Name: ______________________ Student #: ______________

Thompson Rivers University

MATH 212
Linear Algebra I

Instructor: Richard Taylor

FINAL EXAM

24 April 2006 19:00–22:00

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 9 pages.
4. Organize and write your solutions neatly.
5. You may use the backs of pages for calculations.
6. You must clearly show your work to receive full credit.
7. You may use a calculator.

PROBLEM	GRADE	OUT OF
1		6
2		6
3		6
4		6
5		10
6		5
7		5
8		6
TOTAL:		50

***** THIS EXAM CONTAINS SOLUTIONS *****

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Problem 1: Find all solutions of the linear system

$$x_1 + 2x_2 + 2x_3 + x_4 = 14$$
$$-x_1 - 2x_2 - 3x_4 = -4$$
$$2x_1 + 4x_2 + 8x_3 - 2x_4 = 48.$$

SOLUTION:

Gauss-Jordan method gives: $\left[\begin{array}{rrrr|r} 1 & 2 & 2 & 1 & 14 \\ -1 & -2 & 0 & -3 & -4 \\ 2 & 4 & 8 & -2 & 48 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

so the general solution is: $\begin{cases} x_1 = 4 - 2x_2 - 3x_4 \\ x_3 = 5 + x_4 \\ x_2,\ x_4 \text{ are free} \end{cases}$

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Problem 2: Consider the linear system

$$x_1 + x_3 = q$$
$$x_2 + 2x_4 = 0$$
$$x_1 + 2x_3 + 3x_4 = 0$$
$$2x_2 + 3x_3 + px_4 = 3$$

in which the numbers p and q are parameters. Under what conditions (i.e. for what values of p and q) does this system have: (a) a unique solution?

SOLUTION:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & q \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & p & 3 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & q \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & -q \\ 0 & 0 & 0 & p-13 & 3+3q \end{array}\right]$$

so there is a unique solution as long as $p - 13 \neq 0$. Therefore $p \neq 13$.

(b) no solution?

SOLUTION:

$p = 13$ and $3 + 3q \neq 0$. Therefore $p = 13$ and $q \neq -1$.

(c) an infinite number of solutions?

SOLUTION:

$p = 13$ and $q = -1$.

/6 **Problem 3:**(a) Use the Gauss-Jordan method to find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$.

SOLUTION:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 1 & 0 \\ 0 & 0 & 1 & 16 & -4 & 1 \end{array}\right]. \text{ Therefore } A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 16 & -4 & 1 \end{bmatrix}$$

(b) Find a matrix X such that $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

SOLUTION:

$$AX = B \implies X = A^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 16 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \\ 17 & -3 \end{bmatrix}.$$

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Problem 4:(a) Calculate the determinant of the matrix $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$.

SOLUTION:

$$\det A = (a)\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + (1)\begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix}$$
$$= (a)(a^2 - 1) - (1)(a - 1) + (1)(1 - a)$$
$$= a^3 - 3a + 2$$

(b) Under what conditions (i.e. for what value(s) of a) is A *not* invertible?

SOLUTION:

$$\det A = a^3 - 3a + 2 = 0 \implies a = -2 \text{ or } 1$$

/10 **Problem 5:** Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$.

(a) Find the characteristic polynomial for A.

SOLUTION:

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 5 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - 10 = \lambda^2 - 3\lambda - 10$$

(b) Find the eigenvalues for A, and the corresponding eigenvectors.

SOLUTION:

$\det(A - \lambda I) = 0 \implies \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0 \implies \lambda = 5$ or $\lambda = -2$.

case $\lambda_1 = 5$: $(A - \lambda I)\mathbf{v} = \mathbf{0} \implies \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 5 & -5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_1 = (1, 1)$

case $\lambda_2 = -2$: $(A - \lambda I)\mathbf{v} = \mathbf{0} \implies \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 5 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 2/5 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_2 = (2, -5)$

Problem 5 continued...

