# *The Calculus of Variations*

*All You Need to Know in One Easy Lesson*

Richard Taylor

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TRU Math Seminar

## Putnam 2006: Problem B5

For each continuous function $f : [0,1] \to \mathbb{R}$ let

$$I(f) = \int_0^1 \left( x^2 f(x) - x f(x)^2 \right) dx.$$

Find the maximum value of $I(f)$ over all such functions $f$.

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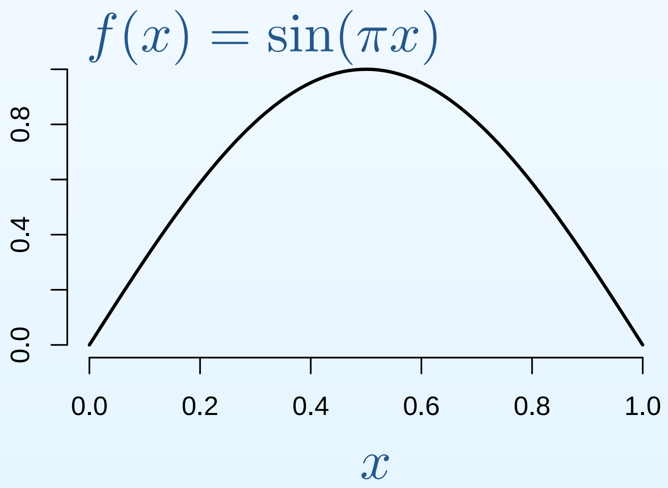


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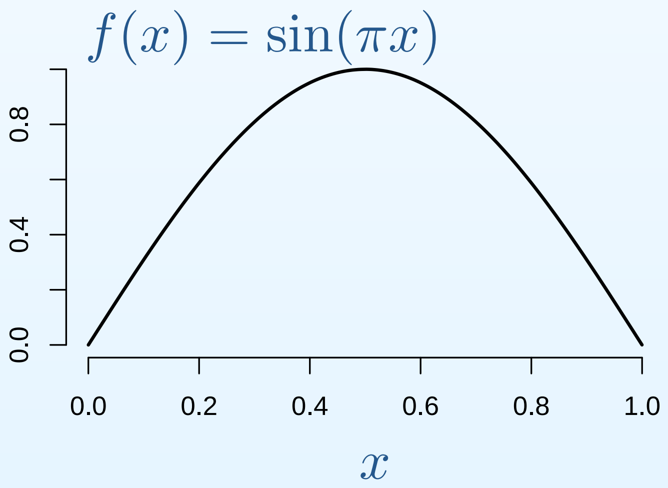


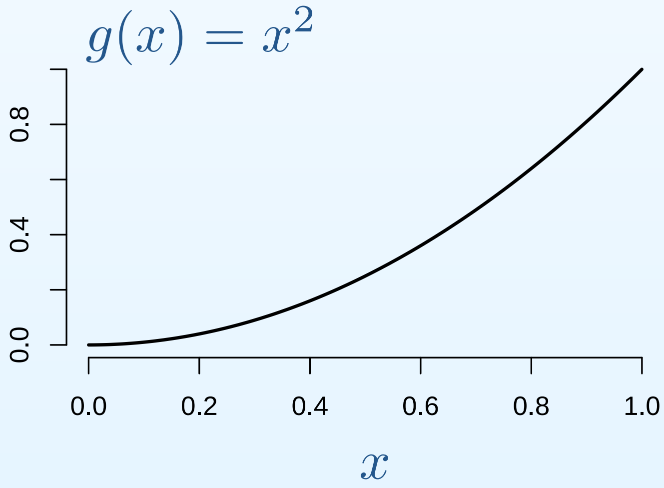


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$I(g) = \frac{1}{30} \approx 0.033$

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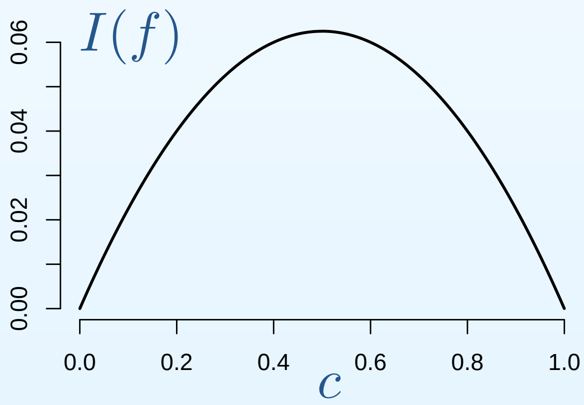
Then $I(f) = \int_0^1 \left(x^2(cx) - x(cx)^2\right) dx = \frac{1}{4}(c - c^2).$

So, restricted to this family, $I(f)$ is maximized with $c = \frac{1}{2}$.



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So the maximum of $I(f)$, achieved with $g(x) = 0$, is $\frac{1}{16}$. □

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A more reliable method uses ideas from multivariable calculus:

**Definition.** *Given a function* $f : \mathbb{R}^n \to \mathbb{R}$*, the* ***directional derivative*** *at* $\mathbf{x}$*, in the direction of a unit vector* $\mathbf{u}$*, is*

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} = \frac{d}{dh} f(\mathbf{x} + h\mathbf{u}) \Big|_{h=0}$$

