The threshold dimension of a graph

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Plan

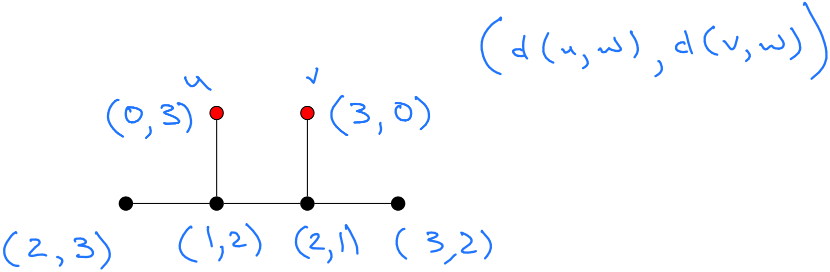
- Metric Dimension
- Threshold Dimension
- Bounds
- A Geometrical Description

Resolving Sets

Let G be a graph.

- A set $S \subseteq V(G)$ is a *resolving set* of G if every vertex of G is uniquely determined by its vector of distances to vertices in S.

Example:

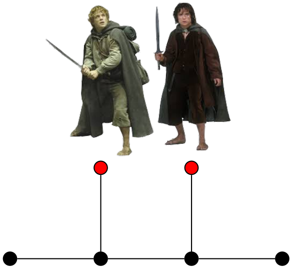


Motivation

- Place a good guy at each vertex in a resolving set.
- Give each good guy a distance-detecting device.
- Suppose a bad guy enters the graph.
- By pooling information, the good guys can determine the exact location of the bad guy!

A FANTASTIC APPLICATION

It is dark in Shelob's lair! But suppose that Sam and Frodo can tell how far away Shelob is by the brightness of their swords.



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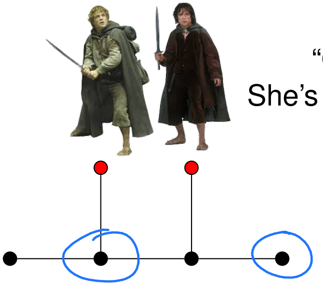

"Oh Sam!
She's two from me."

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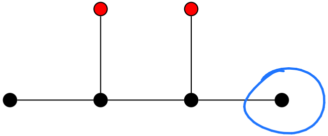
“She's distance three from me, Mr. Frodo!”

“Oh Sam! She's two from me.”



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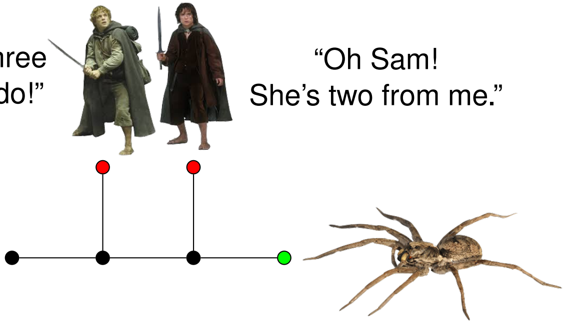

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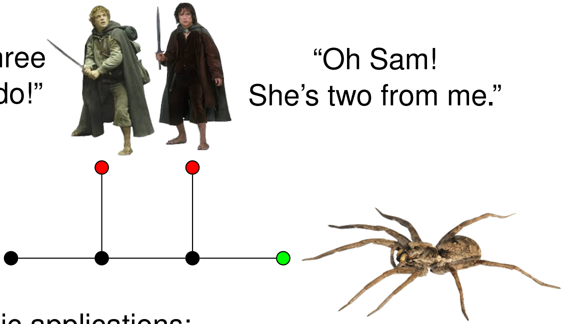


Some more realistic applications:

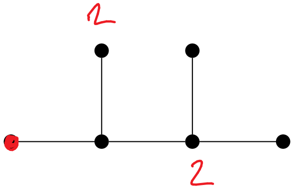
- Locating an intruder by placement of electronic devices.
- Robot navigation.

Distance-detecting devices might be expensive...

Metric Dimension

- The minimum cardinality of a resolving set of G is called the *metric dimension* of G, denoted $\beta(G)$.

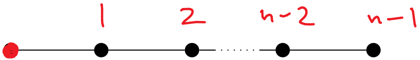
Example:



- We have already seen a resolving set of cardinality 2 in this graph.
- One checks that there is no resolving set of cardinality 1.
- Therefore, this graph has metric dimension 2.

