THOMPSON RIVERS UNIVERSITY

MATH 212
Linear Algebra I

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MIDTERM EXAM #2
SOLUTIONS

28 March 2008 08:30–09:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		7
3		7
4		4
5		5
TOTAL:		29

/6

Problem 1: Consider the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix}$ where $\theta, \alpha \in \mathbb{R}$.

(a) Calculate $\det A$.

/4

expand in minors along the first column:

$$\det A = \cos\theta \begin{vmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix} - (-\sin\theta) \begin{vmatrix} \sin\theta & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix}$$

$$= \cos^2\theta \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} + \sin^2\theta \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$$

$$= \cos^2\theta(\underbrace{\cos^2\alpha + \sin^2\alpha}_{1}) + \sin^2\theta(\underbrace{\cos^2\alpha + \sin^2\alpha}_{1})$$

$$= \cos^2\theta + \sin^2\theta$$

$$= \boxed{1}$$

(b) Prove that A has an inverse for any values of θ and α.

/2

since $\det A = 1 \neq 0$ we have that A^{-1} exists $\forall \alpha, \theta \in \mathbb{R}$.

/7

Problem 2: Let $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (2, \alpha)$, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) If $\alpha = 2$, explain why $\mathcal{B}$ is *not* a basis for $\mathbb{R}^2$.

/3

if $\alpha = 2$ then $\mathbf{v}_2 = 2\mathbf{v}_1$

$\Longrightarrow$ $\mathbf{v}_1$, $\mathbf{v}_2$ are not linearly independent

$\Longrightarrow$ $\mathcal{B}$ is not a basis

(b) Prove that if $\alpha \neq 2$ then $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

/4

– if $\alpha \neq 2$ then $\det \begin{bmatrix} 1 & 2 \\ 1 & \alpha \end{bmatrix} = \alpha - 2 \neq 0$ so the columns $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent

– then $\mathcal{B}$ is a basis, since *any* two linearly independent vectors form a basis for $\mathbf{R}^2$

/7

Problem 3: Let A be a given $n \times n$ matrix.

(a) Let $V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$. (i.e. V is the set of solutions of the linear system $A\mathbf{x} = \mathbf{0}$.) Prove that V is a subspace of $\mathbb{R}^n$.

/4

Clearly $V \subseteq \mathbb{R}^n$, so we need only check the closure axioms:

1. suppose $\mathbf{x}_1, \mathbf{x}_2 \in V$; then $\mathbf{x}_1 + \mathbf{x}_2 \in V$, since:

$$A(\mathbf{x}_1 + \mathbf{x}_2) = \underbrace{A\mathbf{x}_1}_{\mathbf{0}} + \underbrace{A\mathbf{x}_2}_{\mathbf{0}} = \mathbf{0}$$

2. suppose $\mathbf{x} \in V$, $\alpha \in \mathbb{R}$; then $\alpha\mathbf{x} \in V$, since:

$$A(\alpha\mathbf{x}) = \alpha \underbrace{(A\mathbf{x})}_{\mathbf{0}} = \mathbf{0}$$

(b) Let $\mathbf{b} \in \mathbb{R}^n$ be a given non-zero vector, and let $W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$. (i.e. W is the set of solutions of the linear system $A\mathbf{x} = \mathbf{b}$.) Prove that W is *not* a vector space.

/3

solution #1:

W is not closed under addition: if $\mathbf{x}_1, \mathbf{x}_2 \in W$ then $\mathbf{x_1} + \mathbf{x}_2 \notin W$ since:

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}.$$

solution #2:

W is not closed under scalar multiplication: if $\mathbf{x} \in W$ and $\alpha \neq 0$ then $\alpha\mathbf{x} \notin W$ since:

$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\mathbf{b} \neq \mathbf{b}.$$

solution #3:

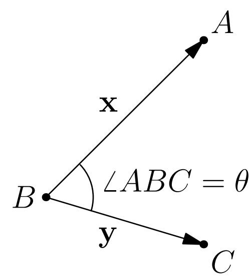
W does not contain the $\mathbf{0}$ element since

$$A\mathbf{0} = \mathbf{0} \neq \mathbf{b},$$

hence W is not a vector space.

(**Note:** Any one of these solutions would be sufficient; e.g. it is *not* necessary to show failure of closure under *both* addition and scalar multiplication; failure of just one of these suffices to demonstrate that W is not a vector space.)

/4 **Problem 4:** The vertices $A(3,0,2)$, $B(4,3,0)$ and $C(8,1,-1)$ form a triangle in $\mathbb{R}^3$. Calculate the angle $\angle ABC$ to the nearest degree.



Let $\mathbf{x} = \overline{BA} = (3,0,2) - (4,3,0) = (-1,-3,2)$.
Let $\mathbf{y} = \overline{BC} = (8,1,-1) - (4,3,0) = (4,-2,-1)$.

Then

$$\mathbf{x} \cdot \mathbf{y} = -4 + 6 - 2 = 0 = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta \implies \cos\theta = 0 \implies \boxed{\theta = 90^\circ}$$

/5 **Problem 5:** The planes $x + 2y + 13z = 8$ and $x + 3y + 18z = 10$ intersect in a line. Find the equation of this line:

/3 (a) in parametric (vector) form.

Solve the linear system:

$$\begin{bmatrix} 1 & 2 & 13 & 8 \\ 1 & 3 & 18 & 10 \end{bmatrix} \xrightarrow{R_2 - R_2} \begin{bmatrix} 1 & 2 & 13 & 8 \\ 0 & 1 & 5 & 2 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 5 & 2 \end{bmatrix}$$

The general solution (which gives all points on the line of intersection) is

$$\begin{cases} x = 4 - 3t \\ y = 2 - 5t \qquad (t \in \mathbb{R}) \\ z = t \end{cases}$$

which can be written in vector form as

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad \text{with} \quad \mathbf{x}_0 = (4,2,0), \; \mathbf{v} = (-3,-5,1)$$

/2 (b) in symmetric form.

Solve for the parameter t:

$$t = \boxed{\frac{x-4}{-3} = \frac{y-2}{-5} = \frac{z-0}{-1}}$$