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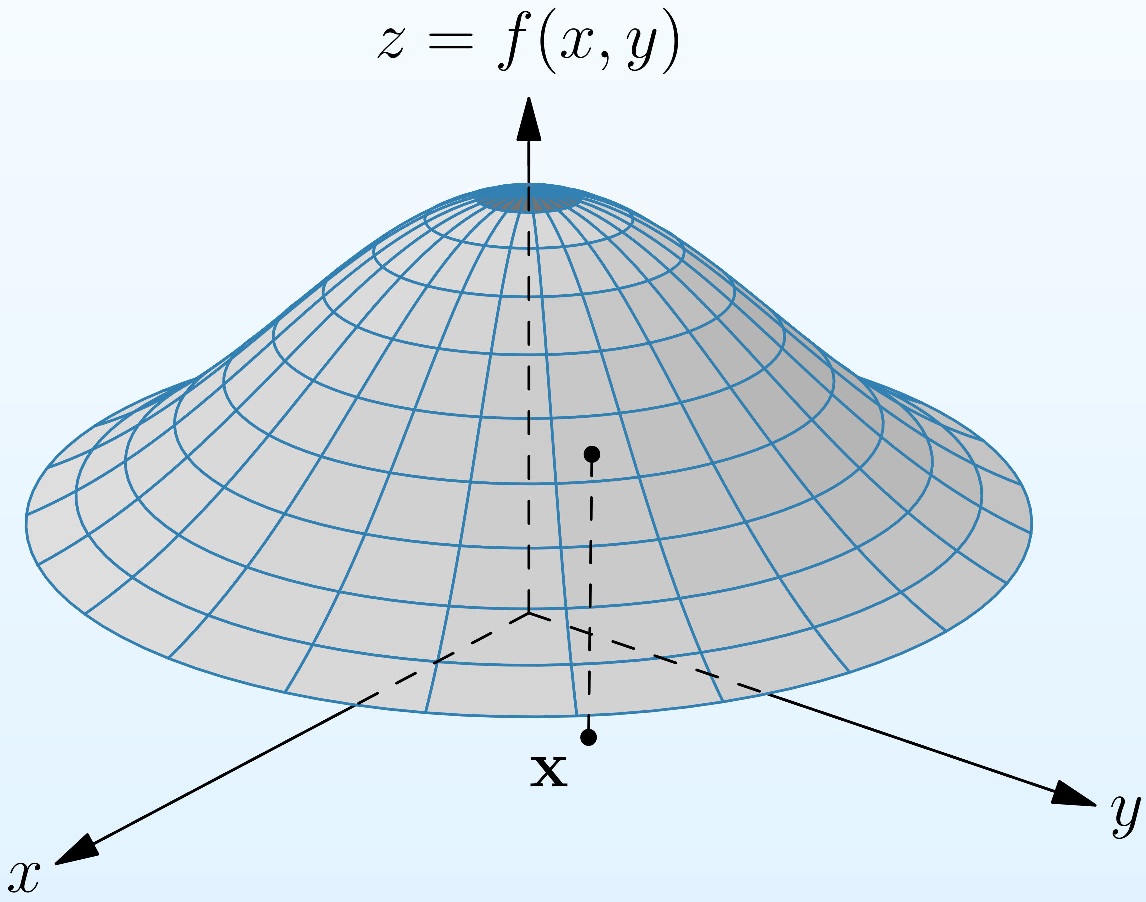
$D_{\mathbf{u}} f(\mathbf{x})$ gives the rate of change of $f(\mathbf{x})$ as we move in the direction $\mathbf{u}$ at unit speed. (e.g. rate of change of temperature along a given direction).

## Some Multivariable Calculus

For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of $y = f(\mathbf{x})$ is a surface, and $D_{\mathbf{u}} f(\mathbf{x})$ is the slope of this surface along the direction $\mathbf{u}$:



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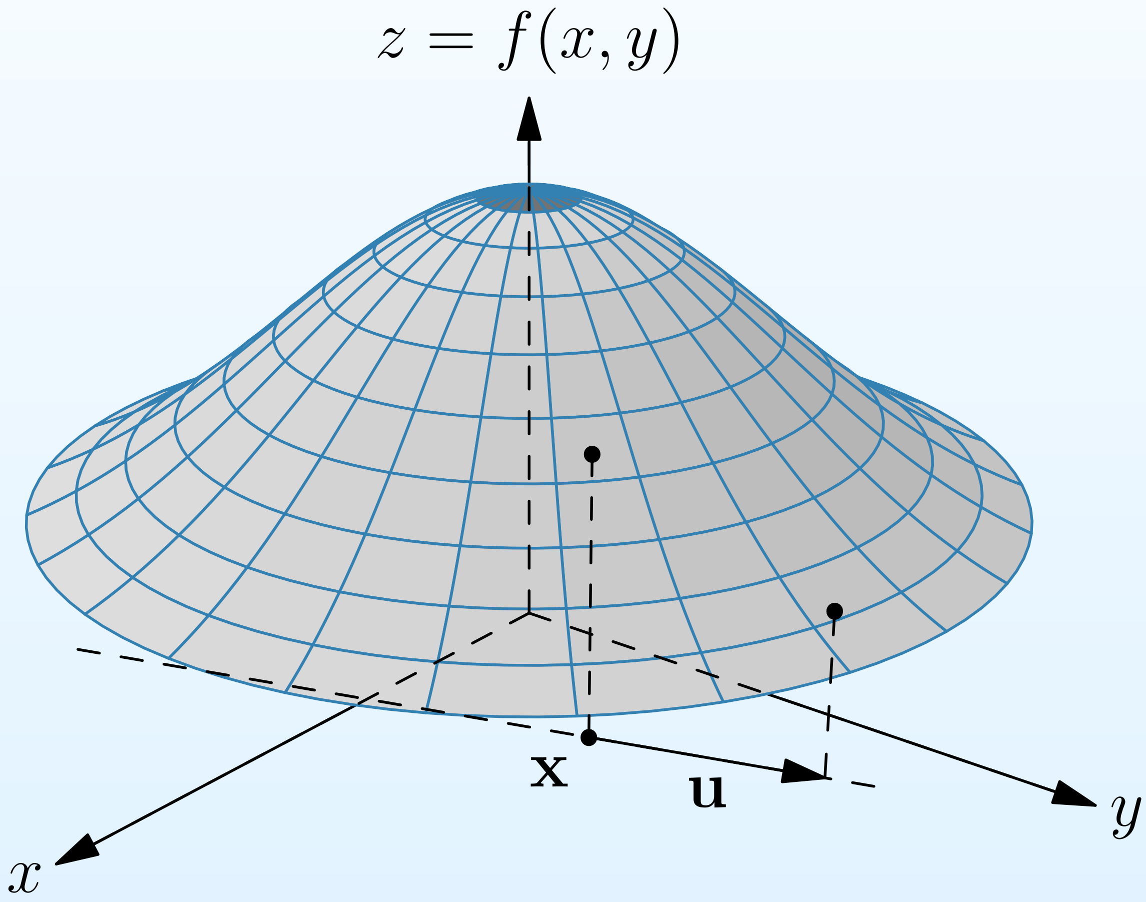

