases} x = -1 \\ y = 1 \end{cases}}$$

/7 **Problem 6:** Suppose that two competing television stations, Station1 and Station2, each initially have 50% of the viewer market. Over each 1-year period, Station1 captures 5% of Station2's market share, and Station2 captures 10% of Station1's market share.

(a) What is each station's market share after 2 years?

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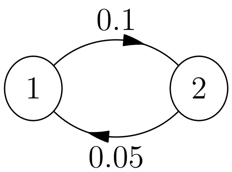

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $\mathbf{x}^{n+1} = \begin{bmatrix} 0.9x_1 + 0.05x_2 \\ 0.1x_1 + 0.95x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}}_{P} \mathbf{x}^n$, so:

$$\mathbf{x}^2 = P^2\mathbf{x}^0 = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}^2 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix} \begin{bmatrix} 0.475 \\ 0.525 \end{bmatrix} \approx \boxed{\begin{bmatrix} 0.454 \\ 0.546 \end{bmatrix}}$$

(b) Write (but do not evaluate) a matrix expression that gives each station's market share after 15 years.

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$$\mathbf{x}^{15} = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}^{15} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

(c) What will each station's market share eventually be (i.e. when the market distribution reaches a steady state)?

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At steady state: $P\mathbf{x} = \mathbf{x} \implies (P - I)\mathbf{x} = \mathbf{0}$

$$\implies \left[\begin{array}{rr|r} -0.1 & 0.05 & 0 \\ 0.1 & -0.05 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rr|r} -0.1 & 0.05 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$0.05x_2 = 0.1x_1 \implies \begin{cases} x_1 \text{ is free} \\ x_2 = 2x_1 \end{cases}$$

Total market share is 100% :

$$1 = x_1 + x_2 = x_1 + 2x_1 = 3x_1 \implies \boxed{\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{2}{3} \end{cases}}$$