The undirected repetition threshold

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THE UNIVERSITY OF WINNIPEG

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PLAN

REPETITIONS

BOUNDS

A TERNARY ENCODING

CONSTRUCTIONS

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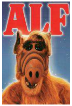

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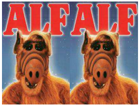

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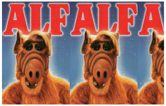

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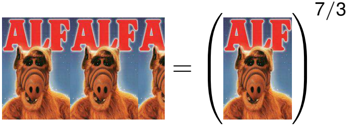

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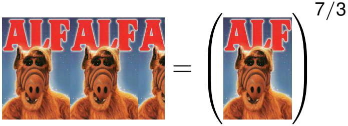

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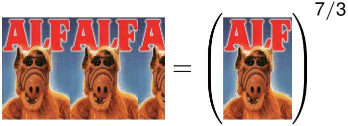

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    - We could consider different types of repetitions.

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- The *repetition threshold up to* $\sim$, denoted $\mathrm{RT}_\sim(k)$, is defined by

$$\mathrm{RT}_\sim(k) = \inf\left\{1 < \alpha \leq 2 \colon \substack{\text{there is an infinite word on } k \text{ letters} \\ \text{that is } \alpha\text{-free up to } \sim}\right\}$$

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- $\mathrm{ART}(k) := \mathrm{RT}_{\approx}(k)$ is the *Abelian repetition threshold*.
- Conjecture (Samsonov and Shur, 2012):

$$\mathrm{ART}(k) = \begin{cases} 9/5, & \text{if } k = 4; \\ (k-2)/(k-3) & \text{if } k \geq 5. \end{cases}$$

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- We refer to r-powers up to $\simeq$ as *undirected* r*-powers*, and call $\mathrm{URT}(k) := \mathrm{RT}_{\simeq}(k)$ the *undirected repetition threshold*.
- Straightforward fact: $\mathrm{RT}(k) \leq \mathrm{URT}(k) \leq ART(k)$

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Conjecture: $\mathrm{URT}(k) = \frac{k-1}{k-2}$ for all $k \geq 4$.

SOME CONTEXT

k	$\mathrm{RT}(k)$	$\mathrm{URT}(k)$	$\mathrm{ART}(k)$
2	2		
3	$7/4$	$7/4$	
4	$7/5$	$3/2$	$9/5$
5	$5/4$	$4/3$	$3/2$
6	$6/5$	$5/4$	$4/3$
7	$7/6$	$6/5$	$5/4$
8	$8/7$	$7/6$	$6/5$
9	$9/8$	$8/7$	$7/6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$k/(k-1)$	$(k-1)/(k-2)$	$(k-2)/(k-3)$

—- Confirmed —- Conjectured

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4	$7/5$	$3/2$	$9/5$
5	$5/4$	$4/3$	$3/2$
6	$6/5$	$5/4$	$4/3$
7	$7/6$	$6/5$	$5/4$
8	$8/7$	$7/6$	$6/5$
9	$9/8$	$8/7$	$7/6$
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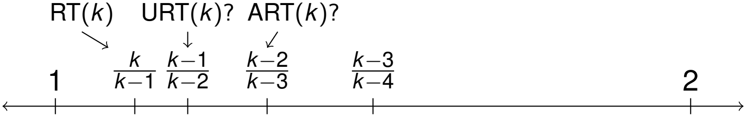
Consider α-free words on *k* letters, where α is in one of the intervals below.



RT(*k*)

URT(*k*)?

ART(*k*)?

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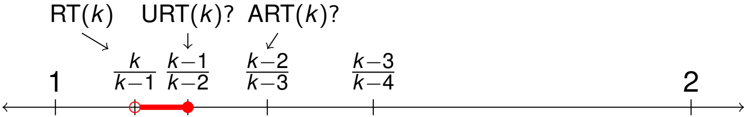

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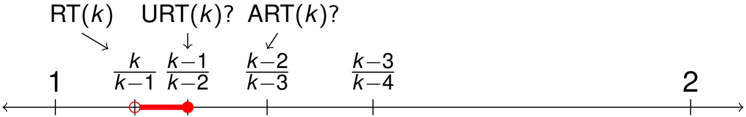

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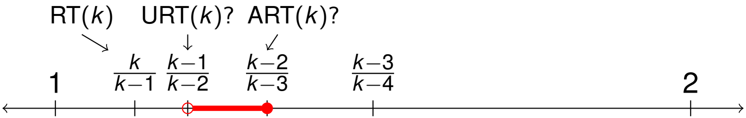

RT(k) URT(k)? ART(k)?