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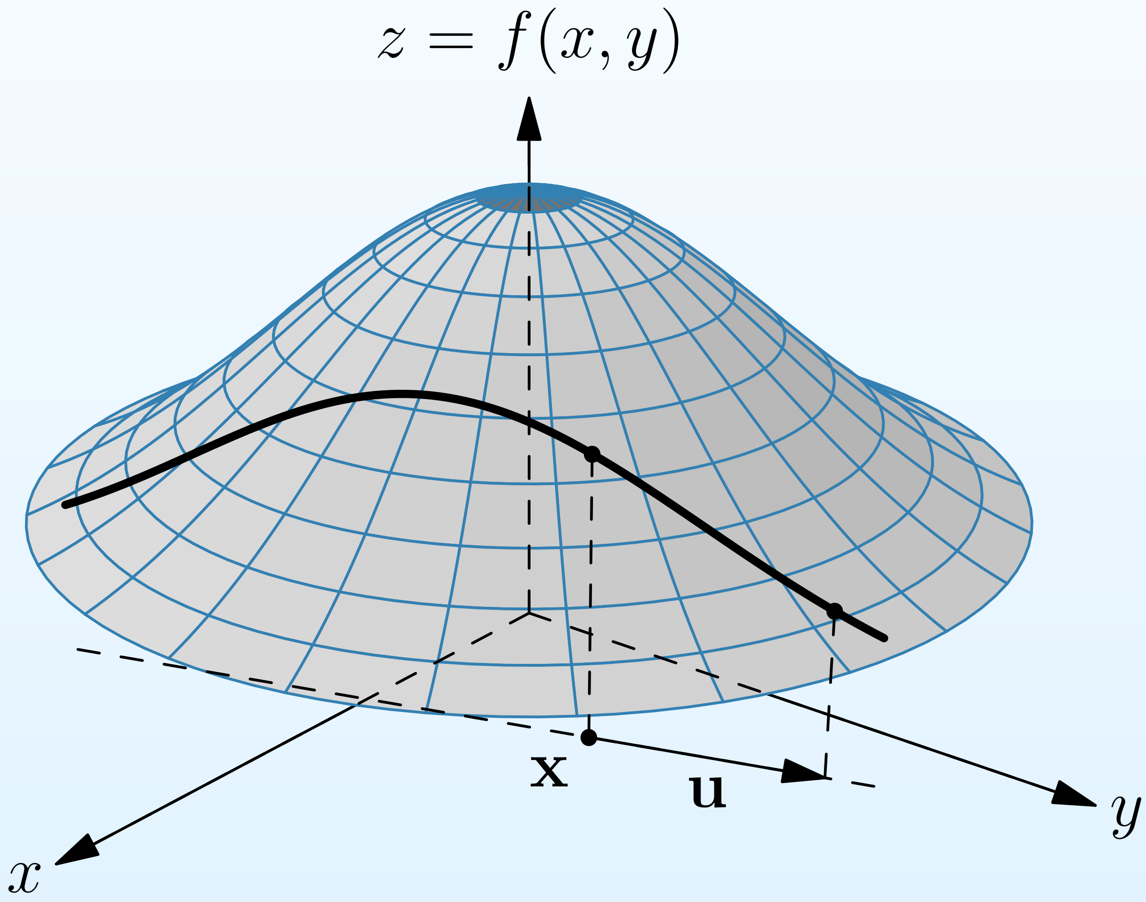

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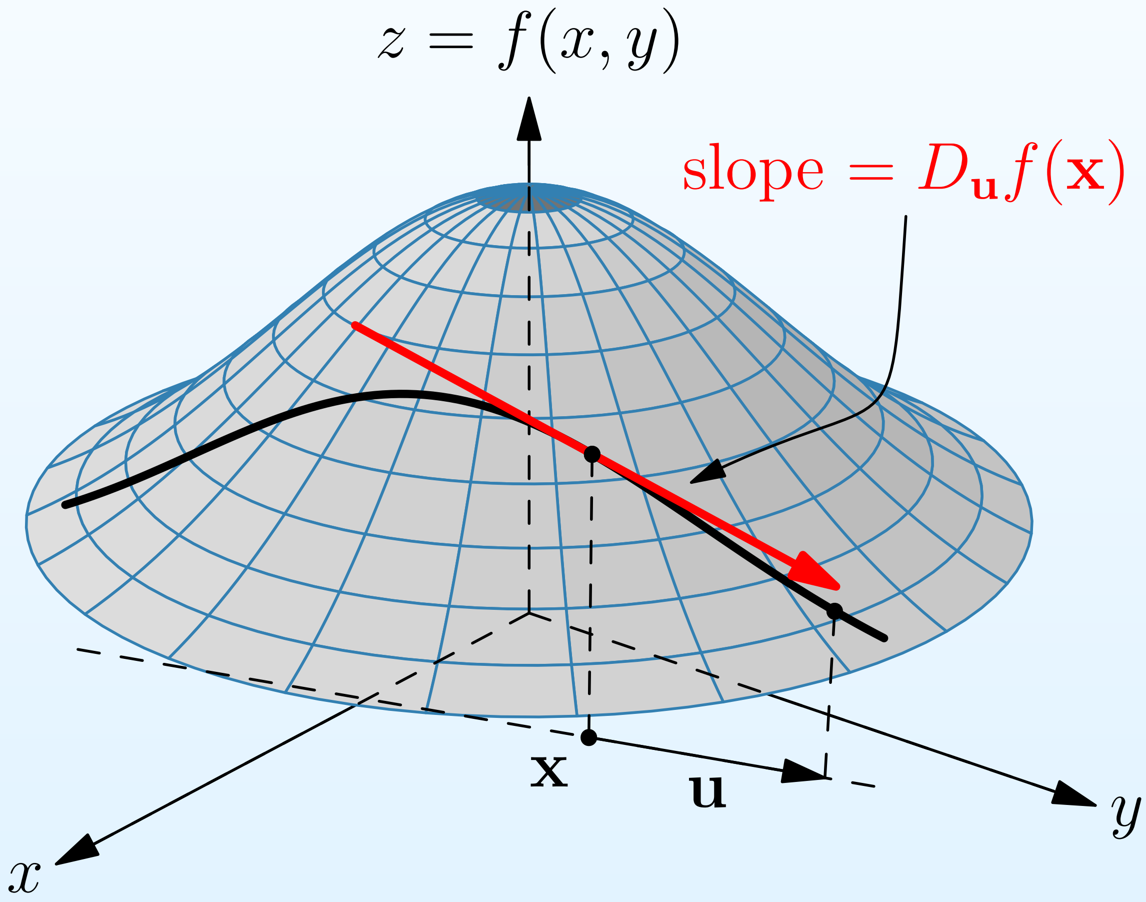

