THOMPSON RIVERS UNIVERSITY

MATH 1300
Linear Algebra for Engineers

Instructor: Richard Taylor

FINAL EXAM
SOLUTIONS

5 December 2012   14:00–17:00

IB 1020

PROBLEM	GRADE	OUT OF
1		8
2		10
3		10
4		10
5		5
6		4
7		9
8		8
9		7
10		5
11		6
12		8
TOTAL:		90

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 11 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/8

Problem 1: Find the solution(s) of the following linear system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 5 \end{bmatrix} \xrightarrow[R_4-2R_1;\ \frac{1}{5}R_3]{R_2-2R_1} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{6}(R_4+4R_2)]{R_3+R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{-(R_2+3R_4)} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{R_1+2R_2)} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\Longrightarrow \begin{cases} x_1 = -2t - 4s - 3w \\ x_2 = w \in \mathbb{R} \\ x_3 = -2s \\ x_4 = s \in \mathbb{R} \\ x_5 = t \in \mathbb{R} \\ x_6 = \frac{1}{3} \end{cases}$$

/10

Problem 2: Consider the following linear system in which a, $b \in \mathbb{R}$ are constants:

$$\begin{cases} ax_1 + bx_3 = 2 \\ ax_1 + ax_2 + 4x_3 = 4 \\ ax_2 + 2x_3 = b \end{cases}$$

Determine the value(s) of a and b such that this system has:

(a) A unique solution.

/4

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{bmatrix}$$

To get a unique solution we require both $b - 2 \neq 0$ and $a \neq 0$:

$$\Longrightarrow \boxed{a \neq 0,\ b \neq 2}$$

(b) No solution.

/2

To get no solution we require either:

both $b - 2 = 0$ and $b - 2 \neq 0$ (clearly impossible), or

$a = 0$, but also and $b - 2 \neq 0$ (otherwise the system ends up being consistent)

$$\Longrightarrow \boxed{a = 0,\ b \neq 2}$$

(c) A one-parameter family of (infinitely many) solutions.

/2

To get a free variable we can have $a = 0$ and/or $b - 2 = 0$. In the former case, we actually get two free variables. So for a one-parameter family of solutions we need

$$\Longrightarrow \boxed{a \neq 0,\ b = 2}$$

(d) A two-parameter family of (infinitely many) solutions.

/2

The reasoning in part (c) gives

$$\Longrightarrow \boxed{a = 0,\ b = 2}$$

/10 **Problem 3:** Let A and B be invertible $n \times n$ matrices. For each of the following, indicate whether the statement is always true, sometimes true, or always false.

(a) $(AB)^2 = A^2B^2$

/2

In general,

$$(AB)^2 = ABAB \neq A^2B^2$$

unless A and B commute (for example if $A = I$, or if $A = B$), so the statement is *sometimes true.*

(b) $(AB^{-1})^{-1} = BA^{-1}$

/2

In general,

$$(AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = BA^{-1}$$

so the statement is *always true.*

(c) $(AB^{-1})(A^{-1}B) = I$

/2

In general,

$$(AB^{-1})(A^{-1}B) = AB^{-1}A^{-1}B \neq I$$

unless A^{-1} and B^{-1} commute (for example if $A = I$, or if $A = B$), so the statement is *sometimes true.*

(d) $(A - B)^2 = (B - A)^2$

/2

We have

$$(A - B)^2 = (A - B)(A - B) = A^2 - AB - BA + B^2$$

and

$$(B - A)^2 = (B - A)(B - A) = B^2 - BA - AB + A^2 = A^2 - AB - BA + B^2$$

so the statement is *always true.*

(e) $AB = BA$

/2

In general this statement isn't true, unless A and B commute. So the statement is *sometimes true.*

/10 **Problem 4:** You have been contracted by Canadian Blood Services to analyze the donation patterns of blood donors in Canada. You find that 80% of people who donate blood in a given year will also donate the next year, while 10% of those who do *not* donate in a given year *will* donate the next. In 2012 there were 1 million blood donors and 26 million non-donors.

Assume that these system dynamics are unchanged from year to year, and that the population of Canada remains constant.

(a) Model this situation as a Markov chain: define (in words) the components of your state vector $\mathbf{x}$, and determine the corresponding transition matrix.

/2

With $\mathbf{x} = \begin{bmatrix} x_D \\ x_N \end{bmatrix} = \begin{bmatrix} \text{\# of donors} \\ \text{\# of non-donors} \end{bmatrix}$ the transition matrix is $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$.

(b) How many blood donors will there be in 2015?

/2

$$\mathbf{x}^{(3)} = A^3\mathbf{x}^{(0)} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 26 \end{bmatrix} = \begin{bmatrix} 6.256 \\ 20.744 \end{bmatrix}$$

so there will be 6.256 million donors.

