# MATH 1300 Problem Set: Differential Equations

26 Nov. 2012

1. Solve the system of differential equations

$$\begin{cases} y_1' = 0.5y_1 + 0.5y_2 \\ y_2' = 0.5y_1 + 0.5y_2 \end{cases}$$

with initial condition $y_1(0) = 2$, $y_2(0) = 0$.

Write the system in matrix form:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}}_{A} \mathbf{y}$$

Calculate eigenvalues and eigenvectors for $A$:

$$\lambda_1 = 0,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda_2 = 1,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So the general solution is

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

where the coefficients $c_1$, $c_2$ are determined by the initial conditions:

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \implies c_1 = c_2 = 1$$

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \implies \boxed{\begin{cases} x_1(t) = 1 + e^t \\ x_2(t) = -1 + e^t \end{cases}}$$

2. Solve the system of differential equations

$$\begin{cases} y_1' = -y_2 \\ y_2' = -y_1 \end{cases}$$

with initial condition $y_1(0) = 2$, $y_2(0) = 0$.

Write the system in matrix form:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{A} \mathbf{y}$$

Calculate eigenvalues and eigenvectors for $A$:

$$\lambda_1 = -1,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 1,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So the general solution is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

where the coefficients $c_1$, $c_2$ are determined by the initial conditions:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \implies c_1 = c_2 = 1$$

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} \implies \boxed{\begin{cases} x_1(t) = e^{-t} + e^{t} \\ x_2(t) = e^{-t} - e^{t} \end{cases}}$$

3. At time $t = 0$ there is an accidental spill of 500 kg of arsenic into Lake A, which contains $10^{12}$ litres of water. Fresh water flows into Lake A at a rate of $3 \times 10^{11}$ litres/year, and (well-mixed) water from Lake A flows into Lake B at a rate of $3 \times 10^{11}$ litres/year. There is an additional flow of fresh water into Lake B, at a rate of $10^{11}$ litres/year. Well-mixed water flows out of Lake B at $4 \times 10^{11}$ litres/year. The volume of Lake B is $2 \times 10^{12}$ litres.

   (a) Let $\mathbf{x}(t) = \big(x_1(t), x_2(t)\big)$ be the vector of quantities of arsenic (in kg) in lakes A and B, respectively. Show that the situation described above can be modeled by the following system of differential equations:

$$\begin{cases} \frac{dx_1}{dt} = -0.3x_1 \\ \frac{dx_2}{dt} = 0.3x_1 - 0.2x_2 \end{cases}$$

   (b) Solve this system of differential equations. Sketch the graphs of $x_1(t)$ and $x_2(t)$.

   (c) Determine the maximum quantity of arsenic in lake $B$, and time at which it occurs.

(b) Write the system in matrix form:

$$\frac{d\mathbf{y}}{dt} = \underbrace{\begin{bmatrix} -0.3 & 0 \\ 0.3 & -0.2 \end{bmatrix}}_{A} \mathbf{y}$$

Calculate eigenvalues and eigenvectors for $A$:

$$\lambda_1 = -0.3,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \lambda_2 = -0.2,\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

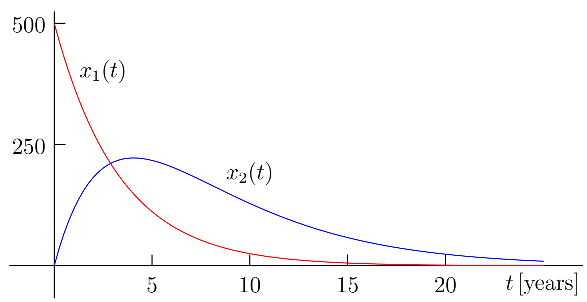
So the general solution is

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-0.3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-0.2t}$$

where the coefficients $c_1$, $c_2$ are determined by the initial conditions:

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 500 \\ 0 \end{bmatrix} \implies c_1 = 500,\ c_2 = 1500$$

$$\mathbf{x}(t) = 500 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-0.3t} + 1500 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-0.2t} \implies \boxed{\begin{cases} x_1(t) = 500e^{-0.3t} \\ x_2(t) = 1500(e^{-0.2t} - e^{-0.3t}) \end{cases}}$$



(c) At the local max of $x_2(t)$ we have:

$$\frac{dx_2}{dt} = 0 = 1500(-0.2e^{-0.2t} + 0.3e^{-0.3t}) \implies 0.2e^{-0.2t} = 0.3e^{-0.3t}$$

$$\implies \frac{0.2}{0.3} = e^{-0.1t}$$

$$\implies \boxed{t = \frac{\ln(0.2/0.3)}{-0.1} \approx 4.05\,\text{years}}$$

At this time the amount of arsenic in Lake B is

$$1500\left[\left(\frac{0.2}{0.3}\right)^2 - \left(\frac{0.2}{0.3}\right)^3\right] = \boxed{1500 \cdot \frac{4}{27} \approx 222\,\text{kg}}$$