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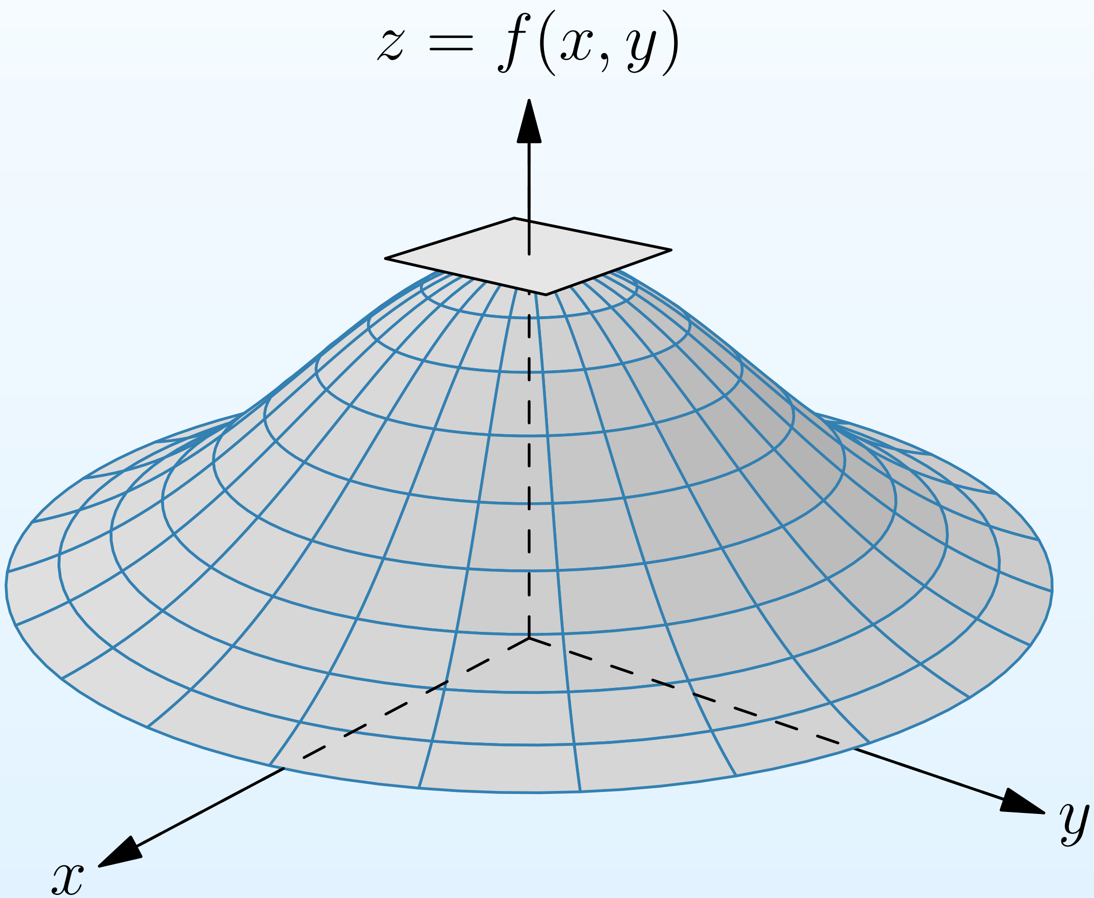
**Theorem.** *If $f : \mathbb{R} \to \mathbb{R}^n$ has a local extremum at $\mathbf{x}$, then $D_{\mathbf{u}} f(\mathbf{x}) = 0$ for every direction $\mathbf{u}$.*



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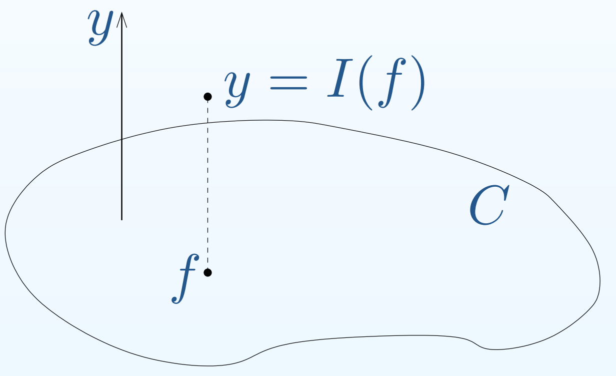

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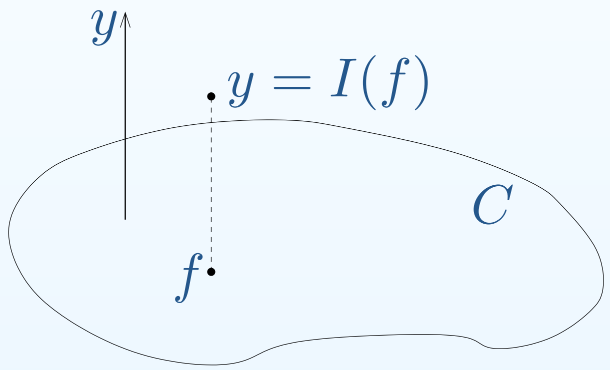