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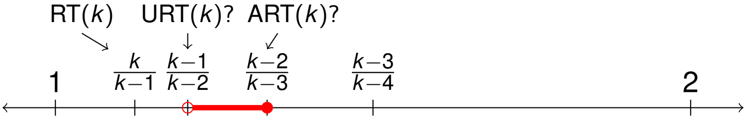

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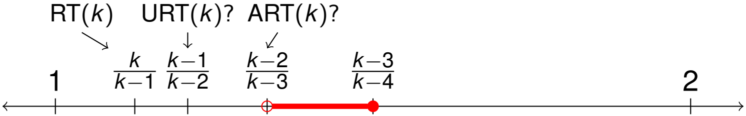

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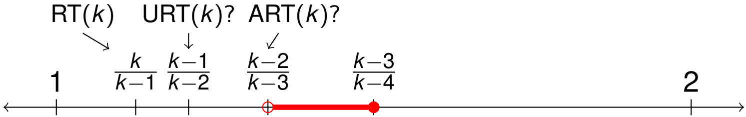
SOME CONTEXT

Consider α-free words on *k* letters, where α is in one of the intervals below.



- $\frac{k}{k-1}^{+}$ to $\frac{k-1}{k-2}$
  - After `12` $\cdots$ `(k-1)`, we have *two* options for the next letter.
- $\frac{k-1}{k-2}^{+}$ to $\frac{k-2}{k-3}$
  - After `12` $\cdots$ `(k-2)`, we have *three* options for the next letter.
- $\frac{k-2}{k-3}^{+}$ to $\frac{k-3}{k-4}$
  - After `12` $\cdots$ `(k-3)`, we have *four* options for the next letter.

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A LOWER BOUND

Proposition (Currie and Mol): For all $k \geq 4$, we have $\mathrm{URT}(k) \geq \frac{k-1}{k-2}$.

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Proof: There are only finitely many undirected $\frac{k-1}{k-2}$-free words.

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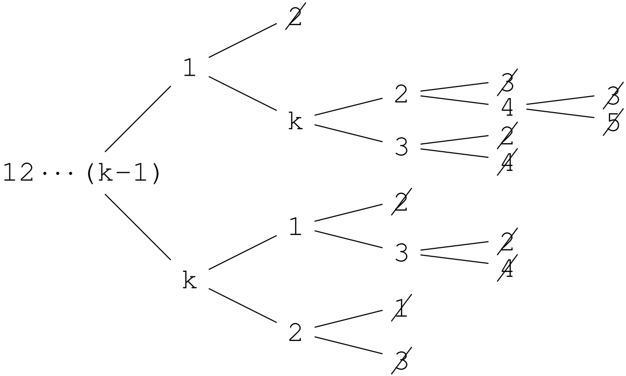

AN UPPER BOUND

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Proof:

- Take an infinite RT(k)$^+$-free word on $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$.

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Proof:

- Take an infinite RT(k)$^+$-free word on $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}\}$.
- Apply the morphism

$$\begin{aligned} \mathtt{1} &\mapsto \mathtt{1}\overline{\mathtt{1}} \\ \mathtt{2} &\mapsto \mathtt{2}\overline{\mathtt{2}} \\ &\vdots \\ \mathtt{k} &\mapsto \mathtt{k}\overline{\mathtt{k}} \end{aligned}$$

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- The only *reversible factors* in the resulting word are single letters.

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A TERNARY ENCODING

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A TERNARY ENCODING

Goal: Construct an infinite undirected $\frac{k-1}{k-2}^{+}$-free word on k letters for every $k \geq 4$.

- Idea: Use a generalized “Pansiot encoding”.
- Rank the letters by the order of their last appearance.
- Only *three* choices for the next letter – those three that appeared longest ago.

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

`123243414212324`

AN EXAMPLE

Consider the following $3/2^{+}$-free word on 4 letters.

$$\underline{\texttt{123}}\texttt{2}\texttt{43414212324}$$

Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{123}2434142123 24$$

$$3$$

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{1232}\,4\,3414212324$$

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AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$\underline{1232}4341421232 4$

31

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

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AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{\texttt{12324}}\texttt{3414212324}$$

$$\texttt{312}$$

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{123243}4\,14212324$$

312

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^{+}$-free word on 4 letters.

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$$\texttt{3123}$$

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{\texttt{12324341}}\texttt{1}\texttt{4212324}$$

$$\texttt{3123}$$

Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

$$\underline{\texttt{12324341}}\texttt{14212324}$$

$$\texttt{31231}$$

Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{pmatrix}$

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

`1232434142123 24`

`31231`

And so on…

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

123243414212324

312313123131

AN EXAMPLE

Consider the following $3/2^+$-free word on 4 letters.

```
123243414212324
   312313123131
```

- Given the prefix and the encoding, we can recover the word.

