The threshold dimension of a graph

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PLAN

METRIC DIMENSION

THRESHOLD DIMENSION

BOUNDS

EMBEDDINGS

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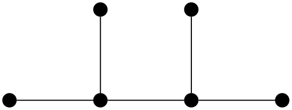

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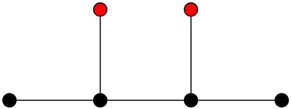

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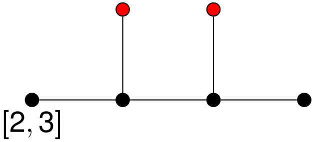

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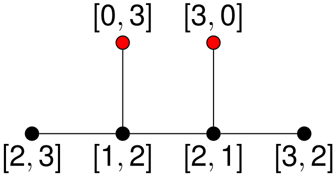

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- Applications: locating an intruder, robot navigation, etc.
- If there is a cost associated with establishing or maintaining landmark vertices, then one would be interested in finding the smallest possible resolving set.

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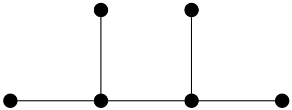

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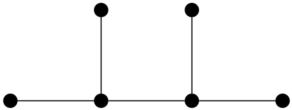

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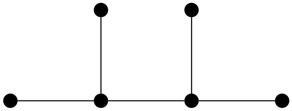

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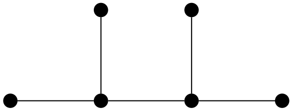

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- One checks that there is no resolving set of cardinality 1.
- Therefore, this graph has metric dimension 2.

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Fact: If G has order n, then $1 \leq \beta(G) \leq n - 1$.

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- Upper bounds in terms of diameter (Khuller et al., 1996, sharpened by Hernando et al., 2010).

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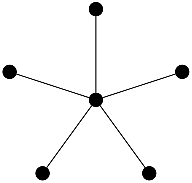
- Imagine that we can add edges to a graph cheaply (relative to landmark vertices).
- Then we would want to find the smallest resolving set across all graphs H that can be obtained from G by adding edges.

- The *threshold dimension* of G, denoted $\tau(G)$, is the size of such a smallest resolving set:

$$\tau(G) = \min\{\beta(H) : H \text{ contains } G \text{ as a spanning subgraph}\}$$

AN EXAMPLE

Consider the graph $G = K_{1,5}$.



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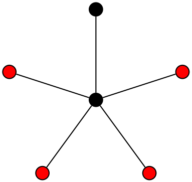

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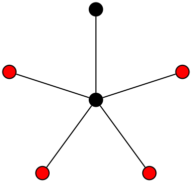

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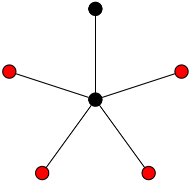

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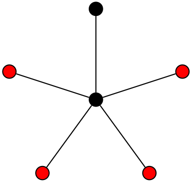

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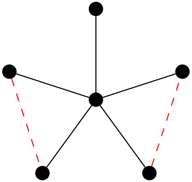

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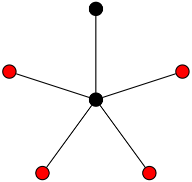

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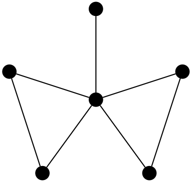

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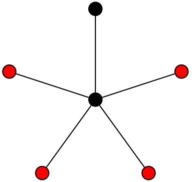

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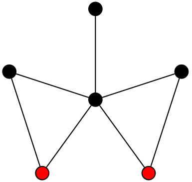

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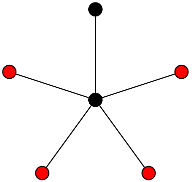

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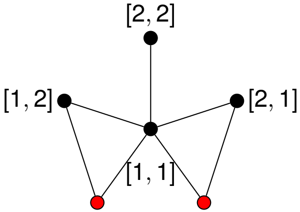

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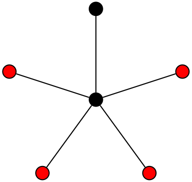

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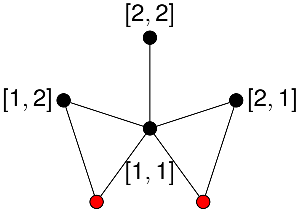

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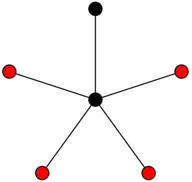

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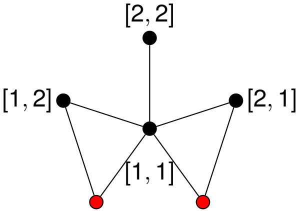

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- Attach each vertex not in W to a unique subset of W.

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- For every i, take d_{n_i} vertices from V_i – together, these will form a resolving set.
- Use ideas like we did for trees.
- Finally, show that the worst case is when the n_i's are approximately equal.

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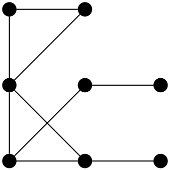

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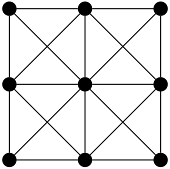

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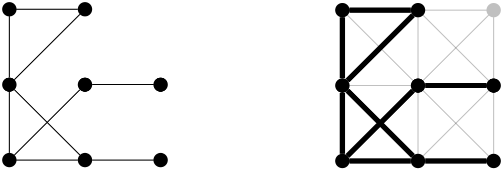

G H

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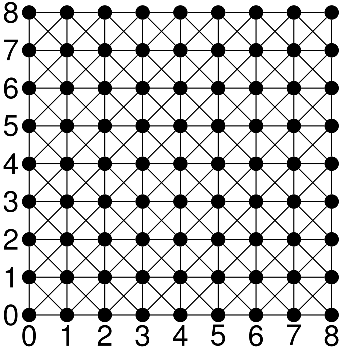

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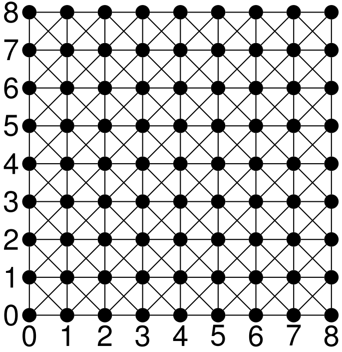

- The strong product of *b* paths is an analogous *b*-dimensional grid.