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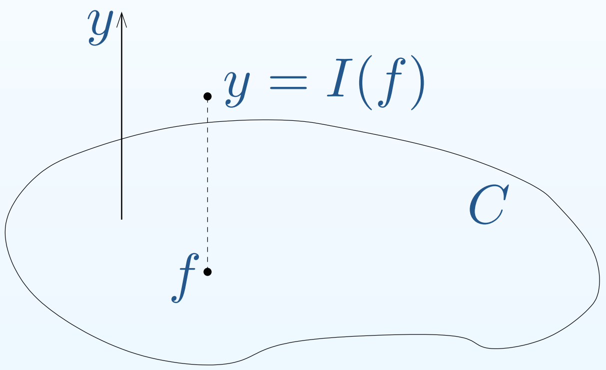


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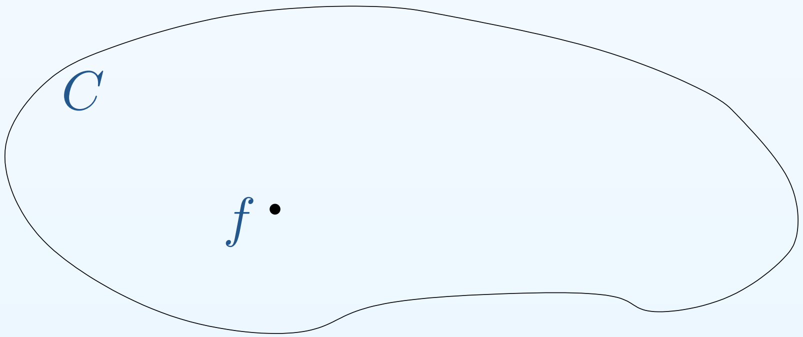
$I(f)$ depends continuously (even differentiably) on $f \in C$.

## Some Infinite-Dimensional Calculus

Directional derivative of $I(f)$ on the function space $C$:

Pick a function $f \in C$



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Directional derivative of $I(f)$ on the function space $C$:

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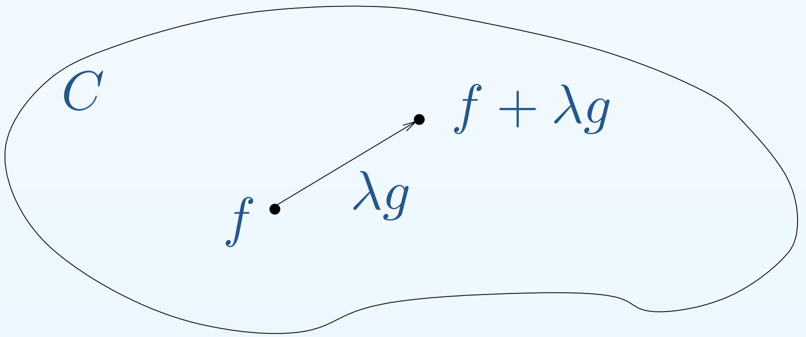

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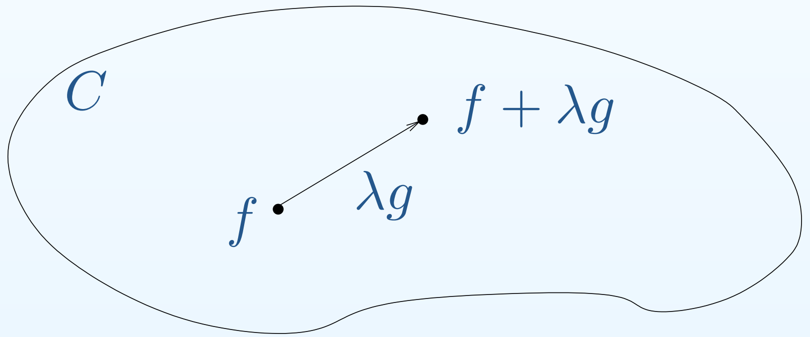

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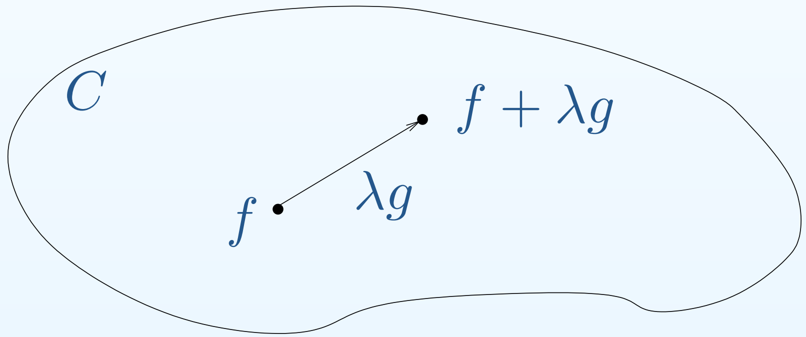


The rate of change of $I(f)$ in the $g$-direction is the directional or **Gâteaux derivative**:

$$\boxed{D_g I(f) = \lim_{\lambda \to 0} \frac{I(f + \lambda g) - I(f)}{\lambda} = \frac{d}{d\lambda} I(f + \lambda g)\Big|_{\lambda = 0}}$$

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So the directional derivative of $I(f)$ in the “direction” of the function $g$ is:

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If this derivative is to be zero for *every* direction $g$, then