(c) The eigenvalues of the transition matrix are $\lambda_1 = 0.7$ and $\lambda_2 = 1$. Find an expression for the number of blood donors in year n (where $n = 0$ corresponds to the year 2012).

/4

In general we have $\mathbf{x}^{(n)} = A^n\mathbf{x}^{(0)} = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2$, where $\mathbf{x}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. So first find the eigenvectors:

$$\lambda_1 = 1: \quad (A - (1)I)\mathbf{v} = \mathbf{0} \implies \left[\begin{array}{cc|c} -0.2 & 0.1 & 0 \\ 0.2 & -0.1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 0.7: \quad (A - (0.7)I)\mathbf{v} = \mathbf{0} \implies \left[\begin{array}{cc|c} 0.1 & 0.1 & 0 \\ 0.2 & 0.2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We also require

$$c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 26 \end{bmatrix} \implies c_1 = 9,\ c_2 = -8.$$

So

$$\mathbf{x}^{(n)} = 9(1)^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 8(0.7)^2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and in particular the number of donors in year n is $\boxed{9 - 8(0.7)^n \text{ (million).}}$

(d) Determine the long-term equilibrium number of blood donors at the steady state of this Markov chain.

/2

In the answer to part (c) we can let $n \to \infty$ to find the steady state number of donors is 9 million. Alternatively, the steady state $\mathbf{x}$ is characterized by

$$\mathbf{x} = A\mathbf{x} \implies (A - I)\mathbf{x} = \mathbf{0}.$$

We could solve this for $\mathbf{x}$, or just recognize this as the same eigenvector equation we solved for $\mathbf{v}_1$ in part (c), so that

$$\mathbf{x} = t\mathbf{v}_1 = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

We have $t + 2t = 27 \implies t = 9$ so $\mathbf{x} = 9\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$. In particular, the equilibrium number of donors is 9 million.

/5

Problem 5: Show that $\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$ is singular (non-invertible) for all values of the entries.

$$\begin{vmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{vmatrix} = (-a)\begin{vmatrix} b & c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & f & 0 & g \\ 0 & 0 & h & 0 \end{vmatrix}$$

$$= (-a)(b)\begin{vmatrix} 0 & e & 0 \\ f & 0 & g \\ 0 & h & 0 \end{vmatrix}$$

$$= (-a)(b)(-f)\begin{vmatrix} e & 0 \\ h & 0 \end{vmatrix}$$

$$= (-a)(b)(-f)(0) = 0$$

so the matrix is singular, regardless of the values of the entries.

/4

Problem 6: Points $A(3,0,2)$, $B(4,3,0)$ and $C(8,1,-1)$ are the vertices of a right-angled triangle. At which vertex is the right angle?

Side AB is in the direction of $\mathbf{u} = (4,3,0) - (3,0,2) = (1,3,-2)$.

Side AC is in the direction of $\mathbf{v} = (8,1,-1) - (3,0,2) = (5,1,-3)$.

Side BC is in the direction of $\mathbf{w} = (8,1,-1) - (4,3,0) = (4,-2,-1)$.

We can find the angles at the vertices using the dot product:

$$\begin{aligned} \angle A: &\quad \mathbf{u}\cdot\mathbf{v} = 5+3+6 = 14 \\ \angle B: &\quad \mathbf{u}\cdot\mathbf{w} = 4-6+2 = 0 \\ \angle C: &\quad \mathbf{v}\cdot\mathbf{w} = 20-2+3 = 21 \end{aligned}$$

So the right angle is at vertex B.

/9

Problem 7: Suppose A is an upper-triangular matrix whose diagonal entries are all non-zero.

(a) Show that A is invertible.

/3

A is triangular so $\det A$ is the product of its diagonal entries, hence $\det A \neq 0$ and A is invertible.

(b) Show that the columns of A are linearly independent.

/2

Since A is invertible, the columns of A are (by the big theorem discussed in class) linearly independent.

(c) Calculate A^{-1} for $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

/4

$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/5 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{R_2-2R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 1/5 \\ 0 & 1 & 0 & 0 & 1/2 & -2/5 \\ 0 & 0 & 1 & 0 & 0 & 1/5 \end{array}\right]$$

$$\xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & 7/5 \\ 0 & 1 & 0 & 0 & 1/2 & -2/5 \\ 0 & 0 & 1 & 0 & 0 & 1/5 \end{array}\right] = [\,I \mid A^{-1}\,]$$

/8 **Problem 8:** Let $\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$. (a) Are $\mathbf{v}_1$, $\mathbf{v}_2$ linearly independent?

