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Problem 1: (a) Write, in *standard form*, the equation of the plane in $\mathbb{R}^3$ that passes through the point $\mathbf{r} = (-3, 5, 1)$ and has normal vector $\mathbf{n} = (1, -1, 5)$.

$$(\underline{x} - \underline{r}) \cdot \underline{n} = 0 \rightarrow [(x,y,z) - (-3,5,1)] \cdot (1,-1,5) = 0$$

$$\rightarrow (x+3, y-5, z-1) \cdot (1,-1,5) = 0$$

$$\rightarrow (x+3) - (y-5) + 5(z-1) = 0$$

$$\rightarrow \boxed{x - y + 5z = -3}$$

(b) Write the equation of the same plane in *parametric form*.

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 + y - 5z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

where $y, z \in \mathbb{R}$ are free.

$$\rightarrow \boxed{\begin{cases} x = -3 + s - 5t \\ y = s \\ z = t \end{cases} ; \; s, t \in \mathbb{R}}$$

(c) For a given point $\mathbf{x} \in \mathbb{R}^3$, the distance from $\mathbf{x}$ to the plane is equal to the component of $\mathbf{x} - \mathbf{r}$ in the direction $\mathbf{n}$ (i.e. the distance is equal to the length of $\text{proj}_{\mathbf{n}}(\mathbf{x} - \mathbf{r})$). Use this to find the distance from the point $(1, 0, 0)$ to the plane in part (a).

$$\text{dist} = \frac{|(\underline{x} - \underline{r}) \cdot \underline{n}|}{|\underline{n}|} = \frac{|[(-3,0,0) - (1,0,0)] \cdot (1,-1,5)|}{\sqrt{1 + 1 + 25}}$$

$$= \frac{|(-4,0,0) \cdot (1,-1,5)|}{\sqrt{27}}$$

$$= \boxed{\frac{4}{\sqrt{27}} \approx 0.77}$$

$$= \frac{12\sqrt{3}}{27} = \frac{4\sqrt{3}}{9} = \frac{\sqrt{432}}{27}$$

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Problem 2: Two planes in $\mathbb{R}^3$ have the following equations in standard form:

$$x + 2y - z = 3$$
$$2x + 3y + z = 1$$

(a) The planes intersect along a line. In what direction is this line oriented? (i.e., find a direction vector $\mathbf{v}$ so that the points $\mathbf{x}$ on the line are parametrized by $\mathbf{x} = \mathbf{r} + t\mathbf{v}$, $t \in \mathbb{R}$.)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & -7 \\ 0 & 1 & -3 & 5 \end{bmatrix}$$

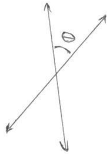
$$\therefore \begin{cases} x = -7 - 5z \\ y = 5 + 3z \\ z \text{ is free} \end{cases} \Rightarrow \underline{x} = \begin{bmatrix} -7-5z \\ 5+3z \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix}}_{\underline{r}} + z \underbrace{\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}_{\underline{v}}, \quad z \in \mathbb{R}.$$

$$\boxed{\underline{v} = (-5, 3, 1)}$$

(b) The angle between two planes is defined by the (acute) angle between the planes' normal vectors. Calculate the angle between the two planes in (a).

$\underline{n}_1 = (1, 2, -1)$
$\underline{n}_2 = (2, 3, 1)$ by inspection.

edge view:



$$\cos\Theta = \frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1|\,|\underline{n}_2|} = \frac{7}{\sqrt{6}\sqrt{14}} = \frac{7}{\sqrt{84}}$$

$$\to \boxed{\Theta = \cos^{-1}\frac{7}{\sqrt{84}} \approx 40.2^\circ}$$

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Problem 3: (a) Let $H \subset \mathbb{R}^2$ be the set of vectors (x, y) such that $y \geq x$. Is H a vector space? Why/why not?

No. $\underline{x} = (1,2) \in H$, but $(-1)\underline{x} = (-1,-2) \notin H$ since $-1 > -2$

$\therefore$ H isn't closed under scalar multiplication, hence not a vector space.

sub-

(b) Let $H \subset C[0, 1]$ be the set of even functions. Is H a subspace of $C[\cancel{0}, 1]$? Why/why not? -1

(Recall that $C[\cancel{0}, 1]$ -1 is the vector space consisting of all continuous functions defined on the interval $[\cancel{0}, 1]$. A function f is even if and only if $f(x) = f(-x)$).)
-1

Yes. Let $f, g \in H$.

- Then $f+g$ is even, so $f+g \in H$, since
$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$.

- αf is even, so $\alpha f \in H$, since
$(\alpha f)(-x) = \alpha f(-x) = \alpha f(x) = (\alpha f)(x)$.

Because H is a subset of a vector space, and closed under addition and scalar multiplication, it is a subspace.

/8 **Problem 4:** (a) For what value(s) of h are the vectors $\mathbf{u} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}0\\1\\h\end{bmatrix}$ linearly independent?

Consider $c_1\underline{u} + c_2\underline{v} + c_3\underline{w} = \underline{0}$.

$$\rightarrow \begin{bmatrix}1&1&0\\0&1&1\\1&0&h\end{bmatrix} \rightarrow \begin{bmatrix}1&1&0\\0&1&1\\0&-1&h\end{bmatrix} \rightarrow \begin{bmatrix}1&1&0\\0&1&1\\0&0&h+1\end{bmatrix}$$

So $\underline{c} = 0$ is the unique sol'n if $h+1 \neq 0$.

$\therefore$ $\underline{u}, \underline{v}, \underline{w}$ are linearly independent iff $\boxed{h \neq -1.}$

(if $h=-1$ then $\underline{w} = -\underline{u} + \underline{v}$ $\Rightarrow$ linearly dependent)

(b) Let $p(x) = x$ and $q(x) = 1 + 2x + 3x^2$. Does the set $\mathcal{B} = \{p, q\}$ span the vector space $\mathcal{P}_2 =$ {polynomials of degree ≤ 2}? Why / why not?

<u>No.</u> Consider $f(x) = x^2$. Then $f \in P_2$ but $f \notin \text{span}\{p, q\}$, since $f \in \text{span}\{p,q\}$ requires

$$c_1(x) + c_2(1 + 2x + 3x^2) = x^2$$

for some c_1, c_2.

Equating coefficients: $\begin{cases} 3c_2 = 1 \\ c_1 + 2c_2 = 0 \\ c_2 = 0 \end{cases}$ inconsistent.

so no solution for c_1, c_2.

/4 **Problem 5:** Prove Pythagoras' Theorem:

Vectors $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ are orthogonal if and only if $|\mathbf{x}+\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$.

$$|\underline{x}+\underline{y}|^2 = (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y})$$

$$= \underline{x}\cdot\underline{x} + 2\underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{y}$$

$$= |\underline{x}|^2 + |\underline{y}|^2 + 2\underline{x}\cdot\underline{y}$$

So $|\underline{x}+\underline{y}|^2 = |\underline{x}|^2 + |\underline{y}|^2$ iff $\underline{x}\cdot\underline{y} = 0$

(ie, iff $\underline{x} \perp \underline{y}$).