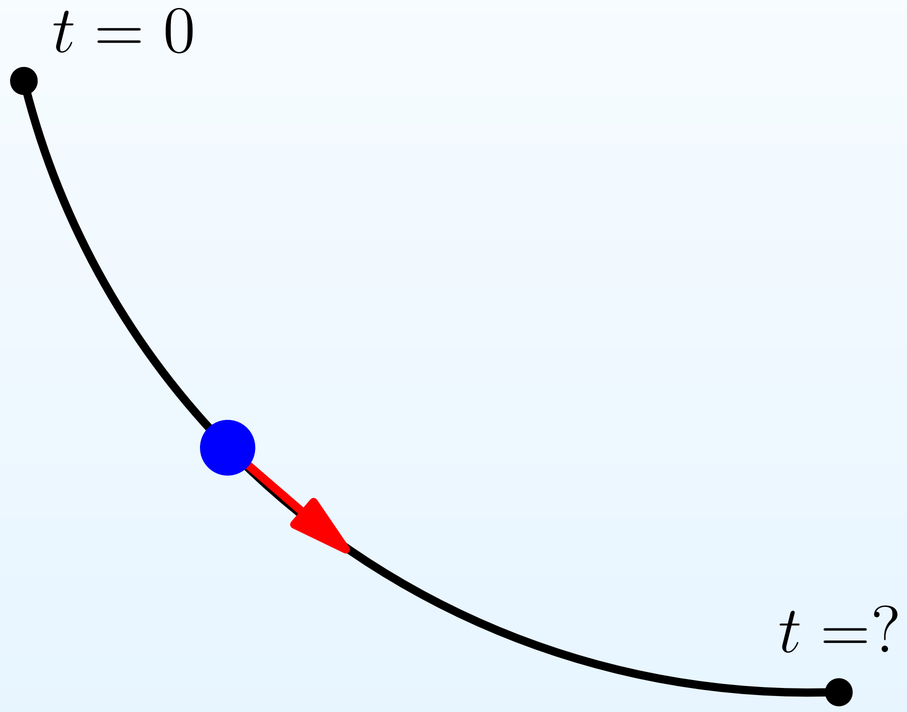
$$x^2 - 2xf(x) = 0 \implies f(x) = \frac{x}{2}$$

## PROCLAMATION:

> "Since it is known with certainty that there is scarcely anything which more greatly excites noble and ingenious spirits to labors which lead to the increase of knowledge than to propose difficult and at the same time useful problems through the solution of which, as by no other means, they may attain to fame and build for themselves eternal monuments among prosperity; so I should expect to deserve the thanks of the mathematical world if . . . I should bring before the leading analysts of this age some problem up which as upon a touchstone they could test their methods, exert their powers, and, in case they brought anything to light, could communicate with us in order that everyone might publicly receive his deserved praise from us."
>
> — Johann Bernoulli, *Acta Eruditorum*, June 1696

## Brachistochrone Problem:

*Problem*: Bead falls from rest along frictionless wire. Find shape of wire to minimize transit time.



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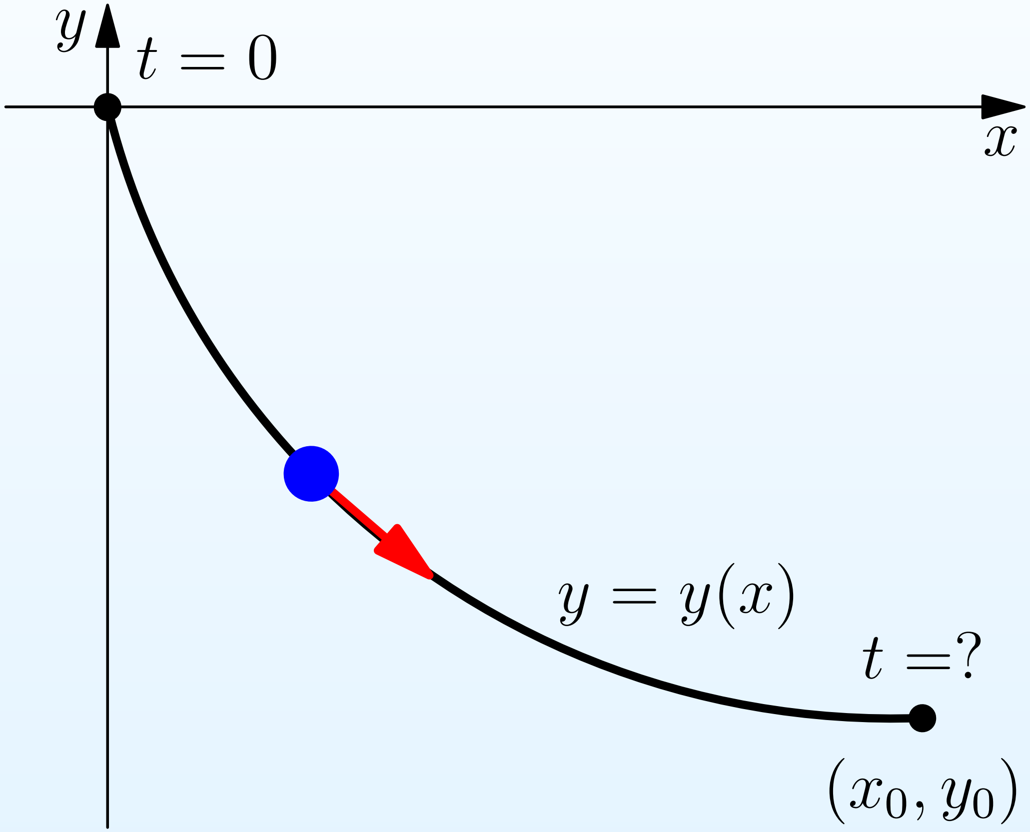

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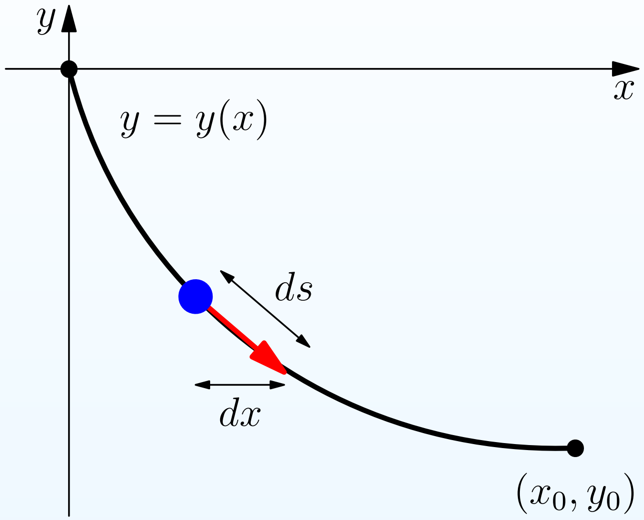


