Thompson Rivers University

MATH 212
Linear Algebra I

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FINAL EXAM
SOLUTIONS

19 April 2008   09:00–12:00

PROBLEM	GRADE	OUT OF
1		6
2		6
3		4
4		3
5		4
6		6
7		7
8		4
9		4
10		4
11		5
TOTAL:		53

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 8 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

/6

Problem 1: Consider the following system of linear equations, in which $a \in \mathbb{R}$ is a given constant.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

For what value(s) of a does this system have:

(a) an infinite number of solutions? Find the solutions in this case.

/4

Row reduce the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-3R_1} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

This system will have a free variable (hence an infinite family of solutions) if and only if both:

$$\begin{cases} a^2 - 16 = 0 \\ a - 4 = 0 \end{cases} \implies \boxed{a = 4}$$

(b) a unique solution? Find the solution in this case.

/1

The system will have three pivots (hence a unique solution) if and only if:

$$a^2 - 16 \neq 0 \implies \boxed{a \neq \pm 4}$$

(c) no solution?

/1

The system will have no solution if and only if

$$\begin{cases} a^2 - 16 = 0 \\ a - 4 \neq 0 \end{cases} \implies \boxed{a = -4}$$

/6 **Problem 2:** Consider the matrix $A = \begin{bmatrix} k & 0 & 0 \\ 1 & k & 0 \\ 0 & 1 & k \end{bmatrix}$ where $k \in \mathbb{R}$ is a given constant.

(a) Prove that A is invertible if and only if $k \neq 0$.

/3

We have:

$$\begin{aligned} \det A &= (k)\begin{vmatrix} k & 0 \\ 1 & k \end{vmatrix} - (1)\begin{vmatrix} 0 & 0 \\ 1 & k \end{vmatrix} + (0)\begin{vmatrix} 0 & 0 \\ k & 0 \end{vmatrix} \\ &= k(k^2 - 0) - 1(0 - 0) + 0 \\ &= k^3 \end{aligned}$$

Thus $\det A \neq 0$ (hence A is invertible) if and only if $k^3 \neq 0$ (equivalently $k \neq 0$).

(b) Assume $k \neq 0$. Use the Gauss-Jordan method to calculate A^{-1}.

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$$[\, A \mid I \,] = \left[\begin{array}{ccc|ccc} k & 0 & 0 & 1 & 0 & 0 \\ 1 & k & 0 & 0 & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - \frac{1}{k}R_1} \left[\begin{array}{ccc|ccc} k & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & -1/k & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{k}R_2} \left[\begin{array}{ccc|ccc} k & 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & -1/k & 1 & 0 \\ 0 & 1 & k & 1/k^2 & -1/k & 1 \end{array}\right]$$

$$\xrightarrow{\times \frac{1}{k}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 0 & 1 & 0 & -1/k^2 & 1/k & 0 \\ 0 & 0 & 1 & 1/k^3 & -1/k^2 & 1/k \end{array}\right] = [\, I \mid A^{-1} \,]$$

Thus

$$\boxed{A^{-1} = \begin{bmatrix} 1/k & 0 & 0 \\ -1/k^2 & 1/k & 0 \\ 1/k^3 & -1/k^2 & 1/k \end{bmatrix}}$$

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Problem 3: Suppose A is an $n \times n$ matrix such that $A^2 - 3A + I = 0$. Show that $A^{-1} = 3I - A$.

solution #1 (part marks):

If we *assume* that A is invertible, we can right-multiply both sides by A^{-1}:

$$(A^2 - 3A + I)A^{-1} = (0)A^{-1} \implies A - 3I + A^{-1} = 0$$
$$\implies A^{-1} = 3I - A.$$

solution #2:

A better approach: we have

$$\begin{aligned} I = 3A - A^2 &= A(3I - A) \\ &= (3I - A)A \end{aligned}$$

hence A is invertible and its inverse is the matrix $3I - A$.

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Problem 4: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$ and $k \in \mathbb{R}$. Which of the following expressions do not make sense? Explain.

(a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

$\mathbf{v} \cdot \mathbf{w}$ is a scalar; the dot-product of a vector with a scalar is undefined.

(b) $(\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$

$\mathbf{u} \cdot \mathbf{v}$ is a scalar; addition of a scalar and a vector is undefined.

(c) $k \cdot (\mathbf{u} + \mathbf{v})$

$\mathbf{u} + \mathbf{v}$ is a vector; the dot-product of a scalar with a vector is undefined.