No. Clearly they are not scalar multiples of each other.

(b) Find a vector $\mathbf{v}_3$ such that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. Prove that your choice really does yield a basis.

/3

Any vector linearly independent of the other 2 will do, e.g. $\mathbf{v}_3 = (0, 0, 1)$.

To prove linearly independence consider the system

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \implies \underbrace{\begin{bmatrix}1 & 1 & 0\\1 & 0 & 0\\0 & 1 & 1\end{bmatrix}}_{A}\mathbf{c} = \mathbf{0}.$$

We have $\det A = (-1)\begin{vmatrix}1 & 0\\1 & 1\end{vmatrix} = -1 \neq 0$ so A is invertible, hence this system has only the trivial solution $\mathbf{c} = A^{-1}\mathbf{0} = \mathbf{0}$.

Since we have 3 linearly independent vectors, they automatically span $\mathbb{R}^3$, hence form a basis for $\mathbb{R}^3$.

(c) Let $\mathbf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathcal{B}$ be the basis you found in part (b). Find $\mathbf{x}_\mathcal{B}$ (components of $\mathbf{x}$ relative to $\mathcal{B}$).

/3

We have $\mathbf{x}_\mathcal{B} = (c_1, c_2, c_3)$ where

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{x} \implies \begin{bmatrix}1 & 1 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix}1 & 1 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

$$\implies c_3 = -1, \quad c_2 = 1, \quad c_1 = 1 - 1 = 0$$

$$\implies \mathbf{x}_\mathcal{B} = (0, 1, -1).$$

/7 **Problem 9:** It can be useful to know that the eigenvalues of any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be found if you know only its determinant and its "trace" (defined as $\text{tr}(A) = a + d$).

(a) Show that the characteristic polynomial of A can be expressed as

/3

$$P(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

$$\begin{aligned} P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - \underbrace{(a + d)}_{\text{tr}(A)}\lambda + \underbrace{ad - bc}_{\det(A)} \end{aligned}$$

(b) Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$. Evaluate $\det(A)$ and $\text{tr}(A)$, form the characteristic polynomial as above, and determine the eigenvalues of A.

/4

We have $\det(A) = (10)(-2) - (-9)(4) = 16$ and $\text{tr}(A) = 10 - 2 = 8$ so

$$\begin{aligned} P(\lambda) &= \lambda^2 - 8\lambda + 16 \\ &= (\lambda - 4)^2 \end{aligned}$$

$$\implies \boxed{\lambda = 4 \quad \text{(with algebraic multiplicity 2)}}$$

/5

Problem 10: Solve by Gaussian elimination with back-substitution:

$$\begin{cases} x + (1+i)y = 0 \\ (1-i)x + 2y = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1+i & 0 \\ 1-i & 2 & 0 \end{bmatrix} \xrightarrow{R_2-(1-i)R_1} \begin{bmatrix} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(note that } (1-i)(1+i) = 2\text{)}$$

$$\implies \boxed{\begin{cases} x = -(1+i)t \\ y = t \in \mathbb{R} \end{cases}}$$

/6

Problem 11: Find all of the complex roots of $z^6 = 64$ and sketch their locations in the complex plane.

Since $1 = 1e^{i0} = e^{i(0+n2\pi)}$ we have

$$z^6 = e^{i(n2\pi)} \implies z = e^{in\frac{2\pi}{6}} = e^{in\frac{\pi}{3}}$$

$$\begin{cases} n=0: & z = e^{i0} = 1 \\ n=1: & z = e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ n=2: & z = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ n=3: & z = e^{i\pi} = -1 \\ n=4: & z = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ n=5: & z = e^{i\frac{5\pi}{3}} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{cases}$$

The roots are symmetrically spaced around the unit circle at angles of $\pi/3$.

/8

Problem 12: Find the functions $x(t)$ and $y(t)$ that satisfy

$$\begin{cases} x' = x + 3y \\ y' = 4x + 5y \end{cases}$$

along with the initial conditions $x(0) = 2$, $y(0) = 1$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} \implies \mathbf{x}' = A\mathbf{x}$$

Eigenvalues:

$$0 = \det(A - \lambda I) = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) \implies \lambda = -1,\ 7$$

Eigenvectors:

$$\lambda_1 = -1: \quad (A - (-1)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\lambda_2 = 7: \quad (A - (7)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} -6 & 3 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Initial conditions:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\implies \mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = \frac{3}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} + \frac{7}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}$$

$$\boxed{\begin{cases} x(t) = \frac{9}{8} e^{-t} + \frac{7}{8} e^{7t} \\ y(t) = -\frac{3}{4} e^{-t} + \frac{7}{4} e^{7t} \end{cases}}$$