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speed: $v = \sqrt{-2gy(x)}$

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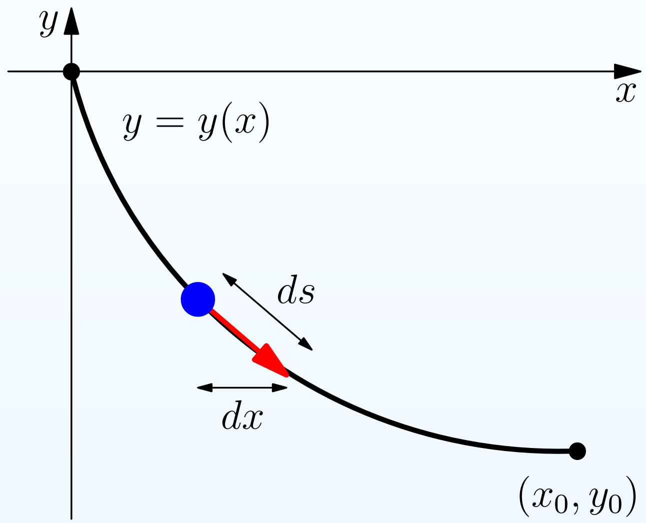


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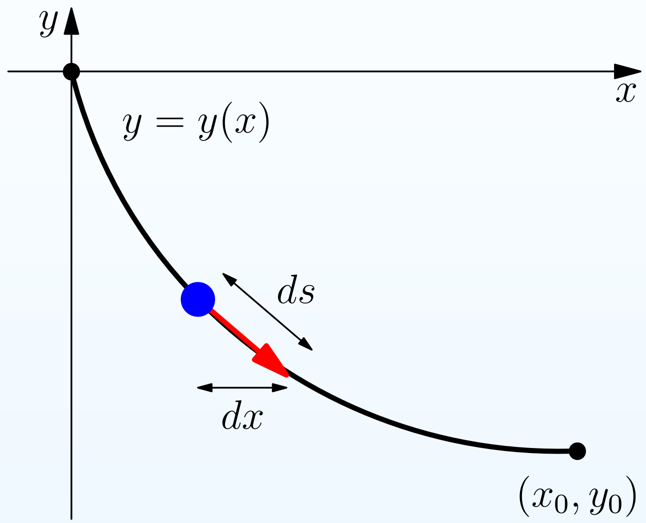


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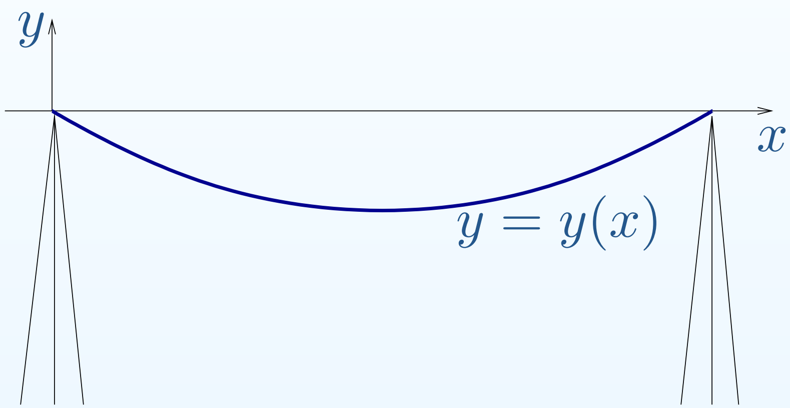

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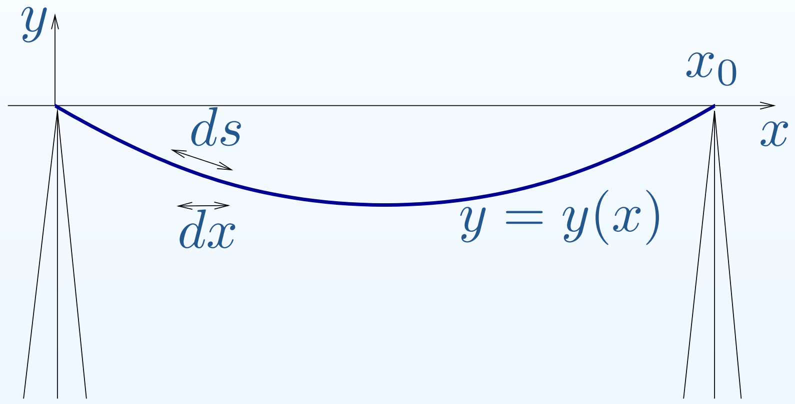


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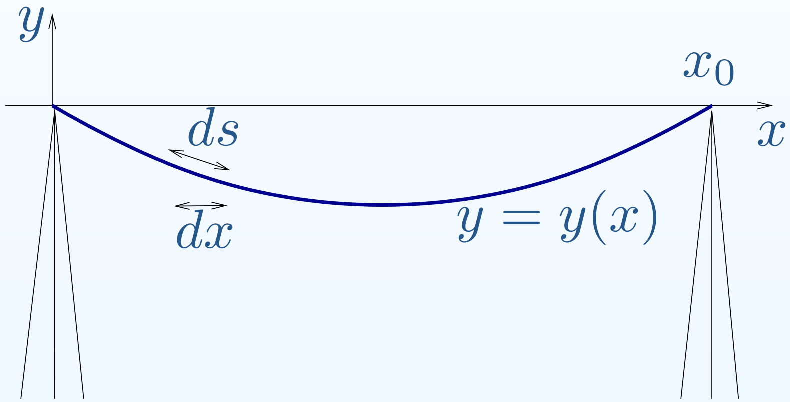