A MAP INTO THE SYMMETRIC GROUP

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$$\mathtt{1} \mapsto \begin{pmatrix} \mathtt{1} & \mathtt{2} & \mathtt{3} & \mathtt{4} & \mathtt{...} & \mathtt{k-1} & \mathtt{k} \\ \mathtt{2} & \mathtt{3} & \mathtt{4} & \mathtt{5} & \mathtt{...} & \mathtt{k} & \mathtt{1} \end{pmatrix}$$

$$\mathtt{2} \mapsto \begin{pmatrix} \mathtt{1} & \mathtt{2} & \mathtt{3} & \mathtt{4} & \mathtt{...} & \mathtt{k-1} & \mathtt{k} \\ \mathtt{1} & \mathtt{3} & \mathtt{4} & \mathtt{5} & \mathtt{...} & \mathtt{k} & \mathtt{2} \end{pmatrix}$$

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- Let's look at the encoding from the previous slide again...

AN EXAMPLE

$$\underline{\texttt{123}}\texttt{243414212324}$$

$$\phantom{\texttt{123}}\texttt{312313123131}$$

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = r_0$

AN EXAMPLE

$$\underline{1232}434142123 24$$

$$3\,1231312313 1$$

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Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = r_0\sigma(3)$

AN EXAMPLE

$$\underline{12324}3414212324$$

$$\textcolor{red}{31}2313123131$$

$$\text{Initial Ranking: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$$

$$\text{Current Ranking: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = r_0\sigma(\textcolor{red}{31})$$

AN EXAMPLE

$\underline{123243}414212324$

312313123131

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = r_0 \sigma(312)$

AN EXAMPLE

$\underline{\texttt{1232434}}\texttt{14212324}$

$\texttt{3123}\texttt{13123131}$

$$\text{Initial Ranking: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$$

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AN EXAMPLE

$$\underline{12324341}4212324$$

$$31231\,3123131$$

Initial Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} = r_0\sigma(31231)$

AN EXAMPLE

$$\underline{\texttt{123243414}}\texttt{212324}$$

$$\texttt{312313123131}$$

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = r_0\sigma(\texttt{312313})$

AN EXAMPLE

$$\underline{\texttt{1232434142}}\texttt{12324}$$

$$\texttt{312313123131}$$

Initial Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix} = r_0\sigma(\texttt{3123131})$

AN EXAMPLE

$\underline{\texttt{12324341421}}\texttt{2324}$

$\texttt{312313123131}$

$$\text{Initial Ranking: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$$

$$\text{Current Ranking: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = r_0 \sigma(\texttt{31231312})$$

AN EXAMPLE

$\underline{123243414212}324$

312313123131

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = r_0\sigma(312313123)$

AN EXAMPLE

$\underline{1232434142123}24$

3123131231 31

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

Current Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = r_0\sigma(3123131231)$

AN EXAMPLE

$$\underline{\texttt{123}}\texttt{2434142}\underline{\texttt{123}}\texttt{24}$$

$$\texttt{312313123131}$$

Initial Ranking: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} =: r_0$

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  - $f_k : \{\texttt{1},\texttt{2}\}^* \to \{\texttt{1},\texttt{2}\}^*$ is a particular uniform morphism depending on k, e.g. f_4 is defined by

$$\texttt{1} \mapsto \texttt{121}$$

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- Recall: We must avoid both ordinary r-powers (xyx) and reverse r-powers (xyx^R) for $r > \frac{k-1}{k-2}$.

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  - η_0 is short.
- The "algebraic property" of Moulin-Ollagnier ensures that π_0 is in the kernel of τ.
  - There is a permutation ϕ such that $\phi \cdot \tau(f_k(\texttt{a})) \cdot \phi^{-1} = \tau(\texttt{a})$ for all $\texttt{a} \in \{\texttt{1}, \texttt{2}\}$.

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  - One shows that the encoding v for this factor is in $\{\mathtt{31}, \mathtt{2}\}^*$.

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      for some word u.
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We conclude that $\mathrm{URT}(k) = \frac{k-1}{k-2}$ for $k \in \{4, 8, 12\}$.

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- Are these generalized Pansiot encodings useful for other problems?
- It would be nice to make some progress on the Abelian repetition threshold.
  - Conjecture (Samsonov and Shur, 2012):

$$\mathrm{ART}(k) = \begin{cases} 9/5, & \text{if } k = 4; \\ (k-2)/(k-3) & \text{if } k \geq 5. \end{cases}$$

Thank you!