MORE EXAMPLES

$\beta(P_n) = 1$



- This is the unique connected graph of order n and metric dimension 1.

$\beta(K_n) = n - 1$

- This is the unique connected graph of order n and metric dimension $n - 1$.

Fact: If G has order n, then $1 \leq \beta(G) \leq n - 1$.

A Brief History

- Introduced independently by Slater (1975), and Harary and Melter (1976). Both gave a linear time algorithm for calculating the metric dimension of a tree.
- Finding the metric dimension is NP-hard in general – deciding whether the metric dimension of a graph is at most a given integer is NP-complete (Garey & Johnson 1979).
- Polynomial time algorithms exist for several restricted classes of graphs:
  - outerplanar graphs (Díaz et al., 2012)
  - graphs of bounded cyclomatic number (Epstein et al., 2012)
  - etc.
- Upper bounds in terms of diameter (Khuller et al., 1996, sharpened by Hernando et al., 2010).

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Threshold Dimension

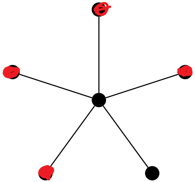
- If distance-detecting devices are expensive, we want to find a smallest possible resolving set.
- Imagine that we can add edges to a graph cheaply (relative to distance-detecting devices).
- Then we would want to find the smallest resolving set across all graphs H that can be obtained from G by adding edges.

- The *threshold dimension* of G, denoted $\tau(G)$, is the size of such a smallest resolving set:

$$\tau(G) = \min\{\beta(H) : \ H \text{ can be obtained from } G \text{ by adding edges}\}$$

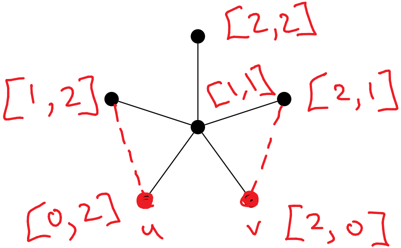
AN EXAMPLE

Consider the graph $G = K_{1,5}$.



$\beta(K_{1,5}) = 4$

Now add a couple of edges:



$\tau(K_{1,5}) = 2$

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TREES

For every positive real number n, let d_n be the smallest positive integer such that $n \leq 2^{d_n} + d_n$.

- Note that $d_n \approx \log_2(n)$.

Theorem (MMO 2020): Let T be a tree of order n. Then $\tau(T) \leq d_n$. Moreover, this bound is sharp.
Sketch of Proof:

- If $\beta(T) \leq d_n$, then we are done.
- Otherwise, it must be the case that T has at least d_n leaves.
- Take any set W of d_n leaves – this is going to be our resolving set.
- Since $n \leq 2^{d_n} + d_n$, there are at most 2^{d_n} vertices outside of W.
- Attach each vertex not in W to a unique subset of W.

A Bound in terms of the Chromatic Number

Theorem (MMO 2020): Let G be a graph of order n with chromatic number k. Then

$$\tau(G) < k(d_{n/k} + 2) \approx k(\log_2(n/k) + 2).$$

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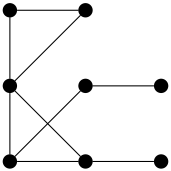
EMBEDDINGS AND STRONG PRODUCTS

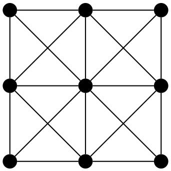
An *embedding* of G in H is an injective function $\phi : V(G) \rightarrow V(H)$ satisfying

$$xy \in E(G) \Rightarrow \phi(x)\phi(y) \in E(H).$$

In other words, an embedding of G in H is an injective homomorphism from G to H.



G



H

Embeddings and Strong Products

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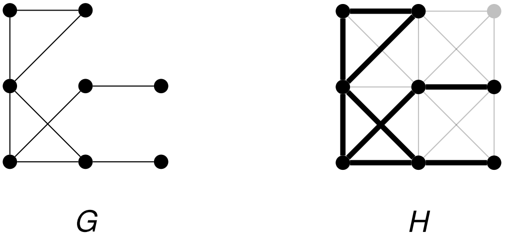
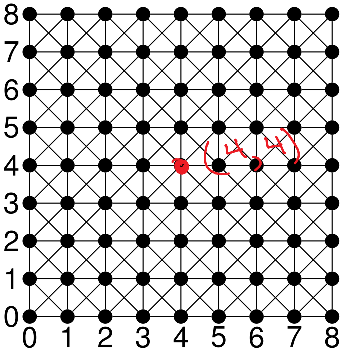


Strong Products

We will be concerned with embeddings of graphs in the strong product of a number of paths.