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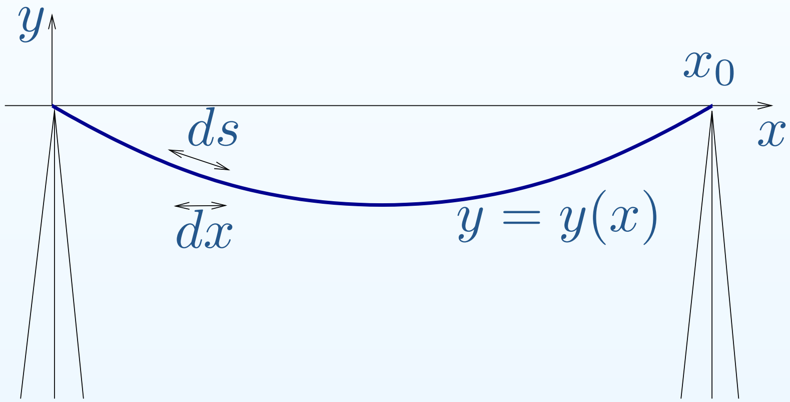


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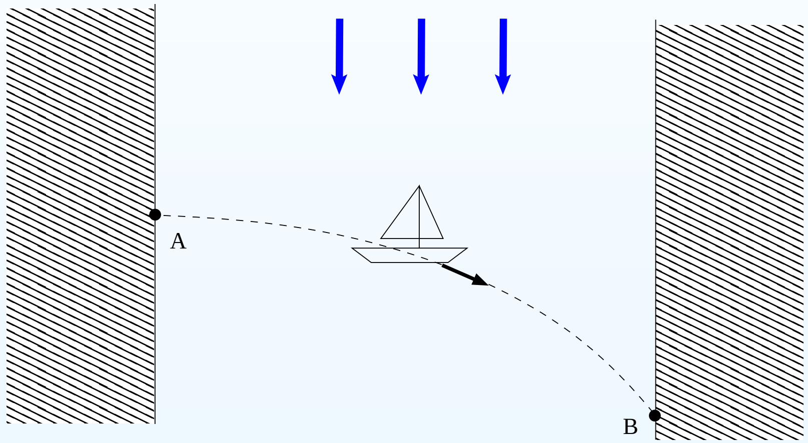
Find route to minimize transit time from $A$ to $B$.



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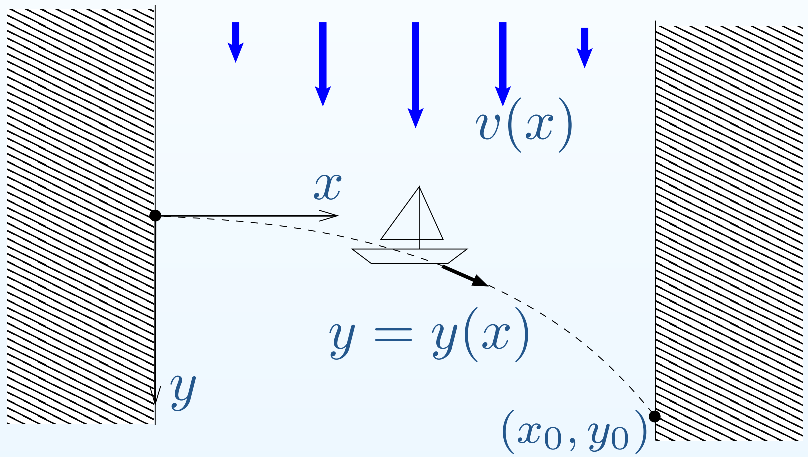
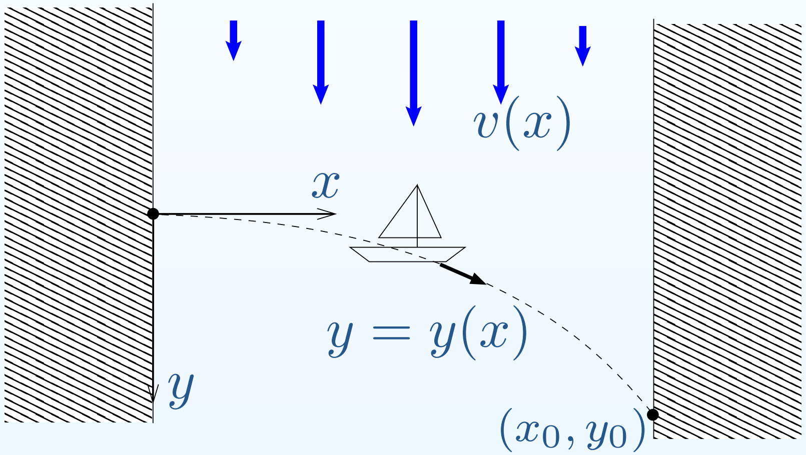


## Navigation Problem:

*Problem*: Boat cross river at constant speed relative to water. Find route to minimize transit time from $A$ to $B$.




Total transit time along route $y = y(x)$:

$$T(y) = \int_0^{x_0} \left[\alpha(x)^2\sqrt{1+\alpha(x)^2 y'(x)^2} - \left(\alpha(x)^2 v(x) y'(x)\right]\, dx$$

$$\text{where } \alpha(x) = \left(1 - v(x)^2\right)^{-1/2}$$

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The company wants to find $R(t)$ to maximize the total profit over some time interval:

$$P(R) = \int_a^b \Big[ \underbrace{p \cdot y\big(R(t)\big)}_{\text{revenue}} - \underbrace{w \cdot R(t)}_{\text{cost of resource}} - \underbrace{c\big(R'(t)\big)}_{\text{adjustment cost}} \Big]\, dt$$

(while perhaps also satisfying various constraints or targets).

# A Collection of Optimization Problems:

brachistochrone: $$T(y) = \int_0^{x_0} \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2gy(x)}}\, dx$$

hanging cable: $$P(y) = -\int_0^{x_0} y(x)\sqrt{1 + y'(x)^2}\, dx$$

river navigation: $$T(y) = \int_0^{x_0} \Big[\alpha(x)^2\sqrt{1 + \alpha(x)^2 y'(x)^2} - \big(\alpha(x)^2 v(x) y'(x)\Big]\, dx$$

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In general: $$\boxed{I(y) = \int_a^b F\big(x, y(x), y'(x)\big)\, dx}$$

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Setting $D_g I(y) = 0$ for *every* direction $g$, we get the **Euler-Lagrange** equation (ca. 1750):

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...a differential equation for the unknown function $y(x)$.

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