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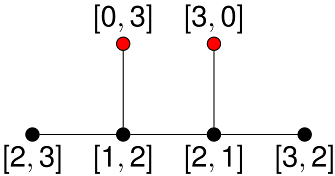

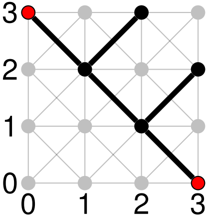

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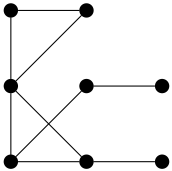
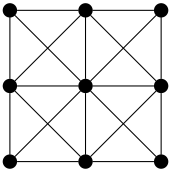


G



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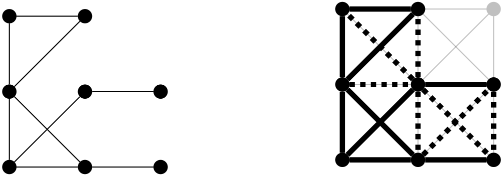

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GOOD EMBEDDINGS

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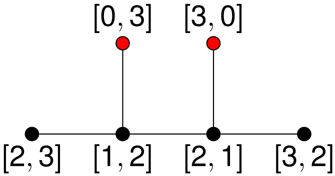
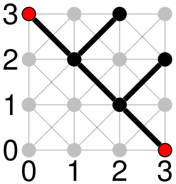
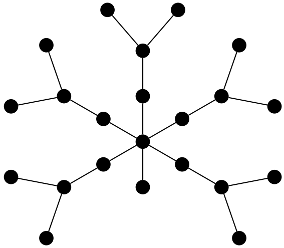
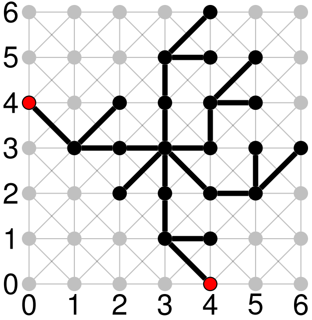




More Good Embeddings

This tree has metric dimension 5, but has a good embedding in the strong product of only 2 paths.





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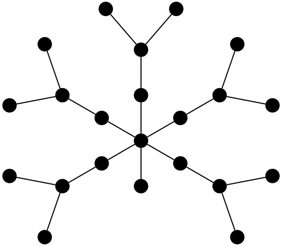

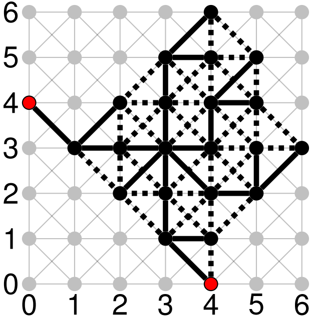

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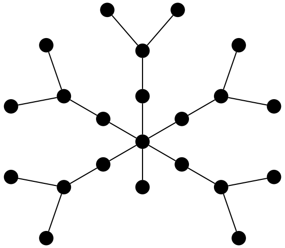

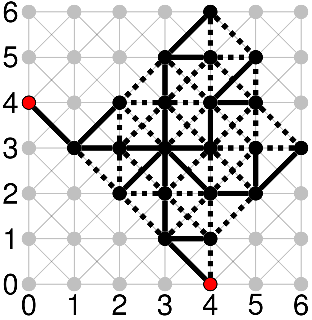

This tree has threshold dimension 2.

A GEOMETRICAL CHARACTERIZATION

Theorem (MMO 2019+): Let G be a graph. Then $\tau(G) = b$ if and only if b is the smallest number such that there is a good embedding of G in the strong product of b paths.

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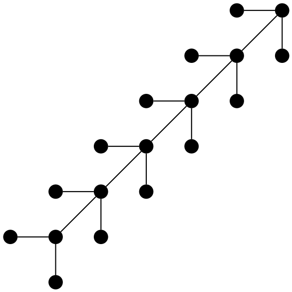

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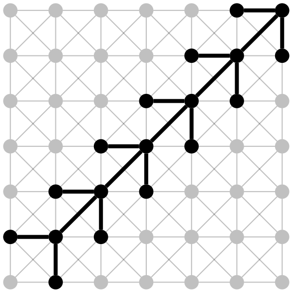

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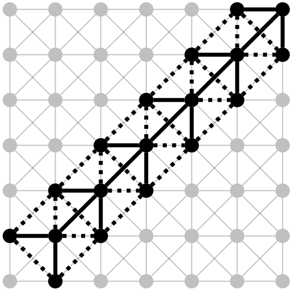

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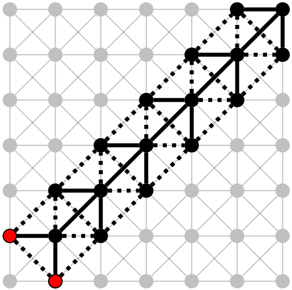

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  - The proof relies heavily on properties of trees. What can be said for general graphs?

Thank you!