(c) Show that -8 is an eigenvalue of A^3.

SOLUTION:

Consider $\mathbf{v}_2$ from part (b):

$(A^3)\mathbf{v}_1 = A(A(A\mathbf{v}_1)) = A(A(\lambda_1\mathbf{v}_1)) = \lambda_1 A(A\mathbf{v}_1) = \lambda_1 A(\lambda_1\mathbf{v}_1) = \cdots = \lambda_1^3\mathbf{v}_1 = (-2)^3\mathbf{v}_1$

Since $(A^3)\mathbf{v}_1 = (-8)\mathbf{v}_1$, by definition -8 is an eigenvalue of A^3 (with eigenvector $\mathbf{v}_1$).

(d) Use your answer to (b) to calculate A^5.

SOLUTION:

We have $A = PDP^{-1}$ with $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix}$.

Therefore

$$\begin{aligned} A^5 = PD^5P^{-1} &= \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 5^5 & 0 \\ 0 & (-2)^5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix}^{-1} \\ &= -\frac{1}{7} \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 5^5 & 0 \\ 0 & (-2)^5 \end{bmatrix} \begin{bmatrix} -5 & -2 \\ -1 & 1 \end{bmatrix} \\ &= -\frac{1}{7} \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} (-5)5^5 & (-2)5^5 \\ -(-2)^5 & (-2)^5 \end{bmatrix} \\ &= -\frac{1}{7} \begin{bmatrix} (-5)5^5 - 2(-2)^5 & (-2)5^5 + 2(-2)^5 \\ (-5)5^5 + 5(-2)^5 & (-2)5^5 - 5(-2)^5 \end{bmatrix} \\ &= \begin{bmatrix} 2223 & 902 \\ 2255 & 870 \end{bmatrix} \end{aligned}$$

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Problem 6: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that effects a counter-clockwise rotation by 30°, followed by a reflection across the y-axis. Find the standard matrix A_T for T (i.e. so that $T(\mathbf{x}) = A_T\mathbf{x}$).

SOLUTION:

$T(\mathbf{e}_1) = (-\cos(30), \sin(30)) = (-\sqrt{3}/2, 1/2)$

$T(\mathbf{e}_2) = (\sin(30), \cos(30)) = (1/2, \sqrt{3}/2)$

Therefore $A_T = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$.

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Problem 7: Recall that P_2 is the vector space consisting of all polynomials of degree ≤ 2. Do the polynomials $f(x) = 1 + x + x^2$, $g(x) = x + x^2$ and $h(x) = x^2$ form a basis for P_2?

SOLUTION:

f, g, h form a basis for P_2 if and only if for any coefficients A, B, C there exist scalars c_1, c_2, c_3 such that

$$c_1 f(x) + c_2 g(x) + c_2 h(x) = A + Bx + Cx^2$$
$$\Longrightarrow c_1(1 + x + x^2) + c_2(x + x^2) + c_3(x^2) = A + Bx + Cx^2.$$

Matching coefficients, this is equivalent to the linear system

$$\begin{aligned} c_1 &= A \\ c_1 + c_2 &= B \\ c_1 + c_2 + c_3 &= C. \end{aligned}$$

The coefficient matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ has $\det A = 1 \neq 0$, so there is a (unique) solution.

Therefore f, g, h form a basis for P_2.

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Problem 8: Let A be a given 4×4 matrix, and let $V \subset \mathbb{R}^4$ be the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. That is, $V = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$. Prove that V is a subspace of $\mathbb{R}^4$. (V is called the *null space* of A.)

SOLUTION:

Let $\mathbf{x}$, $\mathbf{y} \in V$. Then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = 0$.

Then $\mathbf{x} + \mathbf{y} \in V$ since $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$,

and $\alpha\mathbf{x} \in V$ since $A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0}$.

Since V is a subset of the vector space $\mathbb{R}^4$ and is closed under addition and scalar multiplication, V is a subspace of $\mathbb{R}^4$. qed.