- The strong product of 2 paths is a 2-dimensional grid with diagonal edges in addition to horizontal and vertical ones.



- The strong product of b paths is an analogous b-dimensional grid.

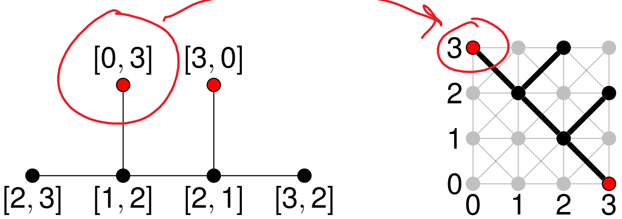
Lemma (MMO 2020): If $\beta(G) = b$, then G can be embedded in the strong product of b paths.

Idea of proof:

- Let $W = \{w_1, \ldots, w_b\}$ be a resolving set of G.
- Define ϕ by

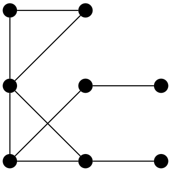
$$\phi(x) = [d(x, w_1), d(x, w_2), \ldots, d(x, w_b)].$$

Example:



NOTATION

- For an embedding ϕ of G in H, let $\phi(G)$ denote the subgraph of H induced by the set $\phi(V(G))$.



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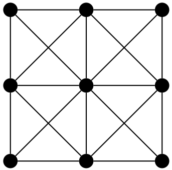

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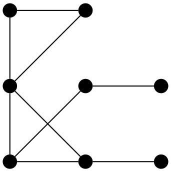

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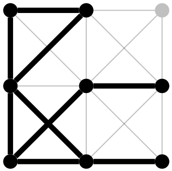

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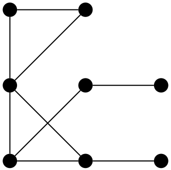

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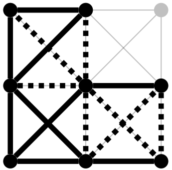
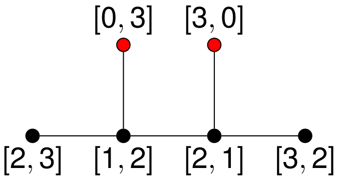
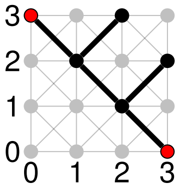


$\phi(G)$

Good Embeddings

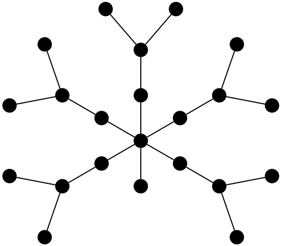
- Call an embedding ϕ of G in the strong product of b paths *good* if there is a set of vertices $W = \{w_1, \ldots, w_b\}$ such that for every vertex x of G, the coordinates of $\phi(x)$ are the distances from $\phi(x)$ to the vertices in the set $\phi(W)$ in the subgraph $\phi(G)$.
- The embeddings constructed in the previous lemma are the prototypical good embeddings.



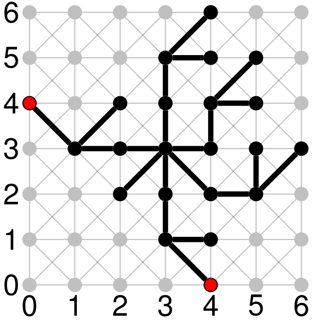


More Good Embeddings

This tree has metric dimension 5, but has a good embedding in the strong product of only 2 paths.



$\beta(T) = 5$



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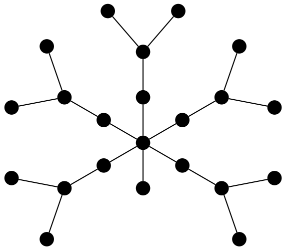
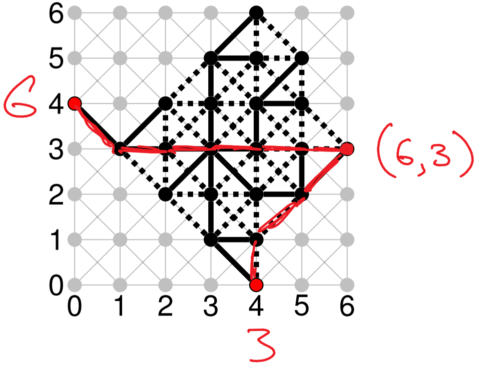