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Problem 5: Prove Pythagoras' Theorem:

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthogonal then $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Assume $\mathbf{x}$, $\mathbf{y}$ are orthogonal (i.e. $\mathbf{x} \cdot \mathbf{y} = 0$). Then:

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + \underbrace{\mathbf{x} \cdot \mathbf{y}}_{0} + \underbrace{\mathbf{y} \cdot \mathbf{x}}_{0} + \mathbf{y} \cdot \mathbf{y} \\
&= \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2
\end{aligned}$$

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Problem 6: Let M_{22} be the set of all 2×2 matrices. (Recall that M_{22} is a vector space.) Let V be the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Prove that V is a subspace of M_{22}.

Clearly $V \subset M_{22}$ so we need only check the closure axioms:

1. Suppose $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in V$. Then $A + B \in V$ since

$$A + B = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$$

has the required form for membership in V.

2. Suppose $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in V$, $\alpha \in \mathbb{R}$. Then $\alpha A \in V$ since

$$\alpha A = \begin{bmatrix} \alpha a & 0 \\ 0 & \alpha b \end{bmatrix}$$

has the required form for membership in V.

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Problem 7: Let $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (2, 2, 0)$, $\mathbf{w} = (3, 3, 3)$.

/3 (a) Prove that $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a basis for $\mathbb{R}^3$.

We have

$$\det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$$

so $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent. Since any 3 linearly independent vectors form a basis for $\mathbb{R}^3$, $\mathcal{B}$ must be such a basis.

/3 (b) Let $\mathbf{x} = (1, 2, 3)$. Find the vector $\mathbf{x}_{\mathcal{B}}$ giving the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$.

We have $\mathbf{x}_{\mathcal{B}} = (c_1, c_2, c_3)$ where $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{x}$. This gives the a linear system with the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

By back-substition this gives $c_3 = 1 \implies c_2 = -\frac{1}{2} \implies c_1 = -1$ so

$$\boxed{\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}}$$

/1 (c) Is $\mathcal{C} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ a basis for $\mathbb{R}^3$? Justify your answer.

No. There can be *at most* 3 linearly independent vectors in $\mathbb{R}^3$; i.e. the linear system with augmented matrix $\left[\begin{array}{cccc|c} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} & \mathbf{0} \end{array}\right]$ must have at least one free variable since the coefficient matrix has 4 columns but at most 3 pivots, hence $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{x} = \mathbf{0}$ has more than just the trivial solution and the vectors are therefore not linearly independent.

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Problem 8: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the (x, y)-plane that effects a counterclockwise rotation by 90° followed by an expansion by a factor of 5 in the x-coordinate. Find the standard matrix for T.

We have

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-5\\0\end{bmatrix}$$

so

$$\boxed{A = \begin{bmatrix}0 & -5\\1 & 0\end{bmatrix}}$$

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Problem 9: Find an equation of the plane passing through the point $(2, -7, 6)$ and parallel to the plane $5x - 2y + z = 9$.

The given plane has normal $\mathbf{n} = (5, -2, 1)$. The plane in question is parallel to the given plane, so must have the same normal. Therefore the equation of the plane sought is, in normal form:

$$\begin{aligned}\big((x, y, z) - (2, -7, 6)\big) \cdot (5, -2, 1) = 0 &\implies (x - 2, y + 7, z - 6) \cdot (5, -2, 1) = 0\\ &\implies (5)(x - 2) + (-2)(y + 7) + (1)(z - 6) = 0\\ &\implies \boxed{5x - 2y + z = 30}\end{aligned}$$

/4 **Problem 10:** Find the eigenvalues of the matrix $A = \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$.

$$\begin{aligned} 0 = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} &= -(2+\lambda)(2-\lambda) + 7 \\ &= -(4-\lambda^2) + 7 \\ &= \lambda^2 + 3 \end{aligned}$$

$$\implies \boxed{\lambda = \pm\sqrt{3}i}$$

/5 **Problem 11:** Consider the matrix $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$.

/3 (a) Show that $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector for A and find the corresponding eigenvalue.

$$A\mathbf{v} = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4\mathbf{v}$$

So $\mathbf{v}$ is an eigenvector, with corresponding eigenvalue $\lambda = 4$.

/2 (b) Calculate $A^5\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$.

$$A^5\mathbf{x} = A^5(3\mathbf{v}) = 3(A^5\mathbf{v}) = 3(4^5\mathbf{v}) = 3072\mathbf{v} = \begin{bmatrix} 9216 \\ 6144 \end{bmatrix}$$