THOMPSON RIVERS UNIVERSITY

MATH 212
Linear Algebra I

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MIDTERM EXAM #1
SOLUTIONS

13 February 2008 08:30–09:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		8
2		8
3		5
4		6
5		8
TOTAL:		35

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Problem 1: Consider the following system of linear equations, in which h and k are constants.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

For what value(s) of h and k does this system have:

(a) an infinite number of solutions? Find the solutions in this case.

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

There will be an infinite family of solutions provided:

$$\begin{cases} 8 - 4h = 0 \\ k - 8 = 0 \end{cases} \implies \boxed{h = 2,\ k = 8}$$

In this case the REF of the augmented matrix is

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the general solution is

$$\boxed{\begin{cases} x_1 = 2 - 2t \\ x_2 = t \end{cases} \qquad (t \in \mathbb{R})}$$

(b) no solution?

There will be no solution provided:

$$\begin{cases} 8 - 4h = 0 \\ k - 8 \neq 0 \end{cases} \implies \boxed{h = 2,\ k \neq 8}$$

(c) one unique solution? Find the solution in this case.

There will be a unique solution provided:

$$8 - 4h \neq 0 \implies \boxed{h \neq 2}$$

In this case we can use back-substitution to find the solution:

$$\boxed{\begin{cases} x_2 = \dfrac{k-8}{8-4h} \\ x_1 = 2 - hx_2 = 2 - h \cdot \dfrac{k-8}{8-4h} \end{cases}}$$

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Problem 2: (a) Use the Gauss-Jordan method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+2R_3]{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = [\,I \mid A^{-1}\,]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}}$$

(b) Use your answer to part (a) to find the solution of the linear system

$$\begin{aligned} x + 2z &= a \\ x + y &= b \\ x + y + z &= c \end{aligned}$$

where a, b and c are constants.

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b - 2c \\ -a - b + 2c \\ -b + c \end{bmatrix}$$

$$\boxed{\begin{cases} x = a + 2b - 2c \\ y = -a - b + 2c \\ z = -b + c \end{cases}}$$

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Problem 3: Assuming that A, B and C are all $n \times n$ invertible matrices. Solve the matrix equation

$$C(A+X)B^{-1} = I + CB$$

for X, where I is the $n \times n$ identity matrix. Simplify your solution as much as possible.

$$C(A+X)B^{-1} = I + CB$$

$$\begin{aligned} \implies A + X &= C^{-1}(I+CB)B \\ &= C^{-1}B + B^2 \end{aligned}$$

$$\implies \boxed{X = C^{-1}B + B^2 - A}$$

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Problem 4: Prove the following:

Theorem. If A is an invertible matrix then $(A^{-1})^{-1} = A$.

approach #1: We have that C is an inverse for A^{-1} if and only if $CA^{-1} = A^{-1}C = I$. Observe that the matrix $C = A$ has this property:

$$\begin{aligned} CA^{-1} &= AA^{-1} = I \\ A^{-1}C &= A^{-1}A = I. \end{aligned}$$

Therefore $C = A$ is the inverse of A^{-1}.

approach #2: *Assuming* that $(A^{-1})^{-1}$ exists, it satisfies the following identity:

$$(A^{-1})^{-1}A^{-1} = I.$$

Multiplying on both sides by A yields:

$$(A^{-1})^{-1}\underbrace{A^{-1}A}_{I} = IA \implies (A^{-1})^{-1} = A.$$

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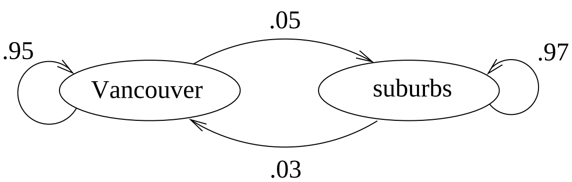
Problem 5: Suppose that every year, 5% of Vancouver's population moves into the suburbs, while 3% of the suburban population moves into Vancouver. Suppose that 1.5 million people now live in Vancouver, and that 1.0 million people live in the suburbs.

(a) Write a matrix expression (but do not evaluate it) that gives the populations of Vancouver and its suburbs after this process has continued for 8 years.



Let $\mathbf{x}^n = \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix}$ where $\begin{cases} x_1^n & = \text{population of Vancouver after } n \text{ years} \\ x_2^n & = \text{population of suburbs after } n \text{ years.} \end{cases}$

Then

$$\begin{cases} x_1^{n+1} & = 0.95x_1^n + 0.03x_2^n \\ x_2^{n+1} & = 0.05x_1^n + 0.97x_2^n \end{cases} \implies \mathbf{x}^{n+1} = \underbrace{\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}}_{A} \mathbf{x}^n$$

so that

$$\mathbf{x}^n = A^n \mathbf{x}^0.$$

After 8 years, the population vector (in millions) is

$$\boxed{\mathbf{x}^8 = A^8 \mathbf{x}^0 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}^8 \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}}$$

(b) If this process continues indefinitely, what will be the equilibrium populations of Vancouver and its suburbs?

At equilibrium we have

$$A\mathbf{x} = \mathbf{x} \implies (A - I)\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} -0.05 & 0.03 \\ 0.07 & -0.03 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is equivalent to the following linear system:

$$\left[\begin{array}{rr|r} -0.05 & 0.03 & 0 \\ 0.05 & -0.03 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{rr|r} -0.05 & 0.03 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

whose general solution is

$$\begin{cases} x_1 = 3t \\ x_2 = 5t \end{cases} \quad (t \in \mathbf{R})$$

We require the total population to be

$$1.5 + 1.0 = x_1 + x_2 = 3t + 5t = 8t \implies t = \frac{2.5}{8} = \frac{5}{16}$$

so at equilibrium the populations are

$$\boxed{\begin{cases} x_1 = 3 \cdot \frac{5}{16} = \frac{15}{16} \approx 0.94 \text{ million} \\ x_2 = 5 \cdot \frac{5}{16} = \frac{25}{16} \approx 1.56 \text{ million} \end{cases}}$$