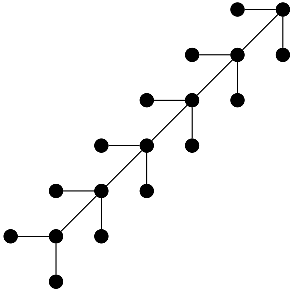
A GEOMETRICAL CHARACTERIZATION

Theorem (MMO 2020): Let G be a graph. Then $\tau(G) = b$ if and only if b is the smallest number such that there is a good embedding of G in the strong product of b paths.

- So the notion of threshold dimension corresponds in some way to our usual geometrical notion of dimension.
- We thought this was cool!
- Is it useful?

Question: Are there trees of arbitrarily large metric dimension whose threshold dimension is 2?

Answer: Yes.



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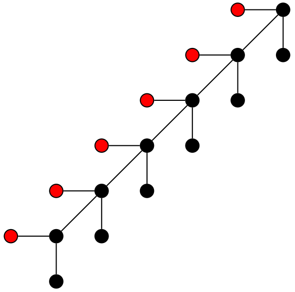

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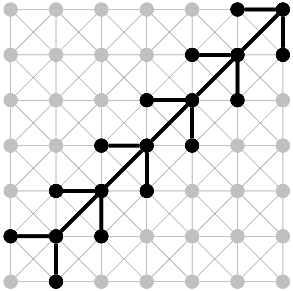

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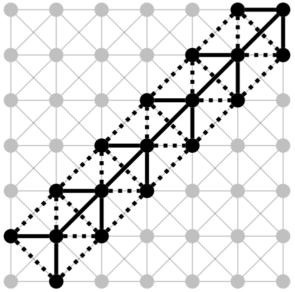

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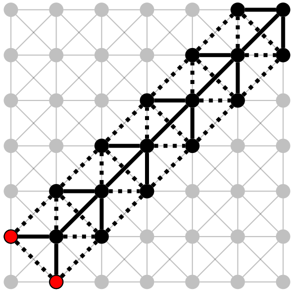

Reducible and Irreducible Graphs

- For which graphs does $\tau(G) = \beta(G)$?
  - If $\tau(G) = \beta(G)$, then we say that G is *irreducible*.
  - Otherwise, we say that G is *reducible*.

- Theorem (MMO 2020+): If T is a tree such that $\beta(T) \geq 3$, then T is reducible.

- Theorem (MMO 2020+): If T is a tree such that $\beta(T) = 4$, then $\tau(T) = 2$.

- Theorem (MMO 2020+): For every $n \geq 1$, and for every $b \in \{1, \ldots, n-1\}$, there is an irreducible graph of order n and dimension b.

FURTHER WORK

- How much difference can a single edge make?
  - Theorem (Chartrand et al., 2000): If H is obtained from a tree T by adding a single edge, then

    $$\beta(T) - 2 \leq \beta(H) \leq \beta(T) + 1.$$

  - Theorem (Mashkaria et al., 2020+): If H is obtained from a graph G by adding a single edge, then

    $$\beta(G) - 2 \leq \beta(H).$$

- The *threshold strong dimension* of a graph (Benakli et al., 2020+).
- Complexity questions are still wide open!


IT'S OVER
THANK YOU FOR YOUR ATTENTION
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References

- N. Benakli, N. H. Bong, S. M. Dueck, L. Eroh, B. Novick, and O. R. Oellermann, The threshold strong dimension of a graph, preprint, 2020. Available at https://arxiv.org/abs/2008.04282.
- S. Mashkaria, G. Ódor, and P. Thiran, On the robustness of the metric dimension to adding a single edge, preprint, 2020. Available at https://arxiv.org/abs/2010.11023.
- L. Mol, M. J. H. Murphy, and O. R. Oellermann, The threshold dimension of a graph, Discrete Appl. Math. **287** (2020), 118–133.
- L. Mol, M. J. H. Murphy, and O. R. Oellermann, The threshold dimension and irreducible graphs, Discuss. Math. Graph Theory, in press, 2020. Available at https://www.dmgt.uz.zgora.pl/publish/view_press.php?ID=38600.