MATH 1300 Problem Set: Differential Equations

26 Nov. 2012

1. Solve the system of differential equations

$$\begin{cases} y_1' = 0.5y_1 + 0.5y_2 \\ y_2' = 0.5y_1 + 0.5y_2 \end{cases}$$

with initial condition $y_1(0) = 2$, $y_2(0) = 0$.

2. Solve the system of differential equations

$$\begin{cases} y_1' = -y_2 \\ y_2' = -y_1 \end{cases}$$

with initial condition $y_1(0) = 2$, $y_2(0) = 0$.

3. At time $t = 0$ there is an accidental spill of 500 kg of arsenic into Lake A, which contains 10^{12} litres of water. Fresh water flows into Lake A at a rate of 3×10^{11} litres/year, and (well-mixed) water from Lake A flows into Lake B at a rate of 3×10^{11} litres/year. There is an additional flow of fresh water into Lake B, at a rate of 10^{11} litres/year. Well-mixed water flows out of Lake B at 4×10^{11} litres/year. The volume of Lake B is 2×10^{12} litres.

   (a) Let $\mathbf{x}(t) = \big(x_1(t), x_2(t)\big)$ be the vector of quantities of arsenic (in kg) in lakes A and B, respectively. Show that the situation described above can be modeled by the following system of differential equations:

$$\begin{cases} \frac{dx_1}{dt} = -0.3x_1 \\ \frac{dx_2}{dt} = 0.3x_1 - 0.2x_2 \end{cases}$$

   (b) Solve this system of differential equations. Sketch the graphs of $x_1(t)$ and $x_2(t)$.

   (c) Determine the maximum quantity of arsenic in lake B, and time at which it occurs.